Ann. Funct. Anal. 10 (2019), no. 1, 106–121
https://doi.org/10.1215/20088752-2018-0012
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

ON J-FRAMES RELATED TO MAXIMAL DEFINITE SUBSPACES

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Communicated by O. Christensen

Abstract. We propose a definition of frames in Krein spaces which generalizes the concept of J-frames defined relatively recently by Giribet, Maestripieri, Martínez-Pería, and Massey. The difference consists in the fact that a J-frame is related to maximal definite subspaces $\mathcal{M}_{\pm}$ which are not assumed to be uniformly definite. The latter allows us to extend the set of J-frames. In particular, some J-orthogonal Schauder bases can be interpreted as J-frames.

1. Introduction

Usually, frames are defined in a Hilbert space setting: let $\mathfrak{H}$ be an (infinite-dimensional) Hilbert space with an inner product $(\cdot,\cdot)$. A frame for $\mathfrak{H}$ is a family of vectors $\mathcal{F}=\{f_n\}$ that satisfies inequalities

$$A\|f\|^2 \le \sum_{n\in\mathbb{N}} \left|(f,f_n)\right|^2 \le B\|f\|^2, \quad f\in\mathfrak{H}, \tag{1.1}$$

for constants $0<A\le B<\infty$, which are called *frame bounds.* Sometimes, it is convenient to work with frames defined in terms of an indefinite inner product. The development of such an ideology leads to the following definition. Denote by $(\mathfrak{H},[\cdot,\cdot])$ a Krein space with indefinite inner product $[\cdot,\cdot]$ and fundamental symmetry J. The associated Hilbert space $(\mathfrak{H},(\cdot,\cdot))$ is endowed with the positive inner product $(\cdot,\cdot)=[J\cdot,\cdot]$ and the norm $\|\cdot\|=\sqrt{(\cdot,\cdot)}$.

Received Jan. 6, 2018; Accepted May 14, 2018.
First published online Nov. 17, 2018.
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2010 *Mathematics Subject Classification.* Primary 47B50; Secondary 46C20.
Keywords. Krein space, frame, J-orthogonal sequence, Schauder basis.

Definition 1.1 ([6, Definition 3.1]). Let $(\mathfrak{H}, [\cdot,\cdot])$ be a Krein space. A family of vectors $\mathcal{F} = \{f_n\}$ is called a *frame* if

$$A\|f\|^2 \le \sum_{n\in\mathbb{N}} \big|[f, f_n]\big|^2 \le B\|f\|^2, \quad f \in \mathfrak{H},$$

for some constants $0 < A \le B < \infty$.

Since the operator of fundamental symmetry J is unitary and self-adjoint in $\mathfrak{H}$, Definition 1.1 implies that $\mathcal{F}$ is a frame in the Krein space $(\mathfrak{H}, [\cdot,\cdot])$ if and only if $\mathcal{F}$ is a frame with the same frame bounds in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$ (see [6, Theorem 3.3]). This means that the definition just rephrases the conventional definition of frames in terms of Krein spaces. It seems natural to consider a more general setting, that is, to define frames in the Hilbert space $\mathfrak{H}_W$ with W-metric $(W\cdot,\cdot)$ (see [4, Definition 2.12]), which is the completion of $\mathfrak{H}$ with respect to $(|W|\cdot,\cdot)$, where a bounded self-adjoint operator W with $\ker W = \{0\}$ is the "light version" of the operator of fundamental symmetry J (see [2, Section 1.6]).

Another reason for studying frames in Krein spaces has to do with the signal processing problem. The existence of various decompositions of a given Krein space $(\mathfrak{H}, [\cdot,\cdot])$ into direct sums of positive and negative subspaces (see Section 2 for the Krein spaces terminology) allows one to construct effective filters for the signals considering vectors $f \in \mathfrak{H}$ as disturbances if their indefinite inner products $[f, f]$ are close to zero (see the discussion at the beginning of Section 3 in [9]). These kind of ideas give rise to the following definition of J-frame. Let $\mathcal{F} = \{f_n\}$ be a Bessel sequence in the Hilbert space $\mathfrak{H}$; that is, $\mathcal{F}$ satisfies the upper bound condition in (1.1). Then the synthesis operator

$$T\{c_n\} = \sum_{n\in\mathbb{N}} c_n f_n, \quad \{c_n\} \in l_2(\mathbb{N}) \tag{1.2}$$

is a bounded map of $l_2(\mathbb{N})$ into $\mathfrak{H}$. Consider the orthogonal decomposition

$$l_2(\mathbb{N}) = l_2(\mathbb{N}_+) \oplus l_2(\mathbb{N}_-), \tag{1.3}$$

where $\mathbb{N}_+ = \{n \in \mathbb{N} : [f_n, f_n] \ge 0\}$, $\mathbb{N}_- = \{n \in \mathbb{N} : [f_n, f_n] < 0\}$, and determine the restrictions of T onto $l_2(\mathbb{N}_\pm)$: $T_\pm = T\upharpoonright_{l_2(\mathbb{N}_\pm)}$.

Definition 1.2 ([9, Definition 3.1]). The Bessel sequence $\mathcal{F} = \{f_n\}$ is called a *J-frame* if the ranges $\mathcal{R}(T_\pm)$ are, respectively, maximal uniformly positive and maximal uniformly negative subspaces of the Krein space $(\mathfrak{H}, [\cdot,\cdot])$.

The requirement of being maximal uniformly definite imposed on $\mathcal{R}(T_\pm)$ is sufficiently strong, and an elementary analysis carried out in [9] shows that each J-frame $\mathcal{F}$ in the Krein space $(\mathfrak{H}, [\cdot,\cdot])$ has to be a conventional frame in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$. For this reason, each J-frame is a frame in the sense of Definition 1.1. The inverse implication is not true. In particular, there exist orthonormal bases of the Hilbert space $\mathfrak{H}$ which cannot be J-frames. Indeed, let $\mathcal{L}$ be a hypermaximal neutral subspace of the Krein space $(\mathfrak{H}, [\cdot,\cdot])$; then $\mathfrak{H} = \mathcal{L} \oplus J\mathcal{L}$. If $\{f_n\}$ is an orthonormal basis of $\mathcal{L}$, then $\mathcal{F} = \{f_n\} \cup \{Jf_n\}$ is an orthonormal basis of $(\mathfrak{H}, (\cdot,\cdot))$. However, $\mathcal{F}$ cannot be a J-frame because $\mathcal{R}(T_+) = \mathfrak{H}$ and $\mathcal{R}(T_-) = \{0\}$.

The last few years have seen the publication of many papers devoted to the development of full-scale frame theory based on Definition 1.1 (see [1], [4]–[6]) as well as on Definition 1.2 (see [8], [9], [12], [15]). In particular, J-fusion frames were defined and studied in [1] and [12]. However, in our opinion, the above definitions do not completely fit the ideology of Krein spaces and some modification that provides deeper insights into the structural subtleties of frames in Krein spaces is still needed. The matter is that the concept of Krein spaces is more reach in contrast to Hilbert ones due to the possibility of generating infinitely many (not necessarily equivalent) definite inner products $(\cdot,\cdot)$ beginning with a given indefinite inner product $[\cdot,\cdot]$. For this reason, it seems natural to define frames in a Krein space $(\mathfrak{H},[\cdot,\cdot])$ in terms of frame inequalities based on the indefinite inner product $[\cdot,\cdot]$ without any relation to frames in the associated Hilbert space $(\mathfrak{H},(\cdot,\cdot))$.

Denote

$$\mathcal{M}_\pm = \overline{\operatorname{span}\{\mathcal{F}_\pm\}}, \qquad \mathcal{F}_\pm = \{f_n\}_{n\in\mathbb{N}_\pm}. \tag{1.4}$$

Due to [9, Theorem 3.9], Definition 1.2 can be rewritten as follows: $\mathcal{F} = \{f_n\}$ is a J-frame if and only if $\mathcal{F}$ is a conventional frame in the associated Hilbert space $(\mathfrak{H},(\cdot,\cdot))$, the conditions $\mathcal{M}_\pm \cap \mathcal{M}_\pm^{[\perp]} = \{0\}$ hold (the J-orthogonal complements $\mathcal{M}_\pm^{[\perp]}$ of $\mathcal{M}_\pm$ are defined in (2.2)), and there exist constants $B_- \le A_- < 0 < A_+ \le B_+$ such that

$$\begin{aligned} A_+[f,f] &\le \sum_{n\in\mathbb{N}_+} \left|[f,f_n]\right|^2 \le B_+[f,f], \quad \forall f \in \mathcal{M}_+, \\ A_-[f,f] &\le \sum_{n\in\mathbb{N}_-} \left|[f,f_n]\right|^2 \le B_-[f,f], \quad \forall f \in \mathcal{M}_-. \end{aligned} \tag{1.5}$$

It is easy to see that the families $\mathcal{F}_\pm$ are conventional frames of the Hilbert spaces $(\mathcal{M}_\pm, \pm[\cdot,\cdot])$ and the subspaces $\mathcal{M}_\pm$ coincide with $\mathcal{R}(T_\pm)$. Therefore, the $\mathcal{M}_\pm$'s have to be maximal uniformly definite in the Krein space $(\mathfrak{H},[\cdot,\cdot])$. Weakening this condition, we generalize the concept of J-frames.

Definition 1.3. Let $(\mathfrak{H},[\cdot,\cdot])$ be a Krein space with fundamental symmetry J. A family of vectors $\mathcal{F} = \{f_n\}$ is called a *J-frame* if the subspaces $\mathcal{M}_\pm$ defined by (1.4) are maximal definite in $(\mathfrak{H},[\cdot,\cdot])$ and there are constants $B_- \le A_- < 0 < A_+ \le B_+$ such that the inequalities (1.5) hold.

The aim of this work is to develop a theory of J-frames which is based on Definition 1.3. The results essentially depend on coinciding the direct sum $\mathcal{D} = \mathcal{M}_+ \dot{+} \mathcal{M}_-$ with $\mathfrak{H}$. If $\mathcal{D} = \mathfrak{H}$, then the subspaces $\mathcal{M}_\pm$ have to be uniformly definite in the Krein space $(\mathfrak{H},[\cdot,\cdot])$, the conditions of Definition 1.2 are satisfied, and our results in Section 3.1 are close to [9] (see Remark 3.3).

If $\mathcal{D} \ne \mathfrak{H}$, then at least one of $\mathcal{M}_\pm$ loses the property of being uniformly definite and the new inner product $(\cdot,\cdot)_1$ defined on $\mathcal{D}$ is not equivalent to the initial one. In this case a J-frame $\mathcal{F} = \{f_n\}$ cannot be a Bessel sequence in $\mathfrak{H}$ and, hence, Definition 1.2 cannot be applied. Moreover, this case cannot be studied within the framework of Hilbert spaces $\mathfrak{H}_W$ with W-metric (see Section 2.2).

We show that each J-frame $\mathcal{F}$ can be realized as a conventional frame in the new Hilbert space $(\widehat{\mathfrak{H}}, (\cdot,\cdot)_1)$ (see Proposition 3.4) and that the reconstruction formula (3.9) holds for elements of $\widehat{\mathfrak{H}}$ (see Proposition 3.6). The results are simplified when the subspaces $\mathcal{M}_\pm$ are J-orthogonal (see Corollary 3.7). Definition 1.3 allows one to consider some J-orthogonal sequences and J-orthogonal Schauder bases as J-frames (see Proposition 4.2, Corollary 4.4). This opens up the possibility of describing the common part $\mathfrak{D} = \mathfrak{H} \cap \widehat{\mathfrak{H}}$ and the subset $\mathcal{D}_{\rm un} \subset \mathfrak{D}$, where the series $f = \sum_{n\in\mathbb{N}} c_n f_n$ converges unconditionally in terms of the corresponding coefficients $\{c_n\}$ (see Proposition 4.6, Theorem 4.7). The eigenfunctions of the shifted harmonic oscillator are considered as examples of J-frames.

This article is organized as follows. We begin with an elementary presentation of the Krein spaces theory. The monograph [2] is recommended as complementary reading on the subject. In Section 3, we show that each J-frame can be considered as a conventional frame in some Hilbert space. The corresponding reconstruction formulas are rewritten in terms of indefinite inner products. Special attention is paid to the case where the corresponding subspaces $\mathcal{M}_\pm$ are J-orthogonal. Section 4 deals with a special class of J-frames: J-orthogonal sequences.

In what follows, $\mathfrak{H}$ is a complex Hilbert space with inner product $(\cdot,\cdot)$ linear in the first argument. Sometimes it is useful to specify the inner product associated with $\mathfrak{H}$. In this case, the notation $(\mathfrak{H}, (\cdot,\cdot))$ is used. All topological notions refer to the Hilbert space topology. For instance, a subspace of $\mathfrak{H}$ is a linear manifold in $\mathfrak{H}$ which is closed with respect to the norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$. The symbols $\mathcal{D}(A)$ and $\mathcal{R}(A)$ denote the domain and the range of a linear operator A. The restriction of A onto a set $\mathcal{D}$ is denoted as $A\restriction_{\mathcal{D}}$. The notation $\mathcal{L}_1 \oplus \mathcal{L}_2$ means the orthogonal (with respect to an inner product) direct sum of two subspaces $\mathcal{L}_i$.

2. Elements of Krein spaces theory

2.1. Notation and terminology. Let $\mathfrak{H}$ be a complex linear space with an indefinite inner product (indefinite metric) $[\cdot,\cdot]$. The space $(\mathfrak{H}, [\cdot,\cdot])$ is called a *Krein space* if $\mathfrak{H}$ admits a decomposition

$$\mathfrak{H} = \mathfrak{H}_+ [\dot{+}] \mathfrak{H}_-, \tag{2.1}$$

which is an orthogonal (with respect to $[\cdot,\cdot]$) direct sum of two Hilbert spaces $(\mathfrak{H}_+, [\cdot,\cdot])$ and $(\mathfrak{H}_-, -[\cdot,\cdot])$. The decomposition (2.1) is considered *fundamental* and it induces the positive inner product

$$(f,g) := [f_+, g_+] - [f_-, g_-], \quad f = f_+ + f_-, g = g_+ - g_-, f_\pm, g_\pm \in \mathfrak{H}_\pm$$

and the associated Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$. The operator

$$Jf = f_+ - f_-, \quad f = f_+ + f_-, f_\pm \in \mathfrak{H}_\pm,$$

called the *operator of fundamental symmetry*, is unitary and self-adjoint in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$. The fundamental symmetry J allows one to express $(\cdot,\cdot)$ in terms of an indefinite metric: $(\cdot,\cdot) = [J\cdot,\cdot]$.

A closed subspace $\mathcal{L}$ of the Krein space $(\mathfrak{H}, [\cdot,\cdot])$ is called *neutral, negative*, or *positive* if all nonzero elements $f \in \mathcal{L}$ are, respectively, neutral $[f,f] = 0$, negative

$[f,f]<0$, or positive $[f,f]>0$. Further, $\mathcal{L}$ is called *uniformly positive, uniformly negative* if, respectively, $[f,f]\geq\alpha(f,f)$, $-[f,f]\geq\alpha(f,f)$ for certain $\alpha>0$ and all $f\in\mathcal{L}$. A subspace $\mathcal{L}$ of $\mathfrak{H}$ is called *definite* if it is either positive or negative. The term *uniformly definite* is defined accordingly.

In each of the above-mentioned classes we can define maximal subspaces. For instance, a closed positive subspace $\mathcal{L}$ is called *maximal positive* if $\mathcal{L}$ is not a proper subspace of a positive subspace in $\mathfrak{H}$. A maximal neutral subspace $\mathcal{L}$ is called *hypermaximal* if the Hilbert space $(\mathfrak{H},(\cdot,\cdot))$ admits the decomposition $\mathfrak{H}=\mathcal{L}\oplus J\mathcal{L}$. Subspaces $\mathcal{L}_1$, $\mathcal{L}_2$ of $\mathfrak{H}$ are said to be *J-orthogonal* if $[f,g]=0$ for all $f\in\mathcal{L}_1$ and $g\in\mathcal{L}_2$. The J-orthogonal complement of a subspace $\mathcal{L}$ of $\mathfrak{H}$ is defined as

$$\mathcal{L}^{[\perp]}=\left\{g\in\mathfrak{H}:[f,g]=0,\forall f\in\mathcal{L}\right\} \tag{2.2}$$

and it is a subspace of the Hilbert space $(\mathfrak{H},(\cdot,\cdot))$. Let A be a densely defined operator in a Krein space $(\mathfrak{H},[\cdot,\cdot])$. Repeating the standard definition of the adjoint operator with the use of an indefinite inner product $[\cdot,\cdot]$, we define the adjoint operator A^+ of A. In this case, $[Af,g]=[f,A^+g]$ for all $f\in\mathcal{D}(A)$ and all $g\in\mathcal{D}(A^+)$. An operator A is called *J-self-adjoint* if $A=A^+$. An operator A is called *J-positive* if $[Af,f]>0$ for $f(\neq 0)\in\mathcal{D}(A)$.

2.2. Positive inner products generated by an indefinite inner product. *Case I.* Let $\mathcal{M}_\pm$ be maximal uniformly definite (positive/negative) subspaces of the Krein space $(\mathfrak{H},[\cdot,\cdot])$. It is easy to see from [10, Lemma 2.1] that

$$\mathfrak{H}=\mathcal{M}_+\dot{+}\mathcal{M}_-. \tag{2.3}$$

The direct sum (2.3) determines the operator

$$J_{\mathcal{M}}f=f_{\mathcal{M}_+}-f_{\mathcal{M}_-},\quad f=f_{\mathcal{M}_+}+f_{\mathcal{M}_-},f_{\mathcal{M}_\pm}\in\mathcal{M}_\pm \tag{2.4}$$

and the new positive inner product

$$(f,g)_1:=[f_{\mathcal{M}_+},g_{\mathcal{M}_+}]-[f_{\mathcal{M}_-},g_{\mathcal{M}_-}], \tag{2.5}$$

which is equivalent to $(\cdot,\cdot)$. The operator $J_{\mathcal{M}}$ is bounded in $\mathfrak{H}$ and $J_{\mathcal{M}}^2=I$.

Lemma 2.1. *Let $(\cdot,\cdot)_1$ be defined by (2.5). Then*

$$(\cdot,\cdot)_1=[\mathcal{C}\cdot,\cdot],\quad \mathcal{C}=\frac{1}{2}(J_{\mathcal{M}}+J_{\mathcal{M}}^+), \tag{2.6}$$

where $J_{\mathcal{M}}^+$ is the adjoint of $J_{\mathcal{M}}$ with respect to the indefinite inner product $[\cdot,\cdot]$.

Proof. It follows from (2.4) and (2.5) that, for all $f,g\in\mathfrak{H}$,

$$\begin{aligned}(f,g)_1&=\frac{1}{4}\big(\big[(I+J_{\mathcal{M}})f,(I+J_{\mathcal{M}})g\big]-\big[(I-J_{\mathcal{M}})f,(I-J_{\mathcal{M}})g\big]\big)\\&=\frac{1}{2}\big([J_{\mathcal{M}}f,g]+[f,J_{\mathcal{M}}g]\big)=\frac{1}{2}\big[(J_{\mathcal{M}}+J_{\mathcal{M}}^+)f,g\big],\end{aligned}$$

and that completes the proof. $\square$

Relation (2.6) implies that $\mathcal{C}$ is a bounded, J-self-adjoint, and J-positive operator in the Krein space $(\mathfrak{H}, [\cdot,\cdot])$. Moreover, $0 \in \rho(\mathcal{C})$.

Case II. Assume that the subspaces $\mathcal{M}_\pm$ are maximal definite but that at least one of them loses the property of being uniformly definite. Then the direct sum

$$\mathcal{D} = \mathcal{M}_+ \dot{+} \mathcal{M}_- \tag{2.7}$$

does not coincide with $\mathfrak{H}$ and $\mathcal{D}$ is a dense set in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$.

Similarly to case I, the direct sum (2.7) generates by the formula (2.5) a new inner product $(\cdot,\cdot)_1$ defined on $\mathcal{D}$. In contrast to the previous case, the inner product $(\cdot,\cdot)_1$ is *not equivalent* to the initial product $(\cdot,\cdot)$, and the linear space $\mathcal{D}$ endowed with $(\cdot,\cdot)_1$ is a pre-Hilbert space. Let the Hilbert space $(\widehat{\mathfrak{H}}, (\cdot,\cdot)_1)$ be the completion of $\mathcal{D}$ with respect to $(\cdot,\cdot)_1$. By construction, it can be decomposed as

$$\widehat{\mathfrak{H}} = \widehat{\mathcal{M}}_+ \oplus_1 \widehat{\mathcal{M}}_-, \tag{2.8}$$

where the subspaces $\widehat{\mathcal{M}}_\pm$ of $\widehat{\mathfrak{H}}$ are the completion of the subspaces $\mathcal{M}_\pm$ and $\oplus_1$ indicates the orthogonality of $\widehat{\mathcal{M}}_\pm$ with respect to $(\cdot,\cdot)_1$.

The Krein space structure of $\widehat{\mathfrak{H}}$ can be introduced by (2.8). Considering (2.8) as the fundamental decomposition of $\widehat{\mathfrak{H}}$, we define the new indefinite inner product

$$[f,g]_1 := [f_{\widehat{\mathcal{M}}_+}, g_{\widehat{\mathcal{M}}_+}] + [f_{\widehat{\mathcal{M}}_-}, g_{\widehat{\mathcal{M}}_-}], \quad f,g \in \widehat{\mathfrak{H}}. \tag{2.9}$$

The Hilbert space associated with the Krein space $(\widehat{\mathfrak{H}}, [\cdot,\cdot]_1)$ coincides with $(\widehat{\mathfrak{H}}, (\cdot,\cdot)_1)$. The corresponding operator of fundamental symmetry $J_{\widehat{\mathcal{M}}}$ is the closure in $(\widehat{\mathfrak{H}}, (\cdot,\cdot)_1)$ of the operator $J_{\mathcal{M}}$ defined by (2.4). The operator $J_{\widehat{\mathcal{M}}}$ is bounded and self-adjoint in $(\widehat{\mathfrak{H}}, [\cdot,\cdot]_1)$ and $(\cdot,\cdot)_1 = [J_{\widehat{\mathcal{M}}}\cdot,\cdot]_1$.

On the other hand, since $(\cdot,\cdot)$ and $(\cdot,\cdot)_1$ are not equivalent, the operator $J_{\mathcal{M}}$ is a closed unbounded operator in $(\mathfrak{H}, (\cdot,\cdot))$. The adjoint $J_{\mathcal{M}}^+$ of $J_{\mathcal{M}}$ with respect to $[\cdot,\cdot]$ in $\mathfrak{H}$ has the domain $\mathcal{D}(J_{\mathcal{M}}^+) = \mathcal{M}_-^{[\perp]} \dot{+} \mathcal{M}_+^{[\perp]}$ and its action is defined similarly to (2.4), where the maximal definite subspaces $\mathcal{M}_\mp^{[\perp]}$ are used instead of $\mathcal{M}_\pm$. In general, the domains $\mathcal{D}(J_{\mathcal{M}})$ and $\mathcal{D}(J_{\mathcal{M}}^+)$ do not coincide. Therefore, the operator $\mathcal{C}$ in (2.6) can only be defined on $\mathcal{D}_0 = \mathcal{D}(J_{\mathcal{M}}) \cap \mathcal{D}(J_{\mathcal{M}}^+)$. By analogy with the proof of Lemma 2.1, we obtain that

$$(f,g)_1 = [\mathcal{C}f, g], \quad \forall f \in \mathcal{D}_0, \forall g \in \mathcal{D}. \tag{2.10}$$

Case III. For the important particular case where the $\mathcal{M}_\pm$'s are *J-orthogonal*, the subspace $\mathcal{D}_0$ coincides with $\mathcal{D}$ and the operator $\mathcal{C}$ is J-self-adjoint. Moreover, $\mathcal{C}$ coincides with $J_{\mathcal{M}}$ and hence, it is characterized by the additional condition $\mathcal{C}^2 f = f$ for all $f \in \mathcal{D}(\mathcal{C}) = \mathcal{D}(J_{\mathcal{M}})$. In this case, the operator $\mathcal{C}$ can be presented as $\mathcal{C} = Je^Q$, where Q is an unbounded self-adjoint operator in $(\mathfrak{H}, (\cdot,\cdot))$ anticommuting with J (see [13, Theorem 2.1]). It follows from above that $e^Q = J\mathcal{C} = JJ_{\mathcal{M}}$. Therefore, $e^{Q/2} = \sqrt{JJ_{\mathcal{M}}}$. The operator Q characterizes the "deviation" of subspaces $\mathcal{M}_\pm$ with respect to the subspaces $\mathfrak{H}_\pm$ in the fundamental decomposition (2.1). Precisely (see [13]),

$$\mathcal{M}_+ = (I - \tanh Q/2)\mathfrak{H}_+, \qquad \mathcal{M}_- = (I - \tanh Q/2)\mathfrak{H}_-. \tag{2.11}$$

Let $\mathfrak{D}$ be the *energetic space* constructed by the self-adjoint operator e^Q. In other words, $\mathfrak{D}$ denotes the completion of $\mathcal{D} = \mathcal{D}(\mathcal{C}) = \mathcal{D}(e^Q)$ with respect to the energetic norm

$$\|f\|_{\mathrm{en}}^2 := \|f\|^2 + \|f\|_1^2 = \|f\|^2 + [\mathcal{C}f, f] = \|f\|^2 + (e^Q f, f), \quad f \in \mathcal{D}.$$

The energetic space $\mathfrak{D}$ coincides with $\mathcal{D}(e^{Q/2})$ and it is a Hilbert space $(\mathfrak{D}, (\cdot,\cdot)_{\mathrm{en}})$ with respect to the energetic scalar product $(\cdot,\cdot)_{\mathrm{en}} = (\cdot,\cdot) + (e^{Q/2}\cdot, e^{Q/2}\cdot)$. In the remainder of this article, we will sometimes consider $\mathfrak{D}$ as the set of elements (without topology). In this context, the term *energetic linear manifold* will be used.

Comparing the definitions of $\widehat{\mathfrak{H}}$ and $\mathfrak{D}$ leads to the conclusion that the *energetic linear manifold* $\mathfrak{D}$ *coincides with the common part of* $\mathfrak{H}$ *and* $\widehat{\mathfrak{H}}$, that is, $\mathfrak{D} = \mathfrak{H} \cap \widehat{\mathfrak{H}}$. Obviously, the formula (2.10) can be extended onto $\mathfrak{D}$ as follows:

$$(f, g)_1 = (e^{Q/2} f, e^{Q/2} g), \quad \forall f, g \in \mathfrak{D}. \tag{2.12}$$

The Hilbert space $\widehat{\mathfrak{H}}$ cannot be interpreted as a Hilbert space $\mathfrak{H}_W$ with W-metric considered in [4, Section 2.1]. Indeed, if $\widehat{\mathfrak{H}} = \mathfrak{H}_W$ for some self-adjoint W, then $\mathfrak{D} = \mathfrak{H}$ (because $\mathfrak{H}_W$ is the completion of $\mathfrak{H}$), which is impossible since $e^{Q/2}$ is an unbounded operator.

Lemma 2.2. *If the* $\mathcal{M}_\pm$*'s are* J*-orthogonal, then the indefinite inner products* $[\cdot,\cdot]$ *and* $[\cdot,\cdot]_1$ *coincide on the energetic linear manifold* $\mathfrak{D}$*. In the Hilbert space* $(\widehat{\mathfrak{H}}, (\cdot,\cdot)_1)$*, the indefinite inner product* $[\cdot,\cdot]$ *(defined originally on* $\mathfrak{D}$*) can be continuously extended onto* $\widehat{\mathfrak{H}}$ *and this extension coincides with* $[\cdot,\cdot]_1$*.*

Proof. By virtue of (2.9) and the J-orthogonality of $\mathcal{M}_\pm$,

$$\begin{aligned}[f, g] &= [f_{\mathcal{M}_+} + f_{\mathcal{M}_-}, g_{\mathcal{M}_+} + g_{\mathcal{M}_-}] \\ &= [f_{\mathcal{M}_+}, g_{\mathcal{M}_+}] + [f_{\mathcal{M}_-}, g_{\mathcal{M}_-}] = [f, g]_1, \quad \forall f, g \in \mathcal{D}.\end{aligned}$$

The obtained relation is extended onto the energetic space $\mathfrak{D}$ by the continuity because $\max\{|[f,f]|, |[f,f]_1|\} \leq \|f\|_{\mathrm{en}}^2$. The second statement of the lemma follows from the coincidence of $[\cdot,\cdot]$ and $[\cdot,\cdot]_1$ on $\mathfrak{D}$ and the definition of $(\widehat{\mathfrak{H}}, (\cdot,\cdot)_1)$. ☐

3. Frames in Krein spaces and reconstruction formulas

Since the subspaces $\mathcal{M}_\pm$ in Definition 1.3 are assumed to be maximal definite, the linear manifold $\mathcal{D}$ in (2.7) is a dense set in $(\mathfrak{H}, (\cdot,\cdot))$. Properties of the corresponding J-frame $\mathcal{F}$ depend on whether $\mathcal{D}$ coincides with $\mathfrak{H}$ or not.

3.1. The set $\mathcal{D}$ coincides with $\mathfrak{H}$.

Proposition 3.1. *Let* $\mathcal{F} = \{f_n\}$ *be a* J*-frame in the sense of Definition 1.3, and let* $\mathcal{D} = \mathfrak{H}$*. Then* $\mathcal{F}$ *is a conventional frame in the Hilbert space* $(\mathfrak{H}, (\cdot,\cdot)_1)$ *with the frame bounds* $A = \max\{A_+, -A_-\}, B = \max\{B_+, -B_-\}$*. Moreover,* $\mathcal{F}$ *is a* J*-frame in the sense of Definition 1.2.*

Proof. If $\mathcal{D} = \mathfrak{H}$, then the subspaces $\mathcal{M}_\pm$ are maximal uniformly definite in the Krein space $(\mathfrak{H}, [\cdot,\cdot])$ and (2.7) coincides with (2.3). The decomposition (2.3) determines the new inner product $(\cdot,\cdot)_1$ on $\mathfrak{H}$ which is equivalent to $(\cdot,\cdot)$. By the construction, $|[f,g]| = |(f,g)_1|$ where $f,g \in \mathcal{M}_+$ or $f,g \in \mathcal{M}_-$. Moreover, the subspaces $\mathcal{M}_\pm$ are orthogonal with respect to $(\cdot,\cdot)_1$. Therefore, the inequalities (1.5) can be rewritten as

$$A\|f\|_1^2 \le \sum_{n\in\mathbb{N}} \left|(f,f_n)_1\right|^2 \le B\|f\|_1^2, \quad \forall f \in \mathfrak{H}, \tag{3.1}$$

where $A = \max\{A_+, -A_-\}$ and $B = \max\{B_+, -B_-\}$. Therefore, $\mathcal{F}$ is a frame in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot)_1)$ with the frame bounds $A \le B$. The concept of J-frame defined in Definition 1.2 corresponds to the case where the $\mathcal{M}_\pm$'s are maximal uniformly definite subspaces. Therefore, the J-frame $\mathcal{F}$ considered above is also a J-frame in the sense of Definition 1.2. □

By virtue of Proposition 3.1, the synthesis operator $T : l_2(\mathbb{N}) \to \mathfrak{H}$ associated to J-frame $\mathcal{F}$ is well defined. Denote by $T^\dagger$ its adjoint as an operator mapping of the Krein space $(\mathfrak{H}, [\cdot,\cdot])$ into the Hilbert space $l_2(\mathbb{N})$, that is, $[T\{c_n\}, f] = (\{c_n\}, T^\dagger f)_{l_2(\mathbb{N})}$, where $\{c_n\} \in l_2(\mathbb{N})$, $f \in \mathfrak{H}$. It is easy to verify that $T^\dagger f = \{[f, f_n]\}$. The operator $S : \mathfrak{H} \to \mathfrak{H}$,

$$Sf = TT^\dagger f = \sum_{n=1}^{\infty}[f, f_n]f_n, \tag{3.2}$$

is called a *J-frame operator associated to J-frame $\mathcal{F}$.*

Proposition 3.2. *The J-frame operator S is a J-positive, J-self-adjoint bounded operator in the Krein space $(\mathfrak{H}, [\cdot,\cdot])$ and $0 \in \rho(S)$. The reconstruction formula*

$$f = \sum_{n=1}^{\infty}[f, S^{-1}f_n]f_n = \sum_{n=1}^{\infty}[f, f_n]S^{-1}f_n \tag{3.3}$$

holds, where the series converge in the Hilbert space $\mathfrak{H}$.

Proof. By Proposition 3.1, the J-frame $\mathcal{F}$ is a conventional frame in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot)_1)$. The corresponding frame operator S_1 has the form

$$S_1 f = \sum_{n=1}^{\infty}(f, f_n)_1 f_n \tag{3.4}$$

and it is a positive self-adjoint operator in $(\mathfrak{H}, (\cdot,\cdot)_1)$ such that $0 \in \rho(S_1)$. By virtue of (2.6) and (3.2),

$$S_1 = S\mathcal{C}, \qquad S^{-1} = \mathcal{C}S_1^{-1}. \tag{3.5}$$

It follows from (3.5) that S^{-1} and S are J-positive, J-self-adjoint bounded operators in the Krein space $(\mathfrak{H}, [\cdot,\cdot])$. The reconstruction formula for the frame $\mathcal{F}$ in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot)_1)$ is

$$f = \sum_{n=1}^{\infty}(f, S_1^{-1}f_n)_1 f_n = \sum_{n=1}^{\infty}(f, f_n)_1 S_1^{-1}f_n, \tag{3.6}$$

and its first part is transformed to the first part of (3.3) since

$$(f, S_1^{-1}f_n)_1 = [\mathcal{C}f, S_1^{-1}f_n] = [f, \mathcal{C}S_1^{-1}f_n] = [f, S^{-1}f_n].$$

Multiplying (3.6) by the operator $\mathcal{C}$ and using the second relation in (3.5), we obtain

$$\mathcal{C}f = \sum_{n=1}^{\infty}(f, f_n)_1 S^{-1}f_n = \sum_{n=1}^{\infty}[\mathcal{C}f, f_n]S^{-1}f_n,$$

which is equivalent to the second relation in (3.3) (since $\mathcal{R}(\mathcal{C}) = \mathfrak{H}$). Thus, the series (3.3) are convergent in $\mathfrak{H}$ with respect to $(\cdot,\cdot)_1$. Obviously, they remain convergent with respect to $(\cdot,\cdot)$ since the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_1$ are equivalent. □

Remark 3.3. In [9], the Hilbert space $l_2(\mathbb{N})$ was considered as a Krein space with indefinite metric generated by the fundamental decomposition (1.3) and the adjoint T^+ of the synthesis operator T was calculated as an operator acting between *two Krein spaces* $\mathfrak{H}$ *and* $l_2(\mathbb{N})$, that is, $[T\{c_n\}, f] = [\{c_n\}, T^+f]_{l_2(\mathbb{N})}$, where $[\{c_n\},\{c_n\}]_{l_2(\mathbb{N})} = \sum_{\mathbb{N}}\sigma_n|c_n|^2$ and $\sigma_n = \operatorname{sgn}[f_n, f_n]$. It is easy to check that

$$T^+f = \{\sigma_n[f, f_n]\}.$$

Then the corresponding J-frame operator $\widetilde{S} = TT^+$ acts as (cf. (3.2))

$$\widetilde{S} = \sum_{n=1}^{\infty}\sigma_n[f, f_n]f_n.$$

(A detailed investigation of this can be found in [7].) By virtue of (2.4) and (3.2), the J-frame operators S and $\widetilde{S}$ are related as follows:

$$J_{\mathcal{M}}S = \widetilde{S}, \qquad SJ_{\mathcal{M}}^+ = \widetilde{S}. \tag{3.7}$$

Hence, $0 \in \rho(\widetilde{S})$ and $\widetilde{S}$ is a J-self-adjoint operator in $(\mathfrak{H}, [\cdot,\cdot])$ since $\widetilde{S}^+ = (J_{\mathcal{M}}S)^+ = SJ_{\mathcal{M}}^+ = \widetilde{S}$. However, in contrast to S, the J-frame operator $\widetilde{S}$ cannot be J-positive. Indeed, $[\widetilde{S}f, f] = -\sum_{n\in\mathbb{N}_-}|[f, f_n]|^2 < 0$ for $f \in \mathcal{M}_+^{[\perp]}$. By virtue of (3.7), the reconstruction formula (3.3) takes the form:

$$f = \sum_{n=1}^{\infty}\sigma_n[f, \widetilde{S}^{-1}f_n]f_n = \sum_{n=1}^{\infty}\sigma_n[f, f_n]\widetilde{S}^{-1}f_n.$$

3.2. The set $\mathcal{D}$ does not coincide with $\mathfrak{H}$.

Proposition 3.4. *Let $\mathcal{F} = \{f_n\}$ be a J-frame in the sense of Definition 1.3, and let $\mathcal{D} \neq \mathfrak{H}$. Then the J-frame $\mathcal{F}$ is a conventional frame in the new Hilbert space $(\widehat{\mathfrak{H}}, (\cdot,\cdot)_1)$ with the frame bounds $A = \max\{A_+, -A_-\}$, $B = \max\{B_+, -B_-\}$.*

Proof. If $\mathcal{D} \neq \mathfrak{H}$, then at least one of the subspaces $\mathcal{M}_\pm$ is maximal definite but not uniformly definite. In this case, the direct sum (2.7) generates the new Hilbert space $(\widehat{\mathfrak{H}}, (\cdot,\cdot)_1)$. Similarly to the proof of Proposition 3.1, we rewrite the inequalities (1.5) as (3.1) for all $f \in \mathcal{D}$. By virtue of [3, Lemma 5.1.7], (3.1) can be extended onto $\widehat{\mathfrak{H}}$ (since $\mathcal{D}$ is dense in $\widehat{\mathfrak{H}}$). Therefore, $\mathcal{F}$ is a conventional frame in the Hilbert space $(\widehat{\mathfrak{H}}, (\cdot,\cdot)_1)$ with the frame bounds $A \leq B$. $\square$

Corollary 3.5. *Let $\mathcal{F}$ satisfy the assumptions of Proposition 3.4. Then $\mathcal{F}$ cannot be a Bessel sequence in $\mathfrak{H}$.*

Proof. Let, for instance, $\mathcal{M}_+$ be maximal definite but not uniformly definite. By virtue of Proposition 3.4, the reconstruction formula (3.6) holds for all $f \in \widehat{\mathfrak{H}}$ (the series converge in the Hilbert space $(\widehat{\mathfrak{H}}, (\cdot,\cdot)_1)$). In particular, if $f \in \widehat{\mathcal{M}_+}$, then

$$f = \sum_{n=1}^{\infty} (f, S_1^{-1} f_n)_1 f_n, \tag{3.8}$$

where $\{(f, S_1^{-1} f_n)_1\} \in l_2(\mathbb{N})$ and $(f, S_1^{-1} f_n)_1 = 0$ if $f_n \in \mathcal{M}_-$ (the latter follows from (2.8) and the fact that the $\widehat{\mathcal{M}}_\pm$'s are invariant subspaces for S_1).

Assume that $\mathcal{F}$ is a Bessel sequence in $\mathfrak{H}$. Then the series in (3.8) converges in $(\mathfrak{H}, (\cdot,\cdot))$ (see [11, Theorem 7.4]) and its limit coincides with f (since $\|g\|_1^2 = [g,g] \leq \|g\|^2$ for all $g \in \mathcal{M}_+$). This means that $\widehat{\mathcal{M}_+} = \mathcal{M}_+$, which is impossible. Therefore, $\mathcal{F}$ cannot be a Bessel sequence in $\mathfrak{H}$. $\square$

Proposition 3.6. *Let $\mathcal{F}$ satisfy the assumptions of Proposition 3.4. Then, for all $f \in \widehat{\mathfrak{H}}$,*

$$f = \sum_{n=1}^{\infty} [f, S^{-1} f_n]_1 f_n = \sum_{n=1}^{\infty} [f, f_n]_1 S^{-1} f_n, \tag{3.9}$$

where $Sf = \sum_{n=1}^{\infty} [f, f_n]_1 f_n$ is a $J_{\widehat{\mathcal{M}}}$-self-adjoint bounded operator in the Krein space $(\widehat{\mathfrak{H}}, [\cdot,\cdot]_1)$. The series (3.9) converge in the Hilbert space $(\widehat{\mathfrak{H}}, (\cdot,\cdot)_1)$.

Proof. By the construction, the indefinite inner product $[\cdot,\cdot]_1$ coincides with $[\cdot,\cdot]$ on $\mathcal{M}_\pm$, and the $\mathcal{M}_\pm$'s are dense sets in the subspaces $\widehat{\mathcal{M}}_\pm$ of the Hilbert space $\widehat{\mathfrak{H}}$. This means that we can replace, in Definition 1.3, $[\cdot,\cdot]$ and $\mathcal{M}_\pm$ by $[\cdot,\cdot]_1$ and $\widehat{\mathcal{M}}_\pm$, respectively. Therefore, *the J-frame $\mathcal{F}$ in the Krein space $(\mathfrak{H}, [\cdot,\cdot])$ is simultaneously a $J_{\widehat{\mathcal{M}}}$-frame in the Krein space* $(\widehat{\mathfrak{H}}, [\cdot,\cdot]_1)$. In the latter case, since decomposition (2.8) holds, we can apply Proposition 3.2 for the $J_{\widehat{\mathcal{M}}}$-frame $\mathcal{F}$, which completes the proof. $\square$

The above results are simplified when the subspaces $\mathcal{M}_\pm$ are J-orthogonal.

Corollary 3.7. *Let $\mathcal{F}$ satisfy the assumptions of Proposition 3.4, and let the subspaces $\mathcal{M}_\pm$ be J-orthogonal. Then for all elements f from the energetic linear manifold $\mathfrak{D}$ (see Section 2.2),*

$$f = \sum_{n=1}^{\infty} [f, f_n] S^{-1} f_n, \tag{3.10}$$

where $Sf = \sum_{n=1}^{\infty} [f, f_n] f_n$ and the series converges in the Hilbert space $(\widehat{\mathfrak{H}}, (\cdot,\cdot)_1)$.

Proof. By virtue of Proposition 3.6, the formula (3.9) holds for all $f \in \widehat{\mathfrak{H}}$. In a particular case where $f \in \mathfrak{D}$, Lemma 2.2 allows one to modify the second part of (3.9) and the operator S by the replacement of $[f, f_n]_1$ with $[f, f_n]$. $\square$

Remark 3.8. The additional assumption that $S^{-1}f_n \in \mathfrak{D}$ and Lemma 2.2 lead to the conclusion that $f = \sum_{n=1}^{\infty}[f, S^{-1}f_n]f_n$ for all $f \in \mathfrak{D}$.

Proposition 3.9. *Let $\mathcal{F} = \{f_n\}$ satisfy the assumptions of Proposition 3.4, and let the subspaces $\mathcal{M}_\pm$ be J-orthogonal. Then there exists an unbounded self-adjoint operator Q in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$, which anticommutes with J and such that the sequence $\{e^{Q/2}f_n\}$ is a conventional frame in $(\mathfrak{H}, (\cdot,\cdot))$ with the frame bounds $A = \max\{A_+, -A_-\}, B = \max\{B_+, -B_-\}$.*

Proof. Each direct sum of J-orthogonal maximal definite subspaces $\mathcal{M}_\pm$ generates an unbounded self-adjoint operator Q in the Hilbert space $\mathfrak{H}$ which anticommutes with J (see Section 2.2). In this case, the relation (2.12) holds.

According to Proposition 3.4, $\mathcal{F}$ is a frame in the Hilbert space $(\widehat{\mathfrak{H}}, (\cdot,\cdot)_1)$. Therefore, the inequalities (3.1) hold for all $f \in \mathfrak{D}$. Using (2.12), we rewrite them as follows: $A\|\gamma\|^2 \le \sum_{n\in\mathbb{N}}|(\gamma, e^{Q/2}f_n)|^2 \le B\|\gamma\|^2$, where $\gamma = e^{Q/2}f$ runs the dense set $\mathcal{R}(e^{Q/2})$ in $(\mathfrak{H}, (\cdot,\cdot))$. The obtained inequalities can be extended onto $\mathfrak{H}$ due to [3, Lemma 5.1.7]. Therefore, $\{e^{Q/2}f_n\}$ is a frame in $(\mathfrak{H}, (\cdot,\cdot))$. $\square$

The inverse statement is also true.

Proposition 3.10. *Assume that $\{g_n\}$ is a frame in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$ such that each g_n belongs to one of the subspaces $\mathfrak{H}_+$ or $\mathfrak{H}_-$ of the fundamental decomposition (2.1) and there exists an unbounded self-adjoint operator Q in $\mathfrak{H}$ which anticommutes with J and such that $\{\cosh Q/2 g_n\}$ is a complete set in $\mathfrak{H}$. Then the sequence $\{e^{-Q/2}g_n\}$ is a J-frame in the sense of Definition 1.3.*

Proof. The operator $\cosh Q/2 = \frac{1}{2}(e^{Q/2} + e^{-Q/2})$ commutes with J (since Q anticommutes with J). Therefore, the vector $x_n = \cosh Q/2 g_n$ belongs to the same subspace ($\mathfrak{H}_+$ or $\mathfrak{H}_-$) that g_n does. Denote $x_n^\pm = x_n$ if $x_n \in \mathfrak{H}_\pm$. According to Section 2.2, the operator Q uniquely determines a J-orthogonal pair of maximal definite subspaces $\mathcal{M}_\pm$ (see (2.11)). Denote

$$\mathcal{M}'_+ = \overline{\operatorname{span}\{(I - \tanh Q/2)x_n^+\}}, \qquad \mathcal{M}'_- = \overline{\operatorname{span}\{(I - \tanh Q/2)x_n^-\}}.$$

In view of (2.11) and the fact that the set $\{x_n = \cosh Q/2 g_n\}$ is complete in $\mathfrak{H}$, we decide that $\mathcal{M}'_\pm = \mathcal{M}_\pm$. Moreover,

$$\begin{aligned} f_n &= \left(I - \tanh\frac{Q}{2}\right)x_n = \left(I - \tanh\frac{Q}{2}\right)\cosh\frac{Q}{2}g_n \\ &= \cosh\frac{Q}{2}g_n - \sinh\frac{Q}{2}g_n = e^{\frac{-Q}{2}}g_n. \end{aligned}$$

Therefore, $g_n = e^{Q/2}f_n$, where f_n belongs to one of the sets $\mathcal{M}_\pm$ and $\operatorname{sgn}[g_n, g_n] = \operatorname{sgn}[f_n, f_n]$. The frame inequalities for the frame $\{g_n\}$ can be rewritten as

$$A\|\gamma\|^2 \le \sum_{n\in\mathbb{N}}\left|(\gamma, e^{Q/2}f_n)\right|^2 \le B\|\gamma\|^2, \quad \gamma \in \mathfrak{H}. \tag{3.11}$$

Assuming in (3.11) that $\gamma = e^{Q/2}f$, where $f \in \mathcal{M}_\pm$, and using (2.5) and (2.12), we obtain the inequalities (1.5) with the constants $A_\pm = \pm A$ and $B_\pm = \pm B$ for all $f \in \mathcal{M}_\pm$. Therefore, $\{f_n = e^{-Q/2}g_n\}$ is a J-frame in the sense of Definition 1.3. □

Remark 3.11. If the operator Q in the statement above is *bounded*, then the subspaces $\mathcal{M}_\pm$ in (2.11) are maximal uniformly definite and the sequence $\{e^{-Q/2}g_n\}$ turns out to be a J-frame in the sense of Definition 1.2.

4. J-orthogonal sequences and J-frames

4.1. J-orthogonal sequences. Let a sequence $\{f_n\}_{n=1}^{\infty}$ be J-orthogonal (i.e., $[f_n, f_m] = 0$ for $n \neq m$). Then the subspaces $\mathcal{M}_\pm$ defined by (1.4) are J-orthogonal in the Krein space $(\mathfrak{H}, [\cdot,\cdot])$.

Proposition 4.1. *For a J-orthogonal sequence $\mathcal{F} = \{f_n\}$ the following are equivalent:*

(i) *$\mathcal{F}$ is a Riesz basis in $\mathfrak{H}$;*
(ii) *$\mathcal{F}$ is a J-frame in the sense of Definition 1.2.*

Proof. If $\mathcal{F}$ is a Riesz basis, then the synthesis operator (1.2) is well defined and $\mathcal{R}(T_\pm) = \mathcal{M}_\pm$, where the $\mathcal{M}_\pm$'s are defined by (1.4). Moreover, (2.3) holds and therefore, $\mathcal{M}_\pm$ must be maximal uniformly definite in $(\mathfrak{H}, [\cdot,\cdot])$. Implication (i) $\to$ (ii) is proved. Conversely, if $\mathcal{F}$ is a J-frame in the sense of Definition 1.2, then $\mathcal{F}$ should be an orthogonal system in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot)_1)$ satisfying (3.1). The latter means that $\mathcal{F}$ is a Riesz basis in $\mathfrak{H}$. □

Proposition 4.2. *A J-orthogonal sequence $\mathcal{F} = \{f_n\}$ is a J-frame in the sense of Definition 1.3 if and only if $\mathcal{F}$ is J-bounded, that is,*

$$0 < A \le \big|[f_n, f_n]\big| \le B < \infty, \quad \forall f_n \in \mathcal{F}, \tag{4.1}$$

and the subspaces $\mathcal{M}_\pm$ defined by (1.4) are maximal definite in the Krein space $(\mathfrak{H}, [\cdot,\cdot])$.

Proof. The J-orthogonality of $\mathcal{F}$ and (4.1) allow one to verify the inequalities (1.5) with constants $A_\pm = \pm A, B_\pm = \pm B$ for all $f \in \operatorname{span}\{\mathcal{F}_\pm\}$ (see (1.4)). By the construction, $\operatorname{span}\{\mathcal{F}_+\}$ is a dense set in $\mathcal{M}_+$. Therefore, the first inequality in (1.5) can be extended onto $\mathcal{M}_+$ (see [3, Lemma 5.1.7]). Similarly, since $\operatorname{span}\{\mathcal{F}_-\}$ is dense in $\mathcal{M}_-$, the second inequality in (1.5) holds for all $f \in \mathcal{M}_-$. Therefore, $\mathcal{F}$ is a J-frame in the sense of Definition 1.3. Conversely, if $\mathcal{F}$ is a J-frame, then $\mathcal{M}_\pm$ must be maximal definite and (1.5) with $f = f_n$ gives (4.1), where $A = \max\{A_+, -A_-\}$ and $B = \max\{B_+, -B_-\}$. □

Remark 4.3. The maximality of $\mathcal{M}_\pm$ ensures the completeness of the corresponding J-orthogonal sequence $\{f_n\}$ in $(\mathfrak{H}, (\cdot,\cdot))$. The inverse statement is not true (see, e.g., [13]). We can only state that the subspaces $\mathcal{M}_\pm$ are definite in $(\mathfrak{H}, [\cdot,\cdot])$ and their direct sum is dense in $(\mathfrak{H}, (\cdot,\cdot))$.

Let $\{f_n\}$ be a complete J-orthogonal sequence. Its biorthogonal sequence $\{\gamma_n\}$ consists of the elements

$$\gamma_n = \frac{Jf_n}{[f_n, f_n]}. \tag{4.2}$$

Hence, $[\gamma_n, \gamma_n] = 1/[f_n, f_n]$. By Proposition 4.2, if one of sequences $\{f_n\}$ or $\{\gamma_n\}$ is a J-frame, then another one is also a J-frame.

Let a complete J-orthogonal sequence $\mathcal{F} = \{f_n\}$ be a J-frame in the sense of Definition 1.3. Then Corollary 3.7 holds and the operator $Sf = \sum_{n=1}^{\infty}[f, f_n]f_n$ is well defined on the energetic linear manifold $\mathfrak{D}$. It is easy to see that $S^{-1}f_n = f_n/[f_n, f_n]$ and therefore, the decomposition (3.10) can be rewritten as

$$f = \sum_{n=1}^{\infty} \frac{[f, f_n]}{[f_n, f_n]} f_n = \sum_{n=1}^{\infty} (f, \gamma_n) f_n, \quad f \in \mathfrak{D}, \tag{4.3}$$

where the series (4.3) converges with respect to the norm $\|\cdot\|_1 = \sqrt{(e^Q \cdot, \cdot)}$, where e^Q is a positive self-adjoint operator in $\mathfrak{H}$ such that $e^Q f_n = JJ_{\mathcal{M}} f_n = \sigma_n J f_n = \sigma_n [f_n, f_n]\gamma_n = |[f_n, f_n]|\gamma_n$.

4.2. *J*-orthogonal Schauder bases. In what follows, we suppose that $\mathcal{F} = \{f_n\}$ is a Schauder basis of $(\mathfrak{H}, (\cdot, \cdot))$.

Corollary 4.4. *A J-orthogonal Schauder basis $\mathcal{F}$ turns out to be a J-frame if and only if $\mathcal{F}$ is J-bounded.*

Proof. The proof follows from Proposition 4.2 and the fact that the subspaces $\mathcal{M}_{\pm}$ are maximal definite in the case of a J-orthogonal Schauder basis (see [2]). □

Corollary 4.5. *If a J-orthogonal Schauder basis $\mathcal{F} = \{f_n\}$ is a J-frame, then the series (4.3) is convergent in the energetic space $(\mathfrak{D}, (\cdot, \cdot)_{\mathrm{en}})$, that is, it is convergent with respect to the energetic norm $\|\cdot\|_{\mathrm{en}}$.*

Proof. It follows from the definition of energetic norm in Section 2.2 and the convergence of (4.3) with respect to $\|\cdot\|$ (since $\mathcal{F}$ is Schauder basis). □

The energetic linear manifold $\mathfrak{D}$ can be easily described.

Proposition 4.6. *Let a J-orthogonal Schauder basis $\mathcal{F} = \{f_n\}$ be a J-frame. Then $f \in \mathfrak{D}$ if and only if the sequence $\{[f, f_n]\}$ belongs to $l_2(\mathbb{N})$.*

Proof. If $f \in \mathfrak{D}$, then (4.3) converges simultaneously in $\mathfrak{H}$ and $\widehat{\mathfrak{H}}$. Moreover, (4.1) holds and $\{f_n\}$ is a bounded orthogonal basis in $\widehat{\mathfrak{H}}$. This means that the sequence $\{[f, f_n]\}$ belongs to $l_2(\mathbb{N})$. Conversely, assume that $f \in \mathfrak{H}$ and $\{[f, f_n]\} \in l_2(\mathbb{N})$. Then the sequence $\{c_n\}$, where

$$c_n = \frac{[f, f_n]}{[f_n, f_n]} = (f, \gamma_n) = \frac{(f, f_n)_1}{(f_n, f_n)_1},$$

also belongs to $l_2(\mathbb{N})$. Hence, $g_m = \sum_{n=1}^{m} c_n f_n$ is a Cauchy sequence in $(\widehat{\mathfrak{H}}, (\cdot, \cdot)_1)$ since $\{f_n\}$ is a bounded orthogonal basis in $\widehat{\mathfrak{H}}$. The same is true for the space $\mathfrak{H}$ since $\{f_n\}$ is a Schauder basis in $\mathfrak{H}$. Therefore, $\{g_m\}$ is a Cauchy sequence in $(\mathfrak{D}, (\cdot, \cdot)_{\mathrm{en}})$ and f belongs to $\mathfrak{D}$ (since $f = \lim g_m = \sum_{n=1}^{\infty}(f, \gamma_n) f_n$). □

Generally, we cannot state that the convergence of (4.3) is unconditional in $\mathfrak{D}$. Let us discuss this point in detail. Denote

$$\mathcal{D}_{\mathrm{un}} = \left\{ f \in \mathfrak{H} : \text{the series (4.3) converges unconditionally in } \mathfrak{H} \right\}.$$

Theorem 4.7. *Let a J-orthogonal Schauder basis $\mathcal{F} = \{f_n\}$ be a J-frame. Then*

$$\mathcal{D}_{\mathrm{un}} \subset \mathcal{M}_+ \dot{+} \mathcal{M}_- \subset \mathfrak{D},$$

where the inclusions are strict and $f \in \mathcal{D}_{\mathrm{un}}$ if $\{[f, f_n]\}$ belongs to $l_1(\mathbb{N})$.

Proof. First of all, we note that $\mathcal{F}$ is bounded in $\mathfrak{H}$, that is, $0 < C \le \|f_n\|^2 \le D < \infty$. Indeed, the J-orthonormal Schauder basis $\{f_n/[f_n, f_n]\}$ is bounded in $\mathfrak{H}$ (see [2, p. 76]). This means that $\{f_n\}$ is bounded too (since (4.1) holds due to Corollary 4.4).

Let $f \in \mathcal{D}_{\mathrm{un}}$. Then, simultaneously with (4.3), the series $\sum_{n\in\mathbb{N}_\pm} \frac{[f,f_n]}{[f_n,f_n]} f_n$ converge to elements $f_\pm$ in the Hilbert space $\mathfrak{H}$ (see, e.g., [11, Theorem 3.10]). By the construction, $f_\pm \in \mathcal{M}_\pm$. Therefore, $f = f_+ + f_-$ belongs to $\mathcal{M}_+ \dot{+} \mathcal{M}_-$. The sequence $\{f_n\}_{\mathbb{N}_+}$ is a basis for the Hilbert spaces $(\mathcal{M}_+, (\cdot, \cdot))$ and $(\widehat{\mathcal{M}}_+, [\cdot, \cdot])$. In the last case, $\{f_n\}_{\mathbb{N}_+}$ is a bounded orthogonal basis in $\widehat{\mathcal{M}}_+$. Hence, the relation

$$f = \sum_{n\in\mathbb{N}_+} c_n f_n, \quad \{c_n\} \in l_2(\mathbb{N}_+) \tag{4.4}$$

establishes a one-to-one correspondence between Hilbert spaces $\widehat{\mathcal{M}}_+$ and $l_2(\mathbb{N}_+)$.

Suppose that $\mathcal{M}_+ \subset \mathcal{D}_{\mathrm{un}}$. Then the basis $\{f_n\}_{\mathbb{N}_+}$ of $(\mathcal{M}_+, (\cdot, \cdot))$ is unconditional and bounded. Hence, $\{f_n\}_{\mathbb{N}_+}$ is a Riesz basis of $(\mathcal{M}_+, (\cdot, \cdot))$ (see [11, Theorem 7.13]) and the formula (4.4) describes all vectors of $\mathcal{M}_+$. The latter means that $\mathcal{M}_+ = \widehat{\mathcal{M}}_+$, which is impossible. Therefore, $\mathcal{M}_+$ does not belong to $\mathcal{D}_{\mathrm{un}}$. The same statement holds for $\mathcal{M}_-$. The strict inclusion $\mathcal{D}_{\mathrm{un}} \subset \mathcal{M}_+ \dot{+} \mathcal{M}_-$ is proved.

Let $f \in \mathfrak{H}$ be such that $\{[f, f_n]\} \in l_1(\mathbb{N})$. Then the sequence $\{[f, f_n]/[f_n, f_n]\}$ also belongs to $l_1(\mathbb{N})$ (since (4.1) holds). This means that the series (4.3) converges unconditionally (see [11, Lemma 3.5]). Therefore, $f \in \mathcal{D}_{\mathrm{un}}$. □

4.3. Example. Examples of J-frames can be easily constructed with the use of Proposition 3.10. The principal ingredients are a self-adjoint operator Q anticommuting with J and a conventional frame $\{g_n\}$ such that each g_n belongs to one of the subspaces $\mathfrak{H}_\pm$ of the fundamental decomposition (2.1).

Let us consider a concrete example. In the space $\mathfrak{H} = L_2(\mathbb{R})$, we define a fundamental symmetry $J = \mathcal{P}$ as the space parity operator $\mathcal{P}f(x) = f(-x)$. The subspaces $\mathfrak{H}_\pm$ of the fundamental decomposition (2.1) coincide with the subspaces of even and odd functions of $L_2(\mathbb{R})$. The Hermite functions

$$g_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-x^2/2}, \quad H_n(x) = e^{x^2/2}\Big(x - \frac{d}{dx}\Big)^n e^{-x^2/2}$$

form an orthonormal basis of $L_2(\mathbb{R})$ (particular case of a frame). The functions g_n are either odd or even functions. Therefore, $g_n \in \mathfrak{H}_\pm$.

Since the g_n's are entire functions, the complex shift of g_n's can be defined as

$$f_n(x) = g_n(x + ia), \quad a \in \mathbb{R} \setminus \{0\}, n = 0, 1, 2, \ldots.$$

The functions $\{f_n\}$ are eigenvectors of the shifted harmonic oscillator

$$H = -\frac{d^2}{dx^2} + x^2 + 2iax.$$

Applying the Fourier transform $Ff = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,dx$ to f_n, we get $Ff_n = e^{-a\xi}Fg_n$. Therefore, $f_n = F^{-1}e^{-a\xi}Fg_n$. The last relation can be rewritten as

$$f_n = e^{-Q/2}g_n,$$

where $Q = -2ai\frac{d}{dx}$ is an unbounded self-adjoint operator in $L_2(\mathbb{R})$ which anticommutes with $\mathcal{P}$. Since $Fg_n = (-i)^n g_n(\xi)$, the completeness of $\{\cosh Q/2g_n\}$ is equivalent to the completeness of functions $\{\cosh(a\xi)g_n(\xi)\}$ in $L_2(\mathbb{R})$. The latter can be established by analogy with the proof of [14, Lemma 2.5].

By Proposition 3.10, $\mathcal{F} = \{f_n\}$ is a J-frame in the sense of Definition 1.3. Moreover, $\mathcal{F}$ is a J-orthogonal sequence since

$$[f_n, f_m] = (Je^{-Q/2}g_n, e^{-Q/2}g_m) = (e^{Q/2}Jg_n, e^{-Q/2}g_m) = \operatorname{sgn}[g_n, g_n]\delta_{nm}.$$

However, $\mathcal{F}$ cannot be a Schauder basis. Indeed, if $\mathcal{F}$ is a basis, then $\|f_n\|\|\gamma_n\| \leq C$, where $\{\gamma_n\}$ is the biorthogonal sequence (see [11, Theorem 4.13]). This inequality can be transformed to $\|f_n\|^2 \leq C'$ by virtue of (4.1) and (4.2). On the other hand, $\lim_{n\to\infty}\frac{1}{\sqrt{n}}\log\|f_n\|^2 = 2^{3/2}|a|$ (see [14, Theorem 2.6]) which contradicts the inequality above.

The subspaces

$$\mathcal{M}_+ = \overline{\operatorname{span}\{f_n : n = 2k\}}, \qquad \mathcal{M}_- = \overline{\operatorname{span}\{f_n : n = 2k+1\}}, \quad k \in \mathbb{N}\cup\{0\}$$

are maximal positive/negative in the Krein space $(L_2(\mathbb{R}), [\cdot,\cdot])$ with the indefinite inner product $[f,g] = \int_{\mathbb{R}} f(-x)\overline{g(x)}\,dx$. Their direct sum is a dense set in $L_2(\mathbb{R})$ due to [14, Lemma 2.5].

Acknowledgments. The authors would like to express their gratitude to Professor F. Martínez Pería for his inspiring lecture on J-frames delivered during the STA-2017 conference and for forwarding articles which were helpful in these investigations.

The authors' work was partially supported by the Polish Ministry of Science and Higher Education.

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