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THE TYCHONOFF THEOREM AND INVARIANT PSEUDODISTANCES

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Abstract. In this article we introduce a method of constructing functions with claimed properties by using the Tychonoff theorem. As an application of this method we show that the Carathéodory distance c_D of convex domains D in a complex, locally convex, Hausdorff, and infinite-dimensional topological vector space is approximated by the Carathéodory distances $c_{D\cap Y}$ in finite-dimensional linear subspaces Y. Originally this result is due to Dineen, Timoney, and Vigué who apply ultrafilters in their proof.

Introduction

In this article we introduce a method which allows us to avoid using ultrafilters and ultranets in some proofs. We demonstrate this method in the proof of the Dineen–Timoney–Vigué result, which states that the Carathéodory distance c_D of convex domains D in a complex, locally convex, Hausdorff, and infinite-dimensional topological vector space is approximated by the Carathéodory distances $c_{D\cap Y}$ in finite-dimensional linear subspaces Y. This general result and the Lempert theorem are the basic tools in the proof of the fact that the Carathéodory pseudodistance, the Kobayashi pseudodistance, and the Lempert function are equal on each convex domain in every complex, locally convex, and Hausdorff topological vector space. It seems that ultrafilters (or ultranets) are too sophisticated a notion to be used in the proof of the above-mentioned result. In our proof of Theorem 2.1 we simply apply the Tychonoff theorem and nets, which is a more

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accessible approach since there exist proofs of the Tychonoff theorem which do not use ultrafilters and ultranets. It is worth noting here that our method can be applied to Banach space theory, for example, to a new proof of the Hahn–Banach theorem, to a new definition of ultraproducts of Banach spaces, to constructions of new norms with claimed properties, and so on. We describe these applications in a forthcoming work.

1. Basic notions and facts

In this article we use the standard definitions of net and subnet and their properties connected with compactness, which can be found in [8]. As we mentioned in the Introduction, one of the basic tools used in this article is the Tychonoff theorem. The statement of this theorem can be found in [14], where Tychonoff mentions that the proof is the same as the one he gave for a product of bounded and closed intervals in [13].

Theorem 1.1 ([14, p. 772])**.** *Let $\{(X_j, \mathcal{T}_j)\}_{j\in\mathcal{J}}$ be a family of compact topological spaces. Then the product $\prod_{j\in\mathcal{J}} X_j$ with the product topology is compact.*

We use the standard definition of a locally bounded set of functions. Now we recall the following facts (see [2], [3]) which will be used in this article. If $(X, \mathcal{T}_X)$ is a complex, locally convex, and Hausdorff topological vector space, and if D_1 and D_2 are domains in $(X, \mathcal{T}_X)$ and $\mathbb{C}$, respectively, then we denote the set of all Gâteaux differentiable functions from D_1 to D_2 by $H_G(D_1, D_2)$. Next $f : D_1 \to D_2$ is said to be *holomorphic* if $f \in H_G(D_1, D_2)$ and f is continuous. We denote the set of all holomorphic functions from D_1 to D_2 by $H(D_1, D_2)$.

The following elementary fact is very useful. If $(X, \mathcal{T}_X)$ is a complex, locally convex, and Hausdorff topological vector space, and if D_1 and D_2 are domains in $(X, \mathcal{T}_X)$ and $\mathbb{C}$, respectively, then $f \in H_G(D_1, D_2)$ is holomorphic if and only if it is locally bounded. This result combined with the Montel theorem shows that we may use the finite-dimensional criteria to prove that a function $f : D \to \Delta$ is holomorphic, where Δ is the unit open disk in $\mathbb{C}$. Therefore, we recall the definition of the compact open topology on $H(D, \mathbb{C})$ and the Montel theorem. Assume that $(X, \mathcal{T}_X)$ is a complex, locally convex, and Hausdorff topological vector space, and that D is a domain in $(X, \mathcal{T}_X)$. The compact open topology on $H(D, \mathbb{C})$ (or the topology of uniform convergence on the compact subsets of D) is the locally convex topology generated by the seminorms $p_K(f) := \max_{x\in K} |f(x)|$, where K ranges over all compact subsets of D. We denote this topology by τ_0. The Montel theorem states that if D is a domain in $\mathbb{C}^n$ and a family $\mathcal{F} \subset H(D, \mathbb{C})$ is locally bounded, then the compact open topology τ_0 and the topology τ_p of pointwise convergence coincide on $\mathcal{F}$ and $\mathcal{F}$ is a relatively compact subset of $H(D, \mathbb{C})$. We will also apply the well-known maximum principle for a holomorphic function $f : D \to \mathbb{C}$, where $D \subset \mathbb{C}^n$ is a domain.

The definition of the Poincaré metric ρ in Δ and the definitions of the Carathéodory pseudodistance c_D, the Kobayashi pseudodistance k_D, and the Lempert function δ_D on a domain D in a complex, locally convex, and Hausdorff topological vector space $(X, \mathcal{T}_X)$ can be found in [1], [5], [9], and [12]. In particular, we have

$\rho(0,z) = \tanh^{-1}|z|$ for $z \in \Delta$. Directly from these definitions we get that if $(X_1, \mathcal{T}_{X_1})$ and $(X_2, \mathcal{T}_{X_2})$ are complex, locally convex, and Hausdorff topological vector spaces, and if D_1 and D_2 are domains in $(X_1, \mathcal{T}_{X_1})$ and $(X_2, \mathcal{T}_{X_2})$, respectively, then each holomorphic $f : D_1 \to D_2$ is nonexpansive with respect to the Carathéodory and Kobayashi pseudodistances and the Lempert function. Additionally, for a domain D in a complex, locally convex, and Hausdorff topological vector space $(X, \mathcal{T}_X)$ we also have

$$c_D \le k_D \le \delta_D.$$

Remark 1.2. In [6], Harris introduced the so-called *Schwarz–Pick systems* of pseudometrics in domains in Banach spaces which include the Carathéodory and Kobayashi pseudodistances as the smallest and the largest pseudometrics, respectively.

Remark 1.3. Observe that directly from the definitions of the Carathéodory pseudodistance, the Montel theorem, and the maximum principle we get that for a convex domain $D \subset \mathbb{C}^n$, $\tilde{w}, \tilde{z} \in D$, and $\tilde{w} \neq \tilde{z}$, there exists a holomorphic $f : D \to \Delta$ such that $f(\tilde{w}) = 0$, $f(\tilde{z}) = \sigma > 0$, and

$$c_D(\tilde{w}, \tilde{z}) = \rho(0, \sigma) = \tanh^{-1} \sigma.$$

In the case of $\mathbb{C}^n$, Lempert proved the following fundamental theorem.

Theorem 1.4 ([10, p. 259])**.** *Let $D \subset \mathbb{C}^n$ be a convex domain. Then*

$$c_D = k_D = \delta_D.$$

2. Modifications of the Dineen–Timoney–Vigué proof

In [4], Dineen, Timoney, and Vigué generalized Theorem 1.4. They showed that the Carathéodory pseudodistance, the Kobayashi pseudodistance, and the Lempert function coincide on each convex domain in every complex, locally convex, and Hausdorff topological vector space. The crucial role in their proof is played by Theorem 2.1 given below. In the original proof of Theorem 2.1, Dineen, Timoney, and Vigué used the ultrafilter technique to build a suitable holomorphic function. In our proof we present a different construction of such a function.

Theorem 2.1 ([4, Théorème 2.1])**.** *Let $(X, \mathcal{T})$ be a complex, locally convex, and Hausdorff topological vector space, and let D be a convex domain in $(X, \mathcal{T})$. We denote by $\mathcal{Y}$ the family of all finite-dimensional linear subspaces Y of X, and for $x, y \in D$ we denote by $\mathcal{Y}_{x,y}$ the family of all $Y \in \mathcal{Y}$ containing both x and y. Then for $\tilde{w}, \tilde{z} \in D$ there exists a holomorphic function $f : D \to \Delta$ such that*

$$\rho\big(f(\tilde{w}), f(\tilde{z})\big) = \inf_{Y \in \mathcal{Y}_{\tilde{w},\tilde{z}}} c_{D\cap Y}(\tilde{w}, \tilde{z}) = c_D(\tilde{w}, \tilde{z}).$$

Proof. Choose $\tilde{w}, \tilde{z} \in D$, and assume that $\tilde{w} \neq \tilde{z}$. Setting $Y_1 \le Y_2$, if $Y_1, Y_2 \in \mathcal{Y}_{\tilde{w},\tilde{z}}$ and $Y_1 \subset Y_2$, then we get a directed set $(\mathcal{Y}_{\tilde{w},\tilde{z}}, \le)$. Clearly, the set $\mathcal{Y}_{\tilde{w},\tilde{z}}$ can be treated as a net $\{\psi(Y)\}_{Y \in \mathcal{Y}_{\tilde{w},\tilde{z}}}$, where $\psi(Y) = Y$ for $Y \in \mathcal{Y}_{\tilde{w},\tilde{z}}$.

Now we can construct a holomorphic function $f : D \to \Delta$ such that $f(\tilde{w}) = 0$, $f(\tilde{z}) = \sigma > 0$, and

$$c_D(\tilde{w}, \tilde{z}) = \rho\big(f(\tilde{w}), f(\tilde{z})\big)$$

in the following way. Let $Y \in \mathcal{Y}_{\tilde{w},\tilde{z}}$. First, observe that for each $x \in D \cap Y$ we have

$$D \cap Y_{\tilde{w},x} \subset D \cap Y \subset D,$$

where $Y_{\tilde{w},x} = \operatorname{span}\{\tilde{w}, x\} \subset Y$. This implies that

$$c_D(\tilde{w}, x) \leq c_{D\cap Y}(\tilde{w}, x) \leq c_{D\cap Y_{\tilde{w},x}}(\tilde{w}, x) < \infty.$$

Next, by Remark 1.3 there exists a holomorphic function $f_{D\cap Y} : D\cap Y \to \Delta$ such that $f_{D\cap Y}(\tilde{w}) = 0$, $f_{D\cap Y}(\tilde{z}) = \sigma_Y > 0$, and

$$c_{D\cap Y}(\tilde{w}, \tilde{z}) = \rho\big(f_{D\cap Y}(\tilde{w}), f_{D\cap Y}(\tilde{z})\big) = \rho\big(0, f_{D\cap Y}(\tilde{z})\big).$$

Hence the function $f_Y : D \to \Delta$ given by

$$f_Y(x) = \begin{cases} f_{D\cap Y}(x) & \text{if } x \in D \cap Y, \\ 0 & \text{otherwise} \end{cases}$$

is an element of

$$\prod_{x\in D} \overline{B\big(0, \tanh\big(c_{D\cap Y_{\tilde{w},x}}(\tilde{w}, x)\big)\big)}$$

with the product topology, where

$$\overline{B\big(0, \tanh\big(c_{D\cap Y_{\tilde{w},x}}(\tilde{w}, x)\big)\big)} = \big\{z \in \Delta : |z| \leq \tanh\big(c_{D\cap Y_{\tilde{w},x}}(\tilde{w}, x)\big)\big\}$$

and

$$\tanh\big(c_{D\cap Y_{\tilde{w},x}}(\tilde{w}, x)\big) < 1$$

for each $x \in D$. By the Tychonoff theorem $\prod_{x\in D} \overline{B(0, \tanh(c_{D\cap Y_{\tilde{w},x}}(\tilde{w}, x)))}$ with the product topology is compact. In this way we obtain a net $\{f_Y\}_{Y\in\mathcal{Y}_{\tilde{w},\tilde{z}}}$ in the compact space; therefore, there exists a subnet $\{f_{Y_s}\}_{s\in\mathcal{S}}$ of $\{f_Y\}_{Y\in\mathcal{Y}_{\tilde{w},\tilde{z}}}$ such that $\{f_{Y_s}\}_{s\in\mathcal{S}}$ converges to some f in

$$\prod_{x\in D} \overline{B\big(0, \tanh\big(c_{D\cap Y_{\tilde{w},x}}(\tilde{w}, x)\big)\big)}.$$

It is obvious that $f(D) \subset \Delta$, $f(\tilde{w}) = 0$, and $f(\tilde{z}) \geq 0$. Now observe that for each finite-dimensional subspace $Y \in \mathcal{Y}$ the functions $f_{Y_s|D\cap Y} : D \cap Y \to \Delta$ are holomorphic for all sufficiently large s and therefore, after applying consecutively the Montel theorem and the boundedness of the limit function f, we obtain that f is holomorphic. Finally, we get

$$\begin{aligned} c_D(\tilde{w}, \tilde{z}) &\geq \rho\big(f(\tilde{w}), f(\tilde{z})\big) = \lim_{s\in\mathcal{S}} \rho\big(f_{Y_s}(\tilde{w}), f_{Y_s}(\tilde{z})\big) = \lim_{s\in\mathcal{S}} \rho\big(0, f_{Y_s}(\tilde{z})\big) \\ &= \lim_{s\in\mathcal{S}} c_{D\cap Y_s}(\tilde{w}, \tilde{z}) = \inf_{Y\in\mathcal{Y}_{\tilde{w},\tilde{z}}} c_{D\cap Y}(\tilde{w}, \tilde{z}) \geq c_D(\tilde{w}, \tilde{z}). \end{aligned}$$ □

Remark 2.2. Let $D = B$ be the unit open ball in an infinite-dimensional Banach space. For some spaces the equality

$$c_B(\tilde{w}, \tilde{z}) = \inf_{Y\in\mathcal{Y}_{\tilde{w},\tilde{z}}} c_{B\cap Y}(\tilde{w}, \tilde{z})$$

can be obtained in a simpler way. As an example, consider a Hilbert space H and a Hilbert ball B_H. Then it suffices to apply Möbius transformations (see [5]).

The second example is the following. Let $(X, \|\cdot\|)$ be a Banach space having a Schauder basis $\{e_i\}_{i=1}^{\infty}$ with the basis constant 1; that is, $\sup_{1\le k<\infty} \|P_k\| = 1$, where $P_k x = \sum_{i=1}^{k} x^i e_i$ for $x = \sum_{i=1}^{\infty} x^i e_i \in X$ and $k = 1, 2, \ldots$ (see [11]). Consider the unit open ball $B \subset X$. Then for each natural $1 \le k < \infty$ and each $w, z \in X_k = P_k(X)$, we have $c_B(w,z) = c_{B\cap X_k}(w,z)$. Now take arbitrary $\tilde{w}, \tilde{z} \in X$. Then there exists $0 < r < 1$ such that $\|\tilde{w}\| \le r$ and $\|\tilde{z}\| \le r$. For sequences $\{w_k\}_{k=1}^{\infty} = \{P_k\tilde{w}\}_{k=1}^{\infty}$, $\{z_k\}_{k=1}^{\infty} = \{P_k\tilde{z}\}_{k=1}^{\infty}$, we have $\lim_k w_k = \tilde{w}$, $\lim_k z_k = \tilde{z}$ in the norm $\|\cdot\|$ and $w_k \in X_k$, $\|w_k\| \le r$, $z_k \in X_k$, $\|z_k\| \le r$ for $k = 1, 2, \ldots$. Let $X_{k,\tilde{w},\tilde{z}} = \operatorname{span}(X_k \cup \{\tilde{w}, \tilde{z}\})$ for $k = 1, 2, \ldots$. Let us take k_0 such that $\|\tilde{w} - w_k\| < 1 - r$ and $\|\tilde{z} - z_k\| < 1 - r$ for each $k \ge k_0$. Then for each $k \ge k_0$, we have (see [5])

$$\begin{aligned} c_{B\cap X_{k,\tilde{w},\tilde{z}}}(\tilde{w}, w_k) &\le \tanh^{-1}\Big(\frac{\|\tilde{w} - w_k\|}{1-r}\Big) \underset{k}{\longrightarrow} 0,\\ c_B(w_k, \tilde{w}) &\le \tanh^{-1}\Big(\frac{\|w_k - \tilde{w}\|}{1-r}\Big) \underset{k}{\longrightarrow} 0,\\ c_{B\cap X_{k,\tilde{w},\tilde{z}}}(z_k, \tilde{z}) &\le \tanh^{-1}\Big(\frac{\|z_k - \tilde{z}\|}{1-r}\Big) \underset{k}{\longrightarrow} 0,\\ c_B(\tilde{z}, z_k) &\le \tanh^{-1}\Big(\frac{\|\tilde{z} - z_k\|}{1-r}\Big) \underset{k}{\longrightarrow} 0 \end{aligned}$$

and therefore

$$\begin{aligned} c_B(\tilde{w}, \tilde{z}) &\le c_{B\cap X_{k,\tilde{w},\tilde{z}}}(\tilde{w}, \tilde{z})\\ &\le c_{B\cap X_{k,\tilde{w},\tilde{z}}}(\tilde{w}, w_k) + c_{B\cap X_{k,\tilde{w},\tilde{z}}}(w_k, z_k) + c_{B\cap X_{k,\tilde{w}_k,\tilde{z}}}(z_k, \tilde{z})\\ &\le \tanh^{-1}\Big(\frac{\|\tilde{w} - w_k\|}{1-r}\Big) + c_{B\cap X_k}(w_k, z_k) + \tanh^{-1}\Big(\frac{\|z_k - \tilde{z}\|}{1-r}\Big)\\ &= \tanh^{-1}\Big(\frac{\|\tilde{w} - w_k\|}{1-r}\Big) + c_B(w_k, z_k) + \tanh^{-1}\Big(\frac{\|z_k - \tilde{z}\|}{1-r}\Big)\\ &\le 2\tanh^{-1}\Big(\frac{\|\tilde{w} - w_k\|}{1-r}\Big) + c_B(\tilde{w}, \tilde{z}) + 2\tanh^{-1}\Big(\frac{\|z_k - \tilde{z}\|}{1-r}\Big) \underset{k}{\longrightarrow} c_B(\tilde{w}, \tilde{z}). \end{aligned}$$

So we get the claimed result.

Finally, for the purpose of completeness of this article, we give the Dineen–Timoney–Vigué theorem with their proof.

Theorem 2.3 ([4, Théorème 2.5])**.** *Let $(X, \mathcal{T}_X)$ be a complex, locally convex, and Hausdorff topological vector space, and let D be a convex domain in $(X, \mathcal{T}_X)$. Then we have*

$$c_D = k_D = \delta_D.$$

Proof ([4]). For each $w, z \in D$ and each $Y \in \mathcal{Y}_{w,z}$, we have

$$\delta_D(w,z) \le \delta_{D\cap Y}(w,z)$$

and therefore

$$\delta_D(w,z) \le \inf_{Y\in\mathcal{Y}_{w,z}} \delta_{D\cap Y}(w,z).$$

Now direct applications of Theorem 2.1 and the inequalities

$$c_D \leq k_D \leq \delta_D$$

give

$$\begin{aligned} c_D(w,z) \leq k_D(w,z) \leq \delta_D(w,z) &\leq \inf_{F \in \mathcal{Y}_{w,z}} \delta_{D\cap F}(w,z) \\ &= \inf_{F \in \mathcal{Y}_{w,z}} c_{D\cap F}(w,z) = c_D(w,z) \end{aligned}$$

and the proof is complete. □

Remark 2.4. Observe that all pseudometrics assigned to a convex domain D in a Banach space by the Schwarz–Pick systems coincide (see Remark 1.2). If, in addition, D is assumed to be bounded, then this unique distance is sometimes called the *hyperbolic metric*. (For more information on this metric and its applications, see, e.g., the books [5] and [12].) We also note that, in the case of bounded symmetric domains in Banach spaces, another proof of the coincidence of all metrics in the Schwarz–Pick systems can be found in [12, Corollary 3.2, Remark 3.3]. This proof is based on the following deep results: the Riemann mapping theorem due to Kaup (Theorem 4.9 in [7]) and homogeneity of symmetric domains due to Vigué (Théorème 3.2.6 in [15]).

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