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UNITARY REPRESENTATIONS OF INFINITE WREATH PRODUCTS

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Abstract. Using C^*-algebraic techniques and especially AF-algebras, we present a complete classification of the continuous unitary representations for a class of infinite wreath product groups. These nonlocally compact groups are realized by a topological completion of the semidirect product of the countably infinite symmetric group acting on the countable direct product of a finite Abelian group.

1. Introduction

The continuous unitary representations for the unitary group $U(H)$ (with the strong operator topology) of a separable Hilbert space H or the full symmetric group $\overline{S}(\infty)$ (with the topology of pointwise convergence) on a countable set are equivalent to the tame representations of dense direct limit subgroups. For $U(H)$, the subgroup is $U(\infty)$, the inductive limit unitary group $\lim_{\rightarrow} U(n)$; while for $\overline{S}(\infty)$ it is $S(\infty)$, the group of finitely supported permutations on positive integers.

These results suggest the following general problem: if $\overline{G}$ is a topological completion of a countable discrete subgroup G, determine the representation theory of

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$\overline{G}$ through the restriction of its representations to G. In this context, a tame representation of G is exactly the restriction of a continuous unitary representation of $\overline{G}$. In general, $\overline{G}$ may not be locally compact.

In this article, we study the representation theory of an infinite wreath product group G relative to a fixed finite Abelian group A. The subgroup G is a countable locally finite group obtained as a canonical semidirect product of $S(\infty)$ with the countably infinite direct sum $\bigoplus A$. In Section 2.3, we give a natural topology on G whose completion $\overline{G}$ is a nonlocally compact topological group.

We classify the tame representations of G, that is, the restriction to G of the continuous unitary representations of the completion group $\overline{G}$ using C^*-algebraic techniques applied to the group algebra of G. Since the completion $\overline{G}$ is not locally compact, classical methods do not directly apply. For the unitary group and the full symmetric group, Olshanski [8] introduced the method of semigroups to classify tame representations. The C^*-algebra approach we use here provides an alternative method to semigroups. (See [1], [2] for other applications of C^*-algebraic techniques to study the representation theory of nonlocally compact groups.)

2. Background

2.1. Wreath product representations. Let $S(n)$ be the symmetric group on n letters. As usual we identify its irreducible representations $\lambda \in S(n)^\wedge$ (the dual space of $S(n)$) with the partitions of n; that is, $\lambda : \lambda_1 \geq \cdots \geq \lambda_\ell > 0$, where $\sum \lambda_j = n$ and ℓ is the number of parts of the partition. The representation λ has an alternate description as a shape or (Ferrers) diagram $D(\lambda)$, that is, a finite subset of $\mathbb{N}^2$ where $(i, j) \in D(\lambda)$ if and only if $1 \leq j \leq \lambda_i$, for $1 \leq i \leq \ell$. As a diagram, ℓ is the number of its rows. The diagram of the partition of 0 is simply $\emptyset$. Note that the trivial representation of $S(n)$ is given by the diagram with a single row ($\ell = 1$). The restriction of $\lambda \in S(n)^\wedge$ to $S(n-1)$ decomposes simply as a direct sum of $\mu \in S(n-1)^\wedge$, where $|D(\lambda) \setminus D(\mu)| = 1$.

2.1.1. Classification. Let A be a finite Abelian group with dual group $\widehat{A} = \{\omega_1, \ldots, \omega_{|A|}\}$, where ω_1 denotes the trivial character which is identically 1. Let $G(n)$ be the wreath product of $S(n)$ with $\prod_{j=1}^n A$; that is, $G(n)$ is the semidirect product of the natural action of the symmetric group $S(n)$ on the product of Abelian groups. Explicitly, if $\sigma \in S(n)$ and $a = (a_1, \ldots, a_n) \in \prod_{j=1}^n A$, then $\sigma \cdot a = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)})$. The dual space $G(n)^\wedge$ of irreducible representations of $G(n)$ is given by $\widehat{A}$-tuples $(\lambda(\omega) : \omega \in \widehat{A})$ such that $\lambda(\omega)$ is a partition of $|\lambda(\omega)|$ and $\sum\{|\lambda(\omega)| : \omega \in \widehat{A}\} = n$ (see [11]). We allow $|\lambda(\omega)| = 0$ which corresponds to a partition whose diagram $D(\lambda(\omega)) = \emptyset$. The restriction $\lambda \in G(n)^\wedge$ to $G(n-1)$ decomposes without multiplicity; its components $\mu \in G(n-1)^\wedge$ must satisfy the following condition: there exists $\omega' \in \widehat{A}$ such that

$$\mu(\omega) = \begin{cases} \lambda(\omega), & \omega \neq \omega', \\ \mu(\omega'), & |D(\lambda') \setminus D(\omega')| = 1. \end{cases}$$

The trivial representation $1_{G(n)} = 1$ of $G(n)$ has the form

$$1_{G(n)} = \begin{cases} \lambda_1(\omega), & \omega = \omega_1, \\ \emptyset, & \omega \neq \omega_1, \end{cases}$$

where ω_1 is the trivial character of A and the diagram $D(\lambda_1(\omega))$ consists of the single row: $\{(1, j) : 1 \leq j \leq n\}$.

2.1.2. Induction product. For a finite group H, we use the notation $\mathrm{Rep}(H)$ for the Grothendieck group generated by the unitary equivalence classes of finite-dimensional representations of H. For the wreath product groups $G(n)$, there is a natural induction product on $\bigoplus_{n=0}^{\infty} \mathrm{Rep}(G(n))$ given by

$$\mu \circ \nu = \mathrm{Ind}_{G(n_1) \times G(n_2)}^{G(n_1+n_2)} (\mu \boxtimes \nu),$$

where $\mu \in \mathrm{Rep}(G(n_1))$, $\nu \in \mathrm{Rep}(G(n_2))$, $\boxtimes$ denotes the outer tensor product of representations, and $G(n_1)$ and $G(n_2)$ are the natural subgroups of $G(n)$. (We refer the reader to [11] for a detailed discussion.) If μ and ν are also irreducible, then there is an explicit irreducible decomposition of $\mu \circ \nu$,

$$\mathrm{Ind}_{G(n_1) \times G(n_2)}^{G(n_1+n_2)} (\mu \boxtimes \nu) = \bigoplus \{ c_{\mu,\nu}^{\gamma} \gamma : \gamma \in G(n_1 + n_2)^{\wedge} \},$$

where the integers $c_{\mu,\nu}^{\gamma}$ are given by

$$c_{\mu,\nu}^{\gamma} = \prod \{ c_{\mu(\omega),\nu(\omega)}^{\gamma(\omega)} : \omega \in \widehat{A} \},$$

where $c_{\mu(\omega),\nu(\omega)}^{\gamma(\omega)}$ are the usual Littlewood–Richardson coefficients for the symmetric group induction product (see [4, Theorem 4.3]).

For $m < n$, let $G_m(n)$ be the subgroup of $G(n)$ given by $S_m(n) \cdot \prod_{j=m+1}^{n} A$, where $S_m(n)$ is the centralizer of the natural copy of $S(m)$ in $S(n)$. Further, we set $G_m(\infty) = \bigcup_{n=m}^{\infty} G_m(n)$, the corresponding subgroup of the infinite wreath product G. If $\lambda \in G(n)^{\wedge}$ and $\mu \in G(m)^{\wedge}$ appears as an irreducible component of the restriction $\lambda | G(m)$, write $\mu < \lambda$.

2.2. AF C^*-algebras. Let $G = \bigcup_{n=0}^{\infty} G(n)$, the infinite wreath product group. Since G is a countable discrete locally compact group, its group C^*-algebra $C^*(G)$ is an approximately finite-dimensional (AF) C^*-algebra; that is, it is the inductive limit of $\lim_{\to} C^*(G(n))$ where each $C^*(G(n))$ is isomorphic to the direct sum $\bigoplus \{ M(H(\lambda)) : \lambda \in G(n)^{\wedge} \}$, where the embedding of $C^*(G(n))$ into $C^*(G(n+1))$ is consistent with the restriction rule. (See [9] and [10] for a general discussion of AF-algebras.) We remark that G is also the semidirect product of $S(\infty)$ with $\bigoplus_{n=1}^{\infty} A$. If T is a unitary representation of G, then we write $\tilde{T}$ for the corresponding representation of $C^*(G)$.

2.2.1. Primitive ideal parameterization for wreath products. A primitive ideal J of $C^*(G)$ is parameterized by a diagram D_J which is an $\widehat{A}$-tuple, where each $D_J(\omega)$ is a certain infinite subset of $\mathbb{N}^2$. Either $D_J(\omega) = \mathbb{N}^2$ or is a proper subset of the form $D_J(\omega) = D_J(k(\omega), \ell(\omega), Y(\omega))$:

$$\begin{aligned} D_J\big(k(\omega), \ell(\omega), Y(\omega)\big) &= \big\{(i,j) : 1 \le j < \infty \text{ for } 1 \le i \le k(\omega)\big\} \\ &\quad \cup \big\{(i,j) : 1 \le i < \infty \text{ for } 1 \le j \le \ell(\omega)\big\} \\ &\quad \cup \big(\big(k(\omega), \ell(\omega)\big) + Y(\omega)\big), \end{aligned}$$

where $Y(\omega)$ is the diagram of some partition and $(k(\omega), \ell(\omega)) \in \mathbb{N}^2$ (see [3]).

2.2.2. Primitive quotient spectrum. The diagram D_J of a primitive ideal J naturally describes the Bratteli diagram of the primitive quotient $A_J = C^*(G)/J$, where $A_J = \lim_{\to} (A_J)_n$, where $(A_J)_n$ is a subalgebra of $C^*(G(n))$. Its spectrum $(A_J)_n^\wedge \subset G(n)^\wedge$ consists of those irreducible representations λ such that $D(\lambda(\omega)) \subset D_J(\omega)$ for all $\omega \in \widehat{A}$.

2.2.3. Tame primitive ideals. We are particularly concerned with primitive ideals for which $k(\omega) = \ell(\omega) = 0$ for $\omega \neq \omega_1$ and $\ell(\omega_1) = 0, k(\omega_1) = 1$. We will call these primitive ideals *tame* and simplify their notation. If J is tame, then we write

$$D_{J,T} = \big(D_{J,T}(\omega) : \omega \in \widehat{A}\big),$$

where

$$D_{J,T}(\omega) = \begin{cases} Y(\omega), & \omega \neq \omega_1, \\ ((0,1) + Y(\omega_1)) \cup R_0, & \omega = \omega_1, \end{cases}$$

where $Y(\omega)$ is a diagram for some partition and $R_0 = \{(1,j) : 1 \le j < \infty\}$.

In other words, there is a bijection between tame primitive ideals and $\widehat{A}$-tuples of finite diagrams.

2.2.4. Postliminary primitive quotient. If π is a factor representation of a separable C^*-algebra A, then its kernel $J = \operatorname{Ker}(\pi)$ is a primitive ideal. Moreover, every primitive ideal is the kernel of some irreducible representation. If π_1 and π_2 are irreducible representations with the same kernel, then it is not true in general that they are unitarily equivalent. This holds if the primitive quotient A/J is postliminary or type I.

For AF C^*-algebras, every primitive ideal is the kernel of an irreducible representation π which is a direct limit of irreducible representations; that is, if $A = \lim_{\to} A_n$ and π_n is an irreducible representation of A_n, then $\pi = \lim_{\to} \pi_n$ (see [9, Chapter 1]). If the primitive quotient A/J is postliminary, then every irreducible representation with kernel J is, in particular, unitarily equivalent to a direct limit of irreducible representations.

An AF C^*-algebra $A = \lim_{\to} A_n$ has finite rank if there is a constant K such that $|\widehat{A_n}| \le K$ for all n. In general, a finite rank AF-algebra may either be postliminary or not.

2.3. Tame representations and topological completion $\overline{G}$ of G. Suppose that a group K has the form $\bigcup_{n=1}^{\infty} K(n)$, where $\{K(n)\}_{n=1}^{\infty}$ is an increasing sequence of subgroups. Neeb [6, Definition 3.4] says that the $\{K(n)\}$ are well complemented by a decreasing sequence $\{K_n\}_{n=1}^{\infty}$ of subgroups if $\bigcap_{n=1}^{\infty} K_n = \{1\}$ and $K(n)$ commutes with K_n for all n. He further observed that $\{K_n\}$ is a basis of neighborhoods of the identity for a unique Hausdorff group topology τ for K.

A unitary representation T of K is *tame* (see [8], [6]) if the space $\bigcup_{n=0}^{\infty} H_n(T)$ is dense in $H(T)$, where $H_n(T)$ is the subspace

$$H_n(T) = \big\{v \in H(T) : T(g)v = v \text{ for all } g \in K_n\big\}.$$

If T is an irreducible representation, then T is tame if and only if there exists some m such that $H_m(T) \neq \{0\}$. Then the representation T of K is tame if and only if it is continuous relative to the topology τ (see [6, Section 3]).

If G is the infinite product group, its subgroups $G(n)$ are well-complemented by the subgroups $G_n(\infty)$. The resulting topology τ is the topology of pointwise convergence. To make this explicit, we treat G as the set $S(\infty) \times \bigoplus_{n=1}^{\infty} A$. For $\sigma \in S(\infty)$ and $a = (a_n) \in \bigoplus_{n=1}^{\infty} A$, map (σ, a) to $(\sigma(n), a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty}(\mathbb{Z}^+ \times A)$. The topology τ is identical to the pullback of the restriction of the topology of pointwise convergence (or product topology) of the image of G. Furthermore, the image of G is dense in the full product.

Let $\overline{G}$ be a semidirect product of the full symmetric group $\overline{S}(\infty)$ (of $\mathbb{Z}^+$) with the Abelian infinite product group $\prod_{n=1}^{\infty} A$, where the topology is given by the identification with $\prod_{n=1}^{\infty}(\mathbb{Z}^+ \times A)$. Since G is dense in $\overline{G}$, the classification of the tame representations of G is equivalent to classifying the continuous unitary representations of $\overline{G}$.

3. Irreducible tame representations of G

Proposition 3.1. *Let $m < n$. Then there is an injection of $G(m)^\wedge$ into $G(n)^\wedge$ given by*

$$\nu \mapsto \mu = \nu \circ 1,$$

where $\mu \circ 1$ denotes the induction product. Further, $\lambda \in G(n)^\wedge$ is in the image if and only if the restriction of λ to $G(m) \times G_m(n)$ is equivalent to $\nu \boxtimes 1$ for some irreducible representation $\nu \in G(m)^\wedge$. For $m < k < n$, $\pi \in G(k)^\wedge$ lies in the restriction $\mu|G(k)$ if and only if $D(\pi(\omega)) \subset D(\mu(\omega))$ for $\omega \in \widehat{A}$.

Proof. This follows immediately using the Littlewood–Richardson rule extended to wreath products. □

Proposition 3.2. *Let μ_m be an irreducible representation of $G(m)$. If $\mu_{m+j} \in G(m+j)^\wedge$ such that its restriction $\mu_{m+j}|G(m) \times G_m(m+j) = \mu_m \boxtimes 1$ and $\mu_{m+j} < \mu_{m+j+1}$, then the direct limit representation $\mu = \lim_{\to} \mu_{m+j}$ is irreducible and tame.*

Proof. We know that μ is irreducible since it is a direct limit of irreducible representations (see [5, Theorem 1]). To verify that μ is tame it is enough to show that $H_m(\mu) \neq \{0\}$. By construction, $\mu_{m+j}|G(m) \times G_m(m+j) = \mu_m \boxtimes 1$ for any j. Hence, $H(\mu) \subset H_m(\mu)$; in particular, it is nonzero. □

Proposition 3.3. *Let J be a primitive tame ideal of $C^*(G)$. Then the primitive quotient $A_J = C^*(G)/J$ is postliminary.*

Proof. Write $A_J = \lim_{\to}(A_J)_n$. Then $\lambda \in G(n)^\wedge$ lies in $(A_J)_n^\wedge$ if and only if $D(\lambda(\omega)) \subset D_{J,T}(\omega)$ for $\omega \neq \omega_1$ and $D(\lambda(\omega_1)) \subset \{(1,j) : 1 \leq j < \infty\} \cup D_{J,T}(\omega_1)$. The primitive quotient A_J is postliminary provided that it contains an ideal isomorphic to the algebra $\mathcal{K}$ of compact operators. Let $\mu_n \in (A_J)_n^\wedge$ such that $D(\mu_n(\omega)) \subset D_{J,T}(\omega)$, $\omega \neq \omega_1$, and $D(\mu_n(\omega_1)) \subset (0,1) + D_{J,T}(\omega_1)$. Consider $\lambda \in (A_J)_{n+1}^\wedge$ such that μ_n is contained in the restriction $\lambda_{n+1}|G(n)$. In terms of diagrams, we have $D(\mu_n(\omega)) \subset D(\lambda(\omega))$ for $\omega \in \widehat{A}$. By the set containments forced by the primitive ideal condition, we conclude that $\lambda = \mu_{n+1}$.

To finish, we look at the central decompositions $(A_J)_j = \bigoplus\{M(\pi) : \pi \in (A_J)_n^\wedge\}$ (for $j = n, n+1$). Then the matrix algebra $M(\mu_n) \subset (A_J)_n$ embeds simply into $M(\mu_{n+1}) \subset (A_J)_{n+1}$. This is exactly the embedding that gives an ideal isomorphic to $\mathcal{K}$. □

Corollary 3.4. *Let J be a primitive tame ideal of $C^*(G)$. Then $J = \mathrm{Ker}(\tilde{\mu})$ for some irreducible tame representation μ of G.*

Proof. This follows easily from the Proposition 3.3. Consider the direct limit representation $\mu = \lim_{\to} \mu_n$. By Proposition 3.2, μ is irreducible and tame. Furthermore, $\tilde{\mu}$ is a faithful representation of $A_J = C^*(G)/J$ because any two nonzero ideals in a primitive algebra intersect. In particular, if P is the kernel of $\tilde{\mu}$, then it has a nonzero intersection with Q, where Q is the ideal generated by $\bigcup_n M(\mu_n)$. This contradicts that μ is nonzero on Q by construction. □

Theorem 3.5. *Let $\tilde{T}$ be an irreducible representation of $C^*(G)$, where T is an irreducible tame representation of G. Let $J = \mathrm{Ker}(\tilde{T}) \in \mathrm{Prim}(G)$. Then J is a primitive tame ideal.*

Proof. Assume that T is an irreducible tame representation. Then there exists a positive integer m such that $H_m(T) \neq \{0\}$. Consider $T|G(m)$ acting on $H_m(T)$. Since $G(m)$ is a finite group, $H_m(T)$ decomposes discretely into a direct sum of irreducible representations; pick one, say, $\mu \in G(m)^\wedge$ and the corresponding subspace V_m. Note that by construction, $T(g) = I$ for $G_m(\infty)$. Next consider the cyclic subspace V_{m+1} generated by $T(G(m+1))$ acting on V_m. We decompose V_{m+1} into a direct sum of subspaces where $G(m+1)$ acts irreducibly.

Now $V_m \subset V_{m+1}$ by construction. Furthermore, $T|(G(m) \times G_m(m+1)) = \mu \boxtimes 1$ on the subspace V_m. But this condition uniquely determines an irreducible representation, say, $mu_{m+1} \in G(m+1)^\wedge$. By the minimality of the cyclic subspace V_{m+1}, this subspace supports the irreducible representation μ_{m+1}. Consequently, we have constructed a chain $\mu_m < \mu_{m+1} < \cdots$ such that $\mu_{m+j} \in G(m+j)^\wedge$, $H(\mu_{m+j}) \subset H(\mu_{m+j+1})$, $H(\mu_{m+j}) \subset H_{m+j}(T)$. By construction, the union $\bigcup_{j=0}^{\infty} H(\mu_{m+j})$ is invariant under G. By the irreducibility of T, the infinite union is dense in $H(T)$.

Consider the primitive quotient $A_J = C^*(G)/J$. Then its Bratteli diagram is given by the minimal choice of sets X_{m+j} that satisfy: (1) $X_{m+j} \subset G(m+j)^\wedge$; (2) $\mu_{m+j} \in X_{m+j}$; (3) if $\mu \in X_{m+j+1}$, then all the irreducible components of its restriction $\mu|G(m+j)$ belong to X_{m+j}.

We will be able to identify X_{m+j} by means of the irreducible representations μ_{m+j} and their restrictions. First note that

$$D\big(\mu_{m+j}(\omega)\big) = D\big(\mu_m(\omega)\big), \quad \omega \neq \omega_1$$

for all $j \geq 0$. We write $D(\mu_m(\omega_1))$ in the form

$$D\big(\mu_m(\omega_1)\big) = \big((0,1) + Y_0\big) \cup R_0,$$

where

$$R_0 = \big\{(1,i) : 1 \leq i \leq \big|\mu_m(\omega_1)\big| - |Y_0|\big\}.$$

Then we write the diagram $D(\mu_{m+j}(\omega_1))$ in the form

$$D\big(\mu_{m+j}(\omega_1)\big) = \big((0,1) + Y_0\big) \cup R_j,$$

where

$$R_j = \big\{(1,i) : 1 \leq i \leq \big|\mu_{m+j}(\omega_1)\big| - |Y_0|\big\};$$

that is, it is independent of j except for the length of its first row.

We now choose diagrams $D(\omega)$, $\omega \neq \omega_1$ such that

$$D(\omega) \subset D\big(\omega_m(\omega)\big)$$

and a diagram Y such that $Y \subset Y_0$. These diagrams specify uniquely the irreducible representation $\mu \in G(m+i)^\wedge$ given by

$$D\big(\mu(\omega)\big) = \begin{cases} D(\omega) & \omega \neq \omega_1, \\ ((0,1) + Y) \cup R & \omega = \omega_1, \end{cases}$$

where $R = \{(1,\alpha) : 1 \leq \alpha \leq k\}$ and $k = m + i - |Y| - \sum\{|D(\omega)| : \omega \neq \omega_1\}$. Then the set $X_{m+i} \subset G(m+i)^\wedge$ is the collection of all such $\mu \in G(m+i)^\wedge$. It is straightforward to verify that these sets satisfy the above properties (1)–(3). Since $\tilde{T}$ is a faithful irreducible representation of A_J, we identify that the diagrams for J are those for a tame ideal. $\square$

Corollary 3.6. *Let T_1 and T_2 be irreducible tame representations of G. Then T_1 and T_2 are unitarily equivalent if and only if* $\mathrm{Ker}(\tilde{T}_1) = \mathrm{Ker}(\tilde{T}_2)$.

Corollary 3.7. *Every irreducible tame representation T is unitarily equivalent to a direct limit* $\lim_{\to} T_n$ *of irreducible representations $T_n \in G(n)^\wedge$.*

Remark 3.8. By construction, the cardinality of the collection of tame primitive ideals is countable since they are uniquely determined by $\widehat{A}$-tuples of (finite) diagrams. In turn, there are only countably many distinct unitary equivalence classes of tame irreducible representations of G.

By [3], the finite-rank primitive quotients of $C^*(G)$ are all postliminary. This is not true for general C^*-algebras. It would be interesting to have a deeper group-theoretic understanding of this relationship.

Proposition 3.9. *Every tame representation T of G decomposes uniquely as a direct sum of irreducible tame representations; that is, let $\mathcal{T}$ be the collection of the unitary equivalence classes of tame irreducible representations. Then*

$$T \simeq \bigoplus \{n_\tau \tau : \tau \in \mathcal{T}\},$$

where the multiplicity of τ in T is unique.

Proof. The proof in [2, p. 385] applies word for word here since the cardinality of the set of equivalence classes of the irreducible tame representations is countable. □

4. Induced representation realization

In this section, we show that the tame irreducible representations of G can be realized as induced representations just as the case for the infinite symmetric group (see [8]). Let $(D(\omega))$ be an $\widehat{A}$-tuple, and set $m = \sum |D(\omega)|$. Corresponding to $(D(\omega))$ there are two objects: $\lambda \in G(m)^\wedge$ and a primitive tame ideal J. Consider the subgroup $G(n)$ of G with $m < n$. For the sake of notation, set $K = G(m) \times G_m(\infty) \subset G$ and $K(n) = G(n) \cap K$. We can extend λ trivially to an irreducible representation to either K or $K(n)$ by $\lambda \boxtimes 1$, where 1 is the trivial representation of either K or $K(n)$. Let

$$T = \mathrm{Ind}_K^G(\lambda \boxtimes 1), \qquad T_n = \mathrm{Ind}_{K(n)}^{G(n)}(\lambda \boxtimes 1);$$

that is, $T_n = \lambda \circ 1$ (induction product) in the notation of Proposition 3.1. We will identify the representation spaces $H(\lambda \boxtimes 1)$ and $H(\lambda)$ so the representation space $H(T_\lambda)$ consists of all functions $f : G \to H(\lambda)$ such that $f(gk) = (\lambda \boxtimes 1)(k)^{-1} = \lambda(k')^{-1} f(g)$, where $k = k'k''$, $k' \in G(m)$ and $k'' \in G_m(\infty)$, and satisfies the L^2-condition $\sum \|f(\dot{g})\|^2_{H(\lambda)}$ (sum over the cosets $\dot{g} = gK$). This is similarly the case for $H(T_n)$: $f_0 : G(n) \to H(\lambda)$ such that $f_0(g_0 k_0) = \lambda(k_0')^{-1} f_0(g_0)$, where $g_0 \in G(n)$, $k_0 \in K(n)$, and $k_0 = k_0' k_0''$, where $k_0' \in G(m)$ and $k_0'' \in G_m(n)$. A function $f_0 \in H(T_n)$ has a unique extension to $\tilde{f}_0 \in H(T)$:

$$\tilde{f}_0(x) = \begin{cases} f_0(g_0 k'), & x = g_0 k, g_0 \in G(n), k = k'k'', k' \in G(m), k'' \in G_m(\infty), \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that $\tilde{f}_0$ lies in $H(T)$ and the L^2-norm is preserved. We conclude that the induced representation T is the direct limit of T_n. An important consequence of the direct limit form of T is that it allows us to find the primitive ideal corresponding to T by Proposition 3.1. We summarize this discussion.

Proposition 4.1. *Let $\lambda \in G(m)^\wedge$. Then the induced representation*

$$T = \mathrm{Ind}_{G(m)\times G_m(\infty)}^G(\lambda \boxtimes 1)$$

is an irreducible tame representation of G with tame primitive ideal $\mathrm{Ker}(\tilde{T}) = J$ *given by $D_{J,T}(\omega) = \lambda(\omega)$ for $\omega \in \widehat{A}$.*

The induced representation T can be studied directly using the results of [7, Theorems 1, 2] to deduce their irreducibility and their equivalence. The classification of the irreducible representations of $\overline{G}$ is consistent with Mackey's results on semidirect products; unfortunately, they did not apply since $\overline{G}$ is not locally compact.

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