

## NONEXPANSIVE BIJECTIONS BETWEEN UNIT BALLS OF BANACH SPACES

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ABSTRACT. It is known that if  $M$  is a finite-dimensional Banach space, or a strictly convex space, or the space  $\ell_1$ , then every nonexpansive bijection  $F: B_M \rightarrow B_M$  of its unit ball  $B_M$  is an isometry. We extend these results to nonexpansive bijections  $F: B_E \rightarrow B_M$  between unit balls of two different Banach spaces. Namely, if  $E$  is an arbitrary Banach space and  $M$  is finite-dimensional or strictly convex, or the space  $\ell_1$ , then every nonexpansive bijection  $F: B_E \rightarrow B_M$  is an isometry.

### 1. INTRODUCTION

Let  $M$  be a metric space. A map  $F: M \rightarrow M$  is called *nonexpansive* if  $\rho(F(x), F(y)) \leq \rho(x, y)$  for all  $x, y \in M$ . The space  $M$  is called *expand-contract plastic* (or simply, an *EC-space*) if every nonexpansive bijection from  $M$  onto itself is an isometry.

This definition was introduced in [8], where an extensive study of this notion was performed. Among other results it was shown that “an EC-space need not be compact, complete, or bounded” and it was observed that “it is an open question whether there exists a simple characterization of these spaces.” Theorem 1.1 from the same source states that every compact (or even totally bounded) metric space is expand-contract plastic, so in particular every bounded subset of  $\mathbb{R}^n$  is an EC-space.

The situation with bounded subsets of infinite-dimensional spaces is different. On the one hand, there is a non-expand-contract plastic bounded closed convex

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subset of a Hilbert space (see [2, Example 2.7]; in fact, that set is an ellipsoid), but on the other hand, the unit ball of a Hilbert space, and in general the unit ball of every strictly convex Banach space, is an EC-space (see [2, Theorem 2.6]). It is unknown whether the same result remains valid for the unit ball of an arbitrary Banach space—in other words, the following problem arises.

**Problem 1.1.** *For which Banach spaces  $Y$  is every nonexpansive bijection  $F : B_Y \rightarrow B_Y$  an isometry?*

Outside of strictly convex spaces, Problem 1.1 is solved positively for all finite-dimensional spaces (because of the compactness of the unit ball), and has been proved for the space  $\ell_1$  in [6, Theorem 1]. To the best of our knowledge, the following natural extension of Problem 1.1 is also open.

**Problem 1.2.** *For which pairs  $(X, Y)$  of Banach spaces is every bijective nonexpansive map  $F : B_X \rightarrow B_Y$  an isometry?*

An evident bridge between these two problems is the following one, which we also are not yet able to solve in full generality.

**Problem 1.3.** *Let  $X, Y$  be Banach spaces that admit a bijective nonexpansive map  $F : B_X \rightarrow B_Y$ . Is it true that they are linearly isometric?*

Indeed, if one solves Problem 1.2 for a fixed pair  $(X, Y)$  in the positive, one may also solve Problem 1.3 for this pair applying a classical theorem by Mankiewicz (see Proposition 3.2). On the other hand, for this fixed pair the positive answers to Problems 1.1 and 1.3 would imply the positive solution for Problem 1.2.

The aim of this article is to demonstrate that for all spaces  $Y$  where Problem 1.1 is known to have a positive solution (i.e., strictly convex spaces,  $\ell_1$ , and finite-dimensional spaces), Problem 1.2 can be solved in the positive for all pairs of the form  $(X, Y)$  (see Theorems 3.1, 3.5, and 3.8). In fact, our result for pairs  $(X, Y)$  with  $Y$  being strictly convex repeats the arguments given for the case  $X = Y$  from [2, Theorem 2.6] almost word-to-word. The proof of Theorem 3.5 needs additional work compared to its particular case  $X = \ell_1$  from [6, Theorem 1]. The most difficult one is the finite-dimensional case, because the approach from [8, Theorem 1.1] uses iterations of the map and consequently is not applicable for maps between two different spaces. Our proof relies on duality technique and uses some differentiability argument from [2] and topological ideas from [6].

There is another similar circle of problems that motivates our study. In 1987, Tingley [11] proposed the following question: let  $f$  be a bijective isometry between the unit spheres  $S_X$  and  $S_E$  of real Banach spaces  $X, E$  respectively. Is it then true that  $f$  extends to a linear (bijective) isometry  $F : X \rightarrow E$  of the corresponding spaces? Let us mention that this is equivalent to the fact that the following natural positive-homogeneous extension  $F : X \rightarrow E$  of  $f$  is linear:

$$F(0) = 0, \quad F(x) = \|x\| f(x/\|x\|) \quad (x \in X \setminus \{0\}).$$

Since, according to Mankiewicz's theorem [7, Main Theorem] every bijective isometry between convex bodies can be uniquely extended to an affine isometry of the whole spaces, Tingley's problem can be reformulated as follows

**Problem 1.4.** *Let  $F: B_X \rightarrow B_E$  be a positive-homogeneous map, whose restriction to  $S_X$  is a bijective isometry between  $S_X$  and  $S_E$ . Is it true that  $F$  is an isometry itself?*

Various publications are devoted to Tingley's problem (see [3] for a survey of corresponding results), and, in particular, the problem is solved in the positive for many concrete classical Banach spaces. Surprisingly, for general spaces this innocent-looking question remains open even in dimension 2. For finite-dimensional polyhedral spaces the problem was solved in the positive by Kadets and Martín [5] in 2012, and the positive solution for the class of generalized lush spaces was given by Tan, Huang, and Liu [10] in 2013. A step in the proof of the latter result was a lemma (see [10, Proposition 3.4]) which in our terminology says that if the map  $F$  in Problem 1.4 is nonexpansive, then the problem has a positive solution. So, the problem which we address in our paper (Problem 1.2) can be considered as a much stronger variant of that lemma.

## 2. PRELIMINARIES

In the following, the letters  $X$  and  $Y$  stand for real Banach spaces. We denote by  $S_X$  and  $B_X$  the unit sphere and the closed unit ball of  $X$ , respectively. For a convex set  $A \subset X$ , denote by  $\text{ext}(A)$  the set of extreme points of  $A$ ; that is,  $x \in \text{ext}(A)$  whenever  $x \in A$  and for every  $y \in X \setminus \{0\}$  either  $x + y \notin A$  or  $x - y \notin A$ . Recall that  $X$  is called *strictly convex* when all elements of  $S_X$  are extreme points of  $B_X$ , or in other words, when  $S_X$  does not contain nontrivial line segments. Strict convexity of  $X$  is equivalent to the strict triangle inequality  $\|x + y\| < \|x\| + \|y\|$  holding for all pairs of vectors  $x, y \in X$  that do not have the same direction. For subsets  $A, B \subset X$  we use the standard notation  $A + B = \{x + y: x \in A, y \in B\}$  and  $aA = \{ax: x \in A\}$ .

Now let us reformulate the results of [2] on the case of two different spaces. The following theorem generalizes [2, Theorem 2.3], where the case  $X = Y$  was considered. It can be demonstrated repeating the proof there almost word to word.

**Theorem 2.1.** *Let  $F: B_X \rightarrow B_Y$  be a nonexpansive bijection. The following hold.*

- (1)  $F(0) = 0$ .
- (2)  $F^{-1}(S_Y) \subset S_X$ .
- (3) If  $F(x)$  is an extreme point of  $B_Y$ , then  $F(ax) = aF(x)$  for all  $a \in (0, 1)$ .
- (4) If  $F(x)$  is an extreme point of  $B_Y$ , then  $x$  is also an extreme point of  $B_X$ .
- (5) If  $F(x)$  is an extreme point of  $B_Y$ , then  $F(-x) = -F(x)$ .

Moreover, if  $Y$  is strictly convex, then

- (i)  $F$  maps  $S_X$  bijectively onto  $S_Y$ ;
- (ii)  $F(ax) = aF(x)$  for all  $x \in S_X$  and  $a \in (0, 1)$ ;
- (iii)  $F(-x) = -F(x)$  for all  $x \in S_X$ .

Following the notation in [2], for every  $u \in S_X$  and  $v \in X$ , denote by  $u^*(v)$  the directional derivative of the function  $x \mapsto \|x\|_X$  at the point  $u$  in the direction  $v$ :

$$u^*(v) = \lim_{a \rightarrow 0^+} \frac{1}{a} (\|u + av\|_X - \|u\|_X).$$

Since the norm is a convex function, its directional derivative exists. Let  $M \subset X$  be a subspace, let  $u$  be a smooth point of  $S_M$ , and let  $u^*|_M$  be the restriction of  $u^*$  to  $M$ ;  $u^*|_M$  is known to be the supporting functional at point  $u$ , that is, the unique linear functional on  $M$  that satisfies  $u^*|_M(u) = 1$ ,  $\|u^*|_M\| = 1$ . If  $u$  is a nonsmooth point, the map  $u^*: X \rightarrow \mathbb{R}$  is not linear. However, it turns out to be subadditive, positively homogeneous, and satisfying the following inequality: for any  $y_1, y_2 \in X$ ,

$$u^*(y_1) - u^*(y_2) \leq \|y_1 - y_2\|_X. \quad (1)$$

The next lemma generalizes in a straightforward way [2, Lemma 2.4].

**Lemma 2.2.** *Let  $F: B_X \rightarrow B_Y$  be a bijective nonexpansive map, and suppose that for some  $u \in S_X$  and  $v \in B_X$  we have  $u^*(-v) = -u^*(v)$ ,  $\|F(u)\| = \|u\|$  and  $F(av) = aF(v)$  for all  $a \in [-1, 1]$ . Then  $(F(u))^*(F(v)) = u^*(v)$ .*

The following result and Corollary 2.4 are extracted from the proof of [2, Lemma 2.5].

**Lemma 2.3.** *Let  $F: B_X \rightarrow B_Y$  be a bijective nonexpansive map such that  $F(S_X) = S_Y$ . Let  $V \subset S_X$  be a subset such that  $F(av) = aF(v)$  for all  $a \in [-1, 1]$ ,  $v \in V$ . Denote  $A = \{tx : x \in V, t \in [-1, 1]\}$ ; then  $F|_A$  is a bijective isometry between  $A$  and  $F(A)$ .*

*Proof.* Fix arbitrary  $y_1, y_2 \in A$ . Let  $E = \text{span}\{y_1, y_2\}$ , and let  $W \subset S_E$  be the set of smooth points of  $S_E$  (which is dense in  $S_E$ ). All the functionals  $x^*$ , where  $x \in W$ , are linear on  $E$ , so  $x^*(-y_i) = -x^*(y_i)$ , for  $i = 1, 2$ . Also, according to our assumption,  $F(ay_i) = aF(y_i)$  for all  $a \in [-1, 1]$ . Now we can apply Lemma 2.2:

$$\begin{aligned} \|F(y_1) - F(y_2)\|_Y &\leq \|y_1 - y_2\|_X \\ &= \sup\{x^*(y_1 - y_2) : x \in W\} \\ &= \sup\{x^*(y_1) - x^*(y_2) : x \in W\} \\ &= \sup\{(F(x))^*(F(y_1)) - (F(x))^*(F(y_2)) : x \in W\} \\ &\leq \|F(y_1) - F(y_2)\|_Y, \end{aligned}$$

where on the last inequality we used (1). So  $\|F(y_1) - F(y_2)\| = \|y_1 - y_2\|$ .  $\square$

**Corollary 2.4.** *If  $F: B_X \rightarrow B_Y$  is a bijective nonexpansive function that satisfies (i), (ii), and (iii) of Theorem 2.1, then  $F$  is an isometry.*

*Proof.* We can apply Lemma 2.3 with  $V = S_X$  and  $A = B_X$ .  $\square$

## 3. MAIN RESULTS

The first of our goals, mentioned in the Introduction, can be now achieved by using the results of Section 2.

**Theorem 3.1.** *Let  $F: B_X \rightarrow B_Y$  be a bijective nonexpansive map. If  $Y$  is strictly convex, then  $F$  is an isometry.*

*Proof.* If  $Y$  is strictly convex, then  $F$  satisfies (i), (ii), and (iii) of Theorem 2.1, and so Corollary 2.4 is applicable.  $\square$

Our next goal is to show that each nonexpansive bijection from the unit ball of arbitrary Banach space to the unit ball of  $\ell_1$  is an isometry. In the proof we will use the following three known results.

**Proposition 3.2** ([7, Main Theorem]). *If  $A \subset X$  and  $B \subset Y$  are convex with nonempty interior, then every bijective isometry  $F: A \rightarrow B$  can be extended to a bijective affine isometry  $\tilde{F}: X \rightarrow Y$ .*

Taking into account that in the case of  $A, B$  being the unit balls, every isometry maps 0 to 0, this result implies that every bijective isometry  $F: B_X \rightarrow B_Y$  is the restriction of a linear isometry from  $X$  onto  $Y$ .

**Proposition 3.3** (Brower's invariance of domain principle [1]). *Let  $U$  be an open subset of  $\mathbb{R}^n$  and let  $f: U \rightarrow \mathbb{R}^n$  be an injective continuous map; then  $f(U)$  is open in  $\mathbb{R}^n$ .*

**Proposition 3.4** ([6, Proposition 4]). *Let  $X$  be a finite-dimensional normed space and let  $V$  be a subset of  $B_X$  with the following two properties:  $V$  is homeomorphic to  $B_X$  and  $V \supset S_X$ . Then  $V = B_X$ .*

Now we give the promised theorem.

**Theorem 3.5.** *Let  $X$  be a Banach space, and let  $F: B_X \rightarrow B_{\ell_1}$  be a bijective nonexpansive map. Then  $F$  is an isometry.*

*Proof.* Denote by  $e_n = (\delta_{i,n})_{i \in \mathbb{N}}$ ,  $n = 1, 2, \dots$ , the elements of the canonical basis of  $\ell_1$  (here, as usual,  $\delta_{i,n} = 0$  for  $n \neq i$  and  $\delta_{n,n} = 1$ ). It is well known and easy to check that  $\text{ext}(B_{\ell_1}) = \{\pm e_n, n = 1, 2, \dots\}$ .

Denote  $g_n = F^{-1}e_n$ . According to item (4) of Theorem 2.1 each of  $g_n$  is an extreme point of  $B_X$ .

One more notation: for every  $N \in \mathbb{N}$  and  $X_N = \text{span}\{g_k\}_{k \leq N}$ , denote by  $U_N$  and  $\partial U_N$  the unit ball and the unit sphere of  $X_N$ , respectively, and analogously for  $Y_N = \text{span}\{e_k\}_{k \leq N}$  denote by  $V_N$  and  $\partial V_N$  the unit ball and the unit sphere of  $Y_N$ , respectively.

**Claim.** *For every  $N \in \mathbb{N}$  and every collection  $\{a_k\}_{k \leq N}$  of reals, it holds that*

$$(A) \quad \left\| \sum_{k=1}^N a_k g_k \right\| = \sum_{k=1}^N |a_k| \quad \text{and}$$

$$(B) \quad \text{if } \left\| \sum_{k=1}^N a_k g_k \right\| \leq 1 \quad \text{then } F\left(\sum_{k=1}^N a_k g_k\right) = \sum_{k=1}^N a_k e_k.$$

*Proof of the Claim.* We will prove (A) and (B) simultaneously, using induction on  $N$ . If  $N = 1$ , then the statement (A) is obvious and (B) follows from items (3) and (5) of Theorem 2.1. Now assume the validity of (A) and (B) for  $N - 1$ , and let us prove it for  $N$ . At first, we will prove (A). Note that, due to the positive homogeneity of the norm, it is sufficient to consider the case of  $\sum_{k=1}^N |a_k| \leq 1$ . In such a case

$$\left\| \sum_{k=1}^{N-1} a_k g_k \right\| \leq \sum_{k=1}^{N-1} \|a_k g_k\| = \sum_{i=1}^{N-1} |a_k| \leq \sum_{i=1}^N |a_k| \leq 1,$$

and  $\sum_{k=1}^{N-1} a_k g_k \in U_N$ . On the one hand, denoting  $x := \sum_{k=1}^N a_k g_k$  we have

$$\|x\| = \left\| \sum_{k=1}^N a_k g_k \right\| \leq \sum_{k=1}^N |a_k|.$$

On the other hand, by the induction hypothesis  $F(\sum_{k=1}^{N-1} a_k g_k) = \sum_{k=1}^{N-1} a_k e_k$ . Also, by items (3) and (5) of Theorem 2.1  $F(-a_N g_N) = -a_N e_N$ . Consequently,

$$\begin{aligned} \|x\| &= \left\| \sum_{k=1}^{N-1} a_k g_k + a_N g_N \right\| \\ &= \left\| \sum_{k=1}^{N-1} a_k g_k - (-a_N g_N) \right\| \\ &\geq \left\| F\left(\sum_{k=1}^{N-1} a_k g_k\right) - F(-a_N g_N) \right\| \\ &= \left\| \sum_{k=1}^{N-1} a_k e_k + a_N e_N \right\| \\ &= \sum_{k=1}^N |a_k|, \end{aligned}$$

and (A) is demonstrated. That means that

$$\begin{aligned} U_N &= \left\{ \sum_{n \leq N} a_n g_n : \sum_{n \leq N} |a_n| \leq 1 \right\}, \\ \partial U_N &= \left\{ \sum_{n \leq N} a_n g_n : \sum_{n \leq N} |a_n| = 1 \right\}. \end{aligned}$$

The remaining part of the proof of the Claim, and of the whole theorem repeats almost literally the corresponding part of the proof of [6, Theorem 1], so we present it here only for the reader's convenience. Let us show that

$$F(U_N) \subset V_N. \quad (2)$$

To this end, consider  $x \in U_N$ . If  $x$  is of the form  $a g_N$  the statement follows from Theorem 2.1. So we must consider  $x = \sum_{k=1}^N a_k g_k$ ,  $\sum_{k=1}^N |a_k| \leq 1$ , with

$\sum_{k=1}^{N-1} |a_k| \neq 0$ . Denote the expansion of  $F(x)$  by  $F(x) = \sum_{k=1}^{\infty} y_k e_k$ . For the element

$$x_1 = \frac{\sum_{k=1}^{N-1} a_k g_k}{\sum_{k=1}^{N-1} |a_k|}$$

we have by the induction hypothesis

$$F(x_1) = \frac{\sum_{k=1}^{N-1} a_k e_k}{\sum_{k=1}^{N-1} |a_k|}.$$

So we may write the following chain of inequalities:

$$\begin{aligned} 2 &= \left\| F(x_1) - \frac{a_N}{|a_N|} e_N \right\| \\ &\leq \left\| F(x_1) - \sum_{k=1}^N y_k e_k \right\| + \left\| \sum_{k=1}^N y_k e_k - \frac{a_N}{|a_N|} e_N \right\| \\ &= \|F(x_1) - F(x)\| + \left\| F(x) - \frac{a_N}{|a_N|} e_N \right\| - 2 \sum_{k=N+1}^{\infty} |y_k| \\ &\leq \|F(x_1) - F(x)\| + \left\| F(x) - F\left(\frac{a_N}{|a_N|} g_N\right) \right\| \\ &\leq \|x_1 - x\| + \left\| x - \frac{a_N}{|a_N|} g_N \right\| \\ &= \sum_{j=1}^{N-1} \left| a_j - \frac{a_j}{\sum_{k=1}^{N-1} |a_k|} \right| + |a_N| + \sum_{j=1}^{N-1} |a_j| + \left| a_N - \frac{a_N}{|a_N|} \right| \\ &= \sum_{j=1}^{N-1} |a_j| \left( 1 + \left| 1 - \frac{1}{\sum_{k=1}^{N-1} |a_k|} \right| \right) + |a_N| \left( 1 + \left| 1 - \frac{1}{|a_N|} \right| \right) = 2. \end{aligned}$$

This means that all the inequalities in between are in fact equalities, so in particular  $\sum_{k=N+1}^{\infty} |y_k| = 0$  (i.e.,  $F(x) = \sum_{k=1}^N y_k e_k \in V_N$ ) and (2) is proved.

Now, let us demonstrate that

$$F(U_N) \supset \partial V_N. \tag{3}$$

Assume on the contrary, that there is  $y \in \partial V_N \setminus F(U_N)$ . Denote  $x = F^{-1}(y)$ . Then,  $\|x\| = 1$  (by (2) of Theorem 2.1) and  $x \notin U_N$ . For every  $t \in [0, 1]$ , consider  $F(tx)$ . Let  $F(tx) = \sum_{n \in \mathbb{N}} b_n e_n$  be the corresponding expansion. Then,

$$\begin{aligned} \|y\| &= \|0 - tx\| + \|tx - x\| \\ &\geq \|0 - F(tx)\| + \|F(tx) - y\| \\ &= 2 \sum_{n>N} |b_n| + \left\| \sum_{n \leq N} b_n e_n \right\| + \left\| y - \sum_{n \leq N} b_n e_n \right\| \\ &\geq 2 \sum_{n>N} |b_n| + 1, \end{aligned}$$

so  $\sum_{n>N} |b_n| = 0$ , since  $\|y\| = 1$ . This means that  $F(tx) \in V_N$  for every  $t \in [0, 1]$ . On the other hand,  $F(U_N)$  contains a relative neighborhood of 0 in  $V_N$  (here we use the fact that  $F(0) = 0$  and Proposition 3.3), so the continuous curve  $\{F(tx) : t \in [0, 1]\}$  in  $V_N$  which connects 0 and  $y$  has a nontrivial intersection with  $F(U_N)$ . This implies that there is a  $t \in (0, 1)$  such that  $F(tx) \in F(U_N)$ . Since  $tx \notin U_N$  this contradicts the injectivity of  $F$ . Inclusion (3) is proved.

Now, inclusions (2) and (3) allow us to apply Proposition 3.4 to the finite-dimensional Banach space  $Y_N$ , its unit ball  $V_N$ , and to the subset  $F(U_N) \subset V_N$ , which is a homeomorphic copy of an  $n$ -dimensional ball. This implies that  $F(U_N) = V_N$ . Observe that by (A),  $U_N$  is isometric to  $V_N$ , and by finite dimensionality,  $U_N$  and  $V_N$  are compacta. So,  $U_N$  and  $V_N$  can be considered as two copies of the same compact metric space, and [8, Theorem 1.1] implies that every bijective nonexpansive map from  $U_N$  onto  $V_N$  is an isometry. In particular,  $F$  maps  $U_N$  onto  $V_N$  isometrically. Finally, the application of Proposition 3.2 gives us that the restriction of  $F$  to  $U_N$  extends to a linear map from  $X_N$  to  $Y_N$ , which completes the proof of (B) and that of the Claim.

Now let us complete the proof of the theorem. At first, passing in (A) to the limit as  $N \rightarrow \infty$ , we get

$$\|z\| = \sum_{k=1}^{\infty} |z_k|$$

for every  $z = \sum_{k=1}^{\infty} z_k g_k$  with  $\sum_{k=1}^{\infty} |z_k| < \infty$ . The continuity of  $F$  and the statements (A) and (B) imply that, for every  $x = \sum_{k=1}^{\infty} x_k e_k \in B_{\ell_1}$ ,

$$(A') \quad \left\| \sum_{k=1}^{\infty} x_k g_k \right\| = \sum_{k=1}^{\infty} |x_k| \quad \text{and} \quad (B') \quad F\left(\sum_{k=1}^{\infty} x_k g_k\right) = \sum_{k=1}^{\infty} x_k e_k.$$

Let  $T: \ell_1 \rightarrow X$  be the unique bounded operator satisfying that  $T(e_n) = g_n$  for every  $n \in \mathbb{N}$ . Then (A') gives that  $T$  is a linear isometry (in general not onto) and (B') gives that  $T|_{B_{\ell_1}} = F^{-1}$ .  $\square$

So,  $F^{-1}$  is an isometry, and consequently the same is true for  $F$ .  $\square$

Our next (and last) goal is to demonstrate that each nonexpansive bijection between the unit balls of two different finite-dimensional Banach spaces is an isometry. Below we recall the definitions and well-known properties of total and 1-norming subsets of dual spaces that we will need further.

A subset  $V \subset S_{X^*}$  is called *total* if for every  $x \neq 0$  there exists  $f \in V$  such that  $f(x) \neq 0$ .  $V$  is called *1-norming* if  $\sup |f(x)|_{f \in V} = \|x\|$  for all  $x \in X$ . We will use the following easy exercise.

**Lemma 3.6** ([4, Exercise 9, p. 538]). *Let  $A \subset S_X$  be dense in  $S_X$ , and for every  $a \in A$  let  $f_a$  be a supporting functional at  $a$ . Then  $V = \{f_a : a \in A\}$  is 1-norming (and consequently total).*

The following known fact is an easy consequence of the bipolar theorem.

**Lemma 3.7.** *Let  $X$  be a reflexive space. Then  $V \subset S_{X^*}$  is 1-norming if and only if  $\overline{\text{conv}}(V) = B_{X^*}$ .*



Now we can demonstrate the promised result.

**Theorem 3.8.** *Let  $X, Y$  be Banach spaces, let  $Y$  be finite-dimensional, and let  $F: B_X \rightarrow B_Y$  be a bijective nonexpansive map. Then  $F$  is an isometry.*

*Proof.* Take an arbitrary finite-dimensional subspace  $Z \subset X$ . Then the restriction of  $F$  to  $B_Z$  is a bijective and continuous map between two compact sets  $B_Z$  and  $F(B_Z)$ , so  $B_Z$  and  $F(B_Z)$  are homeomorphic. Thus, Brower’s invariance of domain principle (Proposition 3.3) implies that  $\dim Z \leq \dim Y$ . By arbitrariness of  $Z \subset X$  this implies that  $\dim X \leq \dim Y$ . Consequently,  $F$  being bijective and a continuous map between compact sets  $B_X$  and  $B_Y$ , it is a homeomorphism. Another application of Proposition 3.3 says that  $\dim X = \dim Y$ ,  $F$  maps interior points in interior points, and  $F(S_X) = S_Y$ .

Let  $G$  be the set of all  $x \in S_X$  such that the norm is differentiable both at  $x$  and  $F(x)$ . According to [9, Theorem 25.5], the complement to the set of differentiability points of the norm is meager. Consequently, since  $G$  is an intersection of two comeager sets, it is dense in  $S_X$ . Recall that  $F$  is a homeomorphism, so  $F(G)$  is dense in  $S_Y$ . Given a smooth point  $x \in S_X$ , we will denote by  $x^* \in S_{X^*}$  the unique supporting functional of  $B_X$  at  $x$ . Let us introduce  $A := \{x^* : x \in G\}$  and  $B := \{F(x)^* : x \in G\} = \{y^* : y \in F(G)\}$  the sets of the supporting functionals of  $x$  and  $F(x)$  accordingly. Thus, Lemma 3.6 ensures that  $A$  and  $B$  are 1-norming subsets of  $X^*$  and  $Y^*$ , respectively, and consequently by Lemma 3.7,

$$\overline{\text{aconv}}(A) = B_{X^*}, \quad \overline{\text{aconv}}(B) = B_{Y^*}. \tag{4}$$

Denote  $K = F^{-1}(\text{ext } B_Y) \subset \text{ext } B_X$ . Note that for all  $x \in G$  the corresponding  $(F(x))^*$  and  $x^*$  are linear, and Lemma 2.2 implies that for all  $x \in G$  and  $z \in K$  the following equality holds true:

$$(F(x))^*(F(z)) = x^*(z).$$

Let us define the map  $H: A \rightarrow B$  such that  $H(x^*) = (F(x))^*$ . For the correctness of this definition it is necessary to verify for all  $x_1, x_2 \in G$  the implication

$$(x_1^* = x_2^*) \implies (F(x_1)^* = F(x_2)^*).$$

Assume for given  $x_1, x_2 \in G$  that  $x_1^* = x_2^*$ . In order to check equality  $F(x_1)^* = F(x_2)^*$  it is sufficient to verify that  $F(x_1)^*y = F(x_2)^*y$  for  $y \in \text{ext } B_Y$  (i.e., for  $y$  of the form  $y = F(x)$  with  $x \in K$ ). Indeed,

$$F(x_1)^*(F(x)) = x_1^*(x) = x_2^*(x) = F(x_2)^*(F(x)).$$

Let us extend  $H$  by linearity to  $\tilde{H}: X^* = \text{span}(x^*, x \in G) \rightarrow Y^*$ . For  $x^* = \sum_{k=1}^N \lambda_k x_k^*$ ,  $x_k \in G$  let  $\tilde{H}(x^*) = \sum_{k=1}^N \lambda_k H(x_k^*)$ . To verify the correctness of this extension, we will prove that

$$\left( \sum_{k=1}^N \lambda_k x_k^* = \sum_{k=1}^M \mu_k y_k^* \right) \implies \left( \sum_{k=1}^N \lambda_k H(x_k^*) = \sum_{k=1}^M \mu_k H(y_k^*) \right).$$

Again we will prove equality  $\sum_{k=1}^N \lambda_k H(x_k^*) = \sum_{k=1}^M \mu_k H(y_k^*)$  of functionals only on elements of the form  $y = F(x)$  with  $x \in K$ .

$$\begin{aligned} \left( \sum_{k=1}^N \lambda_k H(x_k^*) \right) F(x) &= \sum_{k=1}^N \lambda_k F(x_k)^*(F(x)) \\ &= \sum_{k=1}^N \lambda_k x_k^*(x) \\ &= \sum_{k=1}^M \mu_k y_k^*(x) \\ &= \sum_{k=1}^M \mu_k F(y_k)^*(F(x)) \\ &= \left( \sum_{k=1}^M \mu_k H(y_k^*) \right) F(x). \end{aligned}$$

Observe that, according to (4),  $\tilde{H}(X^*) = \text{span } H(A) = \text{span } B = Y^*$ , so  $\tilde{H}$  is surjective, and consequently, by equality of corresponding dimensions, it is bijective. Recall, that  $\tilde{H}(A) = H(A) = B$ , so  $\tilde{H}$  maps  $A$  to  $B$  bijectively. Applying again (4), we deduce that  $\tilde{H}(B_{X^*}) = B_{Y^*}$  and that  $X^*$  is isometric to  $Y^*$ . Passing to the duals we deduce that  $Y^{**}$  is isometric to  $X^{**}$  (with  $\tilde{H}^*$  being the corresponding isometry), that is  $X$  and  $Y$  are isometric. So,  $B_X$  and  $B_Y$  are two copies of the same compact metric space, and the application of EC-plasticity of compacts [8, Theorem 1.1] completes the proof.  $\square$

Although, we made some progress in solving Problem 1.2, it remains open, as does Problem 1.1. These problems need further consideration and research.

*Remark.* One may get some improvements and corollaries that we list below.

- Theorem 3.5 can be extended to the space  $\ell_1(\Gamma)$ . The argument is similar to that given in the proof of that theorem.
- Theorem 2.1 implies that there is no bijective nonexpansive function from the unit ball of  $c_0$  or  $L_1[0, 1]$  onto the unit ball of a dual Banach space because the unit balls of  $c_0$  and  $L_1[0, 1]$  do not have extreme points, but a dual ball is  $w^*$ -compact, so it has extreme points by the Krein–Milman theorem.
- From Corollary 2.4 and Proposition 3.2 can be deduced that a bijective nonexpansive function between balls is an isometry if and only if (i) to (iii) of Theorem 2.1 hold.

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