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UNIQUE EXPECTATIONS FOR DISCRETE CROSSED PRODUCTS

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Abstract. Let G be a discrete group acting on a unital C^*-algebra $\mathcal{A}$ by $*$-automorphisms. We characterize (in terms of the dynamics) when the inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ has a unique conditional expectation, and when it has a unique pseudoexpectation in the sense of Pitts; we do likewise for the inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes G$. As an application, we re-prove (and potentially extend) some known C^*-simplicity results for $\mathcal{A} \rtimes_r G$.

1. Introduction

Let $\mathcal{B}$ be a unital C^*-algebra and let $\mathcal{A} \subseteq \mathcal{B}$ be a unital C^*-subalgebra, with $1_{\mathcal{A}} = 1_{\mathcal{B}}$. In short, let $\mathcal{A} \subseteq \mathcal{B}$ be a *C^*-inclusion*. Recently we have been concerned with characterizing when a C^*-inclusion admits a unique conditional expectation and when it admits a unique pseudoexpectation in the sense of Pitts, because significant structural consequences often ensue in both cases (see [14], [19]). The present article continues the program, with $\mathcal{B}$ equal to the crossed product of $\mathcal{A}$ by a discrete group G.

A *conditional expectation* for a C^*-inclusion $\mathcal{A} \subseteq \mathcal{B}$ is a *unital completely positive (UCP)* map $E : \mathcal{B} \to \mathcal{A}$ such that $E|_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$. Conditional expectations are automatically $\mathcal{A}$-bimodular, so that $E(ax) = aE(x)$ and $E(xa) = E(x)a$ whenever $x \in \mathcal{B}$ and $a \in \mathcal{A}$. Unfortunately, a C^*-inclusion often admits no conditional expectations at all.

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In [13], Pitts introduced pseudoexpectations as a substitute for possibly nonexistent conditional expectations. A *pseudoexpectation* for a C^*-inclusion $\mathcal{A} \subseteq \mathcal{B}$ is a UCP map $\theta : \mathcal{B} \to I(\mathcal{A})$ such that $\theta|_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$. Here $I(\mathcal{A})$ is Hamana's *injective envelope* of $\mathcal{A}$ (discussed in detail below). Every conditional expectation is a pseudoexpectation, but the converse is false. Just like conditional expectations, pseudoexpectations are $\mathcal{A}$-bimodular. Unlike conditional expectations, pseudoexpectations need not be idempotent. Indeed, if $\theta : \mathcal{B} \to I(\mathcal{A})$ is a pseudoexpectation for $\mathcal{A} \subseteq \mathcal{B}$, then the composition $\theta \circ \theta$ is typically undefined, since it is rarely the case that $I(\mathcal{A}) \subseteq \mathcal{B}$. Furthermore, pseudoexpectations are difficult to describe explicitly, since $I(\mathcal{A})$ only admits a concrete description in exceptional situations.

In spite of their drawbacks, pseudoexpectations enjoy two tremendous technical advantages over conditional expectations, both related to the fact that $I(\mathcal{A})$ is injective. First, pseudoexpectations always exist for any C^*-inclusion $\mathcal{A} \subseteq \mathcal{B}$. Indeed, the identity map $\mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ always has a UCP extension $\theta : \mathcal{B} \to I(\mathcal{A})$, by injectivity. Second, and more generally, pseudoexpectations always extend. That is, if $\theta : \mathcal{B} \to I(\mathcal{A})$ is a pseudoexpectation for $\mathcal{A} \subseteq \mathcal{B}$, and if $\mathcal{B} \subseteq \mathcal{C}$, then there is a pseudoexpectation $\tilde{\theta} : \mathcal{C} \to I(\mathcal{A})$ for $\mathcal{A} \subseteq \mathcal{C}$ such that $\tilde{\theta}|_{\mathcal{B}} = \theta$.

In our experience, for the reasons detailed above, it is easier to characterize when a C^*-inclusion admits a unique pseudoexpectation than to characterize when it admits a unique (or at most one) conditional expectation. Of course, if a C^*-inclusion admits a unique pseudoexpectation, then it admits at most one conditional expectation. So it can be profitable to consider pseudoexpectations, even if one is ultimately interested in conditional expectations. Moreover, because having a unique pseudoexpectation is a stronger condition than having at most one conditional expectation, it usually imposes tougher structural constraints on the inclusion.

In [14], Pitts and the present author investigated the unique pseudoexpectation property for C^*-inclusions, pursuing two complementary directions. On the one hand, we related the unique pseudoexpectation property to other structural properties of the inclusion. For example, we showed that if a C^*-inclusion admits a unique pseudoexpectation which is faithful, then the inclusion is hereditarily essential (see [14, Theorem 3.5]). (A C^*-inclusion $\mathcal{A} \subseteq \mathcal{B}$ is *essential* if every nontrivial ideal $\mathcal{J} \subseteq \mathcal{B}$ intersects $\mathcal{A}$ nontrivially. It is *hereditarily essential* if the C^*-inclusion $\mathcal{A} \subseteq \mathcal{B}_0$ is essential, for every intermediate C^*-algebra $\mathcal{A} \subseteq \mathcal{B}_0 \subseteq \mathcal{B}$.) On the other hand, in [14] we characterized when various special classes of C^*-inclusions admit a unique pseudoexpectation, and in particular, we showed that if $(\mathcal{A}, G, \alpha)$ is a C^*-dynamical system with $\mathcal{A}$ Abelian and G discrete, then the inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ (reduced crossed product) admits a unique pseudoexpectation (necessarily a faithful conditional expectation) if and only if the induced action of G on $\widehat{\mathcal{A}}$ is *topologically free* (for more details, see [14, Theorem 4.6]).

In the present paper, we substantially generalize that theorem. For C^*-dynamical systems $(\mathcal{A}, G, \alpha)$ with $\mathcal{A}$ arbitrary and G discrete, we characterize (in terms of the dynamics) when $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ admits a unique pseudoexpectation, as well as when it admits a unique conditional expectation. There is a unique pseudoexpectation if and only if the action of G is *properly outer* (Theorem 3.5), and there is a unique conditional expectation if and only if G *acts freely* (Theorem 3.2). The same statements hold for the inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes G$ (full crossed product). If the action of G is properly outer, then G acts freely, but the converse is false. Thus we can systematically produce C^*-inclusions with a unique conditional expectation, but multiple pseudoexpectations. (The first such example appears in [19].) Additionally, by combining Theorem 3.5 with the aforementioned [14, Theorem 3.5], we quickly re-prove (and potentially extend) C^*-simplicity results for reduced crossed products, originally due to Kishimoto [11] and Archbold–Spielberg [1].

Remark 1.1. Recently, Kennedy and Schafhauser have independently obtained similar results in a slightly different context in [10]. In particular, they define and analyze pseudoexpectations for discrete C^*-dynamical systems $(\mathcal{A}, G, \alpha)$. These are G-equivariant UCP maps $\phi : \mathcal{A} \rtimes_r G \to I_G(\mathcal{A})$ such that $\phi|_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$, where $I_G(\mathcal{A})$ is Hamana's G-injective envelope of $\mathcal{A}$ [9]. In contrast, we work with (ordinary) pseudoexpectations for the C^*-inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$, which are UCP maps $\theta : \mathcal{A} \rtimes_r G \to I(\mathcal{A})$ such that $\theta|_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$. In general, $I(\mathcal{A}) \subsetneq I_G(\mathcal{A})$, so that a pseudoexpectation for $(\mathcal{A}, G, \alpha)$ (in the sense of Kennedy–Schafhauser) need not be a pseudoexpectation for $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ (in the sense of Pitts). Likewise, a pseudoexpectation for $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ is not G-equivariant in general, and therefore need not be a pseudoexpectation for $(\mathcal{A}, G, \alpha)$.

2. Preliminaries

2.1. Discrete crossed products. Let $\mathcal{A}$ be a unital C^*-algebra, let G be a discrete group, and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be a homomorphism. Briefly, let $(\mathcal{A}, G, \alpha)$ be a discrete C^*-dynamical system. We denote by $\mathcal{A} \rtimes_r G$ (resp., $\mathcal{A} \rtimes G$) the reduced (resp., full) crossed product of $\mathcal{A}$ by G with respect to α. That is, $\mathcal{A} \rtimes_r G$ is the completion of the α-twisted convolution algebra $C_c(G, \mathcal{A})$ with respect to the norm induced by the regular representation, while $\mathcal{A} \rtimes G$ is the completion of $C_c(G, \mathcal{A})$ with respect to the norm induced by the universal representation. Evidently, there exists a $*$-homomorphism $\lambda : \mathcal{A} \rtimes G \to \mathcal{A} \rtimes_r G$ which fixes $C_c(G, \mathcal{A})$. There is also a faithful conditional expectation $\mathbb{E} : \mathcal{A} \rtimes_r G \to \mathcal{A}$ such that $\mathbb{E}(e) = 1$ and $\mathbb{E}(g) = 0$ for all $e \neq g \in G$. This, in turn, gives rise to a canonical conditional expectation $\tilde{\mathbb{E}} = \mathbb{E} \circ \lambda : \mathcal{A} \rtimes G \to \mathcal{A}$.

2.2. Hamana's injective envelope. For every unital C^*-algebra $\mathcal{A}$, there exists a minimal injective operator system $I(\mathcal{A})$ containing $\mathcal{A}$, called the *injective envelope* of $\mathcal{A}$ (see [7]). That is, $I(\mathcal{A})$ is an injective operator system containing $\mathcal{A}$ as an operator subsystem, and if $\mathcal{S}$ is an injective operator system with $\mathcal{A} \subseteq \mathcal{S} \subseteq I(\mathcal{A})$, then $\mathcal{S} = I(\mathcal{A})$. The minimality of $I(\mathcal{A})$ is equivalent to the *rigidity* of the inclusion $\mathcal{A} \subseteq I(\mathcal{A})$ (see [4, Theorem 6.2.1]). That is, if $\Phi : I(\mathcal{A}) \to I(\mathcal{A})$ is a UCP map such that $\Phi|_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$, then $\Phi = \mathrm{id}_{I(\mathcal{A})}$. Using rigidity, it is easy to see that $I(\mathcal{A})$ is uniquely determined up to a complete order isomorphism which fixes $\mathcal{A}$.

A priori, $I(\mathcal{A})$ is just an operator system. However, it turns out that $I(\mathcal{A})$ has a wealth of algebraic and analytical structure. It is a monotonically complete C^*-algebra (and thus an AW^*-algebra) containing $\mathcal{A}$ as a unital C^*-subalgebra. As such, it enjoys many of the nice features one normally associates with von Neumann algebras (see [16, Chapters 2, 8]). In particular, the following apply.

- The projections in $I(\mathcal{A})$ form a complete lattice.
- For every $x \in I(\mathcal{A})$, there exists a smallest projection $\mathrm{LP}(x) \in I(\mathcal{A})$ such that $\mathrm{LP}(x)x = x$. Likewise, there exists a smallest projection $\mathrm{RP}(x) \in I(\mathcal{A})$ such that $x\,\mathrm{RP}(x) = x$.
- For every $x \in I(\mathcal{A})$, there exists a partial isometry $v \in I(\mathcal{A})$ such that $x = v|x|$, $vv^* = \mathrm{LP}(x)$, and $v^*v = \mathrm{RP}(x)$.

It is not true in general that $I(\mathcal{A})$ is a dual Banach space, and so weak-$*$ convergence does not make sense in $I(\mathcal{A})$. On the other hand, there is a well-behaved mode of convergence which often plays the same role (see [16, Chapter 2]). We say that $x \in I(\mathcal{A})$ is the *order limit* of a net $\{x_j\} \subseteq I(\mathcal{A})$, and we write $x = \mathrm{LIM}_j\, x_j$, provided there are increasing nets $\{a_j\}, \{b_j\}, \{c_j\}, \{d_j\} \subseteq I(\mathcal{A})_{\mathrm{sa}}$ with suprema $a, b, c, d \in I(\mathcal{A})_{\mathrm{sa}}$, respectively, such that $x_j = (a_j - b_j) + i(c_j - d_j)$ for all j and $x = (a - b) + i(c - d)$. (It can be shown that this definition is independent of which increasing nets one uses.) From the basic properties of order convergence, we will need the following:

- if $\mathrm{LIM}_j\, x_j = x$ and $\mathrm{LIM}_j\, y_j = y$, then $\mathrm{LIM}_j(x_j + y_j) = x + y$;
- if $\mathrm{LIM}_j\, x_j = x$ and $s, t \in I(\mathcal{A})$, then $\mathrm{LIM}_j\, sx_jt = sxt$;
- if $\mathrm{LIM}_j\, x_j = x$, then $\mathrm{LIM}_j\, x_j^* = x^*$;
- if $\mathrm{LIM}_j\, x_j = x$ and $x_j \to y$ (in norm), then $y = x$;
- if $\{x_j\} \subseteq I(\mathcal{A})_+$ and $\mathrm{LIM}_j\, x_j = x$, then $x \in I(\mathcal{A})_+$;
- if $\{X_j\} \subseteq M_n(I(\mathcal{A}))$, then $\mathrm{LIM}_j\, X_j = X$ if and only if $\mathrm{LIM}_j\, X_j(k, \ell) = X(k, \ell)$ for all $1 \leq k, \ell \leq n$.

2.3. Dynamics. For a unital C^*-algebra $\mathcal{A}$, we denote by $\mathrm{Aut}(\mathcal{A})$ the $*$-automorphisms of $\mathcal{A}$. We say that $\alpha \in \mathrm{Aut}(\mathcal{A})$ is *inner* provided that $\alpha(a) = uau^*$, $a \in \mathcal{A}$, where $u \in \mathcal{A}$ is unitary. Otherwise, we say that α is *outer*. A *dependent element* for $\alpha \in \mathrm{Aut}(\mathcal{A})$ is an element $d \in \mathcal{A}$ such that $da = \alpha(a)d$, $a \in \mathcal{A}$. We say that α is *freely acting* if it has no nonzero dependent elements. Clearly, a freely acting automorphism must be outer.

Every $\alpha \in \mathrm{Aut}(\mathcal{A})$ has a unique extension $\tilde{\alpha} \in \mathrm{Aut}(I(\mathcal{A}))$ (see [7, Corollary 4.2]). This allows one to rephrase dynamical properties of α in terms of dynamical properties of $\tilde{\alpha}$, where the situation is usually simpler. In particular, the definitions of quasi-innerness and proper outerness below are much more tractable when stated for $\tilde{\alpha}$ rather than α, as follows.

- We say that α is *quasi-inner* if its Borchers spectrum is trivial (i.e., if $\Gamma_{\mathrm{Bor}}(\alpha) = \{1\} \subseteq \mathbb{T}$). Equivalently, α is quasi-inner if $\tilde{\alpha}$ is inner (see [9, Theorem 7.4]).
- We say that α is *properly outer* if there does not exist a nonzero α-invariant ideal $\mathcal{J} \subseteq \mathcal{A}$ such that $\alpha|_{\mathcal{J}}$ is quasi-inner. Equivalently, α is properly outer if there does not exist a nonzero $\tilde{\alpha}$-invariant central projection $z \in Z(I(\mathcal{A}))$ such that $\tilde{\alpha}|_{I(\mathcal{A})z}$ is inner (see [9, Remark 7.5]). Equivalently, α is properly outer if $\tilde{\alpha}$ is freely acting (see [8, Proposition 5.1]).

Remark 2.1. Our use of the term "properly outer" coincides with its use by Hamana in [9], who in turn attributes it to Kishimoto. There is another definition of proper outerness in the literature, due to Elliott [5, Definition 2.1]. Kishimoto's condition implies Elliott's condition, and they agree if the C^*-algebra is separable (see [9, p. 477]; see also [10, Section 2.5] and [12, Section 2]).

It follows from the discussion above that for automorphisms of $I(\mathcal{A})$, proper outerness and acting freely are equivalent. For automorphisms of $\mathcal{A}$, proper outerness is in general the stronger condition. Put another way, if $\tilde{\alpha}$ acts freely, then so does α. Indeed, as implied by the following technical lemma, dependent elements for α are also dependent elements for $\tilde{\alpha}$.

Lemma 2.2. *Let $\mathcal{A}$ be a unital C^*-algebra, let $\alpha \in \mathrm{Aut}(\mathcal{A})$, and let $x \in I(\mathcal{A})$. If*

$$xa = \alpha(a)x, \quad a \in \mathcal{A},$$

then

$$xt = \tilde{\alpha}(t)x, \quad t \in I(\mathcal{A}).$$

Proof. We may assume that $\|x\| \leq 1$. We claim that $x^*x = xx^* \in Z(I(\mathcal{A}))$. Indeed, arguing as in the proof of [3, Lemma 1], we see that $x^*x, xx^* \in \mathcal{A}' \cap I(\mathcal{A})$. By [7, Corollary 4.3], $\mathcal{A}' \cap I(\mathcal{A}) = Z(I(\mathcal{A}))$, and so $x^*x, xx^* \in Z(I(\mathcal{A}))$. Then the proof of [3, Lemma 2] shows that $x^*x = xx^*$. It follows immediately from the claim that $|x| \in Z(I(\mathcal{A}))$. Now let $v \in I(\mathcal{A})$ be a partial isometry such that $x = v|x|$, $vv^* = \mathrm{LP}(x)$, and $v^*v = \mathrm{RP}(x)$. We have that $\mathrm{LIM}_n |x|^{1/n} = v^*v$. For all $a \in \mathcal{A}$,

$$\begin{aligned} v|x|a = \alpha(a)v|x| &\implies v|x|^n a = \alpha(a)v|x|^n, \quad n \in \mathbb{N} \\ &\implies v|x|^{1/n} a = \alpha(a)v|x|^{1/n}, \quad n \in \mathbb{N} \\ &\implies vv^*va = \alpha(a)vv^*v \\ &\implies va = \alpha(a)v. \end{aligned}$$

Thus, as before,

$$v^*v = vv^* \in Z\big(I(\mathcal{A})\big).$$

Set $p = v^*v$, a projection in $Z(I(\mathcal{A}))$, and define a UCP map $\theta : I(\mathcal{A}) \to I(\mathcal{A})$ by the formula

$$\theta(t) = v^*\tilde{\alpha}(t)v + p^{\perp}t, \quad t \in I(\mathcal{A}).$$

For all $a \in \mathcal{A}$, we have

$$\theta(a) = v^*\alpha(a)v + p^{\perp}a = v^*va + p^{\perp}a = pa + p^{\perp}a = a.$$

By rigidity, $\theta = \mathrm{id}_{I(\mathcal{A})}$, and so

$$v^*vt = v^*\tilde{\alpha}(t)v, \quad t \in I(\mathcal{A}).$$

Premultiplying by v yields

$$vt = vv^*\tilde{\alpha}(t)v = \tilde{\alpha}(t)v, \quad t \in I(\mathcal{A}).$$

It follows that

$$xt = \tilde{\alpha}(t)x, \quad t \in I(\mathcal{A}),$$

as desired. □

Remark 2.3. We extend the definitions of "outer," "freely acting," and "properly outer" from single automorphisms to actions of discrete groups by insisting that the conditions hold pointwise. More precisely, for a discrete C^*-dynamical system $(\mathcal{A}, G, \alpha)$, we say that α is outer (resp., freely acting, properly outer) provided that α_g is outer (resp., freely acting, properly outer) for all $e \neq g \in G$.

3. Unique expectations

3.1. Unique conditional expectations. In this section we show that $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ admits a unique conditional expectation if and only if G acts freely on $\mathcal{A}$. We begin with a proposition of independent interest, which was inspired by [17, Proposition 3.1.4].

Proposition 3.1. *Let $\mathcal{A} \subseteq \mathcal{B}$ be a C^*-inclusion. Assume that there exists a unique conditional expectation $E : \mathcal{B} \to \mathcal{A}$. Then E is multiplicative on $\mathcal{A}^c = \mathcal{A}' \cap \mathcal{B}$, the relative commutant of $\mathcal{A}$ in $\mathcal{B}$. If, in addition, E is faithful, then $\mathcal{A}^c = Z(\mathcal{A})$.*

Proof. Since E is $\mathcal{A}$-bimodular, $E(\mathcal{A}^c) = Z(\mathcal{A})$. Let $x \in (\mathcal{A}^c)_{\mathrm{sa}}$, with $\|x\| < 1$. Then $1 - x$ is a positive invertible element of $\mathcal{A}^c$, and $1 - E(x)$ is a positive invertible element of $Z(\mathcal{A})$. Define a UCP map $\theta : \mathcal{B} \to \mathcal{A}$ by the formula

$$\theta(b) = E\big((1-x)^{1/2} b (1-x)^{1/2}\big)\big(1 - E(x)\big)^{-1}, \quad b \in \mathcal{B}.$$

It is easy to see that $\theta(a) = a$, $a \in \mathcal{A}$, so that θ is a conditional expectation. By assumption, $\theta = E$, and so

$$E(x)\big(1 - E(x)\big) = E\big((1-x)^{1/2} x (1-x)^{1/2}\big),$$

which implies that $E(x^2) = E(x)^2$. It follows that x is in the multiplicative domain of E. Since the choice of x was arbitrary, $E|_{\mathcal{A}^c} : \mathcal{A}^c \to Z(\mathcal{A})$ is a $*$-homomorphism. If E is faithful, then $E|_{\mathcal{A}^c}$ is injective. In that case, $x = E(x) \in Z(\mathcal{A})$ for all $x \in \mathcal{A}^c$, since $E(x - E(x)) = 0$. □

Theorem 3.2. *Let $(\mathcal{A}, G, \alpha)$ be a discrete C^*-dynamical system. Then the following are equivalent:*

i. *$\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ admits a unique conditional expectation,*
ii. *$\mathcal{A}^c = Z(\mathcal{A})$,*
iii. *G acts freely on $\mathcal{A}$.*

Proof. (i $\implies$ ii) See Proposition 3.1.

(ii $\implies$ iii) Suppose that $\mathcal{A}^c = Z(\mathcal{A})$. Let $e \neq g \in G$ and $d \in \mathcal{A}$, and assume that $da = \alpha_g(a)d$ for all $a \in \mathcal{A}$. Then $g^{-1}d \in \mathcal{A}^c$, which implies that $d = 0$.

(iii $\implies$ i) Suppose that G acts freely on $\mathcal{A}$. Let $\theta : \mathcal{A} \rtimes_r G \to \mathcal{A}$ be a conditional expectation. Fix $e \neq g \in G$. For all $a \in \mathcal{A}$,

$$\theta(g)a = \theta(ga) = \theta\big(\alpha_g(a)g\big) = \alpha_g(a)\theta(g).$$

It follows that $\theta(g) = 0$. Since the choice of g was arbitrary, $\theta = \mathbb{E}$. □

Corollary 3.3. *Let $(\mathcal{A}, G, \alpha)$ be a discrete C^*-dynamical system. Then $\mathcal{A} \subseteq \mathcal{A} \rtimes G$ (full crossed product) admits a unique conditional expectation if and only if G acts freely on $\mathcal{A}$.*

Proof. ($\Rightarrow$) Let $\theta : \mathcal{A} \rtimes_r G \to \mathcal{A}$ be a conditional expectation. Then $\theta \circ \lambda : \mathcal{A} \rtimes G \to \mathcal{A}$ is a conditional expectation, so that $\theta \circ \lambda = \mathbb{E} \circ \lambda$, by uniqueness. Thus $\theta = \mathbb{E}$. By Theorem 3.2, G acts freely on $\mathcal{A}$.

($\Leftarrow$) Conversely, suppose that G acts freely on $\mathcal{A}$. Let $\Theta : \mathcal{A} \rtimes G \to \mathcal{A}$ be a conditional expectation. Then repeating the proof of (iii $\implies$ i) in Theorem 3.2 above, with θ replaced by Θ, we see that $\Theta(g) = 0$ for all $g \neq e$. Hence, $\Theta = \mathbb{E} \circ \lambda$. □

3.2. Unique pseudoexpectations. In this section, we show that $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ (resp., $\mathcal{A} \subseteq \mathcal{A} \rtimes G$) admits a unique pseudoexpectation if and only if the action of G on $\mathcal{A}$ is properly outer. We begin with a technical lemma, similar in spirit to [4, Lemma 5.1.6].

Lemma 3.4. *Let $\mathcal{A} \subseteq \mathcal{B}$ be a C^*-inclusion and let $\theta : \mathcal{B} \to I(\mathcal{A})$ be a completely positive $\mathcal{A}$-bimodule map. Then there exists a UCP $\mathcal{A}$-bimodule map $\tilde{\theta} : \mathcal{B} \to I(\mathcal{A})$ (i.e., a pseudoexpectation for $\mathcal{A} \subseteq \mathcal{B}$) such that $\theta(x) = \theta(1)\tilde{\theta}(x)$, $x \in \mathcal{B}$.*

Proof. Since

$$a\theta(1) = \theta(a) = \theta(1)a, \quad a \in \mathcal{A},$$

we see that $\theta(1) \in \mathcal{A}' \cap I(\mathcal{A})$. By [7, Corollary 4.3], $\theta(1) \in Z(I(\mathcal{A}))$. We claim that $\mathrm{LIM}_n(\theta(1) + 1/n)^{-1}\theta(x)$ exists for all $x \in \mathcal{B}$. Indeed, for all $x \in \mathcal{B}_+$, $\{(\theta(1) + 1/n)^{-1}\theta(x)\} \subseteq I(\mathcal{A})_+$ is an increasing sequence bounded above by $\|x\|$. In particular, $\mathrm{LIM}_n(\theta(1) + 1/n)^{-1}\theta(1) = p$, where $p = \mathrm{LP}(\theta(1)) = \mathrm{RP}(\theta(1)) \in Z(I(\mathcal{A}))$. Now define a unital positive linear map $\tilde{\theta} : \mathcal{B} \to I(\mathcal{A})$ by the formula

$$\tilde{\theta}(x) = \mathrm{LIM}_n\big(\theta(1) + 1/n\big)^{-1}\theta(x) + p^\perp\Phi(x), \quad x \in \mathcal{B},$$

where $\Phi : \mathcal{B} \to I(\mathcal{A})$ is any fixed UCP $\mathcal{A}$-bimodule map (i.e., any pseudoexpectation for $\mathcal{A} \subseteq \mathcal{B}$). In fact, $\tilde{\theta}$ is completely positive, since

$$\tilde{\theta}_k(X) = \mathrm{LIM}_n\big(I_k \otimes \big(\theta(1) + 1/n\big)^{-1}\big)\theta_k(X) + (I_k \otimes p^\perp)\Phi_k(X), \quad X \in M_k(\mathcal{B}).$$

Because θ and Φ are $\mathcal{A}$-bimodular, so is $\tilde{\theta}$. Furthermore,

$$\theta(1)\tilde{\theta}(x) = p\theta(x), \quad x \in \mathcal{B}.$$

But $p\theta(x) = \theta(x)$, $x \in \mathcal{B}$. Indeed, for all $x \in \mathcal{B}_{\mathrm{sa}}$,

$$-\|x\| \leq x \leq \|x\| \implies -\|x\|\theta(1) \leq \theta(x) \leq \|x\|\theta(1) \implies p^{\perp}\theta(x) = 0.$$

Thus

$$\theta(1)\tilde{\theta}(x) = \theta(x), \quad x \in \mathcal{B}.$$

□

Theorem 3.5. *Let $(\mathcal{A}, G, \alpha)$ be a discrete C^*-dynamical system. Then $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ admits a unique pseudoexpectation if and only if the action of G on $\mathcal{A}$ is properly outer.*

Proof. ($\Rightarrow$) Suppose that $\alpha_g \in \mathrm{Aut}(\mathcal{A})$ is not properly outer for some $g \neq e$. Then $\tilde{\alpha}_g \in \mathrm{Aut}(I(\mathcal{A}))$ is not freely acting, and so there exists $0 \neq v \in I(\mathcal{A})$ such that $vt = \tilde{\alpha}_g(t)v$, $t \in I(\mathcal{A})$. In particular, $va = \alpha_g(a)v$, $a \in \mathcal{A}$. Define a completely bounded map $\theta : \mathcal{A} \rtimes_r G \to I(\mathcal{A})$ by the formula

$$\theta(x) = \mathbb{E}(xg^{-1})v, \quad x \in \mathcal{A} \rtimes_r G,$$

where $\mathbb{E} : \mathcal{A} \rtimes_r G \to \mathcal{A}$ is the canonical conditional expectation. Note that $\theta(g) = v \neq 0$. Obviously, θ is a left $\mathcal{A}$-bimodule map, since $\mathbb{E}$ is. It is also a right $\mathcal{A}$-bimodule map, since for all $x \in \mathcal{A} \rtimes_r G$ and all $a \in \mathcal{A}$, we have

$$\begin{aligned}
\theta(xa) &= \mathbb{E}(xag^{-1})v = \mathbb{E}(xg^{-1}gag^{-1})v \\
&= \mathbb{E}\big(xg^{-1}\alpha_g(a)\big)v = \mathbb{E}(xg^{-1})\alpha_g(a)v \\
&= \mathbb{E}(xg^{-1})va = \theta(x)a.
\end{aligned}$$

By [18, Satz 4.5], $\theta = (\theta_1 - \theta_2) + i(\theta_3 - \theta_4)$, where $\theta_j : \mathcal{A} \rtimes_r G \to I(\mathcal{A})$ is a completely positive $\mathcal{A}$-bimodule map, $1 \leq j \leq 4$. Without loss of generality, $\theta_1(g) \neq 0$. By Lemma 3.4, there exists a pseudoexpectation $\tilde{\theta}_1 : \mathcal{A} \rtimes_r G \to I(\mathcal{A})$ for $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ such that $\theta_1(x) = \theta_1(1)\tilde{\theta}_1(x)$, $x \in \mathcal{A} \rtimes_r G$. In particular, $\tilde{\theta}_1(g) \neq 0$, so that $\tilde{\theta}_1 \neq \mathbb{E}$.

($\Leftarrow$) Conversely, suppose that $\alpha_g \in \mathrm{Aut}(\mathcal{A})$ is properly outer for all $g \neq e$. Then $\tilde{\alpha}_g \in \mathrm{Aut}(I(\mathcal{A}))$ is freely acting for all $g \neq e$. Let $\theta : \mathcal{A} \rtimes_r G \to I(\mathcal{A})$ be a pseudoexpectation for $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$. For $g \in G$, we have

$$gag^{-1} = \alpha_g(a) \implies ga = \alpha_g(a)g \implies \theta(g)a = \alpha_g(a)\theta(g), \quad a \in \mathcal{A}.$$

By Lemma 2.2, we have

$$\theta(g)t = \tilde{\alpha}_g(t)\theta(g), \quad t \in I(\mathcal{A}).$$

Thus $\theta(g) = 0$ for all $g \neq e$. Hence, $\theta = \mathbb{E}$. □

As pointed out to us by David Pitts, the proof of Theorem 3.5 can be repeated verbatim with $\mathcal{A} \rtimes_r G$ replaced by $\mathcal{A} \rtimes G$ and $\mathbb{E} : \mathcal{A} \rtimes_r G \to \mathcal{A}$ replaced by $\tilde{\mathbb{E}} = \mathbb{E} \circ \lambda : \mathcal{A} \rtimes G \to \mathcal{A}$. Thus we have the following.

Corollary 3.6. *Let $(\mathcal{A}, G, \alpha)$ be a discrete C^*-dynamical system. Then $\mathcal{A} \subseteq \mathcal{A} \rtimes G$ (full crossed product) admits a unique pseudoexpectation if and only if the action of G on $\mathcal{A}$ is properly outer.*

4. Applications

4.1. Special inclusions. In this section we specialize Theorems 3.2 and 3.5 and their corollaries to particular cases, namely $\mathcal{A}$ Abelian and $\mathcal{A}$ simple. We begin with the case $\mathcal{A}$ Abelian.

Remark 4.1. If $\mathcal{A}$ is unital Abelian C^*-algebra, then every $\alpha \in \mathrm{Aut}(\mathcal{A})$ induces a homeomorphism $\hat{\alpha} : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$ by the formula $\hat{\alpha}(\sigma) = \sigma \circ \alpha^{-1}$, $\sigma \in \widehat{\mathcal{A}}$. In that case, the following are equivalent:

i. α is properly outer,
ii. α is freely acting,
iii. $\hat{\alpha}$ is topologically free (i.e., $\mathrm{fix}(\hat{\alpha})^\circ = \emptyset$).

Proof. (i $\implies$ ii) This is true in general, not just the Abelian case.

(ii $\implies$ i) Suppose that α is freely acting. Let $\mathcal{J} \subseteq \mathcal{A}$ be an α-invariant ideal such that $\alpha|_{\mathcal{J}}$ is quasi-inner. Then $\widetilde{\alpha|_{\mathcal{J}}}$ is inner, therefore the identity map. Hence $\alpha|_{\mathcal{J}}$ is the identity map. Now let $h \in \mathcal{J}$. For all $a \in \mathcal{A}$, we have

$$ha = \alpha(ha) = \alpha(h)\alpha(a) = h\alpha(a) = \alpha(a)h.$$

Thus $h = 0$. Since the choice of h was arbitrary, $\mathcal{J} = 0$ and α is properly outer.

(ii $\iff$ iii) [6, Theorem 1]. □

Corollary 4.2. *Let $(\mathcal{A}, G, \alpha)$ be a discrete C^*-dynamical system, with $\mathcal{A}$ Abelian. Then the following are equivalent:*

i. *$\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ (or $\mathcal{A} \subseteq \mathcal{A} \rtimes G$) admits a unique pseudoexpectation,*
ii. *$\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ (or $\mathcal{A} \subseteq \mathcal{A} \rtimes G$) admits a unique conditional expectation,*
iii. *$\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ is a MASA,*
iv. *G acts topologically freely on $\widehat{\mathcal{A}}$ (i.e., $\mathrm{fix}(\hat{\alpha}_g)^\circ = \emptyset$ for all $e \neq g \in G$).*

(In particular, we recover [14, Theorem 4.6].)

Now we consider the case $\mathcal{A}$ simple.

Remark 4.3. If $\mathcal{A}$ is a simple unital C^*-algebra and $\alpha \in \mathrm{Aut}(\mathcal{A})$, then the following are equivalent.

i. α is properly outer.
ii. α is freely acting.
iii. α is outer.

Proof. (i $\implies$ ii) This is true in general, not just the simple case.

(ii $\implies$ iii) This is true in general, not just the simple case.

(iii $\implies$ i) Suppose that α is outer. By [15, Theorem 3.6], $\tilde{\alpha}$ is outer. Now $I(\mathcal{A})$ is simple, and therefore a factor [7, Proposition 4.15]. Thus $\tilde{\alpha}$ is properly outer, and therefore α is as well. □

Corollary 4.4. *Let $(\mathcal{A}, G, \alpha)$ be a discrete C^*-dynamical system, with $\mathcal{A}$ simple. Then the following are equivalent.*

i. *$\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ (or $\mathcal{A} \subseteq \mathcal{A} \rtimes G$) admits a unique pseudoexpectation,*
ii. *$\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ (or $\mathcal{A} \subseteq \mathcal{A} \rtimes G$) admits a unique conditional expectation,*
iii. *the action of G on $\mathcal{A}$ is outer.*

4.2. Simplicity of reduced crossed products. In this section, we use Theorem 3.5 and Corollary 3.6 to quickly re-prove (and potentially extend) C^*-simplicity results for reduced crossed products, due to Kishimoto and Archbold–Spielberg.

In [11], Kishimoto proves that if a discrete group G acts on a simple unital C^*-algebra $\mathcal{A}$ by outer automorphisms, then $\mathcal{A} \rtimes_r G$ is simple. It follows that $\mathcal{A} \rtimes_r H$ is simple for any subgroup $H \subseteq G$. Recently, Cameron and Smith obtained the beautiful result that every intermediate C^*-algebra $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{A} \rtimes_r G$ has this form (see [2, Theorem 3.5]). Combining these statements gives the following.

Theorem 4.5 ([11, Theorem 3.1], [2, Theorem 3.5]). *Let G be a discrete group acting on a simple unital C^*-algebra $\mathcal{A}$ by outer automorphisms. Then every intermediate C^*-algebra $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{A} \rtimes_r G$ is simple.*

Proof. By Remark 4.3, the action of G on $\mathcal{A}$ is properly outer, and so by Theorem 3.5 the inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ has a unique pseudoexpectation, which is actually a faithful conditional expectation. By [14, Theorem 3.5], the inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ is *hereditarily essential* (see the Introduction for a reminder of what this means). Now suppose that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{A} \rtimes_r G$ is an intermediate C^*-algebra and $0 \neq \mathcal{J} \subseteq \mathcal{B}$ is an ideal. Then $\mathcal{J} \cap \mathcal{A} \neq 0$, which implies that $\mathcal{J} \cap \mathcal{A} = \mathcal{A}$, which in turn implies that $1 \in \mathcal{J}$. Hence $\mathcal{J} = \mathcal{B}$, and $\mathcal{B}$ is simple. ☐

In [1], Archbold and Spielberg prove that if a discrete group G acts *topologically freely* and *minimally* on a unital C^*-algebra $\mathcal{A}$, then $\mathcal{A} \rtimes_r G$ is simple. We encountered the definition of topological freeness for actions of discrete groups on Abelian C^*-algebras in the preceding section (see the statement of Corollary 4.2). Archbold and Spielberg [1, Definition 1] generalized this definition to actions of discrete groups on arbitrary (non-Abelian) C^*-algebras, as follows: G acts topologically freely on $\mathcal{A}$ if for all finite sets $F \subseteq G \backslash \{e\}$,

$$\Big(\bigcup_{g \in F} \operatorname{fix}(\hat{\alpha}_g)\Big)^{\circ} = \emptyset,$$

where $\hat{\alpha}_g \in \operatorname{Homeo}(\widehat{\mathcal{A}})$ is the homeomorphism of the spectrum of $\mathcal{A}$ induced by the automorphism $\alpha_g \in \operatorname{Aut}(\mathcal{A})$. (Note that for non-Abelian C^*-algebras, topological freeness is no longer a pointwise condition.) On the other hand, minimality of the action means that $\mathcal{A}$ has no nonzero G-invariant ideals.

The aforementioned Archbold–Spielberg C^*-simplicity result is an easy corollary of the following theorem, one of the main results of their paper [1].

Theorem 4.6 ([1, Theorem 1]). *Let G be a discrete group acting topologically freely on a unital C^*-algebra $\mathcal{A}$. If $\mathcal{J} \subseteq \mathcal{A} \rtimes G$ is an ideal such that $\mathcal{J} \cap \mathcal{A} = 0$, then $\mathcal{J} \subseteq \ker(\lambda)$, where $\lambda : \mathcal{A} \rtimes G \to \mathcal{A} \rtimes_r G$ is the canonical $*$-homomorphism.*

We can economically prove Theorem 4.6 under the hypothesis that the action of G on $\mathcal{A}$ is properly outer, instead of topologically free. Simplicity of $\mathcal{A} \rtimes_r G$ when the action is properly outer and minimal then follows as in [1]. If $\mathcal{A}$ is separable, then topological freeness and proper outerness coincide (see [12, Theorem 2.13, Lemma 2.17]), and so we recover the Archbold–Spielberg results in that setting.

In general, the relationship between topological freeness and proper outerness is unclear, so that (potentially) we have extended the Archbold–Spielberg results in the nonseparable case.

Proof of Theorem 4.6 for properly outer actions. Suppose that the action of G on $\mathcal{A}$ is properly outer. Let $\mathcal{J} \subseteq \mathcal{A} \rtimes G$ be an ideal such that $\mathcal{J} \cap \mathcal{A} = 0$. Define a unital $*$-homomorphism $\pi : \mathcal{A} + \mathcal{J} \to \mathcal{A} : a + h \mapsto a$. By injectivity, π extends to a pseudoexpectation $\theta : \mathcal{A} \rtimes G \to I(\mathcal{A})$ for $\mathcal{A} \subseteq \mathcal{A} \rtimes G$. By Corollary 3.6, $\theta = \mathbb{E} \circ \lambda$. Thus

$$\begin{aligned} h \in \mathcal{J} &\implies \text{that } \mathbb{E}\big(\lambda(h)^*\lambda(h)\big) = \mathbb{E}\big(\lambda(h^*h)\big) = \theta(h^*h) = \pi(h^*h) = 0 \\ &\implies \text{that } \lambda(h) = 0. \end{aligned}$$

Hence, $\mathcal{J} \subseteq \ker(\lambda)$. □

4.3. Unique conditional expectation but multiple pseudoexpectations. In [19, Example 4.4], we produce a C^*-inclusion $\mathcal{A} \subseteq \mathcal{B}$ with a unique conditional expectation, but infinitely many pseudoexpectations. In fact, $\mathcal{B}$ is Abelian in our example. Unfortunately, the construction is a bit ad hoc. Also, the conditional expectation is not faithful. Now we can produce many such examples systematically. Indeed, if $\mathcal{A}$ is a unital C^*-algebra and G is a discrete group acting freely but not properly outerly on $\mathcal{A}$, then the C^*-inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_r G$ admits a unique (faithful) conditional expectation, but infinitely many pseudoexpectations. For example, let $\mathcal{A} = \mathbb{C}I + K(\ell^2(\mathbb{Z})) \subseteq B(\ell^2(\mathbb{Z}))$, let $G = \mathbb{Z}$, and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be given by $\alpha_k(T) = S^kTS^{-k}$, where $S \in B(\ell^2(\mathbb{Z}))$ is the bilateral shift.

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