Ann. Funct. Anal. 9 (2018), no. 4, 474–484
https://doi.org/10.1215/20088752-2017-0062
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

ON PSEUDOSPECTRAL RADII OF OPERATORS ON HILBERT SPACES

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Communicated by T. Shulman

Abstract. For $\varepsilon > 0$ and a bounded linear operator T acting on some Hilbert space, the ε-pseudospectrum of T is $\sigma_\varepsilon(T) = \{z \in \mathbb{C} : \|(zI - T)^{-1}\| > 1/\varepsilon\}$ and the ε-pseudospectral radius of T is $r_\varepsilon(T) = \sup\{|z| : z \in \sigma_\varepsilon(T)\}$. In this article, we provide a characterization of those operators T satisfying $r_\varepsilon(T) = r(T) + \varepsilon$ for all $\varepsilon > 0$. Here $r(T)$ denotes the spectral radius of T.

1. Introduction and preliminaries

As usual, we let $\mathbb{N}, \mathbb{C}$ denote, respectively, the set of positive integers and the set of complex numbers, and $\mathcal{H}$ will always denote a complex separable infinitely dimensional Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$. Denote by $\mathcal{B}(\mathcal{H})$ the Banach algebra of all bounded linear operators on $\mathcal{H}$.

The *spectrum* of an operator $T \in \mathcal{B}(\mathcal{H})$, defined as

$$\sigma(T) = \big\{z \in \mathbb{C} : zI - T \text{ is not invertible in } \mathcal{B}(\mathcal{H})\big\},$$

is an important invariant which provides much information about the operator. In general, the operation $T \mapsto \sigma(T)$ is not continuous, which makes it difficult to determine the spectrum of an operator. To estimate spectra of operators, some have proposed the study of pseudospectra of operators. Given $T \in \mathcal{B}(\mathcal{H})$ and

Received May 29, 2017; Accepted Oct. 18, 2017.
First published online May 25, 2018.
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2010 *Mathematics Subject Classification*. Primary 47A10; Secondary 47B20.
Keywords. pseudospectrum, spectrum, normal approximate eigenvalues, normaloid operators.

$\varepsilon > 0$, the *ε-pseudospectrum* of T is defined as

$$\sigma_\varepsilon(T) = \{z \in \mathbb{C} : \|(zI - T)^{-1}\| > \varepsilon^{-1}\}.$$

Conventionally, it is assumed that $\|(zI - T)^{-1}\| = \infty$ if $z \in \sigma(T)$. (See [8] for other equivalent definitions of the ε-pseudospectrum. Pseudospectra can also be defined for elements in a Banach algebra; see [7].)

The properties of pseudospectra differ substantially from those of the spectrum. It is trivial to see that the ε-pseudospectrum of T is always open and that

$$\bigcap_{\varepsilon>0} \sigma_\varepsilon(T) = \sigma(T).$$

Moreover, it is known that the map $(\varepsilon, T) \mapsto \sigma_\varepsilon(T)$ is continuous (see, e.g., [3, Proposition 2.7]). Thus, by examining the ε-pseudospectrum of T, one may give better estimates of its spectrum. Various nice properties of the ε-pseudospectrum, including those mentioned above, make it an effective tool in both matrix analysis and operator theory. Among other things, we mention that the ε-pseudospectrum can be used to give concrete characterizations of special operators, such as nilpotent operators and self-adjoint operators (see [2], [3]).

For $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, it is known that

$$\sigma(T) + B(0, \varepsilon) \subset \sigma_\varepsilon(T),$$

where the converse inclusion in general does not hold. Here $B(0, \varepsilon)$ denotes the set $\{z \in \mathbb{C} : |z| < \varepsilon\}$. For example, if an operator $A \in \mathcal{B}(\mathcal{H})$ is nilpotent of order 2, then $\sigma_\varepsilon(A) = B(0, \sqrt{\varepsilon^2 + \|A\|\varepsilon})$ (see [3, Proposition 2.4]), and

$$\sigma(A) + B(0, \varepsilon) = B(0, \varepsilon) \subsetneq \sigma_\varepsilon(A).$$

In this article, we are interested in the relation between the spectral radius and the pseudospectral radii of a Hilbert space operator. Let $T \in \mathcal{B}(\mathcal{H})$. The *spectral radius* of T is $r(T) = \sup\{|z| : z \in \sigma(T)\}$. For $\varepsilon > 0$, the *ε-pseudospectral radius* of T is $r_\varepsilon(T) = \sup\{|z| : z \in \sigma_\varepsilon(T)\}$. Then, by the discussion in the preceding paragraph, we have $r_\varepsilon(T) \geq r(T) + \varepsilon$ for $\varepsilon > 0$, and the inequality is often strict. Thus a natural question arises.

Question 1.1. When does an operator T satisfy

$$r_\varepsilon(T) = r(T) + \varepsilon \quad \text{for all } \varepsilon > 0? \tag{1.1}$$

We note that each normaloid operator T satisfies (1.1). Recall that T is said to be *normaloid* if $\|T\| = r(T)$. In fact, for any $\varepsilon > 0$, we have $\sigma_\varepsilon(T) \subset B(0, \|T\| + \varepsilon)$. Hence $r_\varepsilon(T) \leq \|T\| + \varepsilon = r(T) + \varepsilon$. It follows that $r_\varepsilon(T) = r(T) + \varepsilon$. Thus many special classes of operators, including normal operators, hyponormal operators, and Toeplitz operators, satisfy (1.1). For convenience, we say that an operator T is *pseudo-normaloid* if $r_\varepsilon(T) = r(T) + \varepsilon$ for all $\varepsilon > 0$. Thus a normaloid operator is always pseudonormaloid.

In this article, we explore and characterize the structure of pseudonormaloid operators on Hilbert spaces. Our result depends on an intensive analysis of normal approximate eigenvalues. A complex number λ is called a *normal approximate eigenvalue* (see [5]) of $A \in \mathcal{B}(\mathcal{H})$ if there exists a sequence $\{x_n\}_{n\geq 1}$ of unit vectors such that

$$\|(A-\lambda)x_n\| + \|(A-\lambda)^* x_n\| \to 0.$$

We will prove that a pseudonormaloid operator has at least one normal approximate eigenvalue.

To state our main result, we need an extra definition. Two operators A and B are said to be *approximately unitarily equivalent* (write $A \cong_a B$) if there exists a sequence of unitary operators U_n such that $\lim_n U_n^* A U_n = B$. If $A \cong_a B$, then it is easy to see that $\sigma_\varepsilon(A) = \sigma_\varepsilon(B)$ for all $\varepsilon > 0$. Our main result here is the following theorem, which gives an answer to Question 1.1.

Theorem 1.2. *An operator T is pseudonormaloid if and only if T is approximately unitarily equivalent to an operator of form $N \oplus A$, where N is normal and $r_\varepsilon(A) \leq r(N) + \varepsilon$ for all $\varepsilon > 0$. In particular, it can be additionally required that $\sigma(N) = \{z \in \sigma(T) : |z| = r(T)\}$.*

By the above result, if T is pseudonormaloid, then T "has" a normal part N satisfying

$$r_\varepsilon(T) = r_\varepsilon(N), \quad \forall \varepsilon > 0;$$

in particular, $r(T) = r(N)$. This shows that pseudonormaloid operators possess a weakened normality. Using Theorem 1.2, one can construct various examples of pseudonormaloid operators. We will provide an example to show that our result is sharp (see Example 2.10). This example also shows that the set of normaloid operators is a proper subset of the set of pseudonormaloid operators.

In the rest of this section, we fix some notation and terminology which will be used later. Let $T \in \mathcal{B}(\mathcal{H})$. We denote by $\ker T$ and $\operatorname{ran} T$ the kernel of T and the range of T, respectively. If $\operatorname{ran} T$ is closed and either $\ker T$ or $\ker T^*$ is of finite dimension, then T is called a *semi-Fredholm operator*. The following set

$$\sigma_{\text{lre}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\}$$

is called the *Wolf spectrum* of T. Denote by $\sigma_0(T)$ the set of *normal eigenvalues* of T, that is,

$$\sigma_0(T) = \big\{\lambda \in \mathbb{C} : \lambda \text{ is an isolated point of } \sigma(T) \text{ and } \lambda \notin \sigma_{\text{lre}}(T)\big\}.$$

(The reader is referred to [1, p. 210] or [6, p. 5] for more details about normal eigenvalues.)

For $T \in \mathcal{B}(\mathcal{H})$, we let $\sigma_\pi(T)$ denote the *approximate point spectrum* of T, that is,

$$\sigma_\pi(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}.$$

2. Proof of Theorem 1.2

Proof of sufficiency for Theorem 1.2. For $\varepsilon > 0$, note that

$$\sigma_\varepsilon(T) = \sigma_\varepsilon(N \oplus A) = \sigma_\varepsilon(N) \cup \sigma_\varepsilon(A).$$

Thus

$$r_\varepsilon(T) = \max\{r_\varepsilon(N), r_\varepsilon(A)\} = r_\varepsilon(N) = r(N) + \varepsilon \le r(T) + \varepsilon.$$

It follows that $r_\varepsilon(T) = r(T) + \varepsilon$. □

To give the proof of necessity for Theorem 1.2, we need several auxiliary results.

Lemma 2.1 ([3, Lemma 2.1]). *Let $T \in \mathcal{B}(\mathcal{H})$ and let $\lambda \in \sigma_p(T)$. If $\ker(\lambda - T)$ is not a reducing subspace of T, then there exists $r > \varepsilon > 0$ such that $B(\lambda, r) \subset \sigma_\varepsilon(T)$; in particular, $r_\varepsilon(T) > |\lambda| + \varepsilon$.*

Corollary 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be pseudonormaloid, and let $\lambda \in \sigma(T)$ with $|\lambda| = r(T)$. If $\ker(T-\lambda) \cup \ker(T-\lambda)^* \neq \{0\}$, then $\ker(T-\lambda) = \ker(T-\lambda)^*$ reduces T.*

Proof. Assume that $\ker(T - \lambda) \neq \{0\}$. If $\ker(T - \lambda)$ does not reduce T, then, by Lemma 2.1, $r_\varepsilon(T) > |\lambda| + \varepsilon = r(T) + \varepsilon$. This contradicts the fact that T is pseudonormaloid. Thus $\ker(T - \lambda)$ reduces T and $\ker(T - \lambda) \subset \ker(T - \lambda)^*$. So $\ker(T - \lambda)^* \neq \{0\}$. Note that T^* is also pseudonormaloid. Using a similar argument as above, one obtains $\ker(T - \lambda)^* \subset \ker(T - \lambda)$.

When $\ker(T - \lambda)^* \neq \{0\}$, the proof follows similar lines. □

For $e, f \in \mathcal{H}$, we let $e \otimes f$ denote the rank 1 operator on $\mathcal{H}$: $x \mapsto \langle x, f\rangle e$.

Lemma 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $\lambda \in \sigma_\pi(T)$. If λ is not a normal approximate eigenvalue of T, then there exists $\delta > 0$ such that, given $\varepsilon > 0$, there exists $K \in \mathcal{B}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K$ can be written as*

$$T + K = \begin{pmatrix} \lambda & e \otimes f \\ 0 & A \end{pmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^\perp \end{matrix},$$

where $e \in \mathcal{H}$ is a unit vector, $f \in (\mathbb{C}e)^\perp$ with $\|f\| \ge \delta$, and A acts on $(\mathbb{C}e)^\perp$.

Proof. Since $\lambda \in \sigma_\pi(T)$, there exist unit vectors $\{e_i\}_{i=1}^\infty$ such that $\|(T-\lambda)e_i\| \to 0$. Note that λ is not a normal approximate eigenvalue of T. Thus $\|(T-\lambda)^* e_i\| \nrightarrow 0$. Then there exists $\delta > 0$ such that $\limsup_i \|(T - \lambda)^* e_i\| > \delta$. For any $\varepsilon > 0$, we can choose n such that $\|(T - \lambda)e_n\| < \varepsilon$ and $\|(T - \lambda)^* e_n\| \ge \delta$. Set $e = e_n$, set $f = (T - \lambda)^* e_n$, and set $h = (T - \lambda)e_n$. Thus $\|h\| < \varepsilon$, $\|f\| \ge \delta$, and

$$T = \begin{pmatrix} \lambda & e \otimes f \\ h \otimes e & A \end{pmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^\perp \end{matrix}.$$

Set

$$K = -\begin{pmatrix} 0 & 0 \\ h \otimes e & 0 \end{pmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^\perp \end{matrix}.$$

Then K is compact with $\|K\| < \varepsilon$ and

$$T + K = \begin{pmatrix} \lambda & e \otimes f \\ 0 & A \end{pmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^\perp \end{matrix}.$$

So K satisfies all requirements. □

Proposition 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $\lambda \in \sigma_\pi(T)$. If λ is not a normal approximate eigenvalue of T, then, given $\varepsilon \in (0, 1)$, there exists $r > \varepsilon > 0$ such that $B(\lambda, r) \subset \sigma_\varepsilon(T)$; in particular, $r_\varepsilon(T) > |\lambda| + \varepsilon$.*

Proof. Set

$$\delta_1 = \sup\big\{\|z-T\|+1 : |z| \le \|T\|+2\big\}.$$

By Lemma 2.3, there exists $\delta_2 > 0$ such that, given $\varepsilon_1 \in (0,1)$, there exists $K \in \mathcal{B}(\mathcal{H})$ with $\|K\| < \varepsilon_1$ such that $T+K$ can be written as

$$T+K = \begin{pmatrix} \lambda & e\otimes f \\ 0 & A \end{pmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^{\perp} \end{matrix},$$

where $e \in \mathcal{H}$ is a unit vector, $f \in (\mathbb{C}e)^{\perp}$ with $\|f\| \ge \delta_2$, and A acts on $(\mathbb{C}e)^{\perp}$. Thus

$$(z-T-K)^{-1} = \begin{pmatrix} (z-\lambda)^{-1} & (z-\lambda)^{-1}(e\otimes f)(z-A)^{-1} \\ 0 & (z-A)^{-1} \end{pmatrix}$$

and

$$\begin{aligned}\big\|(z-T-K)^{-1}\big\| &\ge \left\| \begin{pmatrix} (z-\lambda)^{-1} & (z-\lambda)^{-1}(e\otimes f)(z-A)^{-1} \\ 0 & 0 \end{pmatrix} \right\| \\ &= |z-\lambda|^{-1}\sqrt{1+\big\|(z-A)^{*-1}f\big\|^2}.\end{aligned}$$

Set $\widetilde{\delta} = \min\{3/2, \sqrt{1+(\delta_2/\delta_1)^2}\}$.

For any $\varepsilon \in (0,1)$, choose $\delta \in (1,\widetilde{\delta})$ and set $\widetilde{\varepsilon} = \delta\varepsilon/\widetilde{\delta}$. Then $\widetilde{\varepsilon} < \varepsilon < 1$ and

$$\sigma_{\widetilde{\varepsilon}}(T+K) \subset \big\{z \in \mathbb{C} : |z| \le \|T\|+2\big\}.$$

If $z \in \mathbb{C} \setminus \sigma(T+K)$ and $|z| \le \|T\|+2$, then $z-T-K$ and $z-A$ are both invertible, and

$$\begin{aligned}\big\|(z-A)^{*-1}f\big\| &\ge \frac{\|f\|}{\|(z-A)^*\|} = \frac{\|f\|}{\|z-A\|} \\ &\ge \frac{\|f\|}{\|z-T-K\|} \ge \frac{\|f\|}{\delta_1} \ge \frac{\delta_2}{\delta_1}\end{aligned}$$

and

$$\big\|(z-T-K)^{-1}\big\| \ge |z-\lambda|^{-1}\sqrt{1+(\delta_2/\delta_1)^2} \ge |z-\lambda|^{-1}\widetilde{\delta}. \tag{2.1}$$

Note that

$$\begin{aligned}\sigma_{\widetilde{\varepsilon}}(T+K) &= \big\{z \in \mathbb{C} : \big\|(z-T-K)^{-1}\big\| > 1/\widetilde{\varepsilon}\big\} \\ &= \big\{z \in B\big(0,\|T\|+2\big) : \big\|(z-T-K)^{-1}\big\| > 1/\widetilde{\varepsilon}\big\}.\end{aligned}$$

By (2.1), we have

$$\begin{aligned}\sigma_{\widetilde{\varepsilon}}(T+K) &\supset \sigma(T+K) \cup \big\{z \in B\big(0,\|T\|+2\big)\setminus\sigma(T+K) : |z-\lambda|^{-1}\widetilde{\delta} > 1/\widetilde{\varepsilon}\big\} \\ &\supset \big\{z \in B\big(0,\|T\|+2\big) : |z-\lambda|^{-1}\widetilde{\delta} > 1/\widetilde{\varepsilon}\big\} \\ &= \big\{z \in B\big(0,\|T\|+2\big) : |z-\lambda| < \widetilde{\delta}\widetilde{\varepsilon}\big\} \\ &= \big\{z \in \mathbb{C} : |z-\lambda| < \widetilde{\delta}\widetilde{\varepsilon}\big\} \quad (\text{since } \widetilde{\delta} \le 3/2) \\ &= B(\lambda,\widetilde{\delta}\widetilde{\varepsilon}).\end{aligned}$$

Thus if $|z-\lambda| < \widetilde{\delta}\widetilde{\varepsilon}$, then $\|(z-T-K)^{-1}\| > 1/\widetilde{\varepsilon}$. Since $\|K\| < \varepsilon_1$ and $\varepsilon_1 \in (0,1)$ could be chosen arbitrarily, we deduce that $\|(z-T)^{-1}\| \geq 1/\widetilde{\varepsilon}$. Note also that $\delta\varepsilon = \widetilde{\delta}\widetilde{\varepsilon}$ and that $\varepsilon > \widetilde{\varepsilon}$. By the preceding discussion, if $|z-\lambda| < \delta\varepsilon$, then $\|(z-T)^{-1}\| > 1/\varepsilon$. Thus $B(\lambda, \delta\varepsilon) \subset \sigma_\varepsilon(T)$. Set $r = \delta\varepsilon$. Thus the proof is complete. ☐

Corollary 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be pseudonormaloid. If $\lambda \in \sigma(T)$ with $|\lambda| = r(T)$, then λ is a normal approximate eigenvalue of T.*

Proof. Obviously, λ lies in the boundary of $\sigma(T)$. So $\lambda \in \sigma_\pi(T)$. If λ is not a normal approximate eigenvalue of T, then, by Lemma 2.4, there exists $\varepsilon > 0$ such that $r_\varepsilon(T) > |\lambda| + \varepsilon = r(T) + \varepsilon$, which is a contradiction. ☐

Lemma 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ with $\ker T = \{0\} = \ker T^*$. If 0 is a normal approximate eigenvalue of T, then there exists an orthonormal sequence $\{f_n\}_{n\geq 1}$ in $\mathcal{H}$ such that*

$$\lim_n \big(\|Tf_n\| + \big\|T^*f_n\big\|\big) = 0.$$

Proof. Since 0 is a normal approximate eigenvalue of T, we can find unit vectors $\{e_n\}$ such that

$$\lim_n \big(\|Te_n\| + \big\|T^*e_n\big\|\big) = 0. \tag{2.2}$$

Assume that $T = UP$ is the polar decomposition of T, where $P = |T|$. Denote by $E(\cdot)$ the projection-valued spectral measure corresponding to P. Since $\ker T = \{0\} = \ker T^*$, one can check that U is unitary and $E(\{0\}) = 0$. Thus, given $\varepsilon > 0$ and $x \in \mathcal{H}$, there exists $\delta > 0$ such that

$$\big\|E\big([0,\delta)\big)x\big\| + \big\|E\big([0,\delta)\big)U^*x\big\| < \varepsilon. \tag{2.3}$$

Claim. Given $\delta > 0$ and $\varepsilon > 0$, there exists N large enough such that

$$\big\|E\big(\big[\delta, \|T\|\big]\big)e_N\big\| + \big\|E\big(\big[\delta, \|T\|\big]\big)U^*e_N\big\| < \varepsilon.$$

In fact, we note that

$$\|Te_n\| = \|Pe_n\| \geq \big\|E\big(\big[\delta, \|T\|\big]\big)Pe_n\big\| = \big\|PE\big(\big[\delta, \|T\|\big]\big)e_n\big\| \geq \delta\big\|E\big(\big[\delta, \|T\|\big]\big)e_n\big\|$$

and that

$$\begin{aligned}\|T^*e_n\| = \|PU^*e_n\| &\geq \big\|PE\big(\big[\delta, \|T\|\big]\big)U^*e_n\big\| - \big\|PE\big([0,\delta)\big)U^*e_n\big\| \\ &\geq \delta\big\|E\big(\big[\delta, \|T\|\big]\big)U^*e_n\big\| - \big\|E\big([0,\delta)\big)PU^*e_n\big\| \\ &\geq \delta\big\|E\big(\big[\delta, \|T\|\big]\big)U^*e_n\big\| - \big\|PU^*e_n\big\|.\end{aligned}$$

In view of (2.2), one can see that the claim holds.

Set $s_1 = 1$. By the claim, there exists n_1 such that

$$\big\|E\big(\big[1/s_1, \|T\|\big]\big)e_{n_1}\big\| + \big\|E\big(\big[1/s_1, \|T\|\big]\big)U^*e_{n_1}\big\| < 1/2^2.$$

In view of (2.3), we can find $t_1 > s_1$ such that

$$\big\|E\big([0,1/t_1)\big)e_{n_1}\big\| + \big\|E\big([0,1/t_1)\big)U^*e_{n_1}\big\| < 1/2^2.$$

Set $s_2 = t_1 + 1$. By the claim, there exists $n_2 > n_1$ such that

$$\big\|E\big([1/s_2, \|T\|]\big)e_{n_2}\big\| + \big\|E\big([1/s_2, \|T\|]\big)U^*e_{n_2}\big\| < 1/2^3.$$

In view of (2.3), we can find $t_2 > s_2$ such that

$$\big\|E\big([0, 1/t_2)\big)e_{n_2}\big\| + \big\|E\big([0, 1/t_2)\big)U^*e_{n_2}\big\| < 1/2^3.$$

Then, proceeding by recursion, there exist $\{n_i : i \geq 1\}$ and $\{s_i, t_i : i \geq 1\}$ such that $s_i < t_i < s_{i+1}$,

$$\big\|E\big([1/s_i, \|T\|]\big)e_{n_i}\big\| + \big\|E\big([1/s_i, \|T\|]\big)U^*e_{n_i}\big\| < 1/2^{i+1},$$

and

$$\big\|E\big([0, 1/t_i)\big)e_{n_i}\big\| + \big\|E\big([0, 1/t_i)\big)U^*e_{n_i}\big\| < 1/2^{i+1}.$$

For each $i \geq 1$, we have

$$\begin{aligned}\big\|E\big([1/t_i, 1/s_i)\big)e_{n_i}\big\| &= \big\|e_{n_i} - E\big([1/s_i, \|T\|]\big)e_{n_i} - E\big([0, 1/t_i)\big)e_{n_i}\big\| \\ &\geq 1 - \big\|E\big([1/s_i, \|T\|]\big)e_{n_i}\big\| - \big\|E\big([0, 1/t_i)\big)e_{n_i}\big\| > 1 - 1/2^i.\end{aligned}$$

Similarly,

$$\big\|E\big([1/t_i, 1/s_i)\big)U^*e_{n_i}\big\| > 1 - 1/2^i.$$

That is,

$$\min\big\{\big\|E\big([1/t_i, 1/s_i)\big)e_{n_i}\big\|, \big\|E\big([1/t_i, 1/s_i)\big)U^*e_{n_i}\big\|\big\} > 1 - 1/2^i, \quad \forall i \geq 1. \tag{2.4}$$

For each $i \geq 1$, set

$$f_i = \frac{E([1/t_i, 1/s_i))e_{n_i}}{\|E([1/t_i, 1/s_i))e_{n_i}\|}.$$

Thus $\{f_i\}$ is an orthonormal sequence, since $\{[1/t_i, 1/s_i) : i \geq 1\}$ are pairwise disjoint. In view of (2.4), we compute to see that

$$\begin{aligned}\|Tf_i\| &\leq \frac{\|TE([1/t_i, 1/s_i))e_{n_i}\|}{1 - 1/2^i} = \frac{\|PE([1/t_i, 1/s_i))e_{n_i}\|}{1 - 1/2^i} \\ &= \frac{\|E([1/t_i, 1/s_i))Pe_{n_i}\|}{1 - 1/2^i} \leq \frac{\|Pe_{n_i}\|}{1 - 1/2^i} \to 0\end{aligned}$$

and

$$\begin{aligned}\|T^*f_i\| &\leq \frac{\|PU^*E([1/t_i, 1/s_i))e_{n_i}\|}{1 - 1/2^i} \\ &\leq \frac{\|PU^*e_{n_i}\|}{1 - 1/2^i} + \frac{\|PU^*(E([1/t_i, 1/s_i))e_{n_i} - e_{n_i})\|}{1 - 1/2^i} \\ &\leq \frac{\|PU^*e_{n_i}\|}{1 - 1/2^i} + \frac{(1/2^i)\|T\|}{1 - 1/2^i} \to 0.\end{aligned}$$

Thus the proof is complete. □

We let $\mathcal{K}(\mathcal{H})$ denote the ideal of compact operators on $\mathcal{H}$.

Lemma 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$, and let λ be a normal approximate eigenvalue of T. If $\ker(\lambda I - T) = \{0\} = \ker(\lambda I - T)^*$, then $T \cong_a T \oplus \lambda I$.*

Proof. By Lemma 2.6, we can find an orthonormal sequence $\{e_n\}_{n\geq 1}$ in $\mathcal{H}$ such that

$$\lim_n \big(\|(T-\lambda)e_n\| + \|(T-\lambda)^* e_n\|\big) = 0.$$

Denote by $C^*(T)$ the unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by T and the identity I. Given a polynomial $p(z,w)$ with two free variables z,w, it is easy to verify that $\lim_n \langle p(T^*,T)e_n, e_n\rangle = p(\overline{\lambda},\lambda)$. Hence $\lim_n\langle Xe_n,e_n\rangle$ exists for all $X\in C^*(T)$. Define $\phi(X) = \lim_n\langle Xe_n,e_n\rangle$ for $X\in C^*(T)$. Then this defines a complex $*$-homomorphism of $C^*(T)$ and $\phi(T)=\lambda$. Since $\{e_n\}$ is orthonormal, it is easy to see that

$$\phi(K) = \lim_n \langle Ke_n, e_n\rangle = 0, \quad \forall K\in C^*(T)\cap\mathcal{K}(\mathcal{H}).$$

Using Voiculescu's theorem (see [9, Theorem 1.3] or [4, Corollary II.5.5]), we obtain $T\cong_a \lambda I\oplus T$. □

Theorem 2.8. *Let $T\in\mathcal{B}(\mathcal{H})$ be pseudonormaloid. If N is a normal operator on some Hilbert space with $\sigma(N)\subseteq\{z\in\sigma_{\mathrm{lre}}(T): |z| = r(T)\}$, then $T\cong_a T\oplus N$.*

Proof. Denote $\Gamma = \{z\in\sigma_{\mathrm{lre}}(T): |z|=r(T)\}$. By Corollary 2.5, each $\lambda\in\Gamma$ is a normal approximate eigenvalue of T.

Claim 1: $T\cong_a T\oplus\lambda I$ *for any* $\lambda\in\Gamma$, *where I is the identity operator on* $\mathcal{H}$. Let $\lambda\in\Gamma$. If $\ker(T-\lambda)=\{0\}$, then, by Corollary 2.2, $\ker(T-\lambda)^*=\{0\}$. Since $\lambda\in\Gamma$ is a normal approximate eigenvalue of T, it follows from Lemma 2.7 that $T\cong_a T\oplus\lambda I$.

If $0<\dim\ker(T-\lambda I)<\infty$, then, by Corollary 2.2, $T=\lambda I_1\oplus\widetilde{T}$, where I_1 is the identity on $\ker(T-\lambda I)$ and $\widetilde{T} = T|_{\ker(T-\lambda I)^\perp}$. Hence $\ker(\widetilde{T}-\lambda) = \ker(\widetilde{T}-\lambda)^* = \{0\}$. Still, $\widetilde{T}$ is pseudonormaloid, $\lambda\in\sigma_{\mathrm{lre}}(\widetilde{T})$, and $|\lambda| = r(\widetilde{T})$. By the discussion in the preceding paragraph, we have $\widetilde{T}\cong_a\widetilde{T}\oplus\lambda I_2$, where I_2 is the identity on $\ker(T-\lambda I)^\perp$. Thus

$$T=\lambda I_1\oplus\widetilde{T}\cong_a \lambda I_1\oplus\widetilde{T}\oplus\lambda I_2 = T\oplus\lambda I_2\cong T\oplus\lambda I.$$

If $\dim\ker(T-\lambda I)=\infty$, then, by Corollary 2.2, $T=\lambda I_1\oplus\widetilde{T}$, where I_1 is the identity on $\ker(T-\lambda I)$ and $\widetilde{T} = T|_{\ker(T-\lambda I)^\perp}$. Since $\dim\ker(T-\lambda I)=\infty$, it follows that

$$T=\lambda I_1\oplus\widetilde{T}\cong\lambda I\oplus\lambda I_1\oplus\widetilde{T} = \lambda I\oplus T.$$

This proves Claim 1.

Without loss of generality, we may assume that $\{\lambda_i : i=1,2,\ldots\}$ is a dense subset of Γ. Then, by Claim 1, we have

$$T\cong_a T\oplus\lambda_1 I\cong_a T\oplus\lambda_2 I\oplus\lambda_1 I\cong_a\cdots\cong_a T\oplus\Big(\bigoplus_{i=1}^n\lambda_i I\Big)$$

for each $n\geq 1$. Using an argument similar to that used in the proof of [10, Theorem 3.1], one can prove that $T\cong_a T\oplus N$. For the reader's convenience, we repeat the argument from [10, Theorem 3.1].

Claim 2: $T \cong_a T \oplus (\bigoplus_{i=1}^{\infty} \lambda_i I)$. Given $\varepsilon > 0$, note that $\{B(\lambda_i, \varepsilon)\}_{i=1}^{\infty}$ is an open cover of Γ. Then there exists $k \geq 1$ such that $\{B(\lambda_i, \varepsilon)\}_{i=1}^{k}$ is an open cover of Γ. Then there exists an operator Y on $\bigoplus_{i=1}^{\infty} \mathcal{H}$ with $\|Y\| < \varepsilon$ such that

$$T \oplus \left(\bigoplus_{i=1}^{\infty} \lambda_i I\right) + Y \cong T \oplus \left(\bigoplus_{i=1}^{k} \lambda_i I\right) \cong_a T.$$

It follows that $T \oplus (\bigoplus_{i=1}^{\infty} \lambda_i I) \cong_a T$. This proves Claim 2.

Since

$$\sigma\left(\bigoplus_{i=1}^{\infty} \lambda_i I\right) = \sigma_{\mathrm{lre}}\left(\bigoplus_{i=1}^{\infty} \lambda_i I\right) = \Gamma$$

and N is normal with $\sigma(N) \subseteq \Gamma$, by [4, Theorem II.4.4], we have

$$N \oplus \left(\bigoplus_{i=1}^{\infty} \lambda_i I\right) \cong_a \bigoplus_{i=1}^{\infty} \lambda_i I.$$

Therefore, we conclude that

$$\begin{aligned} T \oplus N &\cong_a \left(T \oplus \left(\bigoplus_{i=1}^{\infty} \lambda_i I\right)\right) \oplus N \\ &\cong_a T \oplus \left(\left(\bigoplus_{i=1}^{\infty} \lambda_i I\right) \oplus N\right) \\ &\cong_a T \oplus \left(\bigoplus_{i=1}^{\infty} \lambda_i I\right) \cong_a T. \end{aligned}$$

Now the proof is complete. $\square$

Lemma 2.9 ([1, p. 366])**.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\partial\sigma(T) \subseteq [\sigma_0(T) \cup \sigma_{\mathrm{lre}}(T)]$.*

Now we are ready to complete the proof of Theorem 1.2.

Proof of necessity for Theorem 1.2. Denote $\Gamma = \{z \in \sigma(T) : |z| = r(T)\}$, $\Gamma_1 = \{z \in \sigma_{\mathrm{lre}}(T) : |z| = r(T)\}$, and $\Gamma_0 = \Gamma \setminus \Gamma_1$. Note that $\Gamma_0 \subset \partial\sigma(T) \setminus \sigma_{\mathrm{lre}}(T)$. Then, by Lemma 2.9, $\Gamma_0 \subset \sigma_0(T)$. Obviously, $\sigma_0(T)$ is at most countable. Without loss of generality, we assume that $\Gamma_0 = \{\lambda_i : i = 1, 2, 3, \ldots\}$. In view of [1, Proposition XI.6.9], each $\lambda \in \Gamma_0$ is an eigenvalue of T. By Corollary 2.2, T has a reducing subspace M such that $T|_M$ is a diagonal operator with eigenvalues $\{\lambda_i : i = 1, 2, 3, \ldots\}$. Denote $N_0 = T|_M$.

On the other hand, choose a normal operator N_1 with $\sigma(N) = \Gamma_1$. Then, by Theorem 2.8, $T \cong_a T \oplus N_1$. Set $N = N_0 \oplus N_1$. Then $\sigma(N) = \Gamma_0 \cup \Gamma_1 = \Gamma$ and

$$T \cong_a T \oplus N_1 = T|_{M^\perp} \oplus N_0 \oplus N_1 = T|_{M^\perp} \oplus N.$$

Set $A = T|_{M^\perp}$. Thus

$$r_\varepsilon(A) \leq r_\varepsilon(T) = r(T) + \varepsilon = r(N) + \varepsilon = r_\varepsilon(N), \quad \forall \varepsilon > 0.$$

Therefore the proof is complete. $\square$

Example 2.10. Let S be the unilateral shift on $l^2(\mathbb{N})$ defined by

$$(\alpha_1, \alpha_2, \alpha_3, \dots) \mapsto (0, \alpha_1, \alpha_2, \alpha_3, \dots).$$

Since S is subnormal (and hence normaloid), we have $r_\varepsilon(S) = r(S) + \varepsilon = 1 + \varepsilon$. Let $R \in \mathcal{B}(\mathbb{C}^2)$ be the operator on $\mathbb{C}^2$ determined by the matrix

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

Then, by [3, Proposition 2.4], $r_\varepsilon(R) = \sqrt{\varepsilon^2 + 2\varepsilon} \le 1 + \varepsilon = r_\varepsilon(S)$.

Set $T = S \oplus R$. Then $\|T\| = 2 > 1 = r(T)$. So T is not normaloid. However, T is pseudonormaloid, since

$$r_\varepsilon(T) = \max\{r_\varepsilon(S), r_\varepsilon(R)\} = r_\varepsilon(S) = 1 + \varepsilon = r(T) + \varepsilon$$

for $\varepsilon > 0$.

Remark 2.11.

(i) The preceding example shows that the equivalence relation "approximate unitary equivalence" in Theorem 1.2 cannot be replaced by "unitary equivalence," since the operator T in Example 2.10 is abnormal; that is, T admits no nonzero reducing subspace M such that $T|_M$ is normal.

(ii) Let T be the pseudonormaloid operator in Example 2.10. By Theorem 1.2, there exists a normal operator N such that $T \cong_a N \oplus A$. We claim that $\sigma(N) \subset \{z \in \sigma(T) : |z| = r(T)\}$. In fact, since T is essentially normal (i.e., $T^*T - TT^*$ is compact), by [6, Proposition 4.27], we have

$$\sigma(N) \subset \sigma_{\mathrm{lre}}(T) = \{z \in \mathbb{C} : |z| = 1\} = \{z \in \sigma(T) : |z| = r(T)\}.$$

This shows that the spectrum of the normal operator N in Theorem 1.2 in general cannot exceed the set $\{z \in \sigma(T) : |z| = r(T)\}$.

We conclude this section with the following observation.

Proposition 2.12. *The set of pseudonormaloid operators is norm-closed.*

Proof. Assume that $\{A_n\}$ are pseudonormaloid operators and that $A_n \to A$. For any $\varepsilon > 0$, by the continuity of the ε-pseudospectrum, we have $r_\varepsilon(A_n) \to r_\varepsilon(A)$ (see [3, Proposition 2.7]). By the upper semicontinuity of the spectrum, we have $\limsup_n r(A_n) \le r(A)$. Thus

$$r_\varepsilon(A) = \lim_n r_\varepsilon(A_n) = \lim_n \big(r(A_n) + \varepsilon\big) \le r(A) + \varepsilon.$$

It follows that $r_\varepsilon(A) = r(A) + \varepsilon$. □

Acknowledgments. The authors thank the anonymous referees for helpful suggestions leading to improvements in the presentation of this article.

Jia's and Feng's work was partially supported by National Natural Science Foundation of China grant 11601181, National Social Science Foundation of China grant 16BTJ020, and Jilin University of Finance and Economics grant 2016Q31.

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