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GENERALIZATIONS OF JENSEN'S OPERATOR INEQUALITY FOR CONVEX FUNCTIONS TO NORMAL OPERATORS

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Abstract. In this article, we generalize a well-known operator version of Jensen's inequality to normal operators. The main techniques employed here are the spectral theory for bounded normal operators on a Hilbert space, and different Jensen-type inequalities. We emphasize the application of a vector version of Jensen's inequality. By applying our results, some classical inequalities obtained for self-adjoint operators can also be extended.

1. Introduction

Throughout this article $(H, \langle \cdot, \cdot \rangle)$ means a complex Hilbert space. The Banach algebra of all bounded linear operators on H is denoted by $\mathcal{B}(H)$. The operator norm on $\mathcal{B}(H)$ is defined as usual by

$$\|A\| := \sup_{\|x\| \leq 1} \|Ax\|, \quad A \in \mathcal{B}(H).$$

An operator $A \in \mathcal{B}(H)$ is said to be *normal* (especially *self-adjoint*) if $AA^* = A^*A$ ($A = A^*$). The spectrum of an operator $A \in \mathcal{B}(H)$ is denoted by $\sigma(A)$. For a set $K \subset \mathbb{C}$, $N(K)$ means the class of all normal operators from $\mathcal{B}(H)$ whose spectra are contained in K. Similarly, if $J \subset \mathbb{R}$ is an interval, then $S(J)$ denotes the class of all self-adjoint operators from $\mathcal{B}(H)$ whose spectra are contained in J.

Different types of inequalities between self-adjoint operators in $\mathcal{B}(H)$ have undergone extensive study and have many applications (see, e.g., Pečarić, Furuta, Mićić, and Seo [9]). The treatment of a large group of such inequalities depends

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on the continuous functional calculus for self-adjoint operators (see Rudin [11]), and the important notions of operator convexity and operator monotonicity. For normal operators in $\mathcal{B}(H)$, on the other hand, there are only a few papers in which convexity or monotonicity is used (see Sookia and Gonpot [12]), although there exists a functional calculus for normal operators too. Other types of inequalities for normal operators have been investigated by various authors (see Conde [1], Dragomir [2], Dragomir and Moslehian [3], and Menkad and Seddik [7]).

In this article, we generalize the following well-known operator version of Jensen's inequality to normal operators.

Theorem 1.1 (see [8, Theorem 1], [9, p. 5]). *Let $J \subset \mathbb{R}$ be an interval. Let $A_i \in S(J)$ and $x_i \in H$ $(i = 1, \ldots, n)$ with $\sum_{i=1}^{n} \|x_i\|^2 = 1$. If $f : J \to \mathbb{R}$ is continuous and convex, then*

$$f\Big(\sum_{i=1}^{n} \langle A_i x_i, x_i \rangle\Big) \le \sum_{i=1}^{n} \big\langle f(A_i) x_i, x_i \big\rangle. \tag{1.1}$$

The main techniques employed here are the spectral theory for normal operators in $\mathcal{B}(H)$ and different Jensen-type inequalities. We emphasize the application of a vector version of Jensen's inequality based on Perlman [10]. By applying our results, some classical inequalities obtained for self-adjoint operators (e.g., Hölder–McCarthy-type inequalities) can also be extended.

2. Preliminaries

In this section, we recall spectral theory and some notions and results corresponding to convexity. First, following Rudin [11] mainly, we briefly summarize the spectral theory for normal operators in $\mathcal{B}(H)$. For every normal operator $A \in \mathcal{B}(H)$ there exists a unique resolution E of the identity (called the *spectral decomposition* of A, and it depends on A) on the Borel subsets of $\sigma(A)$ which satisfies

$$A = \int_{\sigma(A)} \lambda \, dE(\lambda). \tag{2.1}$$

By using E, for every bounded Borel function $f : \sigma(A) \to \mathbb{C}$ we can define the operator

$$\int_{\sigma(A)} f \, dE \tag{2.2}$$

which is denoted by $f(A)$ as usual. The integral (2.2) is the abbreviation for

$$\big\langle f(A)x, y \big\rangle = \int_{\sigma(A)} f \, dE_{x,y}, \quad x, y \in H,$$

where $E_{x,y}$ denotes the complex measure

$$E_{x,y}(\omega) := \big\langle E(\omega)x, y \big\rangle$$

on the Borel subsets of $\sigma(A)$. If $x \in H$ and $\|x\| = 1$, then $E_{x,x}$ is a probability measure.

The following statements about the numerical range of an operator can be found in Gustafson and Rao [4]. The numerical range of an operator $A \in \mathcal{B}(H)$ is defined by

$$W(A) := \big\{\langle Ax, x\rangle \in \mathbb{C} \;\big|\; \|x\| = 1\big\}.$$

By the Toeplitz–Hausdorff theorem, $W(A)$ is convex. The closure of $W(A)$ (denoted by $\overline{W}(A)$) contains $\sigma(A)$. If A is normal, then $\overline{W}(A)$ is the smallest closed and convex set containing $\sigma(A)$. We only need two special cases of Jensen's inequality for convex vector-valued functions (more general results can be found in Perlman [10]).

We briefly discuss partial orderings on $\mathbb{C}$ to formulate these assertions. The following notions and results can be found in a more general context in Kelley and Namioka [5] or Perlman [10]. A binary relation $\preceq$ on $\mathbb{C}$ is called a *partial ordering* on $\mathbb{C}$ if it is reflexive, transitive, and antisymmetric. We say that the partial ordering $\preceq$ on $\mathbb{C}$ is a *closed cone ordering* if it satisfies the following additional conditions.

(i) If z_1, $z_2 \in \mathbb{C}$ such that $z_1 \preceq z_2$, then for every $z_3 \in \mathbb{C}$ and for every $\alpha \geq 0$ we have $\alpha(z_1 + z_3) \preceq \alpha(z_2 + z_3)$.

(ii) If $(z_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ are convergent sequences in $\mathbb{C}$ such that $z_n \preceq w_n$ for all $n \geq 1$, then $\lim_{n\to\infty} z_n \preceq \lim_{n\to\infty} w_n$.

A subset K of $\mathbb{C}$ is called a *cone* if, for every $z \in K$ and for every $\alpha \geq 0$, we have $\alpha z \in K$. The cone $K \subset \mathbb{C}$ is said to be *pointed* if $K \cap (-K) = \{0\}$. There is a one-to-one correspondence between closed cone orderings and pointed closed convex cones on $\mathbb{C}$. If $K \subset \mathbb{C}$ is a pointed closed convex cone, then the binary relation

$$\preceq_K := \big\{(z_1, z_2) \in \mathbb{C}^2 \;\big|\; z_2 - z_1 \in K\big\}$$

is a closed cone ordering. Conversely, if $\preceq$ is a closed cone ordering, then

$$K := \{z \in \mathbb{C} \mid 0 \preceq z\} \tag{2.3}$$

is a pointed closed convex cone and $\preceq_K = \preceq$.

Remark 2.1. Let K be a pointed closed convex cone in $\mathbb{C}$. It is not hard to check that K is either a closed half-line with endpoint 0 or that there are two independent z, $w \in \mathbb{C}$ such that K is spanned by these numbers; that is,

$$K = \{\alpha z + \beta w \in \mathbb{C} \mid \alpha, \beta \geq 0\}.$$

Definition 2.2. Let $C \subset \mathbb{C}$ be a convex set, let $\preceq$ be a closed cone ordering on $\mathbb{C}$, and let $f : C \to \mathbb{C}$. We say that f is *convex* with respect to $\preceq$ if

$$f\big(\lambda z + (1-\lambda)w\big) \preceq \lambda f(z) + (1-\lambda) f(w), \quad z, w \in C, \leq \lambda \leq 1. \tag{2.4}$$

Lemma 2.3. *Let $C \subset \mathbb{C}$ be a convex set, and let $f : C \to \mathbb{R}$ be a real-valued complex function. Then f is convex with respect to a closed cone ordering on $\mathbb{C}$ exactly if f is either convex or concave in the usual sense, that is, either*

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y), \quad x, y \in C, \leq \lambda \leq 1$$

or

$$f\big(\lambda x + (1-\lambda)y\big) \geq \lambda f(x) + (1-\lambda) f(y), \quad x, y \in C, \leq \lambda \leq 1.$$

Proof. Assume that f is convex with respect to a closed cone ordering $\preceq$ on $\mathbb{C}$. Since the restriction of $\preceq$ to $\mathbb{R}$ is either $\leq$ or $\geq$ or $=$, it follows from (2.4) that f is either convex or concave.

Conversely, assume that f is convex. If $\preceq$ is a closed cone ordering on $\mathbb{C}$ such that the corresponding pointed closed convex cone (see (2.3)) contains the closed half-line

$$\{x + yi \in \mathbb{C} \mid x \geq 0, y = 0\},$$

then f is convex with respect to $\preceq$. The concave case can be handled similarly. The proof is complete. □

Example 2.4. Let $m_1 < m_2$ be fixed, and let $K_{m_1}^{m_2} \subset \mathbb{C}$ be defined by

$$K_{m_1}^{m_2} := \{x + yi \in \mathbb{C} \mid m_1 x \leq y \leq m_2 x\}. \tag{2.5}$$

Then $K_{m_1}^{m_2}$ is a pointed closed convex cone, and the closed cone ordering on $\mathbb{C}$ generated by $K_{m_1}^{m_2}$ is

$$u + vi \preceq_{m_1}^{m_2} x + yi \quad \Longleftrightarrow \quad m_1(x-u) \leq y - v \leq m_2(x-u). \tag{2.6}$$

Let $C \subset \mathbb{C}$ be a convex set, and let $f = f_1 + f_2 i : C \to \mathbb{C}$. It follows from (2.6) that f is convex with respect to $\preceq_{m_1}^{m_2}$ if and only if the inequalities

$$\begin{aligned} &m_1\big(\lambda f_1(z) + (1-\lambda) f_1(w) - f_1\big(\lambda z + (1-\lambda) w\big)\big) \\ &\quad \leq \lambda f_2(z) + (1-\lambda) f_2(w) - f_2\big(\lambda z + (1-\lambda) w\big) \\ &\quad \leq m_2\big(\lambda f_1(z) + (1-\lambda) f_1(w) - f_1\big(\lambda z + (1-\lambda) w\big)\big) \end{aligned}$$

hold for every z, $w \in C$ and for all $0 \leq \lambda \leq 1$. By rearranging the previous inequalities, we can see that f is convex with respect to $\preceq_{m_1}^{m_2}$ exactly if the functions

$$f_2 - m_1 f_1 \quad \text{and} \quad m_2 f_1 - f_2$$

are convex. This implies that f_1 must be convex.

It is easy to check that the function

$$f : \mathbb{C} \to \mathbb{C}, \quad f(x + yi) = (f_1 + f_2 i)(x + yi) = x^2 + y^2 + 2xyi$$

is convex with respect to $\preceq_{-1}^{1}$, but f_2 is neither convex nor concave. It is worth noting that K_{-1}^{1} is the smaller cone among the cones in (2.5) for which f is convex.

Finally, we give the aforementioned Jensen-type inequalities.

Theorem 2.5 (Vector version of Jensen's discrete inequality; see [10, p. 55])**.** *Let $\preceq$ be a closed cone ordering on $\mathbb{C}$, and let C be a convex subset of $\mathbb{C}$. If $f : C \to \mathbb{C}$ is a convex function with respect to $\preceq$, $x_i \in C$, $p_i \geq 0$ ($i = 1, \ldots, n$), and $\sum_{i=1}^n p_i = 1$, then*

$$f\Big(\sum_{i=1}^n p_i x_i\Big) \preceq \sum_{i=1}^n p_i f(x_i). \tag{2.7}$$

Theorem 2.6 (Vector version of Jensen's integral inequality; [10, Theorem 3.6])**.** *Let $\preceq$ be a closed cone ordering on $\mathbb{C}$, and let g be an integrable function on a probability space $(X, \mathcal{A}, P)$ taking values in a closed and convex set $C \subset \mathbb{C}$. Then $\int_X g\,dP$ lies in C. If $f : C \to \mathbb{C}$ is a continuous and convex function with respect to $\preceq$ such that $f \circ g$ is P-integrable, then*

$$f\Big(\int_X g\,dP\Big) \preceq \int_X f \circ g\,dP. \tag{2.8}$$

3. Main results

Our main result generalizes Theorem 1.1 for normal operators.

Theorem 3.1. *Let $\preceq$ be a closed cone ordering on $\mathbb{C}$. Assume that C is a closed and convex subset of $\mathbb{C}$, that $A_i \in N(C)$ $(i = 1, \dots, n)$, and that $f : C \to \mathbb{C}$ is a continuous and convex function with respect to $\preceq$.*

(a) *If $x_i \in H$ $(i = 1, \dots, n)$ such that $\sum_{i=1}^n \|x_i\|^2 = 1$, then*

$$f\Big(\sum_{i=1}^n \langle A_i x_i, x_i\rangle\Big) \preceq \sum_{i=1}^n \big\langle f(A_i)x_i, x_i\big\rangle. \tag{3.1}$$

(b) *If $x \in H$ with $\|x\| = 1$, and $p_i \geq 0$ $(i = 1, \dots, n)$ such that $\sum_{i=1}^n p_i = 1$, then*

$$f\Big(\sum_{i=1}^n \langle p_i A_i x, x\rangle\Big) \preceq \Big\langle \sum_{i=1}^n p_i f(A_i)x, x\Big\rangle.$$

Proof. (a) We can obviously suppose that $x_i \neq 0$ $(i = 1, \dots, n)$. By (2.1),

$$\sum_{i=1}^n \langle A_i x_i, x_i\rangle = \sum_{i=1}^n \|x_i\|^2 \Big\langle A_i \frac{x_i}{\|x_i\|}, \frac{x_i}{\|x_i\|}\Big\rangle = \sum_{i=1}^n \|x_i\|^2 \int_{\sigma(A_i)} \lambda\,dE^i_{\frac{x_i}{\|x_i\|}, \frac{x_i}{\|x_i\|}}, \tag{3.2}$$

where E^i denotes the spectral decomposition of A_i $(i = 1, \dots, n)$. Since $\sum_{i=1}^n \|x_i\|^2 = 1$, and $E^i_{\frac{x_i}{\|x_i\|}, \frac{x_i}{\|x_i\|}}$ is a probability measure on the Borel sets of $\sigma(A_i)$ $(i = 1, \dots, n)$, (3.2) shows that

$$\sum_{i=1}^n \langle A_i x_i, x_i\rangle \in C.$$

By applying vector versions of Jensen's discrete and integral inequalities to the last expression in (3.2), we obtain

$$\begin{aligned} f\Big(\sum_{i=1}^n \langle A_i x_i, x_i\rangle\Big) &\preceq \sum_{i=1}^n \|x_i\|^2 f\Big(\int_{\sigma(A_i)} \lambda\,dE^i_{\frac{x_i}{\|x_i\|}, \frac{x_i}{\|x_i\|}}\Big) \\ &\preceq \sum_{i=1}^n \|x_i\|^2 \int_{\sigma(A_i)} f(\lambda)\,dE^i_{\frac{x_i}{\|x_i\|}, \frac{x_i}{\|x_i\|}} \\ &= \sum_{i=1}^n \|x_i\|^2 \Big\langle f(A_i)\frac{x_i}{\|x_i\|}, \frac{x_i}{\|x_i\|}\Big\rangle = \sum_{i=1}^n \big\langle f(A_i)x_i, x_i\big\rangle. \end{aligned}$$

(b) This follows from (a) by choosing $x_i = \sqrt{p_i}x$ $(i = 1, \dots, n)$. The proof is complete. □

Corollary 3.2. *Assume that C is a closed and convex subset of $\mathbb{C}$, that $A_i \in N(C)$ $(i = 1, \dots, n)$, and that $f : C \to \mathbb{R}$ is a continuous and convex function.*

(a) *If $x_i \in H$ $(i = 1, \dots, n)$ such that $\sum_{i=1}^n \|x_i\|^2 = 1$, then*

$$f\Big(\sum_{i=1}^n \langle A_i x_i, x_i\rangle\Big) \le \sum_{i=1}^n \big\langle f(A_i)x_i, x_i\big\rangle.$$

(b) *If $x \in H$ with $\|x\| = 1$, and $p_i \ge 0$ $(i = 1, \dots, n)$ such that $\sum_{i=1}^n p_i = 1$, then*

$$f\Big(\sum_{i=1}^n \langle p_i A_i x, x\rangle\Big) \le \Big\langle \sum_{i=1}^n p_i f(A_i)x, x\Big\rangle.$$

Proof. The proof follows from Theorem 3.1, by using Lemma 2.3 and the fact that $f(A_i)$ $(i = 1, \dots, n)$ is self-adjoint. □

Remark 3.3. Consider the special case $n = 1$ of the previous theorem. If $\preceq$ is a closed cone ordering on $\mathbb{C}$, C is a closed and convex subset of $\mathbb{C}$, $A \in N(C)$, $f : C \to \mathbb{C}$ is a continuous and convex function with respect to $\preceq$, and $x \in H$ such that $\|x\| = 1$, then

$$f\big(\langle Ax, x\rangle\big) \preceq \big\langle f(A)x, x\big\rangle.$$

In this case the closure of the numerical range of A is the smallest closed and convex set containing $\sigma(A)$.

Example 3.4. In Example 2.4 we defined the closed cone ordering $\preceq_{-1}^{1}$ on $\mathbb{C}$, and we have seen that the function

$$f : \mathbb{C} \to \mathbb{C}, \quad f(x + yi) = (f_1 + f_2 i)(x + yi) = x^2 + y^2 + 2xyi$$

is convex with respect to $\preceq_{-1}^{1}$. If $A \in N(\mathbb{C})$ and $x \in H$ such that $\|x\| = 1$, then by Remark 3.3,

$$f\big(\langle Ax, x\rangle\big) \preceq_{-1}^{1} \big\langle f(A)x, x\big\rangle.$$

As a first consequence of the previous theorem, a Hölder–McCarthy-type inequality (see McCarthy [6]) is derived for normal operators.

Corollary 3.5. *Assume that $A_i \in \mathcal{B}(H)$ $(i = 1, \dots, n)$ are normal operators and that $x_i \in H$, $x_i \ne 0$ $(i = 1, \dots, n)$ with $\sum_{i=1}^n \|x_i\|^2 = 1$. Then for every $\alpha \ge 1$*

$$\Big|\sum_{i=1}^n \langle A_i x_i, x_i\rangle\Big|^\alpha \le \sum_{i=1}^n \big\langle |A_i|^\alpha x_i, x_i\big\rangle. \tag{3.3}$$

Proof. It is easy to check that the function

$$z \to |z|^\alpha, \quad z \in \mathbb{C} \tag{3.4}$$

is convex if $\alpha \ge 1$, and therefore Corollary 3.2(a) can be applied. □

Remark 3.6. (a) If $\alpha \in]-\infty, 1[$, $\alpha \neq 0$, then the function (3.4) is neither convex nor concave.

(b) For $\alpha = 2$, (3.3) can be written as

$$\left|\sum_{i=1}^{n} \langle A_i x_i, x_i \rangle\right|^2 \leq \sum_{i=1}^{n} \|A_i x_i\|^2.$$

Really, in this case $|A_i|^2 = A_i^* A_i$ $(i = 1, \dots, n)$.

Next, we apply Theorem 3.1 to get some norm inequalities.

Corollary 3.7. *Assume that $A_i \in \mathcal{B}(H)$ $(i = 1, \dots, n)$ are normal operators and that $p_i \geq 0$ $(i = 1, \dots, n)$ such that $\sum_{i=1}^{n} p_i = 1$. If $\sum_{i=1}^{n} p_i A_i$ is normal, and $f : [0, \infty[\to \mathbb{R}$ is a nonnegative, continuous, increasing, and convex function, then*

$$f\left(\left\|\sum_{i=1}^{n} p_i A_i\right\|\right) \leq \left\|\sum_{i=1}^{n} p_i f\left(|A_i|\right)\right\|. \tag{3.5}$$

Proof. The operator $\sum_{i=1}^{n} p_i f(|A_i|)$ is positive, because f and p_i $(i = 1, \dots, n)$ are nonnegative.

If $A \in \mathcal{B}(H)$ is a normal operator, then $\|A\| = \sup_{\|x\|=1} |\langle Ax, x \rangle|$. By using this, the continuity and the increase of f yield

$$f\left(\left\|\sum_{i=1}^{n} p_i A_i\right\|\right) = f\left(\sup_{\|x\|=1}\left|\left\langle\sum_{i=1}^{n} p_i A_i x, x\right\rangle\right|\right) = \sup_{\|x\|=1} f\left(\left|\left\langle\sum_{i=1}^{n} p_i A_i x, x\right\rangle\right|\right).$$

Since f is convex and increasing, and the function (3.4) with $\alpha = 1$ is convex, the composition

$$z \to f\left(|z|\right), \quad z \in \mathbb{C}$$

is also convex, and therefore Corollary 3.2(b) shows that

$$f\left(\left\|\sum_{i=1}^{n} p_i A_i\right\|\right) \leq \sup_{\|x\|=1}\left\langle\sum_{i=1}^{n} p_i f\left(|A_i|\right) x, x\right\rangle = \left\|\sum_{i=1}^{n} p_i f\left(|A_i|\right)\right\|.$$

The proof is now complete. □

Remark 3.8. For example, a sufficient condition for the normality of the operator $\sum_{i=1}^{n} p_i A_i$ is $A_i A_j = A_j A_i$ $(i, j = 1, \dots, n)$.

We mention some special cases of the previous result.

Remark 3.9. Assume that $A_i \in \mathcal{B}(H)$ $(i = 1, \dots, n)$ are normal operators, that $p_i \geq 0$ $(i = 1, \dots, n)$ such that $\sum_{i=1}^{n} p_i = 1$, and that $\sum_{i=1}^{n} p_i A_i$ is normal.

(a) If $f(x) = x^\alpha$ $(x \geq 0)$ with $\alpha \geq 1$, then (3.5) gives

$$\left\|\sum_{i=1}^{n} p_i A_i\right\|^\alpha \leq \left\|\sum_{i=1}^{n} p_i |A_i|^\alpha\right\|.$$

(b) If $f(x) = e^x$ $(x \geq 0)$, then (3.5) gives

$$\exp\Big(\Big\|\sum_{i=1}^{n} p_i A_i\Big\|\Big) \leq \Big\|\sum_{i=1}^{n} p_i \exp\big(|A_i|\big)\Big\|.$$

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