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BOUNDEDNESS CHARACTERIZATION OF COMPOSITE OPERATOR WITH ORLICZ–LIPSCHITZ NORM AND ORLICZ-BMO NORM

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Abstract. In this paper, we establish the boundedness estimates for the composition of the homotopy operator T and the potential operator T_Φ on differential forms with Orlicz–Lipschitz norm and Orlicz-BMO norm which are defined by a Young function. Moreover, we derive the two-weight norm inequalities for the composite operator $T \circ T_\Phi$ using the Poincaré-type inequality with $A_r^\lambda(\Omega)$-weight. Finally, we demonstrate some applications of our main results.

1. Introduction

The main purpose of this paper is to characterize the boundedness of the composition of homotopy operator T and potential operator T_Φ on differential forms with Orlicz–Lipschitz norm and Orlicz-BMO norm, which were defined by a Young function φ in our recent work [13]. Recall that a systematic study of homotopy operator on differential forms was initiated by Iwaniec and Lutoborski in [11], where the authors showed the famous decomposition theorem for any differential form u by the homotopy operator T. Since then, homotopy operators have been playing a critical role in the theory of differential forms (see, e.g., [1]–[3], [6]–[8], and [10] for more elegant results on homotopy operators). In 2014, Wang and Xing [20] defined the convolution-type potential operator T_Φ on differential forms and proved the basic L^p-norm inequalities for T_Φ, including the

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local strong (p,p)-inequality and the Caccioppoli-type inequality. In particular, the convolution-type potential operator T_Φ is a kind of generalized operator which includes many classical operators when the kernel function Φ takes some special functions or satisfies certain conditions, such as the fractional integral operator I_α with the kernel $\Phi(t) = |t|^{\alpha-n}$, $0 < \alpha < n$, and the Bessel potential $J_{\beta,\lambda}$ with the kernel $\Phi = K_{\beta,\lambda}$ defined by its Fourier transform $\widehat{K_{\beta,\lambda}}(\xi) = (\lambda^2 + |\xi|^2)^{-\frac{\beta}{2}}$, $\beta, \lambda > 0$. (For more applications of the convolution-type potential operator in potential theory, quantum mechanics, and partial differential equations, see [17] and [18].) Recently, we introduced two new spaces in [13], called the *Orlicz–Lipschitz space* and the *Orlicz-BMO space*, which generalize the notions of the traditional Lipchitz space and BMO space by the Young function φ and give the estimates of Orlicz–Lipschitz norm and Orlicz-BMO norm for homotopy operator T. In the present article, we explore the boundedness estimates for the composition of homotopy operator T and convolution-type potential operator T_Φ with Orlicz–Lipschitz norm and Orlicz-BMO norm which are more complicated than that of the single one. We also prove the two-weight norm inequalities for the composite operator $T \circ T_\Phi$ using the Poincaré-type inequality with $A_r^\lambda(\Omega)$-weight (see [4]). It is worth pointing out that our estimates for the composite operator $T \circ T_\Phi$ provide a technique to deal with the Orlicz–Lipschitz norm and Orlicz-BMO norm estimates for other composite operators, such as the composition of homotopy T and projection operator H (see [19]), and the composition of homotopy T and Green's operator G (see [9]). Additionally, the results in this paper still hold when the convolution-type potential operator T_Φ is replaced by the fractional integral operator I_α or Bessel potential $J_{\beta,\lambda}$, which, due to the kernel function Φ, could take some functions as special cases.

Our work here is organized as follows. Section 2 introduces preliminary material including some definitions and the main lemmas. Theorems 3.2 and 3.3 in Section 3 give the estimates for the composite operator $T \circ T_\Phi$ with Orlicz–Lipschitz norm and Orlicz-BMO norm when the Young function φ belongs to the $G(p,q,c)$-class. In particular, the condition in Theorem 3.2 that a differential form u satisfies *weak reverse Hölder (WRH)* class (see [12]) is not required in Theorem 3.3. In Section 4, we first prove the Poincaré-type inequality with $A_r^\lambda(\Omega)$-weight for the composite operator $T \circ T_\Phi$ in Theorem 4.1. Based on this, the two-weight norm inequalities for the composite operator $T \circ T_\Phi$ are derived in Theorem 4.4. Finally, as applications, we give some estimates for other composite operators in Section 5 using the results and methods developed in the preceding sections.

2. Preliminaries

Before specifying the main results precisely, we introduce some notation. Let Ω be a bounded, convex domain in $\mathbb{R}^n$, $n \geq 2$, let B and σB be the balls with the same center, and let $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. We denote by $|E|$ the n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$. Let $\Lambda^l(\mathbb{R}^n) = \Lambda^l$, $l = 1, 2, \ldots, n$, be the set of all l-forms $u(x) = \sum_I u_I(x)\, dx_I = \sum u_{i_1 \cdots i_l}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_l}$ with summation over all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \leq i_1 < \cdots < i_l \leq n$. We use $D'(\Omega, \Lambda^l)$

to denote the space of all differential l-forms on Ω—namely, the coefficient of the l-forms is differential on Ω. The direct sum $\Lambda = \Lambda(\mathbb{R}^n) = \bigoplus_{l=0}^{n} \Lambda^l(\mathbb{R}^n)$ is a graded algebra with respect to the exterior products. The operator $* : \Lambda^l(\mathbb{R}^n) \to \Lambda^{n-l}(\mathbb{R}^n)$ is the Hodge star operator which is an isometric isomorphism on Λ, and the linear operator $d : D'(\Omega, \Lambda^l) \to D'(\Omega, \Lambda^{l+1})$, $0 \le l \le n-1$, is called the *exterior differential*. The Hodge codifferential operator $d^* : D'(\Omega, \Lambda^{l+1}) \to D'(\Omega, \Lambda^l)$, the formal adjoint of d, is defined by $d^* = (-1)^{nl+1} * d *$ (see [16] for more details). We will denote by $L^p(\Omega, \Lambda^l)$ the space of differential l-forms with coefficients in $L^p(\Omega, \mathbb{R}^n)$ and with norm $\|u\|_{p,\Omega} = (\int_\Omega (\sum_I |u_I(x)|^2)^{\frac{p}{2}} \, dx)^{\frac{1}{p}}$. Similarly, we denote by $W^{1,p}(\Omega, \Lambda^l) = L^p(\Omega, \Lambda^l) \cap L_1^p(\Omega, \Lambda^l)$ the Sobolev space of l-forms with norm $\|u\|_{W^{1,p}(\Omega,\Lambda^l)} = (\operatorname{diam}(\Omega))^{-1}\|u\|_{p,\Omega} + \|\nabla u\|_{p,\Omega}$. A nonnegative function w is called a *weight* if $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w > 0$ almost everywhere. Also, the norm of $u \in L^p(\Omega, \Lambda^l, w)$ is defined by $\|u\|_{p,\Omega,w} = (\int_\Omega |u|^p w(x) \, dx)^{1/p}$.

The homotopy operator $T : C^\infty(\Omega, \Lambda^l) \to C^\infty(\Omega, \Lambda^{l-1})$ is a very important operator in differential-form theory, given by

$$Tu = \int_\Omega \psi(y) K_y u \, dy,$$

where $\psi \in C_0^\infty(\Omega)$ is normalized by $\int_\Omega \psi(y) \, dy = 1$, and K_y is a linear operator defined by

$$(K_y u)(x; \xi_1, \ldots, \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + y - ty; x - y; \xi_1, \ldots, \xi_{l-1}) \, dt.$$

From [11], we have the decomposition

$$u = d(Tu) + T(du)$$

for any differential form $u \in L^p(\Omega, \Lambda^l), 1 \le p < \infty$. A closed form u_Ω is defined by $u_\Omega = d(Tu)$, $l = 1, \ldots, n$, and when u is a differential 0-form, $u_\Omega = |\Omega|^{-1} \int_\Omega u(y) \, dy$.

From [20], given a nonnegative, locally integrable function Φ, the convolution-type potential operator T_Φ is defined by a convolution integral as

$$T_\Phi u(x) = \sum_I \Big(\int_{\mathbb{R}^n} \Phi(x - y) u_I(y) \, dy \Big) \, dx_I,$$

provided that the integral exists for almost all $x \in \mathbb{R}^n$, where $u(x)$ is a differential l-form defined on $\mathbb{R}^n$ and the summation is over all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \le i_1 < \cdots < i_l \le n$. Here, the function Φ is a wide class of kernels satisfying the following weak growth condition (D). There are constants $\delta, c > 0$, and $0 \le \varepsilon < 1$ with the property that

$$\sup_{2^k < |x| < 2^{k+1}} \Phi(x) \le \frac{c}{2^{kn}} \int_{\delta(1-\varepsilon)2^k < |y| < 2\delta(1+\varepsilon)2^k} \Phi(y) \, dy$$

for all $k \in \mathbb{Z}$.

As for the weak growth condition, we refer the reader to [17] for details. When $u(x)$ is a 0-form, the operator T_Φ we study in this paper naturally degenerates into the operator discussed by Pérez in [17]. Namely, for any Lebesgue measurable function f,

$$T_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\,dy.$$

The Orlicz space $L^\varphi(\Omega)$ consists of all measurable functions f on Ω such that $\int_\Omega \varphi(\frac{|f|}{\lambda})\,dx < \infty$ for some $\lambda = \lambda(f) > 0$, and is equipped with the nonlinear Luxemburg functional

$$\|f\|_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \int_\Omega \varphi\Big(\frac{|f|}{\lambda}\Big)\,dx \le 1\Big\}.$$

A convex Orlicz function φ is often called a *Young function*. If φ is a Young function, then $\|\cdot\|_{\varphi(\Omega)}$ defines a norm in $L^\varphi(\Omega)$, which is called the *Orlicz norm* or *Luxemburg norm*. We say that the Young function φ belongs to the $G(p,q,c)$-class, $1 \le p < q < \infty, c \ge 1$, if φ satisfies that: (1) $\frac{1}{c} \le \varphi(t^{1/p})/g(t) \le c$; and (2) $\frac{1}{c} \le \varphi(t^{1/q})/h(t) \le c$, for every $t > 0$, where g is a convex increasing function and h is a concave increasing function on $[0,\infty]$. From [5], each of φ, g, and h in the above definition is doubling in the sense that its values at t and $2t$ are uniformly comparable for all $t > 0$, and the consequent fact that

$$c_1 t^q \le h^{-1}\big(\varphi(t)\big) \le c_2 t^q, \qquad c_1 t^p \le g^{-1}\big(\varphi(t)\big) \le c_2 t^p,$$

where c_1 and c_2 are constants.

In [13], the following definitions about the Orlicz–Lipschitz norm and the Orlicz-BMO norm of differential forms were given.

Definition 2.1. For $u \in L^1_{\mathrm{loc}}(\Omega, \Lambda^l), l = 0,1,\dots,n$, φ is a Young function. We write $u \in L^\varphi\text{-Lip}_{\mathrm{loc},k}(\Omega,\Lambda^l), 0 < k < 1$, if

$$\|u\|_{\varphi \mathrm{loc}\,\mathrm{Lip}_k,\Omega} = \sup_{\sigma B \subset \Omega} |B|^{\frac{-(n+k)}{n}} \|u - u_B\|_{\varphi,B} < \infty$$

for some $\sigma > 1$.

Definition 2.2. For $u \in L^1_{\mathrm{loc}}(\Omega, \Lambda^l), l = 0,1,\dots,n$, φ is a Young function. We write $u \in L^\varphi\text{-BMO}(\Omega,\Lambda^l)$ if

$$\|u\|_{\varphi *,\Omega} = \sup_{\sigma B \subset \Omega} |B|^{-1} \|u - u_B\|_{\varphi,B} < \infty$$

for some $\sigma > 1$.

The following definition of $A_r^\lambda(\Omega)$-weight comes from [4]. (For more results on $A_r^\lambda(\Omega)$-weight, see, e.g., [14], [15].)

Definition 2.3. A pair of weights (w_1, w_2) satisfies the $A_r^\lambda(\Omega)$-condition in a domain $\Omega \subset \mathbb{R}^n$, and we write $(w_1,w_2) \in A_r^\lambda(\Omega)$ for some $r > 0$ and $\lambda > 0$ if

$$\sup_B \Big(\frac{1}{|B|}\int_B w_1\,dx\Big)\Big(\frac{1}{|B|}\int_B \Big(\frac{1}{w_2}\Big)^{1/(r-1)} dx\Big)^{\lambda(r-1)} < \infty$$

for any balls $B \subset \Omega$.

The following definition for the $\mathrm{WRH}(\Lambda^l, \Omega)$-class appears in [12].

Definition 2.4. We say that $u(x) \in D'(\Omega, \Lambda^l)$ belongs to the WRH(Λ^l, Ω)-class, $l = 0, 1, \dots, n$, if there exists a constant $C > 0$ such that $u(x)$ satisfies

$$\|u\|_{s,B} \le C|B|^{\frac{t-s}{st}} \|u\|_{t,\rho B}$$

for every $0 < s, t < \infty$, where $B \subset \Omega$ with $\rho B \subset \Omega$ and $\rho > 1$ is a constant.

In order to prove our results, we need the following three lemmas which were proved by Iwaniec and Lutoborski in [11, pp. 39–42].

Lemma 2.5. *Let $u \in L^s(\Omega, \Lambda^l)$, $l = 1, 2, \dots, n$, $1 < s < \infty$, be a differential form, and let $T : L^s(\Omega, \Lambda^l) \to W^{1,s}(\Omega, \Lambda^{l-1})$ be the homotopy operator. Then we have that*

$$\|Tu\|_{s,\Omega} \le C|\Omega| \operatorname{diam}(\Omega)\|u\|_{s,\Omega}$$

holds for any bounded and convex domain Ω, where C is a constant independent of u.

Lemma 2.6. *Let $u \in D'(\Omega, \Lambda^l)$ be such that $du \in L^t(\Omega, \Lambda^{l+1})$. Then $u - u_\Omega$ is in $L^{\frac{nt}{n-t}}(\Omega, \Lambda^l)$ and*

$$\Big(\int_\Omega |u - u_\Omega|^{\frac{nt}{n-t}}\Big)^{\frac{n-t}{nt}} \le C\Big(\int_\Omega |\, du|^t\Big)^{\frac{1}{t}},$$

where $l = 1, 2, \dots, n, 1 < t < n$.

Lemma 2.7. *Let $u \in L^p(\Omega, \Lambda^l)$, $l = 1, 2, \dots, n$. Then $u_\Omega \in L^p(\Omega, \Lambda^l)$ and*

$$\|u_\Omega\|_{p,\Omega} \le C(n,p)|\Omega|\|u\|_{p,\Omega},$$

where C is a constant independent of u and $1 < p < \infty$.

The following strong (p,p)-inequality for potential operator T_Φ was given in [20].

Lemma 2.8 ([20, Corollary 2.1])**.** *Let $u \in L^p(\mathbb{R}^n, \Lambda^l)$, $l = 0, 1, \dots, n$, $1 < p < \infty$, and let T_Φ be the potential operator. We have that Φ satisfies the weak growth condition (D), and there exists a positive constant K such that*

$$\widetilde{\Phi}\big(l(Q)\big) \le K$$

for any cube Q. Then there exists a constant $C > 0$, independent of u, such that

$$\|T_\Phi u\|_{p,B} \le C\|u\|_{p,B}$$

for all balls $B \subset \mathbb{R}^n$, where $\widetilde{\Phi}(t)$ is taken as $\widetilde{\Phi}(t) = \int_{|z|\le t} \Phi(z)\, dz$ for $t > 0$.

The following lemma appears in [5].

Lemma 2.9 ([5, p. 1613])**.** *Let ψ defined on $[0, +\infty)$ be a strictly increasing, convex function, $\psi(0) = 0$, and let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that $u(x) \in D'(\Omega, \Lambda^l)$ satisfies $\psi(k(|u| + |u_\Omega|)) \in L^1(\Omega, \mu)$ for any real number $k > 0$, and let $\mu(x \in \Omega : |\mu - \mu_\Omega| > 0) > 0$, where μ is a Radon measure defined by $d\mu(x) = \omega(x)\, dx$ with a weight $\omega(x)$. Then for any $a > 0$, we obtain*

$$\int_\Omega \psi\big(a|u|\big)\, d\mu \le C \int_\Omega \psi\big(2a|u - u_\Omega|\big)\, d\mu,$$

where C is a positive constant.

3. Boundedness estimates for the composite operator

In this section, we give the boundedness estimates for the composite operator $T \circ T_\Phi$ on differential forms with the Orlicz–Lipschitz and Orlicz-BMO norms. We also establish the comparison theorems between the Orlicz–Lipschitz norm and the Orlicz-BMO norm for the composite operator. In order to prove our results, we first state the Poincaré-type inequality for $T \circ T_\Phi$.

Lemma 3.1. *Let $u \in L^s(\Omega, \Lambda^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, let T be the homotopy operator, and let T_Φ be the potential operator. Then there exists a constant C, independent of u, such that*

$$\left\| T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B \right\|_{s,B} \le C|B|^2 \operatorname{diam}(B)\|u\|_{s,B}$$

for any balls $B \subset \Omega$.

Proof. Applying the decomposition theorem for differential forms to $T(T_\Phi(u))$, we have

$$T\big(T_\Phi(u)\big) = T\,d\big(T\big(T_\Phi(u)\big)\big) + dT\big(T\big(T_\Phi(u)\big)\big). \tag{3.1}$$

Noting that $dT(T(T_\Phi(u))) = (T(T_\Phi(u)))_B$, and combining (3.1) and Lemma 2.5, we get

$$\begin{aligned} \left\| T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B \right\|_{s,B} &= \left\| Td\big(T\big(T_\Phi(u)\big)\big) \right\|_{s,B} \\ &\le C_1|B| \operatorname{diam}(B) \left\| dT\big(T_\Phi(u)\big) \right\|_{s,B}. \end{aligned}$$

Noting that $dT(T_\Phi(u)) = (T_\Phi(u))_B$, by Lemmas 2.7 and 2.8 it follows that

$$\begin{aligned} \left\| T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B \right\|_{s,B} &\le C_1|B| \operatorname{diam}(B) \left\| \big(T_\Phi(u)\big)_B \right\|_{s,B} \\ &\le C_2|B|^2 \operatorname{diam}(B) \left\| T_\Phi(u) \right\|_{s,B} \\ &\le C_3|B|^2 \operatorname{diam}(B) \|u\|_{s,B}. \end{aligned}$$ □

Now we are ready to estimate the Orlicz–Lipschitz norm of the composite operator $T \circ T_\Phi$.

Theorem 3.2. *Let φ be a Young function in the $G(p, q, c)$-class, $1 \le p < q < \infty, c \ge 1$, and let u be a differential form such that $u \in \mathrm{WRH}(\Lambda^l, \Omega)$-class, $l = 1, 2, \ldots, n$, and $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$. Assume that T is the homotopy operator and that T_Φ is the potential operator. Then there exists a constant C, independent of u, such that*

$$\left\| T\big(T_\Phi(u)\big) \right\|_{\varphi\mathrm{loc}\,\mathrm{Lip}_k,\Omega} \le C\|u\|_{\varphi,\Omega},$$

where $0 < k < 1$ is a constant and Ω is a bounded domain.

Proof. By the definition of $G(p,q,c)$-class and Jensen's inequality, we have

$$
\begin{aligned}
&\int_B \varphi\big(\big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|\big)\,dx \\
&\quad = h\Big(h^{-1}\Big(\int_B \varphi\big(\big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|\big)\,dx\Big)\Big) \\
&\quad \le h\Big(\int_B h^{-1}\big(\varphi\big(\big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|\big)\big)\,dx\Big) \\
&\quad \le h\Big(C_1\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^q\,dx\Big) \\
&\quad \le C_2\varphi\Big(\Big(C_1\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^q\,dx\Big)^{1/q}\Big) \\
&\quad \le C_3\varphi\Big(\Big(\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^q\,dx\Big)^{1/q}\Big).
\end{aligned}
\tag{3.2}
$$

Replacing s by q in Lemma 3.1, we get

$$
\begin{aligned}
&\Big(\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^q\,dx\Big)^{1/q} \\
&\qquad \le C_4(n,q)|B|^2\operatorname{diam}(B)\Big(\int_B |u|^q\,dx\Big)^{1/q}.
\end{aligned}
\tag{3.3}
$$

Since $u \in \mathrm{WRH}(\Lambda^l,\Omega)$-class, we have by Definition 2.4 the inequality

$$
\Big(\int_B |u|^q\,dx\Big)^{1/q} \le C_5|B|^{(p-q)/pq}\Big(\int_{\sigma B} |u|^p\,dx\Big)^{1/p}, \tag{3.4}
$$

where $\sigma > 1$ is a constant. Combining (3.3) and (3.4) yields that

$$
\begin{aligned}
&\Big(\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^q\,dx\Big)^{1/q} \\
&\qquad \le C_6|B|^2\operatorname{diam}(B)|B|^{(p-q)/pq}\Big(\int_{\sigma B} |u|^p\,dx\Big)^{1/p}.
\end{aligned}
$$

Taking into account the fact that $1 < p,q < \infty$, and so $1 + (p-q)/pq > 0$, we then obtain

$$
\Big(\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^q\,dx\Big)^{1/q} \le C_7|B|^{1+1/n}\Big(\int_{\sigma B} |u|^p\,dx\Big)^{1/p}. \tag{3.5}
$$

Noting that φ is an increasing function, and using Jensen's inequality, (3.5), and the definition of $G(p,q,c)$-class, we have

$$
\begin{aligned}
&\varphi\Big(\Big(\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^q\,dx\Big)^{1/q}\Big) \\
&\qquad \le \varphi\Big(C_8|B|^{1+1/n}\Big(\int_{\sigma B} |u|^p\,dx\Big)^{1/p}\Big) \\
&\qquad = \varphi\Big(\Big(C_8^p|B|^{p(1+1/n)}\int_{\sigma B} |u|^p\,dx\Big)^{1/p}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le C_9 g\Big(C_8^p|B|^{p(1+1/n)}\int_{\sigma B}|u|^p\,dx\Big)\\
&= C_9 g\Big(\int_{\sigma B} C_8^p|B|^{p(1+1/n)}|u|^p\,dx\Big)\\
&\le C_9\int_{\sigma B} g\big(C_8^p|B|^{p(1+1/n)}|u|^p\big)\,dx\\
&\le C_{10}\int_{\sigma B}\varphi\big(C_8|B|^{1+1/n}|u|\big)\,dx\\
&\le C_{11}\int_{\sigma B}\varphi\big(|B|^{1+1/n}|u|\big)\,dx. \qquad (3.6)
\end{aligned}
$$

Combining (3.2) and (3.6) yields that

$$\int_B \varphi\Big(\big|T\big(T_\Phi(u)\big)-\big(T\big(T_\Phi(u)\big)\big)_B\big|\Big)\,dx \le C_{12}\int_{\sigma B}\varphi\big(|B|^{1+1/n}|u|\big)\,dx.$$

We see that φ is doubling, so we obtain

$$\int_B \varphi\Big(\frac{|T(T_\Phi(u))-(T(T_\Phi(u)))_B|}{\lambda}\Big)\,dx \le C_{12}\int_{\sigma B}\varphi\Big(\frac{|B|^{1+1/n}|u|}{\lambda}\Big)\,dx$$

for any $\lambda > 0$. Then by the definition of the Orlicz norm, we have

$$
\begin{aligned}
\big\|T\big(T_\Phi(u)\big)-\big(T\big(T_\Phi(u)\big)\big)_B\big\|_{\varphi,B} &\le C_{12}\big\|\big(|B|^{1+1/n}u\big)\big\|_{\varphi,\sigma B}\\
&\le C_{12}|B|^{1+1/n}\|u\|_{\varphi,\sigma B}. \qquad (3.7)
\end{aligned}
$$

We see from the definition of the L^φ-Lipschitz norm and (3.7) that

$$
\begin{aligned}
&\big\|T\big(T_\Phi(u)\big)\big\|_{\varphi\text{loc Lip}_k,\Omega}\\
&\quad= \sup_{\sigma' B\subset\Omega}|B|^{\frac{-(n+k)}{n}}\big\|T\big(T_\Phi(u)\big)-\big(T\big(T_\Phi(u)\big)\big)_B\big\|_{\varphi,B}\\
&\quad\le \sup_{\sigma' B\subset\Omega}|B|^{\frac{-(n+k)}{n}}C_{12}|B|^{1+1/n}\|u\|_{\varphi,\sigma B}\\
&\quad\le \sup_{\sigma' B\subset\Omega}C_{12}|B|^{1+\frac{1}{n}+\frac{-(n+k)}{n}}\|u\|_{\varphi,\sigma B}
\end{aligned}
$$

for all balls $\sigma' B\subset\Omega$ with $\sigma' > \sigma$.

Noting that $1+\frac{1}{n}+\frac{-(n+k)}{n} > 0$ since $0 < k < 1$ and $1 < n < \infty$, we can obtain $|B|^{1+\frac{1}{n}+\frac{-(n+k)}{n}} \le |\Omega|^{1+\frac{1}{n}+\frac{-(n+k)}{n}}$ for any ball $B\subset\Omega$. Then it follows that

$$
\begin{aligned}
\big\|T\big(T_\Phi(u)\big)\big\|_{\varphi\text{loc Lip}_k,\Omega} &\le \sup_{\sigma' B\subset\Omega}C_{12}|\Omega|^{1+\frac{1}{n}+\frac{-(n+k)}{n}}\|u\|_{\varphi,\sigma B}\\
&\le C_{13}\sup_{\sigma' B\subset\Omega}\|u\|_{\varphi,\sigma B}\\
&\le C_{13}\sup_{\sigma' B\subset\Omega}\|u\|_{\varphi,\sigma' B}\\
&\le C_{14}\|u\|_{\varphi,\Omega}.
\end{aligned}
$$

We have thus completed the proof of Theorem 3.2. □

Next, we give the estimate of the Orlicz-BMO norm of the composite operator $T \circ T_\Phi$.

Theorem 3.3. *Let φ be a Young function in the $G(p,q,c)$-class, $1<p<q<\infty$, $c \geq 1, q(n-p)<np$, and let $u \in L^p(\Omega, \Lambda^l), l=1,2,\dots,n$, be a differential form such that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$. Assume that T is the homotopy operator and that T_Φ is the potential operator. Then there exists a constant C, independent of u, such that*

$$\left\|T\left(T_\Phi(u)\right)\right\|_{\varphi*,\Omega} \leq C\|u\|_{\varphi,\Omega},$$

where Ω is a bounded domain.

Proof. We first consider the case that $1<p<n$, $q(n-p)<np$ means $q<\frac{np}{n-p}$. Then by the monotonic property of the L^p-space and Lemmas 2.6, 2.7, and 2.8, we have

$$\begin{aligned}
&\left(\int_B\left|T\left(T_\Phi(u)\right)-\left(T\left(T_\Phi(u)\right)\right)_B\right|^q dx\right)^{1/q}\\
&\quad\leq |B|^{\frac{1}{q}-\frac{1}{p}+\frac{1}{n}}\left(\int_B\left|T\left(T_\Phi(u)\right)-\left(T\left(T_\Phi(u)\right)\right)_B\right|^{\frac{np}{n-p}} dx\right)^{\frac{n-p}{np}}\\
&\quad\leq C_1|B|^{\frac{1}{q}-\frac{1}{p}+\frac{1}{n}}\left(\int_B\left|\,dT\left(T_\Phi(u)\right)\right|^p dx\right)^{\frac{1}{p}}\\
&\quad= C_1|B|^{\frac{1}{q}-\frac{1}{p}+\frac{1}{n}}\left(\int_B\left|\left(T_\Phi(u)\right)_B\right|^p dx\right)^{\frac{1}{p}}\\
&\quad\leq C_2|B|^{\frac{1}{q}-\frac{1}{p}+\frac{1}{n}+1}\left(\int_B\left|T_\Phi(u)\right|^p dx\right)^{\frac{1}{p}}\\
&\quad\leq C_3|B|^{\frac{1}{q}-\frac{1}{p}+\frac{1}{n}+1}\left(\int_B |u|^p\, dx\right)^{\frac{1}{p}}.
\end{aligned}\tag{3.8}$$

Next, we consider the case that $n \leq p < q < \infty$. Taking into account that $\frac{ns}{n-s} \to \infty$, as $s \to n$, we can select s with $1<s<n$ such that $q<\frac{ns}{n-s}$. Now, by Lemmas 2.6 and 2.7 and the monotonic property of the L^p space with $s<p$, we have

$$\begin{aligned}
&\left(\int_B\left|T\left(T_\Phi(u)\right)-\left(T\left(T_\Phi(u)\right)\right)_B\right|^{\frac{ns}{n-s}} dx\right)^{\frac{n-s}{ns}}\\
&\quad\leq C_4\left(\int_B\left|dT\left(T_\Phi(u)\right)\right|^s dx\right)^{\frac{1}{s}}\\
&\quad= C_4\left(\int_B\left|\left(T_\Phi(u)\right)_B\right|^s dx\right)^{\frac{1}{s}}\\
&\quad\leq C_5|B|\left(\int_B\left|T_\Phi(u)\right|^s dx\right)^{\frac{1}{s}}\\
&\quad\leq C_5|B|^{1+\frac{1}{s}-\frac{1}{p}}\left(\int_B |u|^p\, dx\right)^{\frac{1}{p}}.
\end{aligned}\tag{3.9}$$

Applying the monotonic property of the L^p space with $q<\frac{ns}{n-s}$ and (3.9) yields

$$\Big(\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^q\, dx\Big)^{1/q}$$
$$\le |B|^{\frac{1}{q}-\frac{1}{s}+\frac{1}{n}}\Big(\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^{\frac{ns}{n-s}}\, dx\Big)^{\frac{n-s}{ns}}$$
$$\le C_6|B|^{\frac{1}{q}-\frac{1}{s}+\frac{1}{n}}|B|^{1+\frac{1}{s}-\frac{1}{p}}\Big(\int_B |u|^p\, dx\Big)^{\frac{1}{p}}$$
$$= C_6|B|^{1+\frac{1}{q}-\frac{1}{p}+\frac{1}{n}}\Big(\int_B |u|^p\, dx\Big)^{\frac{1}{p}}. \tag{3.10}$$

Noting that $\frac{1}{q}-\frac{1}{p}+\frac{1}{n}>0$, we see from (3.8) and (3.10) that

$$\Big(\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^q\, dx\Big)^{1/q} \le C_7|B|\Big(\int_B |u|^p\, dx\Big)^{\frac{1}{p}} \tag{3.11}$$

holds for all $1<p<q<\infty$ with $q(n-p)<np$. Starting with (3.11) and repeating the similar proof from (3.5) to (3.7), we get

$$\begin{aligned}\big\|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big\|_{\varphi,B} &\le C_8\big\||B|u\big\|_{\varphi,B}\\ &\le C_8|B|\|u\|_{\varphi,B}.\end{aligned} \tag{3.12}$$

By the definition of the L^φ-BMO norm and (3.12), we obtain

$$\begin{aligned}\big\|T\big(T_\Phi(u)\big)\big\|_{\varphi*,\Omega} &= \sup_{\sigma B\subset\Omega}|B|^{-1}\big\|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big\|_{\varphi,B}\\ &\le \sup_{\sigma B\subset\Omega}|B|^{-1}C_8|B|\|u\|_{\varphi,B}\\ &= \sup_{\sigma B\subset\Omega}C_8\|u\|_{\varphi,B}\\ &\le C_9\|u\|_{\varphi,\Omega}.\end{aligned}$$ □

When assuming that the Lebesgue measure $|\{x\in B: |u-u_B|>0\}|>0$, we can derive the following comparison theorems for the composite operator by Lemma 2.9 with $\psi(t)=\varphi(t)$, $\omega(x)=1$ over the ball B.

Corollary 3.4. *Let φ be a Young function in the $G(p,q,c)$-class, $1\le p<q<\infty, c\ge 1$, and let u be a differential form such that $u\in \mathrm{WRH}(\Lambda^l,\Omega)$-class, $l=1,2,\dots,n$, $|\{x\in B: |u-u_B|>0\}|>0$ (for any balls $B\subset\Omega$) and $\varphi(|u|)\in L^1_{\mathrm{loc}}(\Omega)$. Assume that T is the homotopy operator and that T_Φ is the potential operator. Then there exists a constant C, independent of u, such that*

$$\big\|T\big(T_\Phi(u)\big)\big\|_{\varphi\mathrm{loc\,Lip}_k,\Omega} \le C\|u\|_{\varphi*,\Omega},$$

where $0<k<1$ is a constant and Ω is a bounded domain.

4. The two-weight norm inequalities

In this section, we establish the comparison theorems with two-weight for the composite operator $T\circ T_\Phi$, which is based on the following Poincaré-type inequality with $A_r^\lambda(\Omega)$-weight for $T\circ T_\Phi$.

Theorem 4.1. *Let $u \in L^s(\Omega, \Lambda^l) \cap \mathrm{WRH}(\Lambda^l, \Omega)$-class, $l = 1, 2, \ldots, n$. Assume that T is the homotopy operator, that T_Φ is the potential operator, and that $(w_1(x), w_2(x)) \in A_r^\lambda(\Omega)$ for some $r > 1$, $\lambda > 0$. Then there exists a constant C, independent of u, such that*

$$\Big(\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^s w_1^\alpha \, dx\Big)^{\frac{1}{s}} \le C|B|^{1+\frac{1}{n}}\Big(\int_{\sigma B} |u|^s w_2^{\alpha\lambda}\, dx\Big)^{\frac{1}{s}}$$

for all balls B with $\sigma B \subset \Omega$, where $0 < \alpha < 1$, $\sigma > 1$, $s > \alpha\lambda(r-1)+1$.

Proof. Choosing $t = \frac{s}{1-\alpha}$ so that $1 < s < t$, and using Hölder's inequality with $\frac{1}{s} = \frac{1}{t} + \frac{t-s}{ts}$, we obtain

$$\begin{aligned}
&\Big(\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^s w_1^\alpha \, dx\Big)^{\frac{1}{s}} \\
&\quad = \Big(\int_B \big(\big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big| w_1^{\frac{\alpha}{s}}\big)^s \, dx\Big)^{\frac{1}{s}} \\
&\quad \le \Big(\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^t \, dx\Big)^{\frac{1}{t}} \Big(\int_B w_1^{\frac{\alpha t}{t-s}}\, dx\Big)^{\frac{t-s}{st}} \\
&\quad = \big\|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big\|_{t,B}\Big(\int_B w_1\, dx\Big)^{\frac{\alpha}{s}}.
\end{aligned} \tag{4.1}$$

Applying Lemma 3.1 and Definition 2.4, we have

$$\begin{aligned}
\big\|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big\|_{t,B} &\le C_1|B|^2 \operatorname{diam}(B)\|u\|_{t,B} \\
&\le C_2|B|^2 \operatorname{diam}(B)|B|^{\frac{m-t}{mt}}\|u\|_{m,\sigma B} \\
&= C_2|B|^{2+\frac{1}{n}}|B|^{\frac{m-t}{mt}}\|u\|_{m,\sigma B}
\end{aligned} \tag{4.2}$$

for all balls B with $\sigma B \subset \Omega$, $\sigma > 1$. Next, selecting $m = \frac{s}{\alpha\lambda(r-1)+1}$ so that $m > 1$ and applying Hölder's inequality with $\frac{1}{m} = \frac{1}{s} + \frac{s-m}{sm}$ gives

$$\begin{aligned}
\|u\|_{m,\sigma B} &= \Big(\int_{\sigma B} \big(|u| w_2^{\frac{\alpha\lambda}{s}} w_2^{-\frac{\alpha\lambda}{s}}\big)^m \, dx\Big)^{\frac{1}{m}} \\
&\le \Big(\int_{\sigma B} |u|^s w_2^{\alpha\lambda}\, dx\Big)^{\frac{1}{s}} \Big(\int_{\sigma B} \Big(\frac{1}{w_2}\Big)^{\frac{m\alpha\lambda}{s-m}} dx\Big)^{\frac{s-m}{sm}} \\
&= \Big(\int_{\sigma B} |u|^s w_2^{\alpha\lambda}\, dx\Big)^{\frac{1}{s}} \Big(\int_{\sigma B} \Big(\frac{1}{w_2}\Big)^{\frac{1}{r-1}} dx\Big)^{\frac{(r-1)\alpha\lambda}{s}}.
\end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3) yields that

$$\begin{aligned}
&\big\|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big\|_{t,B} \\
&\quad \le C_3|B|^{2+\frac{1}{n}}|B|^{\frac{m-t}{mt}}\Big(\int_{\sigma B} |u|^s w_2^{\alpha\lambda}\, dx\Big)^{\frac{1}{s}} \\
&\qquad \times \Big(\int_{\sigma B} \Big(\frac{1}{w_2}\Big)^{\frac{1}{r-1}} dx\Big)^{\frac{(r-1)\alpha\lambda}{s}}.
\end{aligned} \tag{4.4}$$

Substituting (4.4) into (4.1), we have

$$
\begin{aligned}
&\Big(\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^s w_1^\alpha\, dx\Big)^{\frac{1}{s}} \\
&\quad \le C_3 |B|^{2+\frac{1}{n}} |B|^{\frac{m-t}{mt}} \Big(\int_B w_1\, dx\Big)^{\frac{\alpha}{s}} \\
&\qquad \times \Big(\int_{\sigma B} \Big(\frac{1}{w_2}\Big)^{\frac{1}{r-1}} dx\Big)^{\frac{(r-1)\alpha\lambda}{s}} \Big(\int_{\sigma B} |u|^s w_2^{\alpha\lambda}\, dx\Big)^{\frac{1}{s}}. \qquad (4.5)
\end{aligned}
$$

Since $(w_1(x), w_2(x)) \in A_r^\lambda(\Omega)$, it follows that

$$
\begin{aligned}
&\Big(\int_B w_1\, dx\Big)^{\frac{\alpha}{s}} \Big(\int_{\sigma B} \Big(\frac{1}{w_2}\Big)^{\frac{1}{r-1}} dx\Big)^{\frac{(r-1)\alpha\lambda}{s}} \\
&\quad \le \Big(\int_{\sigma B} w_1\, dx\Big)^{\frac{\alpha}{s}} \Big(\int_{\sigma B} \Big(\frac{1}{w_2}\Big)^{\frac{1}{r-1}} dx\Big)^{\frac{(r-1)\alpha\lambda}{s}} \\
&\quad = \Big[\Big(\int_{\sigma B} w_1\, dx\Big)\Big(\int_{\sigma B} \Big(\frac{1}{w_2}\Big)^{\frac{1}{r-1}} dx\Big)^{\lambda(r-1)}\Big]^{\frac{\alpha}{s}} \\
&\quad = \Big[|\sigma B|^{\lambda(r-1)+1} \Big(\frac{1}{|\sigma B|}\int_{\sigma B} w_1\, dx\Big)\Big(\frac{1}{|\sigma B|}\int_{\sigma B} \Big(\frac{1}{w_2}\Big)^{\frac{1}{r-1}} dx\Big)^{\lambda(r-1)}\Big]^{\frac{\alpha}{s}} \\
&\quad \le C_4 |\sigma B|^{\frac{\alpha\lambda(r-1)}{s}+\frac{\alpha}{s}} \\
&\quad \le C_5 |B|^{\frac{\alpha\lambda(r-1)}{s}+\frac{\alpha}{s}}. \qquad (4.6)
\end{aligned}
$$

Combining (4.5) and (4.6), and noting that $\frac{m-t}{mt} = -(\frac{\alpha\lambda(r-1)}{s} + \frac{\alpha}{s})$, we get

$$
\Big(\int_B \big|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big|^s w_1^\alpha\, dx\Big)^{\frac{1}{s}} \le C_6 |B|^{2+\frac{1}{n}} \Big(\int_{\sigma B} |u|^s w_2^{\alpha\lambda}\, dx\Big)^{\frac{1}{s}}.
$$

This completes the proof of Theorem 4.1. □

By selecting $\lambda = 1$ in Theorem 4.1, we can immediately obtain the following symmetric two-weight Poincaré-type inequality for $T \circ T_\Phi$, which will be used to establish the comparison theorems with two-weight in the next theorem.

Corollary 4.2. *Let $u \in L^s(\Omega, \Lambda^l) \cap \mathrm{WRH}(\Lambda^l, \Omega)$-class, $l = 1, 2, \dots, n$. Assume that T is the homotopy operator, that T_Φ is the potential operator, and that $(w_1(x), w_2(x)) \in A_r^1(\Omega)$ for some $r > 1$. Then there exists a constant C, independent of u, such that*

$$
\big\|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big\|_{s,B,w_1^\alpha} \le C|B|^2 \operatorname{diam}(B) \|u\|_{s,\sigma B, w_2^\alpha}
$$

for all balls B with $\sigma B \subset \Omega$, where $0 < \alpha < 1$, $\sigma > 1$, $s > \alpha(r-1)+1$.

Lemma 4.3. *Let φ be a Young function such that $\varphi(x) \le x^t$ for any $x > 0$, and let $u \in L^s(\Omega, \Lambda^l), l = 1, 2, \dots, n$, be a differential form in Ω. Then for any weight ω, we have*

$$
\|u\|_{\varphi, B, \omega} \le C\|u\|_{t,\omega,B},
$$

where $1 < t < s < \infty$ and C is a constant independent of u.

Proof. The Young function $\varphi \geq 0$ gives

$$\begin{aligned}\int_B \varphi\Big(\frac{|u(x)|}{\|u(x)\|_{t,B,\omega}}\Big)\omega(x)\,dx &\leq \int_B \Big(\frac{|u(x)|}{\|u(x)\|_{t,B,\omega}}\Big)^t \omega(x)\,dx\\ &= \frac{\int_B |u(x)|^t\omega(x)\,dx}{\|u(x)\|^t_{t,B,\omega}}\\ &= 1.\end{aligned}$$

That, according to the definition of L^{φ}-norm, then implies that

$$\inf\Big\{\lambda > 0 : \int_B \varphi\Big(\frac{|u(x)|}{\lambda}\Big)\omega(x)\,dx \leq 1\Big\} \leq \big\|u(x)\big\|_{t,B,\omega}.$$

That is,

$$\|u\|_{\varphi,B,\omega} \leq \|u\|_{t,B,\omega}. \qquad \square$$

Now we are ready to state the two-weight comparison theorem using the Poincaré-type inequality derived in Corollary 4.2.

Theorem 4.4. *Let φ be a Young function such that $\varphi(x) \leq x^t$, $u \in L^s(\Omega, \Lambda^l)$ and in $\mathrm{WRH}(\Lambda^l, \Omega)$-class, $t < s < \infty$. Assume that T is the homotopy operator, that T_Φ is the potential operator, that $(w_1(x), w_2(x)) \in A^1_r(\Omega)$ for some $r > 1$ with $w_1(x) \geq \varepsilon > 0$ for any $x \in \Omega$, and that the Radon measures μ and ν are defined by $d\mu = w_1^\alpha\,dx$, $d\nu = w_2^\alpha\,dx$. Then there exist constants C_1 and C_2, independent of u, such that*

$$\big\|T\big(T_\Phi(u)\big)\big\|_{\varphi*,\Omega,w_1^\alpha} \leq C_1\big\|T\big(T_\Phi(u)\big)\big\|_{\varphi\mathrm{loc\,Lip}_k,\Omega,w_1^\alpha} \leq C_2\|u\|_{s,\Omega,w_2^\alpha}, \tag{4.7}$$

where $0 < k < 1$ and $0 < \alpha < 1$ are constants, $s > \alpha(r-1)+1$.

Proof. The first inequality in (4.7) follows directly from the definitions of the weighted L^φ-Lipschitz and L^φ-BMO norms; that is,

$$\begin{aligned}&\big\|T\big(T_\Phi(u)\big)\big\|_{\varphi*,\Omega,w_1^\alpha}\\ &\quad= \sup_{\sigma B\subset\Omega}\big(\mu(B)\big)^{-1}\big\|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big\|_{\varphi,B,w_1^\alpha}\\ &\quad= \sup_{\sigma B\subset\Omega}\big(\mu(B)\big)^{k/n}\big(\mu(B)\big)^{-(n+k)/n}\big\|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big\|_{\varphi,B,w_1^\alpha}\\ &\quad\leq C_1\sup_{\sigma B\subset\Omega}\big(\mu(B)\big)^{-(n+k)/n}\big\|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big\|_{\varphi,B,w_1^\alpha}\\ &\quad\leq C_2\big\|T\big(T_\Phi(u)\big)\big\|_{\varphi\mathrm{loc\,Lip}_k,\Omega,w_1^\alpha}.\end{aligned}$$

We now prove the second inequality in (4.7). Applying Lemma 4.3 and the monotonic property of the L^p space with $t < s < \infty$ and Corollary 4.2, we have

$$\begin{aligned}&\big\|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big\|_{\varphi,B,w_1^\alpha}\\ &\quad\leq \big\|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big\|_{t,B,w_1^\alpha}\\ &\quad\leq |B|^{\frac{1}{t}-\frac{1}{s}}\big\|T\big(T_\Phi(u)\big) - \big(T\big(T_\Phi(u)\big)\big)_B\big\|_{s,B,w_1^\alpha}\\ &\quad\leq C_3|B|^{\frac{1}{t}-\frac{1}{s}+2+\frac{1}{n}}\|u\|_{s,\sigma B,w_2^\alpha}.\end{aligned} \tag{4.8}$$

By the definition of the weighted L^φ-Lipschitz-norm and (4.8), we obtain

$$\begin{aligned}\big\|T\big(T_\Phi(u)\big)\big\|_{\varphi\text{loc Lip}_k,\Omega,w_1^\alpha} &= \sup_{\sigma B\subset\Omega}\big(\mu(B)\big)^{-\frac{n+k}{n}}\big\|T\big(T_\Phi(u)\big)-\big(T\big(T_\Phi(u)\big)\big)_B\big\|_{\varphi,B,w_1^\alpha}\\ &\le C_3\sup_{\sigma B\subset\Omega}\big(\mu(B)\big)^{-1-\frac{k}{n}}|B|^{\frac{1}{t}-\frac{1}{s}+2+\frac{1}{n}}\|u\|_{s,\sigma B,w_2^\alpha}.\end{aligned}$$

Since $\mu(B)=\int_B w_1^\alpha\,dx\ge\int_B\varepsilon^\alpha\,dx=C_4|B|$, then for all balls $B\subset\Omega$, we have

$$\frac{1}{\mu(B)}\le\frac{C_5}{|B|}. \tag{4.9}$$

According to (4.9), we have $(\mu(B))^{-1-\frac{k}{n}}\le C_2|B|^{-1-\frac{k}{n}}$, and it follows that

$$\begin{aligned}\big\|T\big(T_\Phi(u)\big)\big\|_{\varphi\text{loc Lip}_k,\Omega,w_1^\alpha} &\le C_6\sup_{\sigma B\subset\Omega}|B|^{-1-\frac{k}{n}}|B|^{\frac{1}{t}-\frac{1}{s}+2+\frac{1}{n}}\|u\|_{s,\sigma B,w_2^\alpha}\\ &= C_6\sup_{\sigma B\subset\Omega}|B|^{1+\frac{1}{t}-\frac{1}{s}+\frac{1}{n}-\frac{k}{n}}\|u\|_{s,\sigma B,w_2^\alpha}.\end{aligned}$$

Noting that $1+\frac{1}{t}-\frac{1}{s}+\frac{1}{n}-\frac{k}{n}>0$ and $|B|^{1+\frac{1}{t}-\frac{1}{s}+\frac{1}{n}-\frac{k}{n}}\le|\Omega|^{1+\frac{1}{t}-\frac{1}{s}+\frac{1}{n}-\frac{k}{n}}$, we have

$$\begin{aligned}\big\|T\big(T_\Phi(u)\big)\big\|_{\varphi\text{loc Lip}_k,\Omega,w_1^\alpha} &\le C_6\sup_{\sigma B\subset\Omega}|\Omega|^{1+\frac{1}{t}-\frac{1}{s}+\frac{1}{n}-\frac{k}{n}}\|u\|_{s,\sigma B,w_2^\alpha}\\ &\le C_7\sup_{\sigma B\subset\Omega}\|u\|_{s,\sigma B,w_2^\alpha}\\ &\le C_8\|u\|_{s,\Omega,w_2^\alpha},\end{aligned}$$

which completes the proof of Theorem 4.4. □

5. Applications

In this section, we present the estimates for some other composite operators with the Orlicz–Lipschitz norm and the Orlicz-BMO norm as applications. First, we consider the composition of homotopy operator T and Green's operator G. We will need the following lemma from [19].

Lemma 5.1 ([19, p. 2088])**.** *Let u be a smooth differential form defined in Ω and $1<s<\infty$, and let G be the Green's operator. Then there exists a positive constant $C=C(s)$, independent of u, such that*

$$\begin{aligned}&\big\|dd^*G(u)\big\|_{s,B}+\big\|d^*dG(u)\big\|_{s,B}+\big\|dG(u)\big\|_{s,B}+\big\|d^*G(u)\big\|_{s,B}+\big\|G(u)\big\|_{s,B}\\ &\quad\le C(s)\|u\|_{s,B}\end{aligned}$$

for all balls $B\subset\Omega$.

Based on Lemma 5.1 and the similar method in Lemma 3.1, we can derive the following Poincaré-type inequality for the composite operator $T\circ G$.

Lemma 5.2. *Let $u\in L^s(\Omega,\Lambda^l)$, $l=1,2,\dots,n$, $1<s<\infty$, let T be the homotopy operator, and let G be the Green's operator. Then there exists a constant C, independent of u, such that*

$$\big\|T(G)-\big(T(G)\big)_B\big\|_{s,B}\le C|B|^2\operatorname{diam}(B)\|u\|_{s,B}$$

for any balls $B\subset\Omega$.

Now, using Lemma 5.2 and the analogous technique developed in Theorems 3.2, 3.3, and 4.4, we can obtain the following results for the composite operator $T \circ G$.

Theorem 5.3. *Let φ be a Young function in the $G(p,q,c)$-class, $1 \le p < q < \infty, c \ge 1$, and let u be a differential form such that $u \in \mathrm{WRH}(\Lambda^l, \Omega)$-class, $l = 1, 2, \ldots, n$, and $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$. Assume that T is the homotopy operator and that G is the Green's operator. Then there exists a constant C, independent of u, such that*

$$\big\|T\big(G(u)\big)\big\|_{\varphi\mathrm{loc}\,\mathrm{Lip}_k,\Omega} \le C\|u\|_{\varphi,\Omega},$$

where $0 < k < 1$ is a constant and Ω is a bounded domain.

Theorem 5.4. *Let φ be a Young function in the $G(p,q,c)$-class, $1 < p < q < \infty$, $c \ge 1, q(n-p) < np$, and let $u \in L^p(\Omega, \Lambda^l), l = 1, 2, \ldots, n$, be a differential form such that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$. Assume that T is the homotopy operator and that G is the Green's operator. Then there exists a constant C, independent of u, such that*

$$\big\|T\big(G(u)\big)\big\|_{\varphi*,\Omega} \le C\|u\|_{\varphi,\Omega},$$

where Ω is a bounded domain.

Theorem 5.5. *Let φ be a Young function such that $\varphi(x) \le x^t$, and let $u \in L^s(\Omega, \Lambda^l) \cap \mathrm{WRH}(\Lambda^l, \Omega)$-class, $t < s < \infty$. Assume that T is the homotopy operator, that G is the Green's operator, that $(w_1(x), w_2(x)) \in A^1_r(\Omega)$ for some $r > 1$ with $w_1(x) \ge \varepsilon > 0$ for any $x \in \Omega$, and that the Radon measures μ and ν are defined by $d\mu = w_1^\alpha\, dx$, $d\nu = w_2^\alpha\, dx$. Then there exist constants C_1 and C_2, independent of u and du, such that*

$$\big\|T(G)\big\|_{\varphi*,\Omega,w_1^\alpha} \le C_1\big\|T(G)\big\|_{\varphi\mathrm{loc}\,\mathrm{Lip}_k,\Omega,w_1^\alpha} \le C_2\|u\|_{s,\Omega,w_2^\alpha},$$

where $0 < k < 1$, $0 < \alpha < 1$, and $s > \alpha(r-1)+1$.

Remark 5.6. Note that the main results in Theorems 3.2, 3.3, and 4.4 still hold when T_Φ is replaced by the Riesz potential operator I_α, Bessel potential $J_{\beta,\lambda}$, and Calderón–Zygmund singular integral operator on differential forms for the reason that the kernels Φ of these operators also satisfy the conditions in Lemma 2.8. It should be pointed out that the method developed in the present article could also be used to study the Orlicz–Lipschitz and Orlicz-BMO norm estimates for the composition of homotopy operator T and projection operator H (see [19]). We leave these proofs to the reader.

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