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I-CONVEXITY AND Q-CONVEXITY IN ORLICZ–BOCHNER FUNCTION SPACES EQUIPPED WITH THE LUXEMBURG NORM

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Abstract. We study I-convexity and Q-convexity, two geometric properties introduced by Amir and Franchetti. We point out that a Banach space X has the weak fixed-point property when X is I-convex (or Q-convex) with a strongly bimonotone basis. By means of some characterizations of I-convexity and Q-convexity in Banach spaces, we obtain criteria for these two convexities in the Orlicz–Bochner function space $L_{(M)}(\mu, X)$: that $L_{(M)}(\mu, X)$ is I-convex (or Q-convex) if and only if $L_{(M)}(\mu)$ is reflexive and X is I-convex (or Q-convex).

1. Introduction

It is well known that convexity and reflexivity play important roles in Banach space theory. Since *B-convexity* (*B-C* for short) and *uniform nonsquareness* (*U-NS*) were given by Beck [3] and James [10], respectively, these convexities have been widely used in probability theory, fixed-point theory, and many other fields (see [3], [6], [7], [10]). Some relevant convexities, such as *P-convexity* (*P-C*), *O-convexity* (*O-C*), *Q-convexity* (*Q-C*), and *I-convexity* (*I-C*) have been introduced and investigated by many mathematicians (see [2], [4], [12]–[14]). Let X be a Banach space. Denote by $B(X)$ and $S(X)$ the unit ball and the unit sphere of X, respectively. A Banach space X is considered to be I-convex if there exist

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$n \in \mathbb{N}$ ($n \geq 2$) and $\varepsilon > 0$ such that for all $x_1, x_2, \ldots, x_n \in B(X)$, we have

$$\min\Big\{\Big\|x_k - \sum_{\substack{i=1\\ i\neq k}}^{n} x_i\Big\| : 1 \leq k \leq n\Big\} < n - \varepsilon. \tag{1.1}$$

Certainly $B(X)$ above can be replaced by $S(X)$. An equivalent definition about I-convexity was given in [2], namely, there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that for all $x_1, x_2, \ldots, x_n \in B(X)$, the following holds:

$$\min\big\{\mathrm{d}\big(x_k, \mathrm{conv}\{x_i\}_{\substack{i=1\\ i\neq k}}^{n}\big) : k = 1, 2, \ldots, n\big\} < 2 - \varepsilon. \tag{1.2}$$

We consider a Banach space X to be Q-convex if there exist $n \in \mathbb{N}$ ($n \geq 2$) and $\varepsilon > 0$ such that, for no $x_1, x_2, \ldots, x_n \in B(X)$, we have

$$\Big\|\sum_{i=1}^{k-1} x_i - x_k\Big\| \geq k - \varepsilon \tag{1.3}$$

for $k = 2, 3, \ldots, n$. It is proved in [2] that X is Q-convex if and only if there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that, for $x_1, x_2, \ldots, x_n \in B(X)$, it holds that

$$\min\big\{\mathrm{d}\big(x_{k+1}, \mathrm{conv}\{x_i\}_{i=1}^{k}\big) : k = 1, 2, \ldots, n-1\big\} < 2 - \varepsilon. \tag{1.4}$$

From [2] we know that either I-convexity or Q-convexity is a self-dual property; that is, X has it if and only if X^* has it. We also know that

(P-C) or (U-NS) $\Rightarrow$ (O-C) $\Rightarrow$ (Q-C) $\Longrightarrow$ (I-C) and (S-Rfx),

where (S-Rfx) denotes super-reflexivity, and that

(I-C) or (S-Rfx) $\Longrightarrow$ (B-C).

We say that $\{x_n\}$ is a *strongly bimonotone Schauder basis* if $\|P_F\| = \|I - P_F\| = 1$ for every segment $F = [a, b] := \{n \in \mathbb{N} : a \leq n \leq b\}$, where P_F is defined by $P_F(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n\in F} a_n x_n$ for $x = \sum_{n=1}^{\infty} a_n x_n \in X$. From Lemma 1 and Theorem 2 in [5], we can immediately get the following proposition about I-convexity and Q-convexity in the fixed-point theory, which generalizes the relevant result about O-convexity (see [15]).

Proposition 1.1. *If X is a Q-convex (or I-convex) Banach space with a strongly bimonotone basis, then X has the weak fixed-point property.*

Some convexities including (U-N), (P-C), and (B-C) in Orlicz space, Lebesgue–Bochner space, and Orlicz–Bochner space were characterized by Alherk and Hudzik [1], Hudzik [8], [9], Kamínska and Turett [11], Kolwicz and Pluciennik [12], and Smith and Turett [16], among others. In [2] Amir and Franchetti obtained the criteria for $l_p(X_i)$ being Q-convex and I-convex. The characterization of Q-convexity in Orlicz space was given in [4]. Because the structure of the Orlicz–Bochner function space $L_{(M)}(\mu, X)$ is much more complicated than Orlicz space, until now the criteria for Q-convexity and I-convexity in Orlicz–Bochner function spaces have not been discussed. In this article we will give some characterizations about I-convexity and Q-convexity in Banach space, then show the criteria for the Orlicz–Bochner space $L_{(M)}(\mu, X)$ being I-convex and Q-convex.

Let $\mathbb{R}$ be the set of all real numbers. A function $M : \mathbb{R} \to \mathbb{R}^+$ is called an $\mathcal{N}$*-function* if M is convex, even, $M(0) = 0$, $M(u) > 0$ $(u \neq 0)$, and $\lim_{u\to 0} \frac{M(u)}{u} = 0$, $\lim_{u\to\infty} \frac{M(u)}{u} = \infty$. The complemented function N of M is defined in the sense of Young by

$$N(v) = \sup_{u\in\mathbb{R}} \big\{uv - M(u)\big\}.$$

It is known that if M is an $\mathcal{N}$-function, then its complemented function N is also an $\mathcal{N}$-function. The function M is said to satisfy the Δ_2-*condition* for all $u \in \mathbb{R}$ ($M \in \Delta_2(\mathbb{R})$ for short) if for some $K > 0$,

$$M(2u) \leq KM(u) \tag{1.5}$$

holds for all $u \in \mathbb{R}$. We usually denote by $M \in \nabla_2(\mathbb{R})$ if $N \in \Delta_2(\mathbb{R})$. Let $(\Omega, \sum, \mu)$ be a nonatomic infinite measure space. For a measurable function $u(t)$, we call $\rho_M(u) = \int_\Omega M(u(t))\, d\mu$ the *modular* of u. The Orlicz function space $L_{(M)}(\mu)$ generated by M is the Banach space

$$L_{(M)}(\mu) = \big\{u : \rho_M(\lambda u) < \infty \text{ for some } \lambda > 0\big\}$$

equipped with Luxemburg norm

$$\|u\|_{(M)} = \inf\Big\{\lambda > 0 : \rho_M\Big(\frac{u}{\lambda}\Big) \leq 1\Big\}.$$

If $u : \Omega \to X$ is a vector-valued measurable function (i.e., there exists a sequence of vector-valued simple functions $\{u_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \|u_n(t) - u(t)\| = 0, \mu$-a.e.), then we denote by $L_{(M)}(\mu, X)$. We call such spaces *Orlicz–Bochner spaces.*

We have known that an Orlicz function space $L_{(M)}(\mu)$ with the Luxemburg norm is B-convex if and only if it is reflexive (see [4]), and similarly for uniform nonsquareness. Thus $L_{(M)}(\mu)$ is I-convex or Q-convex if and only if it is reflexive. (For more details on Orlicz and Orlicz–Bochner function spaces, P-convexity, O-convexity, I-convexity, and Q-convexity, see [2], [4], [6], and [15].)

2. Lemmas

Lemma 2.1 ([1, Lemma 2], [4, Theorem 1.13], [11, Proposition 1], [12, Lemma 1])**.**

(1) *We have* $N \in \Delta_2(\mathbb{R})$ *if and only if for any* $\eta \in (0,1)$ *there exists* $\gamma_0 \in (0,1)$ *such that*

$$M(\eta u) \leq \eta\gamma_0 M(u) \tag{2.1}$$

holds for all $u \in \mathbb{R}$.

(2) *If* $N \in \Delta_2(\mathbb{R})$, *then there exist* $a \in (0,1)$ *and* $\gamma = \gamma(a) \in (0,1)$ *such that*

$$M\Big(\frac{u+v}{2}\Big) \leq \frac{1}{2}(1-\gamma)\big(M(u) + M(v)\big) \tag{2.2}$$

holds for all u, v *satisfying* $|\frac{u}{v}| \leq a$.

(3) *We have $M \in \Delta_2(\mathbb{R})$ if and only if for each $l > 1$ there exists $k = k(l) > 1$ such that for any $u \in \mathbb{R}$ we have*

$$M(lu) \le kM(u). \tag{2.3}$$

By Theorem 1.13 in [4], we know that if $M \in \Delta_2(\mathbb{R})$, then for any $c > 1$, there exists $a_0 \in (0,1)$ such that

$$M\big((1+a_0)u\big) \le cM(u) \tag{2.4}$$

holds for all $u \in \mathbb{R}$. From the proofs of Lemma 2.3 and Proposition 3.2 in [2], we can easily obtain the following two lemmas.

Lemma 2.2. *Let X be a Banach space. Then X is I-convex if and only if there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that for any $x_1, x_2, \ldots, x_n \in S(X)$, $k_0 \in \{1,2,\ldots,n\}$ can be found such that*

$$\Big\| x_{k_0} - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n} x_i \Big\| < 2 - \varepsilon. \tag{2.5}$$

Lemma 2.3. *Let X be a Banach space. Then X is Q-convex if and only if there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that, for any $x_1, x_2, \ldots, x_n \in S(X)$, $k_0 \in \{1,2,\ldots,n-1\}$ can be found such that*

$$\Big\| x_{k_0+1} - \frac{1}{k_0}\sum_{i=1}^{k_0} x_i \Big\| < 2 - \varepsilon. \tag{2.6}$$

Lemma 2.4. *Let X be a Banach space. Then X is I-convex if and only if there exist $n \in \mathbb{N}$ and $\delta > 0$ such that, for any $x_1, x_2, \ldots, x_n \in X\backslash\{0\}$, $k_0 \in \{1,2,\ldots,n\}$ can be found such that*

$$\begin{aligned}
&\Big\| x_{k_0} - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n} x_i \Big\| \\
&\quad \le \Big(\|x_{k_0}\| + \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n} \|x_i\|\Big)\Big(1 - \frac{2\delta\min\{\|x_i\| : i = 1,2,\ldots,n\}}{\|x_{k_0}\| + \frac{1}{n-1}\sum_{i=1, i\neq k_0}^{n}\|x_i\|}\Big).
\end{aligned} \tag{2.7}$$

Proof. (Necessity) By Lemma 2.2, we may assume without loss of generality that x_n satisfies

$$1 - \delta \ge \frac{1}{2}\Big\| \frac{x_n}{\|x_n\|} - \frac{1}{n-1}\sum_{i=1}^{n-1}\frac{x_i}{\|x_i\|}\Big\|,$$

where $\delta = \frac{\varepsilon}{2}$, independent of $x_1, x_2, \ldots, x_n$. Now setting $y_n = x_n, y_i = -x_i$ for $i = 1,2,\ldots,n-1$, we get

$$1 - \delta \ge \frac{1}{2(n-1)}\Big\| (n-1)\frac{y_n}{\|y_n\|} + \frac{y_1}{\|y_1\|} + \frac{y_2}{\|y_2\|} + \cdots + \frac{y_{n-1}}{\|y_{n-1}\|}\Big\|. \tag{2.8}$$

Case I: $\|y_n\| = \min\{\|y_i\| : i = 1, 2, \dots, n\}$. By (2.8), we have

$$\begin{aligned} 1-\delta &\geq \frac{1}{2(n-1)}\Big\|\frac{1}{\|y_n\|}\big(y_1+y_2+\cdots+y_{n-1}+(n-1)y_n\big) \\ &\qquad - \sum_{i=1}^{n-1}\Big(\frac{1}{\|y_n\|}-\frac{1}{\|y_i\|}\Big)y_i\Big\| \\ &\geq \frac{1}{2(n-1)}\Big(\frac{1}{\|y_n\|}\big\|y_1+y_2+\cdots+y_{n-1}+(n-1)y_n\big\| \\ &\qquad - \sum_{i=1}^{n-1}\Big(\frac{1}{\|y_n\|}-\frac{1}{\|y_i\|}\Big)\|y_i\|\Big). \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{1}{2(n-1)\|y_n\|}\big\|y_1+y_2+\cdots+y_{n-1}+(n-1)y_n\big\| \\ &\qquad \leq 1-\delta-\frac{n-1}{2(n-1)}+\frac{1}{2(n-1)}\cdot\frac{\|y_1\|+\|y_2\|+\cdots+\|y_{n-1}\|}{\|y_n\|} \\ &\qquad = 1-\delta-\frac{1}{2}+\frac{\sum_{i=1}^{n-1}\|y_i\|}{2(n-1)\|y_n\|} \\ &\qquad = -\delta+\frac{(n-1)\|y_n\|+\sum_{i=1}^{n-1}\|y_i\|}{2(n-1)\|y_n\|}. \end{aligned}$$

So

$$\begin{aligned} \Big\|y_n+\frac{1}{n-1}\sum_{i=1}^{n-1}y_i\Big\| &\leq 2\|y_n\|\Big(\frac{(n-1)\|y_n\|+\sum_{i=1}^{n-1}\|y_i\|-2\delta(n-1)\|y_n\|}{2(n-1)\|y_n\|}\Big) \\ &= \|y_n\|+\frac{1}{n-1}\sum_{i=1}^{n-1}\|y_i\|-2\delta\|y_n\| \\ &= \Big(\|y_n\|+\frac{1}{n-1}\sum_{i-1}^{n-1}\|y_i\|\Big)\Big(1-\frac{2\delta\|y_n\|}{\|y_n\|+\frac{1}{n-1}\sum_{i=1}^{n-1}\|y_i\|}\Big). \end{aligned}$$

Case II: $\|y_n\| > \min\{\|y_i\| : i = 1, 2, \dots, n\}$. We may assume that $\|y_1\| = \min\{\|y_i\| : i = 1, 2, \dots, n\}$. In view of (2.8), we obtain

$$\begin{aligned} 1-\delta &\geq \frac{1}{2(n-1)}\Big\|\frac{1}{\|y_1\|}\Big(\sum_{i=1}^{n-1}y_i+(n-1)y_n\Big)-\sum_{i=2}^{n-1}\Big(\frac{1}{\|y_1\|}-\frac{1}{\|y_i\|}\Big)y_i \\ &\qquad -(n-1)\Big(\frac{1}{\|y_1\|}-\frac{1}{\|y_n\|}\Big)y_n\Big\| \\ &\geq \frac{1}{2(n-1)}\Big(\frac{1}{\|y_1\|}\Big\|\sum_{i=1}^{n-1}y_i+(n-1)y_n\Big\|-\sum_{i=2}^{n-1}\frac{\|y_i\|}{\|y_1\|}+(n-2) \\ &\qquad -(n-1)\frac{\|y_n\|}{\|y_1\|}+(n-1)\Big) \end{aligned}$$

$$= \frac{1}{2(n-1)\|y_1\|}\Big\|\sum_{i=1}^{n-1} y_i + (n-1)y_n\Big\| - \frac{1}{2(n-1)}\sum_{i=2}^{n-1}\frac{\|y_i\|}{\|y_1\|} + \frac{n-2}{2(n-1)}$$
$$- \frac{(n-1)\|y_n\|}{2(n-1)\|y_1\|} + \frac{1}{2}$$
$$= \frac{1}{2(n-1)\|y_1\|}\Big\|\sum_{i=1}^{n-1} y_i + (n-1)y_n\Big\| - \frac{(n-1)\|y_n\| + \sum_{i=2}^{n-1}\|y_i\|}{2(n-1)\|y_1\|}$$
$$+ \frac{1}{2} + \frac{n-2}{2(n-1)}.$$

Hence

$$\Big\|y_n + \frac{1}{n-1}\sum_{i=1}^{n-1} y_i\Big\| \le 2\|y_1\|\Big(\frac{1}{2} - \delta - \frac{n-2}{2(n-1)} + \frac{(n-1)\|y_n\| + \sum_{i=2}^{n-1}\|y_i\|}{2(n-1)\|y_1\|}\Big)$$
$$= \|y_n\| + \frac{1}{n-1}\sum_{i=1}^{n-1}\|y_i\| - 2\delta\|y_1\|$$
$$= \Big(\|y_n\| + \frac{1}{n-1}\sum_{i-1}^{n-1}\|y_i\|\Big)\Big(1 - \frac{2\delta\|y_1\|}{\|y_n\| + \frac{1}{n-1}\sum_{i=1}^{n-1}\|y_i\|}\Big).$$

Combining Cases I and II, we know that (2.7) holds true.

(Sufficiency) This is obvious. □

Lemma 2.5. *Let X be a Banach space. Then X is Q-convex if and only if there exist $n \in \mathbb{N}$ and $\delta > 0$ such that, for any $x_1, x_2, \ldots, x_n \in X\backslash\{0\}$, $k_0 \in \{1, 2, \ldots, n-1\}$ can be found such that*

$$\Big\|x_{k_0+1} - \frac{1}{k_0}\sum_{i=1}^{k_0} x_i\Big\|$$
$$\le \Big(\|x_{k_0+1}\| + \frac{1}{k_0}\sum_{i=1}^{k_0}\|x_i\|\Big)\Big(1 - \frac{2\delta\min\{\|x_1\|, \|x_2\|, \ldots, \|x_{k_0+1}\|\}}{\|x_{k_0+1}\| + \frac{1}{k_0}\sum_{i=1}^{k_0}\|x_i\|}\Big). \quad (2.9)$$

Proof. The sufficiency is obvious. We only need to show the necessity. By Lemma 2.3, for $x_1, x_2, \ldots, x_n \in X\backslash\{0\}$, there exists $k_0 \in \{1, 2, \ldots, n-1\}$ such that

$$1 - \delta \ge \frac{1}{2}\Big\|\frac{x_{k_0+1}}{\|x_{k_0+1}\|} - \frac{1}{k_0}\sum_{i=1}^{k_0}\frac{x_i}{\|x_i\|}\Big\|.$$

Then we can get the result by using a method similar to the one in the proof of Lemma 2.4. □

3. I-convexity in Orlicz–Bochner space

Lemma 3.1. *Let X be a I-convex Banach space, and let $M \in \Delta_2(\mathbb{R})$ and $N \in \Delta_2(\mathbb{R})$. Then there exist $n \in \mathbb{N}$ and $\tilde{\gamma} \in (0, 1)$ such that for any $x_1, x_2, \ldots, x_n \in X$,* *we have*

$$\sum_{k=1}^{n} M\Big(\frac{1}{2}\Big\|x_k - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k}}^{n} x_i\Big\|\Big) \le (1-\tilde{\gamma})\sum_{i=1}^{n} M\big(\|x_i\|\big). \tag{3.1}$$

Proof. We may assume without loss of generality that $\|x_1\| \ge \|x_2\| \ge \cdots \ge \|x_n\|$ and that $k_0 \in \{1, 2, \ldots, n\}$ satisfies Lemma 2.4.

Case I: $k_0 = n$. That is,

$$\Big\|x_n - \frac{1}{n-1}\sum_{i=1}^{n-1} x_i\Big\| \le \Big(\|x_n\| + \frac{1}{n-1}\sum_{i=1}^{n-1}\|x_i\|\Big)\Big(1 - \frac{2\delta\|x_n\|}{\|x_n\| + \frac{1}{n-1}\sum_{i=1}^{n-1}\|x_i\|}\Big). \tag{3.2}$$

(Subcase I-I): $\frac{\|x_n\|}{\frac{1}{n-1}\sum_{i=1}^{n-1}\|x_i\|} > a$, where a is defined in Lemma 2.1(2). Certainly, we have

$$\frac{\|x_n\|}{\|x_n\| + \frac{1}{n-1}\sum_{i=1}^{n-1}\|x_i\|} \ge \frac{1}{1+\frac{1}{a}} = \frac{a}{1+a}.$$

Therefore, by (3.2) and the inequality above, we get

$$\frac{1}{2}\Big\|x_n - \frac{1}{n-1}\sum_{i=1}^{n-1} x_i\Big\| \le \frac{1}{2}\Big(1 - \frac{2\delta a}{1+a}\Big)\Big(\|x_n\| + \frac{1}{n-1}\sum_{i=1}^{n-1}\|x_i\|\Big).$$

By the convexity of M, we have

$$M\Big(\frac{1}{2}\Big\|x_n - \frac{1}{n-1}\sum_{i=1}^{n-1} x_i\Big\|\Big) \le \frac{1}{2}\Big(1 - \frac{2\delta a}{1+a}\Big)\Big(M\big(\|x_n\|\big) + \frac{1}{n-1}\sum_{i=1}^{n-1} M\big(\|x_i\|\big)\Big).$$

And so

$$\begin{aligned}
\sum_{k=1}^{n} M\Big(\frac{1}{2}\Big\|x_k - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k}}^{n} x_i\Big\|\Big) &\le \frac{1}{2}\Big(\sum_{k=1}^{n} M\big(\|x_k\|\big) + \frac{1}{n-1}\sum_{k=1}^{n}\sum_{\substack{i=1\\ i\neq k}}^{n} M\big(\|x_i\|\big)\Big) \\
&\qquad - \frac{\delta a}{1+a}\Big(M\big(\|x_n\|\big) + \frac{1}{n-1}\sum_{i=1}^{n-1} M\big(\|x_i\|\big)\Big) \\
&\le \sum_{k=1}^{n} M\big(\|x_k\|\big) - \frac{\delta a}{1+a}\cdot\frac{1}{n-1}\sum_{i=1}^{n} M\big(\|x_i\|\big) \\
&= \Big(1 - \frac{a\delta}{(n-1)(1+a)}\Big)\sum_{i=1}^{n} M\big(\|x_i\|\big).
\end{aligned}$$

(Subcase I-II): $\frac{\|x_n\|}{\frac{1}{n-1}\sum_{i=1}^{n-1}\|x_i\|} \le a$. By (2.2), we have

$$M\Big(\frac{1}{2}\Big\|x_n - \frac{1}{n-1}\sum_{i=1}^{n-1} x_i\Big\|\Big) \le M\Big(\frac{1}{2}\Big(\|x_n\| + \frac{1}{n-1}\sum_{i=1}^{n-1}\|x_i\|\Big)\Big)$$

$$\leq \frac{1}{2}(1-\gamma)\Big(M\big(\|x_n\|\big) + \frac{1}{n-1}\sum_{i=1}^{n-1} M\big(\|x_i\|\big)\Big).$$

It follows that

$$\begin{aligned}
&\sum_{k=1}^{n} M\Big(\frac{1}{2}\Big\|x_k - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k}}^{n} x_i\Big\|\Big) \\
&\quad\leq \frac{1}{2}\Big(\sum_{k=1}^{n} M\big(\|x_k\|\big) + \frac{1}{n-1}\sum_{k=1}^{n}\sum_{\substack{i=1\\ i\neq k}}^{n} M\big(\|x_i\|\big)\Big) \\
&\qquad - \frac{\gamma}{2}\Big(M\big(\|x_n\|\big) + \frac{1}{n-1}\sum_{i=1}^{n-1} M\big(\|x_i\|\big)\Big) \\
&\quad\leq \sum_{k=1}^{n} M\big(\|x_k\|\big) - \frac{\gamma}{2}\cdot\frac{1}{n-1}\sum_{i=1}^{n} M\big(\|x_i\|\big) \\
&\quad= \Big(1 - \frac{\gamma}{2(n-1)}\Big)\sum_{i=1}^{n} M\big(\|x_i\|\big).
\end{aligned}$$

Case II: $k_0 \in \{1, 2, \ldots, n-1\}$. In view of (2.1), for $\eta = \frac{2n-3}{2(n-1)} \in (0,1)$, there exists $\gamma_0 \in (0,1)$ such that

$$M(\eta u) \leq \eta\gamma_0 M(u) \tag{3.3}$$

holds true for all $u \in \mathbb{R}$. By (2.4), for $c = \frac{1}{\sqrt{\gamma_0}}$, there exists $a_0 \in (0,1)$ such that for all $u \in \mathbb{R}$, it holds that

$$M\big((1+a_0)u\big) \leq cM(u). \tag{3.4}$$

(Subcase II-I): $\frac{\|x_n\|}{\frac{1}{n-1}\sum_{i=1}^{n-1}\|x_i\|} > a_0$, that is, $\frac{1}{\|x_n\|}(\|x_{k_0}\| + \sum_{i=1, i\neq k_0}^{n-1}\|x_i\|) < \frac{n-1}{a_0}$. Clearly, we have

$$\begin{aligned}
\frac{\|x_n\|}{\|x_{k_0}\| + \frac{1}{n-1}\sum_{i=1,i\neq k_0}^{n}\|x_i\|} &= \frac{1}{\frac{\|x_{k_0}\|}{\|x_n\|} + \frac{1}{n-1}\big(\frac{1}{\|x_n\|}\sum_{i=1,i\neq k_0}^{n-1}\|x_i\| + 1\big)} \\
&\geq \frac{1}{\frac{n-1}{a_0} + \frac{1}{n-1}\big(\frac{n-1}{a_0} + 1\big)} \\
&= \frac{(n-1)a_0}{n(n-1)+a_0} \in (0,1).
\end{aligned}$$

Therefore, by (2.7) and inequality above, we get

$$\Big\|x_{k_0} - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n} x_i\Big\| \leq \Big(\|x_{k_0}\| + \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n}\|x_i\|\Big)\Big(1 - \frac{2\delta a_0(n-1)}{n(n-1)+a_0}\Big).$$

Owing to the convexity of M, we get

$$
\begin{aligned}
&M\Big(\frac{1}{2}\Big\|x_{k_0} - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n} x_i\Big\|\Big)\\
&\quad \leq \frac{1}{2}\Big(1 - \frac{2\delta a_0(n-1)}{n(n-1)+a_0}\Big)\Big(M\big(\|x_{k_0}\|\big) + \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n} M\big(\|x_i\|\big)\Big).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\sum_{k=1}^{n} M\Big(\frac{1}{2}\Big\|x_k - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k}}^{n} x_i\Big\|\Big)\\
&\quad \leq \frac{1}{2}\Big(\sum_{k=1}^{n} M\big(\|x_k\|\big) + \frac{1}{n-1}\sum_{k=1}^{n}\sum_{\substack{i=1\\ i\neq k}}^{n} M\big(\|x_i\|\big)\Big)\\
&\qquad - \frac{\delta a_0(n-1)}{n(n-1)+a_0}\Big(M\big(\|x_{k_0}\|\big) + \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n} M\big(\|x_i\|\big)\Big)\\
&\quad \leq \sum_{k=1}^{n} M\big(\|x_k\|\big) - \frac{\delta a_0(n-1)}{n(n-1)+a_0}\cdot\frac{1}{n-1}\sum_{i=1}^{n} M\big(\|x_i\|\big)\\
&\quad = \Big(1 - \frac{\delta a_0}{n(n-1)+a_0}\Big)\sum_{i=1}^{n} M\big(\|x_i\|\big).
\end{aligned}
$$

(Subcase II-II): $\frac{\|x_n\|}{\frac{1}{n-1}\sum_{i=1}^{n-1}\|x_i\|} \leq a_0$, that is, $\frac{\|x_n\|}{\|x_{k_0}\| + \sum_{i=1, i\neq k_0}^{n-1}\|x_i\|} \leq a_0$. Combining (3.3) with (3.4), we have

$$
\begin{aligned}
&M\Big(\frac{1}{2}\Big\|x_{k_0} - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n} x_i\Big\|\Big)\\
&\quad \leq M\Big(\frac{1}{2}\Big(\|x_{k_0}\| + \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n}\|x_i\|\Big)\Big)\\
&\quad = M\Big(\frac{1}{2(n-1)}\Big((n-1)\|x_{k_0}\| + \sum_{\substack{i=1\\ i\neq k_0}}^{n}\|x_i\|\Big)\Big)\\
&\quad = M\Big(\frac{1}{2(n-1)}\Big((n-1)\|x_{k_0}\| + \sum_{\substack{i=1\\ i\neq k_0}}^{n-1}\|x_i\| + \|x_n\|\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le M\Big(\frac{1}{2(n-1)}\Big(\Big((n-1)\|x_{k_0}\| + \sum_{\substack{i=1\\ i\neq k_0}}^{n-1}\|x_i\|\Big)(1+a_0)\Big)\Big)\\
&= M\Big(\frac{1}{2n-3}\Big(\Big((n-1)\|x_{k_0}\| + \sum_{\substack{i=1\\ i\neq k_0}}^{n-1}\|x_i\|\Big)(1+a_0)\Big)\cdot\frac{2n-3}{2(n-1)}\Big)\\
&\le \gamma_0\frac{2n-3}{2(n-1)}M\Big(\frac{1+a_0}{2n-3}\Big((n-1)\|x_{k_0}\| + \sum_{\substack{i=1\\ i\neq k_0}}^{n-1}\|x_i\|\Big)\Big)\\
&\le \gamma_0 c\frac{2n-3}{2(n-1)}M\Big(\frac{1}{2n-3}\Big((n-1)\|x_{k_0}\| + \sum_{\substack{i=1\\ i\neq k_0}}^{n-1}\|x_i\|\Big)\Big)\\
&\le \gamma_0 c\frac{2n-3}{2(n-1)}\cdot\frac{1}{2n-3}\Big((n-1)M\big(\|x_{k_0}\|\big) + \sum_{\substack{i=1\\ i\neq k_0}}^{n-1}M\big(\|x_i\|\big)\Big)\\
&= \frac{\sqrt{\gamma_0}}{2}\Big(M\big(\|x_{k_0}\|\big) + \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n-1}M\big(\|x_i\|\big)\Big)\\
&\le \frac{\sqrt{\gamma_0}}{2}\Big(M\big(\|x_{k_0}\|\big) + \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n}M\big(\|x_i\|\big)\Big)\\
&= \frac{1}{2}\big(1-(1-\sqrt{\gamma_0})\big)\Big(M\big(\|x_{k_0}\|\big) + \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n}M\big(\|x_i\|\big)\Big),
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\sum_{k=1}^{n} M\Big(\frac{1}{2}\Big\|x_k - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k}}^{n}x_i\Big\|\Big)\\
&\qquad \le \frac{1}{2}\Big(\sum_{k=1}^{n}M\big(\|x_k\|\big) + \frac{1}{n-1}\sum_{k=1}^{n}\sum_{\substack{i=1\\ i\neq k}}^{n}M\big(\|x_i\|\big)\Big)\\
&\qquad\quad - \frac{1-\sqrt{\gamma_0}}{2}\Big(M\big(\|x_{k_0}\|\big) + \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n}M\big(\|x_i\|\big)\Big)\\
&\qquad \le \Big(1-\frac{1-\sqrt{\gamma_0}}{2(n-1)}\Big)\sum_{i=1}^{n}M\big(\|x_i\|\big).
\end{aligned}
$$

Finally, set

$$\tilde{\gamma} = \min\Big\{\frac{a\delta}{(n-1)(1+a)}, \frac{\gamma}{2(n-1)}, \frac{\delta a_0}{n(n-1)+a_0}, \frac{1-\sqrt{\gamma_0}}{2(n-1)}\Big\}.$$

Then $\tilde{\gamma}$ satisfies the demand. □

Theorem 3.2. *The Orlicz–Bochner function space $L_{(M)}(\mu, X)$ is I-convex if and only if*

(1) $M \in \Delta_2(\mathbb{R})$ and $N \in \Delta_2(\mathbb{R})$ (i.e., $L_{(M)}(\mu)$ is reflexive), and

(2) X is I-convex.

Proof. We need to prove only the sufficiency. Lemma 3.1 shows that there exist $n \in \mathbb{N}$ and $\tilde{\gamma} \in (0, 1)$ such that (3.1) holds for any $x_1, x_2, \dots, x_n \in X$. Hence for any $f_1, f_2, \dots, f_n \in S(L_{(M)}(\mu, X))$, we have, for almost everywhere $t \in \Omega$,

$$\sum_{k=1}^{n} M\Big(\frac{1}{2}\Big\|f_k(t) - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k}}^{n} f_i(t)\Big\|\Big) \le (1-\tilde{\gamma})\sum_{i=1}^{n} M\big(\|f_i(t)\|\big). \tag{3.5}$$

Integrating both sides of the inequality above over Ω, we can get

$$\sum_{k=1}^{n} \rho_M\Big(\frac{1}{2}\Big(f_k - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k}}^{n} f_i\Big)\Big) \le (1-\tilde{\gamma})\sum_{i=1}^{n} \rho_M(f_i) = n(1-\tilde{\gamma}).$$

Therefore, there exists $k_0 \in \{1, 2, \dots, n\}$ such that

$$\rho_M\Big(\frac{1}{2}\Big(f_{k_0} - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n} f_i\Big)\Big) \le (1-\tilde{\gamma}).$$

Then by $M \in \Delta_2(\mathbb{R})$, we know that

$$\Big\|f_{k_0} - \frac{1}{n-1}\sum_{\substack{i=1\\ i\neq k_0}}^{n} f_i\Big\|_{(M)} \le 2 - \varepsilon$$

for some $\varepsilon > 0$ depending only on $\tilde{\gamma}$. So $L_{(M)}(\mu, X)$ is I-convex. □

Corollary 3.3. *Suppose that $1 < p < \infty$. Then the Lebesgue–Bochner function space $L_p(\mu, X)$ is I-convex if and only if X is I-convex.*

4. Q-convexity in Orlicz–Bochner space

Lemma 4.1. *Let X be a Q-convex Banach space, and let $M \in \Delta_2(\mathbb{R})$ and $N \in \Delta_2(\mathbb{R})$. Then there exist $n \in \mathbb{N}$ and $\tilde{\gamma} \in (0, 1)$ such that for any $x_1, x_2, \dots, x_n \in X$, it holds that*

$$\sum_{k=1}^{n-1} kM\Big(\frac{\|x_{k+1} - \frac{1}{k}\sum_{i=1}^{k} x_i\|}{2}\Big) \le \frac{n-1}{2}(1-\tilde{\gamma})\sum_{i=1}^{n} M\big(\|x_i\|\big). \tag{4.1}$$

Proof. Let $a \in (0,1)$ and $\gamma = \gamma(a) \in (0,1)$ satisfy (2.2), and denote $\min\{\|x_i\| : i = 1,2,\dots,n\}$ by $\|\overline{x}\|$, $\max\{\|x_i\| : i = 1,2,\dots,n\}$ by $\|\tilde{x}\|$, where n is defined in Lemma 2.3. For clarity, we will divide the proof into two cases.

Case I: $\frac{\|\overline{x}\|}{\|\tilde{x}\|} > a$.

Clearly, $x_i \neq 0$ for all $i \in \{1,2,\dots,n\}$, and for all $i,j \in \{1,2,\dots,n\}$ with $i \neq j$, it holds that

$$a < \frac{\|x_i\|}{\|x_j\|} < \frac{1}{a}.$$

Suppose that $k_0 \in \{1,2,\dots,n-1\}$ satisfies Lemma 2.5. Then

$$\frac{\min\{\|x_1\|, \|x_2\|, \dots, \|x_{k_0+1}\|\}}{\|x_{k_0+1}\| + \frac{1}{k_0}\sum_{i=1}^{k_0}\|x_i\|} \geq \frac{1}{\frac{1}{a} + \frac{1}{k_0}\sum_{i=1}^{k_0}\frac{1}{a}} = \frac{a}{2},$$

and so

$$\Big\|x_{k_0+1} - \frac{1}{k_0}\sum_{i=1}^{k_0} x_i\Big\| \leq (1-a\delta)\Big(\|x_{k_0+1}\| + \frac{1}{k_0}\sum_{i=1}^{k_0}\|x_i\|\Big).$$

By the convexity of M, we have

$$k_0 M\Big(\frac{\|x_{k_0+1} - \frac{1}{k_0}\sum_{i=1}^{k_0} x_i\|}{2}\Big) \leq \frac{k_0}{2}(1-a\delta)\Big(M\big(\|x_{k_0+1}\|\big) + \frac{1}{k_0}\sum_{i=1}^{k_0} M\big(\|x_i\|\big)\Big).$$

In (2.3), putting $l = \frac{1}{a}$ and denoting $\beta = \frac{1}{k}, t = \frac{u}{a}$, we obtain, for any $t \in \mathbb{R}$, $M(at) \geq \beta M(t)$. Consequently,

$$\begin{aligned}
&\sum_{k=1}^{n-1} kM\Big(\frac{\|x_{k+1} - \frac{1}{k}\sum_{i=1}^{k} x_i\|}{2}\Big) \\
&\quad \leq \sum_{k=1}^{n-1}\frac{k}{2}\Big(M\big(\|x_{k+1}\|\big) + \frac{1}{k}\sum_{i=1}^{k} M\big(\|x_i\|\big)\Big) \\
&\qquad - \frac{k_0 a\delta}{2}\Big(M\big(\|x_{k_0+1}\|\big) + \frac{1}{k_0}\sum_{i=1}^{k_0} M\big(\|x_i\|\big)\Big) \\
&\quad = \frac{n-1}{2}\sum_{k=1}^{n} M\big(\|x_k\|\big) - \frac{a\delta}{2}\Big(k_0 M\big(\|x_{k_0+1}\|\big) + \sum_{i=1}^{k_0} M\big(\|x_i\|\big)\Big) \\
&\quad \leq \frac{n-1}{2}\sum_{k=1}^{n} M\big(\|x_k\|\big) - \frac{a\delta}{2} M\big(\|\overline{x}\|\big) \\
&\quad = \frac{n-1}{2}\sum_{k=1}^{n} M\big(\|x_k\|\big) - \frac{a\delta}{2}\frac{1}{n}\sum_{i=1}^{n} M\big(\|\overline{x}\|\big) \\
&\quad \leq \frac{n-1}{2}\sum_{k=1}^{n} M\big(\|x_k\|\big) - \frac{a\delta}{2n}\sum_{i=1}^{n} M\big(a\|\tilde{x}\|\big)
\end{aligned}$$

$$\leq \frac{n-1}{2}\sum_{k=1}^{n} M\big(\|x_k\|\big) - \frac{a\delta\beta}{2n}\sum_{i=1}^{n} M\big(\|\tilde{x}\|\big)$$
$$\leq \frac{n-1}{2}\sum_{k=1}^{n} M\big(\|x_k\|\big) - \frac{a\delta\beta}{2n}\sum_{i=1}^{n} M\big(\|x_i\|\big)$$
$$= \frac{n-1}{2}\Big(1 - \frac{a\delta\beta}{n(n-1)}\Big)\sum_{i=1}^{n} M\big(\|x_i\|\big).$$

Case II: $\frac{\|\overline{x}\|}{\|\tilde{x}\|} \leq a$.

By Lemma 2.1, we know that $\gamma = \gamma(a) \in (0,1)$ such that

$$M\Big(\frac{\|\tilde{x} - \overline{x}\|}{2}\Big) \leq M\Big(\frac{\|\tilde{x}\| + \|\overline{x}\|}{2}\Big) \leq \frac{1}{2}(1-\gamma)\big(M\big(\|\overline{x}\|\big) + M\big(\|\tilde{x}\|\big)\big). \tag{4.2}$$

Since there exists at least one $x_{k_0+1} \in \{x_2, x_3, \ldots, x_n\}$ such that $\|x_{k_0+1}\| = \min\{\|x_i\| : i = 1,2,\ldots,n\}$ or $\|x_{k_0+1}\| = \max\{\|x_i\| : i = 1,2,\ldots,n\}$, we may assume without loss of generality that $k_0 > 1$, $\|x_{k_0+1}\| = \max\{\|x_i\| : i = 1,2,\ldots,n\}$ and $\|x_{k_0}\| = \min\{\|x_i\| : i = 1,2,\ldots,n\}$. Hence by (4.2) and the convexity of M, we get

$$\sum_{k=1}^{n-1} kM\Big(\frac{\|x_{k+1} - \frac{1}{k}\sum_{i=1}^{k} x_i\|}{2}\Big)$$
$$\leq \sum_{k=1}^{n-1} kM\Big(\frac{\|x_{k+1}\| + \frac{1}{k}\sum_{i=1}^{k}\|x_i\|}{2}\Big)$$
$$= \sum_{\substack{k=1\\k\neq k_0}}^{n-1} kM\Big(\frac{\|x_{k+1}\| + \frac{1}{k}\sum_{i=1}^{k}\|x_i\|}{2}\Big)$$
$$+ k_0 M\Big(\frac{1}{k_0}\cdot\frac{k_0\|x_{k_0+1}\| + \|x_1\| + \|x_2\| + \cdots + \|x_{k_0}\|}{2}\Big)$$
$$\leq \sum_{\substack{k=1\\k\neq k_0}}^{n-1} k\frac{M(\|x_{k+1}\|) + M(\frac{1}{k}\sum_{i=1}^{k}\|x_i\|)}{2}$$
$$+ k_0 M\Big(\frac{1}{k_0}\cdot\frac{\|x_{k_0+1}\| + \|x_{k_0}\|}{2} + \frac{k_0-1}{k_0}\cdot\frac{(k_0-1)\|x_{k_0+1}\| + \sum_{i=1}^{k_0-1}\|x_i\|}{2(k_0-1)}\Big)$$
$$\leq \frac{1}{2}\sum_{\substack{k=1\\k\neq k_0}}^{n-1}\Big(kM\big(\|x_{k+1}\|\big) + \sum_{i=1}^{k} M\big(\|x_i\|\big)\Big)$$
$$+ k_0\Big(\frac{1}{k_0}M\Big(\frac{\|x_{k_0+1}\| + \|x_{k_0}\|}{2}\Big) + \frac{k_0-1}{k_0}M\Big(\frac{\|x_{k_0+1}\| + \frac{1}{k_0-1}\sum_{i=1}^{k_0-1}\|x_i\|}{2}\Big)\Big)$$

$$\leq \frac{1}{2}\sum_{\substack{k=1\\k\neq k_0}}^{n-1}\Big(kM\big(\|x_{k+1}\|\big)+\sum_{i=1}^{k}M\big(\|x_i\|\big)\Big)$$
$$+\frac{1}{2}(1-\gamma)\big(M\big(\|x_{k_0+1}\|\big)+M\big(\|x_{k_0}\|\big)\big)$$
$$+\frac{k_0-1}{2}(M\Big(\|x_{k_0+1}\|+\frac{1}{k_0-1}\sum_{i=1}^{k_0-1}M\big(\|x_i\|\big)\Big)$$
$$\leq \frac{1}{2}\sum_{k=1}^{n-1}\Big(kM\big(\|x_{k+1}\|\big)+\sum_{i=1}^{k}M\big(\|x_i\|\big)\Big)-\frac{1}{2}\gamma\big(M\big(\|x_{k_0+1}\|\big)+M\big(\|x_{k_0}\|\big)\big)$$
$$=\frac{n-1}{2}\sum_{k=1}^{n}M\big(\|x_k\|\big)-\frac{1}{2}\gamma\big(M\big(\|x_{k_0+1}\|\big)+M\big(\|x_{k_0}\|\big)\big)$$
$$\leq \frac{n-1}{2}\sum_{k=1}^{n}M\big(\|x_k\|\big)-\frac{\gamma}{2n}(nM\big(\|x_{k_0+1}\|\big)$$
$$\leq \frac{n-1}{2}\sum_{k=1}^{n}M\big(\|x_k\|\big)-\frac{\gamma}{2n}\sum_{i=1}^{n}M\big(\|x_i\|\big)$$
$$=\frac{n-1}{2}\Big(1-\frac{\gamma}{n(n-1)}\Big)\sum_{i=1}^{n}M\big(\|x_i\|\big).$$

Finally, setting $\tilde{\gamma}=\min\{\frac{a\delta\beta}{n(n-1)},\frac{\gamma}{n(n-1)}\}$, we can then get the inequality (4.1), which finishes the proof. □

Theorem 4.2. *The Orlicz–Bochner function space $L_{(M)}(\mu, X)$ is Q-convex if and only if*

(1) $M\in\Delta_2(\mathbb{R})$ and $N\in\Delta_2(\mathbb{R})$ (i.e., $L_{(M)}(\mu)$ is reflexive) and

(2) X is Q-convex.

Proof. We only need to prove the sufficiency. Lemma 4.1 shows that there exist $n\in\mathbb{N}$ and $\tilde{\gamma}\in(0,1)$ such that (4.1) holds for any $x_1, x_2, \ldots, x_n\in X$. Hence for any $f_1, f_2, \ldots, f_n\in S(L_{(M)}(\mu, X))$, we have, for almost everywhere $t\in\Omega$,

$$\sum_{k=1}^{n-1}kM\Big(\frac{\|f_{k+1}(t)-\frac{1}{k}\sum_{i=1}^{k}f_i(t)\|}{2}\Big)\leq\frac{n-1}{2}(1-\tilde{\gamma})\sum_{i=1}^{n}M\big(\|f_i(t)\|\big). \tag{4.3}$$

Integrating both sides of the inequality above over Ω, we can get

$$\sum_{k=1}^{n-1}k\rho_M\Big(\frac{1}{2}\Big(f_{k+1}-\frac{1}{k}\sum_{i=1}^{k}f_i\Big)\Big)\leq\frac{n-1}{2}(1-\tilde{\gamma})\sum_{i=1}^{n}\rho_M(f_i)=\frac{n(n-1)}{2}(1-\tilde{\gamma}).$$

Therefore, there exists $k_0\in\{1,2,\ldots,n-1\}$ such that

$$k_0\rho_M\Big(\frac{1}{2}\Big(f_{k_0+1}-\frac{1}{k_0}\sum_{i=1}^{k_0}f_i\Big)\Big)\leq k_0(1-\tilde{\gamma}).$$

That is,

$$\rho_M\Big(\frac{1}{2}\Big(f_{k_0+1}-\frac{1}{k_0}\sum_{i=1}^{k_0} f_i\Big)\Big)\le 1-\tilde{\gamma}.$$

Then by $M\in\Delta_2(\mathbb{R})$, we know that

$$\Big\|f_{k_0+1}-\frac{1}{k_0}\sum_{i=1}^{k_0} f_i\Big\|_{(M)}\le 2-\varepsilon$$

for some $\varepsilon>0$ depending only on $\tilde{\gamma}$. So $L_{(M)}(\mu,X)$ is Q-convex. □

Corollary 4.3. *Suppose that $1<p<\infty$. Then the Lebesgue–Bochner function space $L_p(\mu,X)$ is Q-convex if and only if X is Q-convex.*

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