Ann. Funct. Anal. 10 (2019), no. 2, 218–228
https://doi.org/10.1215/20088752-2018-0025
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

A NOTE ON PERIPHERALLY MULTIPLICATIVE MAPS ON BANACH ALGEBRAS

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Communicated by M. de Jeu

Abstract. Let A and B be complex Banach algebras, and let ϕ, ϕ_1, and ϕ_2 be surjective maps from A onto B. Denote by $\partial\sigma(x)$ the boundary of the spectrum of x. If A is semisimple, B has an essential socle, and $\partial\sigma(xy) = \partial\sigma(\phi_1(x)\phi_2(y))$ for each $x, y \in A$, then we prove that the maps $x \mapsto \phi_1(\mathbf{1})\phi_2(x)$ and $x \mapsto \phi_1(x)\phi_2(\mathbf{1})$ coincide and are continuous Jordan isomorphisms. Moreover, if A is prime with nonzero socle and ϕ_1 and ϕ_2 satisfy the aforementioned condition, then we show once again that the maps $x \mapsto \phi_1(\mathbf{1})\phi_2(x)$ and $x \mapsto \phi_1(x)\phi_2(\mathbf{1})$ coincide and are continuous. However, in this case we conclude that the maps are either isomorphisms or anti-isomorphisms. Finally, if A is prime with nonzero socle and ϕ is a peripherally multiplicative map, then we prove that ϕ is continuous and either ϕ or $-\phi$ is an isomorphism or an anti-isomorphism.

1. Introduction

Surjective maps between Banach algebras which preserve spectral properties have been extensively studied in connection with the so-called *Kaplansky's problem* (see [9]) and now constitute an ongoing field of research. Let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators acting on a Banach space X. Jafarian and Sourour [8] showed that a linear spectrum-preserving map from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ must be an (algebra) isomorphism or an anti-isomorphism. This result was then extended in [2], where Aupetit and Mouton established that it

Received Apr. 22, 2018; Accepted Sep. 10, 2018.
First published online Mar. 19, 2019.
2010 *Mathematics Subject Classification.* Primary 47A10; Secondary 47B48, 47B49.
Keywords. Banach algebras, spectrum, peripheral spectrum, nonlinear preservers, peripherally multiplicative maps.

is enough to assume that the algebras in question are primitive Banach algebras with minimal ideals.

In [12], Molnár investigated a similar problem on operator algebras and commutative C^*-algebras by omitting the condition of linearity. Since nonlinear spectrum-preserving maps can be almost arbitrary, Molnár had to impose a much more restrictive spectral condition on the maps he examined. In particular, he assumed that the spectrum of the product of the images of any two elements is equal to the spectrum of the product of those two elements. In some sense, these maps can be thought of as spectrally multiplicative. Molnár also restricted this condition to hold only on certain subsets of the spectrum; in particular, he considered the point and surjectivity spectrum.

Recently, Bourhim, Mashreghi, and Stepanyan in [4] investigated spectrally multiplicative maps in a much broader setting. One aim of the present article is to extend their results by restricting the multiplicative condition to the boundary of the spectrum. Our second aim is to investigate *peripherally multiplicative* maps, that is, maps for which the multiplicative condition has been restricted to hold only on the peripheral spectrum. The aforementioned maps have been investigated between special classes of algebras (see, e.g., [10], [11]). Indeed, we show that a surjective peripherally multiplicative map on a prime Banach algebra with a minimal ideal is either an isomorphism or an anti-isomorphism (up to a plus or minus sign).

2. Notation and terminology

Throughout this paper, A and B will denote complex Banach algebras with identity elements. Additional assumptions to A and B will be indicated as needed. We will use $\mathbf{1}$ to denote the identity element in a Banach algebra under consideration. The group of invertible elements and center of A will be denoted by $G(A)$ and $Z(A)$, respectively, and a similar notational convention will be used for B. For any element x in a Banach algebra, we let $\sigma(x)$ be its spectrum and we denote by $\rho(x)$ its spectral radius. Moreover, the peripheral spectrum of x is the set $\sigma_\pi(x) = \{\lambda \in \sigma(x) : |\lambda| = \rho(x)\}$, and the boundary of $\sigma(x)$ will be denoted by $\partial\sigma(x)$. A Banach algebra A is semisimple if its Jacobson radical, $\mathrm{Rad}(A)$, is $\{0\}$. Incidentally, $x \in \mathrm{Rad}(A)$ if and only if $\sigma(xy) = \{0\}$ for all $y \in A$. An ideal J of a Banach algebra is said to be *essential* if it has a nonzero intersection with every nonzero ideal in the Banach algebra; if the Banach algebra is semisimple, then this is equivalent to saying that the condition $xJ = \{0\}$ implies that $x = 0$. A Banach algebra is said to be *prime* if and only if every nonzero ideal is essential. Prime algebras can also be characterized spectrally (see, e.g., Theorem 2.3).

An important ideal in a Banach algebra is the socle, which is the sum of all minimal left (or right) ideals. If the Banach algebra lacks minimal one-sided ideals, then its socle is trivial (i.e., $\{0\}$). By $\mathrm{Soc}(A)$ we will denote the socle of A (and this notation naturally extends to B). If A is semisimple, Aupetit and Mouton have shown in [3] that $\mathrm{Soc}(A)$ can be conveniently represented using their so-called *finite-rank elements*. An element x in a Banach algebra has rank n if and only if $\sup_{y\in A} \#(\sigma(xy) - \{0\}) = n$, where n is an integer and $\#K$ denotes the (possibly infinite) number of elements in the set K. If A is semisimple, then $x \in \operatorname{Soc}(A)$ if and only if x can be written as a finite sum of rank 1 elements. The set of rank 1 elements of A (resp., B) will be denoted using the notation $\mathcal{F}_1(A)$ (resp., $\mathcal{F}_1(B)$). The following characterization of rank 1 elements was implicitly obtained by Bourhim, Mashreghi, and Stepanyan in [4]. Its proof is an easy consequence of the definition and [4, Lemma 3.3].

Theorem 2.1. *Let A be a semisimple Banach algebra. Then $x \in \mathcal{F}_1(A)$ if and only if $\#\sigma_\pi(xy) = 1$ for all $y \in A$.*

Remark. By Jacobson's lemma, an equivalent formulation of Theorem 2.1 could have required that $\#\sigma_\pi(yx) = 1$ for all $y \in A$. We have already mentioned that, in the semisimple case, the socle coincides with the set of finite-rank elements, where the finite-rank elements have a completely spectral definition. Aupetit and Mouton went one step further and gave a completely spectral definition of the trace. (For particular details, see [3].) We do, however, wish to point out that the trace has the following useful properties (here A is semisimple):

(i) tr is a linear functional on $\operatorname{Soc}(A)$ (see [3, Theorem 3.3] and [14, Lemma 2.1]);
(ii) $\sigma_\pi(x) = \{\operatorname{tr}(x)\}$ for each $x \in \mathcal{F}_1(A)$ (easily deducible from the definition of the trace in [3]).

For literary convenience we state two theorems. The former has been used extensively in spectral preserver problems to obtain linearity (as is the case here) and the latter was mentioned earlier and will be used in the remainder of this article.

Theorem 2.2 ([3, Corollary 3.6])**.** *Let A be semisimple, and let $a \in A$. If $\operatorname{tr}(ax) = 0$ for each $x \in \operatorname{Soc}(A)$, then $a\operatorname{Soc}(A) = \{0\}$. Moreover, if $a \in \operatorname{Soc}(A)$, then $a = 0$.*

Theorem 2.3 ([13, Theorem 2.9])**.** *Suppose that A is semisimple and that $\operatorname{Soc}(A) \neq \{0\}$. Then A is prime if and only if for any $a, b \in A$ the following are equivalent:*

(i) *$\rho(ax) \le \rho(bx)$ for all $x \in A$,*
(ii) *$a = \lambda b$ for some $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$.*

3. Main results

We are now in a position to state our main results and some consequences. Our first result extends [4, Theorem 2.1 and Corollary 2.2].

Theorem 3.1. *Let A be semisimple, and let B be a Banach algebra with an essential socle. Suppose that $\phi_1 : A \to B$ and $\phi_2 : A \to B$ are surjective maps such that*

$$\partial\sigma(xy) = \partial\sigma\big(\phi_1(x)\phi_2(y)\big) \quad \text{for all } x, y \in A. \tag{3.1}$$

Then the maps $x \mapsto \phi_1(\mathbf{1})\phi_2(x)$ and $x \mapsto \phi_1(x)\phi_2(\mathbf{1})$ coincide and are continuous Jordan isomorphisms.

Corollary 3.2. *Let A be prime with nonzero socle. Suppose that the surjective maps $\phi_1 : A \to B$ and $\phi_2 : A \to B$ satisfy the condition in (3.1). Then the maps $x \mapsto \phi_1(\mathbf{1})\phi_2(x)$ and $x \mapsto \phi_1(x)\phi_2(\mathbf{1})$ coincide, and $x \mapsto \phi_1(\mathbf{1})\phi_2(x)$ is continuous and is either an isomorphism or an anti-isomorphism.*

The next result is similar to Corollary 3.2. In particular, we assume from the onset that ϕ_1 and ϕ_2 coincide, but relax condition (3.1) somewhat. The conclusion is quite pleasing.

Theorem 3.3. *Let A be prime with nonzero socle. Suppose that $\phi : A \to B$ is a surjective map such that*

$$\sigma_\pi(xy) = \sigma_\pi\big(\phi(x)\phi(y)\big) \quad \text{for all } x, y \in A. \tag{3.2}$$

Then ϕ is continuous and either ϕ or $-\phi$ is an isomorphism or an anti-isomorphism.

Using Theorem 3.3 in conjunction with [15, Theorem 1.1], we obtain an interesting consequence. Indeed, this result adds to the work done in [12] and improves on [11, Theorem 1.1].

Corollary 3.4. *Suppose that $\phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ is a surjective map which satisfies*

$$\sigma_\pi(TS) = \sigma_\pi\big(\phi(T)\phi(S)\big) \quad \text{for all } T, S \in \mathcal{B}(X). \tag{3.3}$$

Then either ϕ or $-\phi$ is an isomorphism or an anti-isomorphism. Hence, either

(i) *there exists a bounded invertible linear operator $U : X \to Y$ such that $\phi(T) = \pm UTU^{-1}$ for each $T \in \mathcal{B}(X)$, or*
(ii) *there exists a bounded invertible linear operator $V : X' \to Y$ such that $\phi(T) = \pm VT^*V^{-1}$ for each $T \in \mathcal{B}(X)$.*

4. Some properties of peripherally multiplicative maps

Throughout this section we assume that the surjective maps $\phi_1 : A \to B$ and $\phi_2 : A \to B$ satisfy

$$\sigma_\pi(xy) = \sigma_\pi\big(\phi_1(x)\phi_2(y)\big) \quad \text{for all } x, y \in A. \tag{4.1}$$

Some of the properties that we present here have been obtained in one form or another (sometimes implicitly) in [4]. However, we provide short arguments for these lemmas so that the proofs of our main results are essentially self-contained within this article.

Lemma 4.1 ([4, Lemma 3.2(2)])**.** *If A is semisimple, then B is semisimple.*

Proof. Suppose that $y \in \operatorname{Rad}(B)$. Then, by surjectivity, $y = \phi_1(x)$ for some $x \in A$. Moreover, by (4.1), we have $0 = \rho(\phi_1(x)\phi_2(u)) = \rho(xu)$ for all $u \in A$. So $x \in \operatorname{Rad}(A) = \{0\}$. Since y was arbitrary, this shows that $\operatorname{Rad}(B) = \{\phi_1(0)\}$. Thus, since $0 \in \operatorname{Rad}(B)$, it must be the case that $\operatorname{Rad}(B) = \{0\}$ as desired. □

Lemma 4.2 ([4, Lemma 3.4(1)])**.** *Let A be semisimple, and suppose that B has an essential socle. Then $\mathcal{F}_1(B) = \phi_1(\mathcal{F}_1(A)) = \phi_2(\mathcal{F}_1(A))$. Thus,* $\operatorname{Soc}(A) \neq \emptyset$.

Proof. From (4.1) we have that

$$\sigma_\pi(xy) = \sigma_\pi\big(\phi_1(x)\phi_2(y)\big) \quad \text{for each } x, y \in A.$$

The first part therefore follows directly from Theorem 2.1 and Jacobson's lemma. The last part now follows from the fact that $\mathrm{Soc}(B) \neq \{0\}$ (by hypothesis). □

Lemma 4.3. *Let A be semisimple, and suppose that B has an essential socle. Then $\mathrm{tr}(xy) = \mathrm{tr}(\phi_1(x)\phi_2(y))$ for all $x \in A$ and $y \in \mathcal{F}_1(A)$ (resp., for all $x \in \mathcal{F}_1(A)$ and $y \in A$).*

Proof. This is easy if one considers Lemma 4.2 and (4.1). The result then follows from the fact that

$$\sigma_\pi(xy) = \big\{\mathrm{tr}(xy)\big\} \qquad \text{and} \qquad \sigma_\pi\big(\phi_1(x)\phi_2(y)\big) = \big\{\mathrm{tr}\big(\phi_1(x)\phi_2(y)\big)\big\}$$

for all $x \in A$ and $y \in \mathcal{F}_1(A)$; or, in the second case, for all $x \in \mathcal{F}_1(A)$ and $y \in A$. □

Lemma 4.4 ([4, Proof of Theorem 2.1, Step 1])**.** *Let A be semisimple, and suppose that B has an essential socle. Then ϕ_1 and ϕ_2 are linear and injective mappings.*

Proof. The linearity of the mappings follows easily from Lemma 4.3, the linearity of the trace, Lemma 4.2, and Theorem 2.2. The injectivity then follows from the linearity of ϕ_1 and ϕ_2, (4.1), and the semisimplicity of A. □

The aim of the next few lemmas is to establish that B is prime with nonzero socle whenever A has these properties.

Lemma 4.5. *Suppose that A is prime with nonzero socle. Then B is semisimple.*

Proof. By [6, Corollary 4.1], A is semisimple, so the result follows from Lemma 4.1. □

Lemma 4.6. *Suppose that A is prime with nonzero socle. Then ϕ_1 and ϕ_2 are injective.*

Proof. Suppose first that $\phi_1(x) = \phi_1(y)$. Then

$$\rho\big(\phi_1(x)\phi_2(z)\big) = \rho\big(\phi_1(y)\phi_2(z)\big) \quad \text{for all } z \in A.$$

Hence, by the condition in (4.1), we may infer that

$$\rho(xz) = \rho(yz) \quad \text{for all } z \in A. \tag{4.2}$$

Thus, by Theorem 2.3 it follows that $x = \lambda y$ for some $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. If $\sigma(yu) = \{0\}$ for all $u \in \mathcal{F}_1(A)$, then $\mathrm{tr}(yv) = 0$ for all $v \in \mathrm{Soc}(A)$. Consequently, since $\mathrm{Soc}(A)$ is essential, it follows from Theorem 2.2 that $y = 0$. Hence, $x = y = 0$. So assume that $\sigma(yu) \neq \{0\}$ for some $u \in \mathcal{F}_1(A)$. Now, by the condition in (4.1) it follows that

$$\sigma_\pi(\lambda yu) = \sigma_\pi\big(\phi_1(x)\phi_2(u)\big) = \sigma_\pi\big(\phi_1(y)\phi_2(u)\big) = \sigma_\pi(yu).$$

Hence, since $\sigma_\pi(yu) = \{\alpha\}$ for some $\alpha \in \mathbb{C} - \{0\}$, by the spectral mapping theorem we may conclude that $\lambda = 1$. Therefore, $x = y$ and ϕ_1 is injective. Using similar reasoning and Jacobson's lemma, we obtain that ϕ_2 is injective as well. □

The idea in the proof of Lemma 4.6 allows us to say a bit more.

Lemma 4.7. *Suppose that A is prime with nonzero socle. Then ϕ_1 and ϕ_2 are homogeneous maps.*

Proof. Let $x \in A$ and $\lambda \in \mathbb{C}$. By the spectral mapping theorem and the condition in (4.1) it follows that

$$\sigma_\pi\big(\phi_1(\lambda x)\phi_2(z)\big) = \sigma_\pi(\lambda xz) = \lambda\sigma_\pi(xz) = \sigma_\pi\big(\lambda\phi_1(x)\phi_2(z)\big)$$

for all $z \in A$. Since ϕ_1 is surjective, it follows that $\lambda\phi_1(x) = \phi_1(y)$ for some $y \in A$. Arguing as in the proof of Lemma 4.5, we conclude that $y = \lambda x$. Of course a similar argument works for ϕ_2. □

Lemma 4.8. *Suppose that A is prime with nonzero socle. Then $\mathcal{F}_1(B) = \phi_1(\mathcal{F}_1(A)) = \phi_2(\mathcal{F}_1(A))$. Thus, $\mathrm{Soc}(B) \neq \emptyset$.*

Proof. The first part is immediate from (4.1) and Theorem 2.1. The last part then follows since $\mathrm{Soc}(A) \neq \{0\}$. □

Lemma 4.9. *Suppose that A is prime with nonzero socle. Then B is prime.*

Proof. Suppose first that A is prime with nonzero socle. From Lemma 4.5 and Lemma 4.8 it follows that B is a semisimple Banach algebra with $\mathrm{Soc}(B) \neq \emptyset$. Moreover, recall that ϕ_1 is homogeneous and bijective. Thus, since A is prime with nonzero socle, we may use (4.1) and Theorem 2.3 to conclude that B is prime. □

5. Proof of Theorem 3.1

The following notation will be fixed throughout this section. By A we denote a semisimple Banach algebra, and by B we denote a Banach algebra with an essential socle. We also assume that the surjective maps $\phi_1 : A \to B$ and $\phi_2 : A \to B$ satisfy the condition in (3.1). It is easy to see that if ϕ_1 and ϕ_2 satisfy the condition in (3.1), then the condition in (4.1) is also met (but not conversely). We now proceed to establish a further sequence of lemmas which will ultimately prove the desired result.

Lemma 5.1. *Suppose that $z \in G(A)$ and that $\phi_1(z) \in G(B)$. Define $\psi : A \to B$ by*

$$\psi(x) = \phi_1(z)\phi_2(z^{-1}x) \quad \text{for each } x \in A.$$

Then ψ is a surjective linear spectral isometry. Moreover, $\psi(\mathbf{1}) = \mathbf{1}$. Hence, $\phi_2(z^{-1}) = \phi_1(z)^{-1}$.

Proof. By (3.1) it follows that

$$\rho\big(\psi(x)\big) = \rho\big(\phi_1(z)\phi_2(z^{-1}x)\big) = \rho(zz^{-1}x) = \rho(x) \quad \text{for all } x \in A,$$

whence ψ is a spectral isometry. The linearity of ψ follows readily from that of ϕ_2. To see that ψ is surjective, fix any $y \in B$. Since $\phi_1(z) \in G(B)$, there exists some $u \in B$ such that $\phi_1(z)u = \mathbf{1}$. Moreover, since ϕ_2 is surjective, there exists some $w \in A$ such that $\phi_2(w) = uy$. But then

$$\psi(zw) = \phi_1(z)\phi_2(z^{-1}zw) = \phi_1(z)uy = y.$$

This shows that ψ is surjective.

We now proceed to prove that $\psi(\mathbf{1}) = \mathbf{1}$. Using (3.1), we deduce that

$$\begin{aligned}\rho\big(\psi(\mathbf{1}) + \psi(x)\big) &= \rho\big(\phi_1(z)\phi_2(z^{-1}) + \phi_1(z)\phi_2(z^{-1}x)\big)\\ &= \rho\big(\phi_1(z)\phi_2(z^{-1} + z^{-1}x)\big)\\ &= \rho\big(z(z^{-1} + z^{-1}x)\big)\\ &= \rho(\mathbf{1} + x)\end{aligned}$$

for each $x \in A$. But $\mathbf{1} \in Z(A)$, so by [1, Corollary 3.2.10],

$$\rho(\mathbf{1} + x) \leq \rho(\mathbf{1}) + \rho(x) = \rho\big(\psi(\mathbf{1})\big) + \rho\big(\psi(x)\big).$$

Hence, $\rho(\psi(\mathbf{1}) + \psi(x)) \leq \rho(\psi(\mathbf{1})) + \rho(\psi(x))$ for all $x \in A$. Thus, since ψ is surjective, it follows from [1, Theorem 5.2.2] that $\psi(\mathbf{1}) \in Z(B)$. Now, by condition (3.1), we note that $\partial\sigma(\psi(\mathbf{1})) = \{1\}$. But then it must be true that $\sigma(\psi(\mathbf{1})) = \{1\}$. Hence, by [5, Theorem 2.1] we conclude that $\psi(\mathbf{1}) = \mathbf{1}$. The last part of the lemma now readily follows. □

Lemma 5.2. *We have $\phi_2(\mathbf{1}) \in G(B)$ and $\phi_2(\mathbf{1})^{-1} = \phi_1(\mathbf{1})$.*

Proof. We first show that $\phi_1(\mathbf{1})\phi_2(\mathbf{1}) = \mathbf{1}$. From the condition in (3.1), it follows that $\partial\sigma(\phi_1(\mathbf{1})\phi_2(\mathbf{1})) = \sigma(\phi_1(\mathbf{1})\phi_2(\mathbf{1})) = \{1\}$. In particular, this implies that $\phi_1(\mathbf{1})$ is right invertible. Moreover, by [5, Theorem 2.1] it now suffices to prove that $\phi_1(\mathbf{1})\phi_2(\mathbf{1}) \in Z(B)$. By (3.1) and [1, Corollary 3.2.10], for each $y \in A$, we have

$$\begin{aligned}\rho\big(\phi_1(\mathbf{1})\phi_2(\mathbf{1}) + \phi_1(\mathbf{1})\phi_2(y)\big) &= \rho\big(\phi_1(\mathbf{1})\phi_2(\mathbf{1} + y)\big)\\ &= \rho(\mathbf{1} + y)\\ &\leq \rho(\mathbf{1}) + \rho(y)\\ &= \rho\big(\phi_1(\mathbf{1})\phi_2(\mathbf{1})\big) + \rho\big(\phi_1(\mathbf{1})\phi_2(y)\big).\end{aligned}$$

Hence, since $\phi_1(\mathbf{1})$ is right invertible, ϕ_2 is surjective, and $y \in A$ is arbitrary, we may apply [1, Theorem 5.2.2] and conclude that $\phi_1(\mathbf{1})\phi_2(\mathbf{1}) \in Z(B)$. This then establishes that $\phi_1(\mathbf{1})\phi_2(\mathbf{1}) = \mathbf{1}$.

We now proceed as follows. To establish the lemma we need only verify that $\phi_2(\mathbf{1})\phi_1(\mathbf{1}) = \mathbf{1}$. Using the surjectivity of ϕ_2, we can find some $x \in A$ such that $\phi_2(x) = \mathbf{1} - \phi_2(\mathbf{1})\phi_1(\mathbf{1})$. In particular, we see that

$$\phi_1(\mathbf{1})\phi_2(x) = \phi_1(\mathbf{1}) - \phi_1(\mathbf{1})\phi_2(\mathbf{1})\phi_1(\mathbf{1}) = \phi_1(\mathbf{1}) - \phi_1(\mathbf{1}) = 0.$$

Consequently,

$$\rho\big(\phi_1(\mathbf{1})\big(\phi_2(y) + \phi_2(x)\big)\big) = \rho\big(\phi_1(\mathbf{1})\phi_2(y)\big) \quad \text{for all } y \in A.$$

Hence, by (3.1) we may infer that

$$\rho(y + x) = \rho(y) \quad \text{for all } y \in A.$$

Thus, by Zemánek's result (see, e.g., [1, Theorem 5.3.1]) it follows that $x \in \mathrm{Rad}(A) = \{0\}$. But ϕ_2 is linear by Lemma 4.4. Thus,

$$0 = \phi_2(x) = \mathbf{1} - \phi_2(\mathbf{1})\phi_1(\mathbf{1}),$$

and so we get the result. □

Lemma 5.3. *The maps ϕ_1 and ϕ_2 are both continuous.*

Proof. Since $\phi_1(\mathbf{1}), \phi_2(\mathbf{1}) \in G(B)$ by Lemma 5.2, and since ϕ_1 and ϕ_2 are linear bijective mappings, it readily follows that the maps defined by

$$x \mapsto \phi_1(\mathbf{1})\phi_2(x) \qquad \text{and} \qquad x \mapsto \phi_1(x)\phi_2(\mathbf{1}) \quad \text{for } x \in A$$

are linear bijective mappings. Thus, by (3.1), we may actually infer that these mappings are surjective linear spectral isometries. Hence, by [1, Theorem 5.5.2] we may conclude that these maps are continuous. Noting again that $\phi_1(\mathbf{1}), \phi_2(\mathbf{1}) \in G(B)$, we obtain the result since multiplication is continuous. □

Proof of Theorem 3.1. By Lemma 4.4, ϕ_1 and ϕ_2 are linear and bijective mappings. Thus, since $\phi_1(\mathbf{1}), \phi_2(\mathbf{1}) \in G(B)$ by Lemma 5.2, it readily follows that the maps $x \mapsto \phi_1(\mathbf{1})\phi_2(x)$ and $x \mapsto \phi_1(x)\phi_2(\mathbf{1})$ are linear and bijective. The continuity of each of these maps was established in Lemma 5.3. We now proceed to show that these two maps coincide and are Jordan isomorphisms. Fix any $x \in A$. Let $\lambda \in \mathbb{C}$, and suppose that $|\lambda| > \max\{\|x\|, \|\phi_1(x)\phi_2(\mathbf{1})\|\}$. Then $\lambda\mathbf{1} - x \in G(A)$ and $\lambda\mathbf{1} - \phi_1(x)\phi_2(\mathbf{1}) = \phi_1(\lambda\mathbf{1} - x)\phi_2(\mathbf{1}) \in G(B)$, where we have used the fact that $\phi_1(\mathbf{1})\phi_2(\mathbf{1}) = \mathbf{1}$. Since $\phi_2(\mathbf{1}) \in G(B)$, it readily follows that $\phi_1(\lambda\mathbf{1} - x) \in G(B)$. Thus, by Lemma 5.1 we conclude that $\phi_2((\lambda\mathbf{1} - x)^{-1}) = \phi_1(\lambda\mathbf{1} - x)^{-1}$. Consequently,

$$\phi_1(\mathbf{1})\phi_2\big((\lambda\mathbf{1} - x)^{-1}\big) = \big(\phi_1(\lambda\mathbf{1} - x)\phi_2(\mathbf{1})\big)^{-1} = \big(\lambda\mathbf{1} - \phi_1(x)\phi_2(\mathbf{1})\big)^{-1}.$$

For the next step we recall, from basic spectral theory, that our choice of $|\lambda| > \max\{\|x\|, \|\phi_1(x)\phi_2(\mathbf{1})\|\}$ implies that

$$(\lambda\mathbf{1} - x)^{-1} = \frac{1}{\lambda}\sum_{j=0}^{\infty}\Big(\frac{x}{\lambda}\Big)^j$$

and

$$\big(\lambda\mathbf{1} - \phi_1(x)\phi_2(\mathbf{1})\big)^{-1} = \frac{1}{\lambda}\sum_{j=0}^{\infty}\Big(\frac{\phi_1(x)\phi_2(\mathbf{1})}{\lambda}\Big)^j.$$

Fix any bounded linear functional f on B. Since both f and ϕ_2 are linear and continuous, it follows that

$$\sum_{j=0}^{\infty}\frac{f(\phi_1(\mathbf{1})\phi_2(x^j))}{\lambda^{j+1}} = \sum_{j=0}^{\infty}\frac{f((\phi_1(x)\phi_2(\mathbf{1}))^j)}{\lambda^{j+1}}.$$

Noting that $\lambda \mapsto f((\lambda\mathbf{1} - \phi_1(x)\phi_2(\mathbf{1}))^{-1})$ is analytic, we may compare coefficients in the series expansions above and conclude in particular for $j = 1$ and $j = 2$ that

$$f\big(\phi_1(\mathbf{1})\phi_2(x)\big) = f\big(\phi_1(x)\phi_2(\mathbf{1})\big)$$

and

$$f\big(\phi_1(\mathbf{1})\phi_2(x^2)\big) = f\big(\big(\phi_1(x)\phi_2(\mathbf{1})\big)^2\big).$$

Since f and $x \in A$ were arbitrary, it follows that $\phi_1(\mathbf{1})\phi_2(x) = \phi_1(x)\phi_2(\mathbf{1})$ for all $x \in A$, and, consequently, that

$$\phi_1(\mathbf{1})\phi_2(x^2) = \big(\phi_1(x)\phi_2(\mathbf{1})\big)^2 = \big(\phi_1(\mathbf{1})\phi_2(x)\big)^2$$

for all $x \in A$. This shows that $x \mapsto \phi_1(\mathbf{1})\phi_2(x)$ is a Jordan isomorphism. Since the maps coincide, this completes the proof. □

6. Proofs of Corollary 3.2 and Theorem 3.3

We fix the following notation throughout this section. By A we denote a prime Banach algebra with nonzero socle, and by B we denote a Banach algebra. We also assume that the surjective maps $\phi_1 : A \to B$ and $\phi_2 : A \to B$ satisfy the condition in (3.1), and that the map $\phi : A \to B$ satisfies the condition in (3.2).

Proof of Corollary 3.2. From Lemma 4.9 it follows that B is prime, whence $\mathrm{Soc}(B)$ is an essential ideal. Moreover, A is necessarily semisimple. Hence, we may apply Theorem 3.1 and conclude that the maps $x \mapsto \phi_1(\mathbf{1})\phi_2(x)$ and $x \mapsto \phi_1(x)\phi_2(\mathbf{1})$ coincide and are continuous Jordan isomorphisms. The result now follows by [7, Theorem 3.1] which states that any Jordan homomorphism into a prime Banach algebra is either a homomorphism or an antihomomorphism. □

We now proceed to develop a proof of Theorem 3.3. In particular, we note that the results from Section 4 apply to the situation in Theorem 3.3 as a special case. From Lemma 4.9, as we have mentioned, it follows that $\mathrm{Soc}(B)$ is an essential ideal. In addition, we also obtain that $Z(B)$ is trivial. As we will soon see, these properties of B are very useful.

Lemma 6.1. *We have that $\phi(\mathbf{1}) = \mathbf{1}$ or $\phi(\mathbf{1}) = -\mathbf{1}$.*

Proof. By the spectral mapping theorem and the condition in (3.2), it follows that

$$\rho\big(\phi(\mathbf{1})\phi(x)\big)^2 = \rho(x)^2 = \rho(x^2) = \rho\big(\phi(x)^2\big) = \rho\big(\phi(x)\big)^2$$

for all $x \in A$. Consequently,

$$\rho\big(\phi(\mathbf{1})\phi(x)\big) = \rho\big(\phi(x)\big) \quad \text{for all } x \in A,$$

and so, by Theorem 2.3 we may infer that $\phi(\mathbf{1}) = \lambda\mathbf{1}$ for some $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. However, since $\sigma_\pi(\phi(\mathbf{1})^2) = \{1\}$, it follows from the spectral mapping theorem that $\phi(\mathbf{1}) = \mathbf{1}$ or $\phi(\mathbf{1}) = -\mathbf{1}$ are the only possibilities. □

Since $\phi(\mathbf{1}) \in G(B)$, the same argument used in the proof of Lemma 5.3 with $\phi_1 = \phi_2$ proves the following result.

Lemma 6.2. *The map ϕ is continuous.*

Lemma 6.3. *Suppose that $z \in G(A)$ and that $\phi(z) \in G(B)$. Define $\psi : A \to B$ by*

$$\psi(x) = \phi(z)\phi(z^{-1}x) \quad \textit{for each } x \in A.$$

Then ψ is a surjective linear spectral isometry. Moreover, $\psi(\mathbf{1}) = \mathbf{1}$. Hence, $\phi(z^{-1}) = \phi(z)^{-1}$.

Proof. The first part of the proof is essentially the same as the first part of the proof of Lemma 5.1.

We now proceed to prove that $\psi(\mathbf{1}) = \mathbf{1}$. Using the condition in (3.2), we deduce that

$$\begin{aligned}\rho\big(\psi(\mathbf{1}) + \psi(x)\big) &= \rho\big(\phi(z)\phi(z^{-1}) + \phi(z)\phi(z^{-1}x)\big)\\ &= \rho\big(\phi(z)\phi(z^{-1} + z^{-1}x)\big)\\ &= \rho\big(z(z^{-1} + z^{-1}x)\big)\\ &= \rho(\mathbf{1} + x)\end{aligned}$$

for each $x \in A$. But $\mathbf{1} \in Z(A)$, so by [1, Corollary 3.2.10],

$$\rho(\mathbf{1} + x) \leq \rho(\mathbf{1}) + \rho(x) = \rho\big(\psi(\mathbf{1})\big) + \rho\big(\psi(x)\big).$$

Hence, $\rho(\psi(\mathbf{1}) + \psi(x)) \leq \rho(\psi(\mathbf{1})) + \rho(\psi(x))$ for all $x \in A$. Thus, since ψ is surjective, it follows from [1, Theorem 5.2.2] that $\psi(\mathbf{1}) \in Z(B)$. Since $Z(B)$ is trivial and $\sigma_\pi(\psi(\mathbf{1})) = \{1\}$ (by (3.2)), we conclude that $\psi(\mathbf{1}) = \mathbf{1}$. The last part of the lemma now readily follows. □

Proof of Theorem 3.3. From Lemma 6.2 it follows that ϕ is continuous. Moreover, recall that ϕ is linear and bijective (e.g., using Lemma 4.4). We proceed to show that either ϕ or $-\phi$ is an isomorphism or anti-isomorphism. By [7, Theorem 3.1] it suffices to show that either ϕ or $-\phi$ is a Jordan isomorphism. By Lemma 6.1, $\phi(\mathbf{1}) = \mathbf{1}$ or $\phi(\mathbf{1}) = -\mathbf{1}$. Suppose first that $\phi(\mathbf{1}) = \mathbf{1}$. Fix any $x \in A$. Let $\lambda \in \mathbb{C}$, and suppose that $|\lambda| > \max\{\|x\|, \|\phi(x)\|\}$. Then $\lambda\mathbf{1} - x \in G(A)$ and $\lambda\mathbf{1} - \phi(x) = \phi(\lambda\mathbf{1} - x) \in G(B)$, where we have used the fact that $\phi(\mathbf{1}) = \mathbf{1}$. Thus, by Lemma 6.3 we conclude that

$$\phi\big((\lambda\mathbf{1} - x)^{-1}\big) = \big(\lambda\mathbf{1} - \phi(x)\big)^{-1}.$$

As in the proof of Theorem 3.1, we see that $\phi(x^2) = \phi(x)^2$ for all $x \in A$. Hence, we conclude that ϕ is indeed a Jordan isomorphism. If $\phi(\mathbf{1}) = -\mathbf{1}$, then we may apply the same argument to $-\phi$. This completes the proof. □

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