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THE STRUCTURE OF 2-LOCAL LIE DERIVATIONS ON VON NEUMANN ALGEBRAS

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ABSTRACT. In this article we characterize the form of each 2-local Lie derivation on a von Neumann algebra without central summands of type I_1. We deduce that every 2-local Lie derivation δ on a finite von Neumann algebra $\mathcal{M}$ without central summands of type I_1 can be written in the form $\delta(A) = AE - EA + h(A)$ for all A in $\mathcal{M}$, where E is an element in $\mathcal{M}$ and h is a center-valued homogenous mapping which annihilates each commutator of $\mathcal{M}$. In particular, every linear 2-local Lie derivation is a Lie derivation on a finite von Neumann algebra without central summands of type I_1. We also show that every 2-local Lie derivation on a properly infinite von Neumann algebra is a Lie derivation.

1. Introduction and preliminaries

Let $\mathcal{A}$ be a complex linear algebra, and let $\mathcal{X}$ be an $\mathcal{A}$-bimodule. Recall that a linear map d from $\mathcal{A}$ into $\mathcal{X}$ is called a *derivation* if $d(AB) = d(A)B + Ad(B)$ for all A, B in $\mathcal{A}$. Obviously, given an element A in $\mathcal{A}$, if $d_A(X) = [A, X] = AX - XA$ for all X in $\mathcal{A}$, then d_A is a derivation. Such a derivation is called *inner*. As is well known, every derivation on a von Neumann algebra is inner. More generally, a linear mapping δ of an associative algebra $\mathcal{A}$ is said to be a *Lie derivation* if $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ for all A, B in $\mathcal{A}$, where $[A, B] = AB - BA$ is the usual Lie product. By [10], it follows that every Lie derivation δ of a von Neumann algebra $\mathcal{M}$ has the form $\delta(A) = [A, S] + \tau(A)$, where S is an element in

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$\mathcal{M}$ and τ is a center-valued linear mapping which annihilates each commutator of $\mathcal{M}$. Obviously, derivations of associative algebras are Lie derivations.

In the study of derivation theory, we cannot overstate the importance of establishing many sufficient conditions to ensure that a mapping on various algebras is a (Lie) derivation. We mention some remarkable results in this area. In the setting of linear mappings, Kadison in [6] first introduced the notion of a local derivation and also proved that every continuous local derivation from a von Neumann algebra into any of its Banach bimodules is a derivation. In [5], Johnson generalized Kadison's conclusion and showed that every local derivation from a C^*-algebra into any of its Banach bimodules is a derivation. In [3], Essaleh, Peralta, and Ramírez established the definition of a weak-local derivation—extending the notion of a local derivation—and proved that weak-local derivations on C^*-algebras are derivations. In the setting of nonlinear mappings, Šemrl in [14] defined the notion of a 2-local derivation as follows. A (not necessarily linear) mapping Δ on a Banach algebra $\mathcal{A}$ is called a *2-local derivation* if for every A, B in $\mathcal{A}$ there exists a derivation $d_{A,B} : \mathcal{A} \to \mathcal{A}$, depending on A and B, such that $\Delta(A) = d_{A,B}(A)$ and $\Delta(B) = d_{A,B}(B)$. He also showed that every 2-local derivation on $B(H)$ of all linear bounded operators on H is a derivation, where H is an infinite separable Hilbert space. Ayupov and Kudaybergenov showed in [1] that every 2-local derivation on an arbitrary von Neumann algebra is a derivation. (Whether or not a 2-local derivation is a derivation on a general C^*-algebra remains an open question.) In order to solve the problem, and motivated by the definitions of local derivations, weak-local derivations, and 2-local derivations, Niazi and Peralta in [11] introduced the definition of a weak-2-local derivation and proved that every weak-2-local $*$-derivation on $M_n(\mathbb{C})$ is a $*$-derivation. Recently, Cabello and Peralta [2] characterized that every weak-2-local derivation on $B(H)$, $K(H)$ of all compact operators on H, where H is any complex Hilbert space, atomic von Neumann algebras, and compact C*-algebras, is a derivation. In [16], Yang and Fang showed that weak-2-local derivations on finite von Neumann algebras are derivations.

Similarly, the notions of local and 2-local Lie derivations can be defined. Some contributions to local Lie derivations can be found in [7]. A (not necessarily linear) mapping δ on a Banach algebra $\mathcal{A}$ is called a *2-local Lie derivation* if for every A, B in $\mathcal{A}$ there exists a Lie derivation $\delta_{A,B}$ on $\mathcal{A}$, depending on A and B, such that $\delta(A) = \delta_{A,B}(A)$ and $\delta(B) = \delta_{A,B}(B)$. We note some new contributions to 2-local Lie derivations. Liu in [8] characterized 2-local Lie derivations on a semifinite factor von Neumann algebra and showed that every 2-local Lie derivation on a semifinite factor von Neumann algebra with dimension greater than 4 can be written in the form of an inner derivation by adding a center-valued homogenous mapping which annihilates each commutator. In their recent article [4], He, Li, An, and Huang showed that 2-local Lie derivations are Lie derivations on factor von Neumann algebras, uniformly hyperfinite algebras, and the Jiang–Su algebra, and they constructed an example of a (nonlinear) 2-local Lie derivation, but not a Lie derivation on a finite von Neumann algebra which is not a factor. It seems natural to consider the form of a 2-local Lie derivation on a general von Neumann algebra which is not a factor and whether all linear 2-local Lie derivations on finite von Neumann algebras are Lie derivations.

In this article, we expect to obtain a complete characterization for proving problems in several cases. By [15], we know that every element in a properly infinite von Neumann algebra $\mathcal{M}$ has the form $\sum_{i=1}^{n}[A_i, B_i]$ for $A_i, B_i \in \mathcal{M}$; then every 2-local Lie derivation on a properly infinite von Neumann algebra is a 2-local derivation. Therefore, every 2-local Lie derivation on a properly infinite von Neumann algebra is a derivation by [1]. Obviously, each derivation on a properly infinite von Neumann algebra is a Lie derivation. So we only consider each 2-local Lie derivation on a finite von Neumann algebra. In Theorem 2.1 we obtain that every 2-local Lie derivation δ on a finite von Neumann algebra $\mathcal{M}$ without central summands of type I_1 can be written in the form $\delta(A) = AE - EA + h(A)$ for all A in $\mathcal{M}$, where E is an element in $\mathcal{M}$ and h is a center-valued homogenous mapping which annihilates each commutator of $\mathcal{M}$. In particular, every linear 2-local Lie derivation is a Lie derivation on a finite von Neumann algebra without central summands of type I_1.

2. Main results

Let $\mathcal{M}$ be a von Neumann algebra. Recall that the set $Z(\mathcal{M}) = \{T \in \mathcal{M} : ST = TS \text{ for all } S \in \mathcal{M}\}$ is said to be the *center* of $\mathcal{M}$. For every $A \in \mathcal{M}$, the central carrier of A, denoted by $c(A)$, can be defined as the intersection of all central projections $Q \in \mathcal{M}$ such that $QA = A$. For each self-adjoint element $A \in \mathcal{M}$, the core of A is defined to be $\sup\{T \in Z(\mathcal{M}) : T = T^*, T \leq A\}$, denoted by $\underline{A}$. Furthermore, if P a projection and $\underline{P} = 0$, then P is said to be *core-free*. Obviously, $\underline{P} = 0$ if and only if $c(I - P) = I$. It is well known that every finite von Neumann algebra has a separating family of normal tracial states. (We refer the reader to [12] for basic theories of von Neumann algebras involved in this article.)

The following is our main result.

Theorem 2.1. *Let $\mathcal{M}$ be a finite von Neumann algebra without central summands of type I_1. Then every 2-local Lie derivation $\delta : \mathcal{M} \to \mathcal{M}$ can be written in the form $\delta(A) = AE - EA + h(A)$ for all A in $\mathcal{M}$, where E is an element in $\mathcal{M}$ and $h : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ is a homogenous mapping which annihilates each commutator of $\mathcal{M}$.*

To prove Theorem 2.1, we need some lemmas.

Lemma 2.2 ([9, Lemmas 4, 5, 14])**.** *Let $\mathcal{M}$ be a von Neumann algebra.*

(1) *If $\mathcal{M}$ has no central summands of type I_1, then each nonzero central projection of $\mathcal{M}$ is the central carrier of a core-free projection of $\mathcal{M}$.*
(2) *For projections $P, Q \in \mathcal{M}$ with $c(P) = c(Q) \neq 0$, if $A \in \mathcal{M}$ commutes with PXQ and QXP for all $X \in \mathcal{M}$, then A commutes with PXP and QXQ for all $X \in \mathcal{M}$.*
(3) *If P is a core-free projection in $\mathcal{M}$, then $P\mathcal{M}P \cap \mathcal{Z}(\mathcal{M}) = 0$.*

Lemma 2.3 ([12, Lemma 2.6.4])**.** *Let $\mathcal{M}$ be a von Neumann algebra. If A is a self-adjoint operator in $\mathcal{M}$ and Z is a self-adjoint operator in $\mathcal{Z}(\mathcal{M})$, then $c(A + Z) = c(A) + Z$. Moreover, if $Z \geq 0$, then $c(AZ) = c(A)Z$.*

Lemma 2.4 ([13, Lemma 2.1])**.** *Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1 or type I_2. Suppose that $L : \mathcal{M} \to \mathcal{M}$ is an additive map. Then L satisfies $L([A, B]) = [L(A), B] + [A, L(B)]$ whenever $[A, B] = 0$ if and only if there exits an element $Z_0 \in \mathcal{Z}(\mathcal{M})$, an additive derivation $D : \mathcal{M} \to \mathcal{M}$, and an addictive map $h : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ such that $L(A) = D(A) + h(A) + Z_0 A$ for all $A \in \mathcal{M}$.*

Lemma 2.5. *Let $\mathcal{M}$ be a finite von Neumann algebra. If $\delta : \mathcal{M} \to \mathcal{M}$ is a 2-local Lie derivation, then $\delta(\lambda A) = \lambda\delta(A)$ and $\delta(A + B) - \delta(A) - \delta(B) \in \mathcal{Z}(\mathcal{M})$ for each $A, B \in \mathcal{M}$ and $\lambda \in \mathbb{C}$.*

Proof. For each $A \in \mathcal{M}$ and $\lambda \in \mathbb{C}$, by definition we easily have $\delta(\lambda A) = \delta_{A,\lambda A}(\lambda A) = \lambda\delta_{A,\lambda A}(A) = \lambda\delta(A)$.

Let P be a projection in $\mathcal{M}$, and denote $P^\perp = I - P$. Let Γ be the set of a separating family of normal tracial states on $\mathcal{M}$. Given $A, X \in \mathcal{M}$ and $\tau \in \Gamma$, there exist an element $T_{A,PXP^\perp}$, depending on A and $PXP^\perp$, and a linear mapping $h_{A,PXP^\perp}$ from $\mathcal{M}$ into $\mathcal{Z}(\mathcal{M})$ annihilating each commutator such that

$$\delta(A) = [A, T_{A,PXP^\perp}] + h_{A,PXP^\perp}(A)$$

and

$$\delta(PXP^\perp) = [PXP^\perp, T_{A,PXP^\perp}].$$

Noting that

$$\begin{aligned}
&\big(\delta(A) - h_{A,PXP^\perp}(A)\big)PXP^\perp + A\delta(PXP^\perp) \\
&\quad = [A, T_{A,PXP^\perp}]PXP^\perp + A[PXP^\perp, T_{A,PXP^\perp}] \\
&\quad = [APXP^\perp, T_{A,PXP^\perp}],
\end{aligned}$$

we obtain

$$\tau\Big(\big(\delta(A) - h_{A,PXP^\perp}(A)\big)PXP^\perp\Big) + \tau\big(A\delta(PXP^\perp)\big) = 0;$$

that is,

$$\tau\big(\delta(A)PXP^\perp\big) = -\tau\big(A\delta(PXP^\perp)\big).$$

For arbitrary $A, B \in \mathcal{M}$, we then have

$$\begin{aligned}
\tau\big(\delta(A + B)PXP^\perp\big) &= -\tau\big((A + B)\delta(PXP^\perp)\big) \\
&= -\tau\big(A\delta(PXP^\perp)\big) - \tau\big(B\delta(PXP^\perp)\big) \\
&= \tau\big(\delta(A)PXP^\perp\big) + \tau\big(\delta(B)PXP^\perp\big) \\
&= \tau\Big(\big(\delta(A) + \delta(B)\big)PXP^\perp\Big).
\end{aligned}$$

Hence

$$\tau\Big(\big(\delta(A + B) - \delta(A) - \delta(B)\big)PXP^\perp\Big) = 0.$$

Denote $Y = \delta(A + B) - \delta(A) - \delta(B)$. Then

$$\tau(YPXP^\perp) = 0. \tag{2.1}$$

Let $X = Y^*$ in (2.1). Then

$$\tau\big(P^\perp YP(P^\perp YP)^*\big) = \tau(YPY^*P^\perp) = 0.$$

Since the trace τ is arbitrary in Γ, it implies that $P^{\perp}YP = 0$. Therefore, by the arbitrariness of P, $PY = YP$ for all $P \in \mathcal{M}$. Since each self-adjoint operator is the norm limit of finite linear combinations of projections in von Neumann algebras, we obtain that $AY = YA$ for all $A \in \mathcal{M}$. Therefore, $\delta(A+B) - \delta(A) - \delta(B) \in \mathcal{Z}(\mathcal{M})$ for all $A, B \in \mathcal{M}$. □

Proof of Theorem 2.1. By Lemma 2.2, we can find out a nontrivial core-free projection P_1 with $c(P_1) = I$. Denote $P_2 = I - P_1$. Moreover, by the definition of central core and central carrier, we can obtain that P_2 is also core-free and $c(P_2) = I$. Let $\mathcal{M}_{ij} = P_i \mathcal{M} P_j$, $i, j = 1, 2$. Then $\mathcal{M} = \mathcal{M}_{11} + \mathcal{M}_{12} + \mathcal{M}_{21} + \mathcal{M}_{22}$ and each element $A \in \mathcal{M}$ can be represented as $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} = P_i A P_j$, $i, j = 1, 2$. We will finish the proof of our main theorem by considering a number of steps.

Claim 1. *There exists an element $T_0 \in \mathcal{M}$ such that $\delta(P_1) - [P_1, T_0] \in \mathcal{Z}(\mathcal{M})$, and $\delta(A_{ij}) = P_i \delta(A_{ij}) P_j + [A_{ij}, T_0]$ for all $A_{ij} \in \mathcal{M}_{ij}$ $(1 \leq i \neq j \leq 2)$.*

For each $A_{12} \in \mathcal{M}_{12}$, note that $A_{12} = [P_1, A_{12}]$. We have

$$\begin{aligned} \delta(A_{12}) &= \delta_{P_1, A_{12}}\big([P_1, A_{12}]\big) \\ &= \big[\delta_{P_1, A_{12}}(P_1), A_{12}\big] + \big[P_1, \delta_{P_1, A_{12}}(A_{12})\big] \\ &= \big[\delta(P_1), A_{12}\big] + \big[P_1, \delta(A_{12})\big] \\ &= \delta(P_1) A_{12} - A_{12} \delta(P_1) + P_1 \delta(A_{12}) - \delta(A_{12}) P_1. \end{aligned} \tag{2.2}$$

Furthermore, by multiplying (2.2) on left-hand side by P_i and on the right-hand side by P_j $(1 \leq i \neq j \leq 2)$, we deduce that

$$P_1 \delta(P_1) P_1 A_{12} = A_{12} P_2 \delta(P_1) P_2 \tag{2.3}$$

and

$$P_2 \delta(A_{12}) P_1 = 0. \tag{2.4}$$

Similarly, for each $A_{21} \in \mathcal{M}_{21}$, we also have

$$\delta(A_{21}) = \delta(A_{21}) P_1 - P_1 \delta(A_{21}) + A_{21} \delta(P_1) - \delta(P_1) A_{21}. \tag{2.5}$$

By multiplying (2.5) on the left-hand side by P_j and on the right-hand side by P_i $(1 \leq i \neq j \leq 2)$, we also deduce that

$$A_{21} P_1 \delta(P_1) P_1 = P_2 \delta(P_1) P_2 A_{21} \tag{2.6}$$

and

$$P_1 \delta(A_{21}) P_2 = 0.$$

Hence, by (2.3) and (2.6), we obtain that

$$\big[P_1 \delta(P_1) P_1 + P_2 \delta(P_1) P_2, A_{12}\big] = \big[P_1 \delta(P_1) P_1 + P_2 \delta(P_1) P_2, A_{21}\big] = 0.$$

From Lemma 2.2(2), it follows that

$$P_1 \delta(P_1) P_1 + P_2 \delta(P_1) P_2 \in \mathcal{Z}(\mathcal{M}).$$

Now we denote $T_0 = P_1 \delta(P_1) P_2 - P_2 \delta(P_1) P_1$. Then

$$\delta(P_1) - [P_1, T_0] = P_1 \delta(P_1) P_1 + P_2 \delta(P_1) P_2 \in \mathcal{Z}(\mathcal{M}).$$

Since $[A_{12}, T_0] = -A_{12}P_2\delta(P_1)P_1 + P_2\delta(P_1)P_1A_{12}$, (2.3) implies that

$$\begin{aligned}[A_{12}, T_0] &= -A_{12}P_2\delta(P_1)P_1 + P_2\delta(P_1)P_1A_{12} + P_1\delta(P_1)P_1A_{12} - A_{12}P_2\delta(P_1)P_2 \\ &= \delta(P_1)A_{12} - A_{12}\delta(P_1).\end{aligned}$$

Thus, by (2.2) and (2.4), we have proved that $\delta(A_{12}) = P_1\delta(A_{12})P_2 + [A_{12}, T_0]$. By the same argument as that used above, we can get $\delta(A_{21}) = P_2\delta(A_{21})P_1 + [A_{21}, T_0]$.

Remark 1. Let $\delta_{T_0}(A) = [A, T_0]$ for all $A \in \mathcal{M}$. Then δ_{T_0} is an inner derivation on $\mathcal{M}$. Denote $\phi = \delta - \delta_{T_0}$. Then it is easy to verify that ϕ is a 2-local Lie derivation.

Claim 2. *We have $\phi(P_1) \in \mathcal{Z}(\mathcal{M})$ and $\phi(A_{ij}) \in \mathcal{M}_{ij}$ ($1 \leq i \neq j \leq 2$).*

From Claim 1, we can get $\phi(P_1) \in \mathcal{Z}(\mathcal{M})$ and $\phi(A_{ij}) \in \mathcal{M}_{ij}$ $(1 \leq i \neq j \leq 2)$.

Claim 3. *We have $\phi(P_2) \in \mathcal{Z}(\mathcal{M})$.*

Note that $A_{12} = [A_{12}, P_2]$. Then we have

$$\begin{aligned}\phi(A_{12}) &= \phi_{A_{12},P_2}(A_{12}) \\ &= \phi_{A_{12},P_2}\big([A_{12}, P_2]\big) \\ &= \big[\phi_{A_{12},P_2}(A_{12}), P_2\big] + \big[A_{12}, \phi_{A_{12},P_2}(P_2)\big] \\ &= \big[\phi(A_{12}), P_2\big] + \big[A_{12}, \phi(P_2)\big].\end{aligned}$$

Hence, by Claim 2, we deduce that $[A_{12}, \phi(P_2)] = 0$. By the same argument as that used above, we can get $[\phi(P_2), A_{21}] = 0$. Thus, we can get $\phi(P_2) \in \mathcal{Z}(\mathcal{M})$ by Lemma 2.2(2).

Claim 4. *For each $A \in \mathcal{M}$, if $P_iAP_j = 0$ ($1 \leq i \neq j \leq 2$), then $P_i\phi(A)P_j = 0$ ($1 \leq i \neq j \leq 2$).*

For each $A \in \mathcal{M}$, by assumption we have $[P_i, A] = P_iA - AP_i = P_iAP_j - P_jAP_i = 0$. Furthermore, Claims 2 and 3 imply that

$$\begin{aligned}0 &= \phi_{P_i,A}\big([P_i, A]\big) \\ &= \big[\phi_{P_i,A}(P_i), A\big] + \big[P_i, \phi_{P_i,A}(A)\big] \\ &= \big[P_i, \phi(A)\big],\end{aligned}$$

which implies that $P_i\phi(A) - \phi(A)P_i = 0$. Hence, by multiplying the last equality on the right-hand side by P_j, we can get the desired equality.

Claim 5. *We have $\phi(A_{ii}) \in \mathcal{M}_{ii} + \mathcal{Z}(\mathcal{M})$, $i = 1, 2$.*

For every $A_{11} \in \mathcal{M}_{11}$, denote $\phi(A_{11}) = \sum_{i,j=1}^{2} B_{ij}$, where $B_{ij} = P_i\phi(A_{11})P_j$ $(1 \leq i, j \leq 2)$. Then, by Claim 4, $B_{12} = B_{21} = 0$. For each $S_{22} \in \mathcal{M}_{22}$, we obtain

$$\begin{aligned}0 &= \phi_{A_{11},S_{22}}\big([A_{11}, S_{22}]\big) \\ &= \big[\phi_{A_{11},S_{22}}(A_{11}), S_{22}\big] + \big[A_{11}, \phi_{A_{11},S_{22}}(S_{22})\big] \\ &= B_{22}S_{22} - S_{22}B_{22} + A_{11}\phi(S_{22}) - \phi(S_{22})A_{11}.\end{aligned} \tag{2.7}$$

Furthermore, multiplying (2.7) from both sides by P_2, we deduce that

$$B_{22}S_{22} = S_{22}B_{22};$$

that is,

$$B_{22} = P_2 Z_1$$

for some $Z_1 \in \mathcal{Z}(\mathcal{M})$. Thus $\phi(A_{11}) = B_{11} - P_1 Z_1 + Z_1 \in \mathcal{M}_{11} + \mathcal{Z}(\mathcal{M})$. By the same argument as that used above, we can get $\phi(A_{22}) \in \mathcal{M}_{22} + \mathcal{Z}(\mathcal{M})$.

Remark 2. In fact, both B_{11} and Z_1 in the Claim 6 are unique. Indeed, if $\phi(A_{11}) = B_{11} + P_2 Z_1$ and $\phi(A_{11}) = \widetilde{B}_{11} + P_2\widetilde{Z_1}$, then by multiplying the above equalities from both sides by P_1, we get $B_{11} = \widetilde{B}_{11}$, which implies that $P_2(Z_1 - \widetilde{Z_1}) = 0$. By Lemma 2.3, we can get $c(P_2)(Z_1 - \widetilde{Z_1})(Z_1 - \widetilde{Z_1})^* = 0$. Thus $Z_1 - \widetilde{Z_1} = 0$ since $c(P_2) = I$. Then we can define a mapping $f_i : \mathcal{M}_{ii} \to \mathcal{Z}(\mathcal{M})$ by $f_i(A_{ii}) = Z_i$ for all $A_{ii} \in \mathcal{M}_{ii}$ $(i = 1, 2)$. Obviously, f_i is homogeneous. Indeed, it suffices to see that the case $i = 1$ for all $\alpha \in \mathbb{C}$. We easily have $f_1(\alpha A_{11})P_2 = \phi(\alpha A_{11})P_2 = \alpha\phi(A_{11})P_2 = \alpha f_1(A_{11})P_2$. Therefore, by Lemma 2.3 and $c(P_2) = I$, we can get $f_1(\alpha A_{11}) = \alpha f_1(A_{11})$, as desired.

Now we define a homogeneous mapping ω on $\mathcal{M}$ such that

$$\begin{aligned}\omega(A) = \phi(P_1AP_1) + \phi(P_1AP_2) + \phi(P_2AP_1) + \phi(P_2AP_2)\\ - f_1(P_1AP_1) - f_2(P_2AP_2)\end{aligned}$$

for all $A \in \mathcal{M}$. Then, by Claims 2, 3, and 5, we get

(1) $\omega(P_i) = 0$, $i = 1, 2$,
(2) $\omega(A_{ij}) \in \mathcal{M}_{ij}$, $i, j = 1, 2$,
(3) $\omega(A_{ii}) = \phi(A_{ii}) - f_i(A_{ii})$ for each $A_{ii} \in \mathcal{M}_{ii}$, $i = 1, 2$,
(4) $\omega(A_{ij}) = \phi(A_{ij})$, $1 \le i \ne j \le 2$.

Claim 6. *We have $\omega(A_{ii}+B_{ii}) = \omega(A_{ii})+\omega(B_{ii})$ for all $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$ $(i = 1, 2)$.*

For all $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$, by Lemma 2.5, we have

$$\begin{aligned}\omega(A_{ii} + B_{ii}) - \omega(A_{ii}) - \omega(B_{ii}) &= \phi(A_{ii} + B_{ii}) - \phi(A_{ii}) - \phi(B_{ii})\\ &\quad - f_i(A_{ii} + B_{ii}) + f_i(A_{ii}) + f_i(B_{ii})\\ &\in \mathcal{Z}(\mathcal{M}).\end{aligned}$$

Since $\omega(A_{ii}+B_{ii})-\omega(A_{ii})-\omega(B_{ii}) \in \mathcal{M}_{ii}$, it follows that $\omega(A_{ii}+B_{ii})-\omega(A_{ii})-\omega(B_{ii}) \in \mathcal{M}_{ii}\cap\mathcal{Z}(\mathcal{M})$. Thus, by Lemma 2.2(3), we can get $\omega(A_{ii}+B_{ii})-\omega(A_{ii})-\omega(B_{ii}) = 0$.

Claim 7. *We have $\omega(A_{ij} + B_{ij}) = \omega(A_{ij}) + \omega(B_{ij})$ for all $A_{ij}, B_{ij} \in \mathcal{M}_{ij}$ $(1 \le i \ne j \le 2)$.*

For all $A_{12}, B_{12} \in \mathcal{M}_{12}$, by Lemma 2.5 and Claims 2 and 3, we have

$$\begin{aligned}\omega(A_{12} + B_{12}) &= \phi(A_{12} + B_{12})\\ &= \phi_{A_{12}+B_{12},P_2}\big([A_{12} + B_{12}, P_2]\big)\\ &= \big[\phi_{A_{12}+B_{12},P_2}(A_{12} + B_{12}), P_2\big] + \big[A_{12} + B_{12}, \phi_{A_{12}+B_{12},P_2}(P_2)\big]\end{aligned}$$

$$
\begin{aligned}
&= \big[\phi_{A_{12}+B_{12},P_2}(A_{12}) + \phi_{A_{12}+B_{12},P_2}(B_{12}), P_2\big] \\
&\quad + \big[A_{12} + B_{12}, \phi_{A_{12}+B_{12},P_2}(P_2)\big] \\
&= \big[\phi(A_{12}) + \phi(B_{12}), P_2\big] + \big[A_{12} + B_{12}, \phi(P_2)\big] \\
&= \big[\phi(A_{12}) + \phi(B_{12}), P_2\big] \\
&= \phi(A_{12}) + \phi(B_{12}) \\
&= \omega(A_{12}) + \omega(B_{12}).
\end{aligned}
$$

Claim 8. *We have that* $[\omega(A), B] + [A, \omega(B)] = 0$ *whenever* $[A, B] = 0$.

By Lemma 2.5, we have

$$
\begin{aligned}
0 &= \phi_{A,B}\big([A, B]\big) \\
&= \big[\phi_{A,B}(A), B\big] + \big[A, \phi_{A,B}(B)\big] \\
&= \big[\phi(A), B\big] + \big[A, \phi(B)\big] \\
&= \big[\phi(A_{11}) + \phi(A_{12}) + \phi(A_{21}) + \phi(A_{22}), B\big] \\
&\quad + \big[A, \phi(B_{11}) + \phi(B_{12}) + \phi(B_{21}) + \phi(B_{22})\big] \\
&= \big[\omega(A) + f_1(A_{11}) + f_2(A_{22}), B\big] + \big[A, \omega(B) + f_1(B_{11}) + f_2(B_{22})\big] \\
&= \big[\omega(A), B\big] + \big[A, \omega(B)\big].
\end{aligned}
$$

By Claims 6 and 7, we easily obtain that ω is linear. From Lemma 2.4 and Claim 8, it follows that there exists an element $Z_0 \in \mathcal{Z}(\mathcal{M})$, a derivation d, and a linear mapping $h_1 : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ such that $\omega(A) = d(A) + h_1(A) + Z_0 A$. And we claim that $Z_0 = 0$. Indeed, by the properties of ω, $0 = \omega(P_1) = d(P_1) + h_1(P_1) + Z_0 P_1$. Multiplying this identity on both sides by P_i $(i = 1, 2)$, we then obtain $h_1(P_1)P_1 = -Z_0 P_1$ and $h_1(P_1)P_2 = 0$, which yields $Z_0 = 0$ by $c(P_1) = c(P_2) = I$ and Lemma 2.3. Thus we get

$$\omega(A) = d(A) + h_1(A)$$

for all $A \in \mathcal{M}$.

We denote $\delta_E = d + \delta_{T_0}$ and $h(A) = h_1(A) + f_1(P_1AP_1) + f_2(P_2AP_2) + \phi(A) - \phi(P_1AP_1) - \phi(P_1AP_2) - \phi(P_2AP_1) - \phi(P_2AP_2)$. Obviously, by Lemma 2.5, h is a center-valued homogenous mapping. Then the definition of ω implies that

$$
\begin{aligned}
\delta(A) &= \phi(A) + \delta_{T_0}(A) \\
&= \omega(A) + f_1(P_1AP_1) + f_2(P_2AP_2) + \phi(A) - \phi(P_1AP_1) \\
&\quad - \phi(P_1AP_2) - \phi(P_2AP_1) - \phi(P_2AP_2) + \delta_{T_0}(A) \\
&= d(A) + h_1(A) + f_1(P_1AP_1) + f_2(P_2AP_2) + \phi(A) - \phi(P_1AP_1) \\
&\quad - \phi(P_1AP_2) - \phi(P_2AP_1) - \phi(P_2AP_2) + \delta_{T_0}(A) \\
&= \delta_E(A) + h(A)
\end{aligned}
$$

for all $A \in \mathcal{M}$.

Since every finite von Neumann algebra has a unique center-valued trace, it follows that each nonzero element in the center of $\mathcal{Z}(\mathcal{M})$ cannot be the form

$\sum_{i=1}^{n}[A_i, B_i]$ for $A_i, B_i \in \mathcal{M}$. Then for every $A \in \mathcal{M}$ and commutator $X \in \mathcal{M}$, we have

$$\begin{aligned} h(A+X) - h(A) &= \delta(A+X) - \delta_E(A+X) - \delta(A) + \delta_E(A) \\ &= [A+X, S_{A+X,A}] + \tau_{A+X,A}(A+X) - [A+X, E] \\ &\quad - [A, S_{A+X,A}] - \tau_{A+X,A}(A) + [A, E] \\ &= [A+X, S_{A+X,A} - E] - [A, S_{A+X,A} - E] \\ &= [X, S_{A+X,A} - E]. \end{aligned}$$

Thus h is a homogenous mapping of $\mathcal{M}$ into its center which annihilates each commutator of $\mathcal{M}$. □

Corollary 2.6. *Every linear 2-local Lie derivation on a finite von Neumann algebra without central summands of type I_1 is a Lie derivation.*

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