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SPECTRAL PROPERTIES OF THE LAU PRODUCT $\mathcal{A} \times_\theta \mathcal{B}$ OF BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras. Then a multiplicative linear functional θ on $\mathcal{B}$ induces a multiplication on the Cartesian product space $\mathcal{A} \times \mathcal{B}$ given by $(a, b)(c, d) = (ac + \theta(d)a + \theta(b)c, bd)$ for all $(a, b), (c, d) \in \mathcal{A} \times \mathcal{B}$. We show that this Lau product is stable with respect to the spectral properties like the unique uniform norm property, the spectral extension property, the multiplicative Hahn–Banach property, and the unique semisimple norm property under certain conditions on θ.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and let θ be a multiplicative linear functional on $\mathcal{B}$. Then the product space $\mathcal{A} \times \mathcal{B}$ is a Banach algebra with the product

$$(a, b)(c, d) = \big(ac + \theta(d)a + \theta(b)c, bd\big) \quad \big((a, b), (c, d) \in \mathcal{A} \times \mathcal{B}\big)$$

and the norm

$$\big\|(a, b)\big\| = \|a\| + \|b\| \quad \big((a, b) \in \mathcal{A} \times \mathcal{B}\big).$$

This Banach algebra is called the *Lau product* of $\mathcal{A}$ and $\mathcal{B}$ and is denoted by $\mathcal{A} \times_\theta \mathcal{B}$. This product was introduced by Lau [10] for certain classes of Banach algebras and was extended by Sangani Monfared [14] for the general case. Many Banach algebra properties of $\mathcal{A} \times_\theta \mathcal{B}$ are studied in [1], [7], [9], [14], and [15].

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The product $\mathcal{A} \times_\theta \mathcal{B}$ generalizes the unitification process of adjoining identity by taking $\mathcal{B} = \mathbb{C}$ and $\theta(\lambda) = \lambda$. We note that $\mathcal{A} \times_\theta \mathcal{B}$ is commutative if and only if both $\mathcal{A}$ and $\mathcal{B}$ are commutative.

In this paper, we investigate the spectral properties of the commutative Banach algebra $\mathcal{A} \times_\theta \mathcal{B}$ arising out of investigation of incomplete algebra norms (see [4], [5], [11], [12]). These properties include the unique uniform norm property (UUNP), the spectral extension property (SEP), the multiplicative Hahn–Banach property (MHBP), and the unique semisimple norm property (USNP). These properties have turned out to be of relevance in the general theory of Banach algebras (see [8], [13]). This paper is aimed at investigating the stability of $\mathcal{A} \times_\theta \mathcal{B}$ with respect to these properties. Not all the results turned out to be along the expected lines, and the techniques involved working with the permanent Gelfand space consisting of all those multiplicative linear functionals which are continuous with respect to all algebra norms. These spectral properties describe some features of the permanence nature of a normable algebra (see [11], [12], [16]), but it is not known whether or not the UUNP implies SEP (see [5]).

When $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, and $T : \mathcal{B} \to \mathcal{A}$ is an algebra homomorphism with $\|T\| \leq 1$, then $\mathcal{A} \times \mathcal{B}$ is a Banach algebra with the multiplication

$$(a, b)(c, d) = \big(ac + aT(d) + T(b)c, bd\big) \quad \big((a, b), (c, d) \in \mathcal{A} \times \mathcal{B}\big)$$

and the norm

$$\big\|(a, b)\big\| = \|a\| + \|b\| \quad \big((a, b) \in \mathcal{A} \times \mathcal{B}\big).$$

This Banach algebra is denoted by $\mathcal{A} \times_T \mathcal{B}$ [2]. When $\mathcal{A}$ is unital with the identity e, then $\mathcal{A} \times_T \mathcal{B} = \mathcal{A} \times_\theta \mathcal{B}$ with $T(b) = \theta(b)e$ $(b \in \mathcal{B})$.

2. The Shilov boundary

If $\mathcal{A}$ and $\mathcal{B}$ are commutative Banach algebras, then the Gelfand space $\Delta(\mathcal{A} \times_\theta \mathcal{B})$ of $\mathcal{A} \times_\theta \mathcal{B}$ is the union of the sets $E = \{(\varphi, \theta) : \varphi \in \Delta(\mathcal{A})\}$ and $F = \{(0, \psi) : \psi \in \Delta(\mathcal{B})\}$ (see [14, Proposition 2.4]). The topologies on E, F and $E \cup \{(0, \theta)\}$ are the induced topology from $\Delta(\mathcal{A} \times_\theta \mathcal{B})$.

Let $\mathcal{A}$ be a commutative Banach algebra, and let $\mathcal{A}_e = \mathcal{A} \times \mathbb{C}$ be the unitization of $\mathcal{A}$. Hence $\Delta(\mathcal{A}_e) = \{\widetilde{\varphi} : \varphi \in \Delta(\mathcal{A})\} \cup \{\varphi_\infty\}$, where $\widetilde{\varphi}(a, \lambda) = \varphi(a) + \lambda$, and $\varphi_\infty(a, \lambda) = \lambda$ for all $(a, \lambda) \in \mathcal{A}_e$. Define the maps $f : \Delta(\mathcal{A}) \to E$ by $f(\varphi) = (\varphi, \theta)$, $g : \Delta(\mathcal{B}) \to F$ by $g(\psi) = (0, \psi)$, and $\widetilde{f} : \Delta(\mathcal{A}_e) \to E \cup \{(0, \theta)\}$ by $\widetilde{f}(\widetilde{\varphi}) = (\varphi, \theta)$ and $\widetilde{f}(\varphi_\infty) = (0, \theta)$. If $\varphi \in \Delta(\mathcal{A})$, $a_1, a_2, \ldots, a_n \in \mathcal{A}$ and $\epsilon > 0$, then by $U(\varphi, a_1, a_2, \ldots, a_n, \epsilon)$ we mean the set $\{\psi \in \Delta(\mathcal{A}) : |\psi(a_i) - \varphi(a_i)| < \epsilon$ for all $i = 1, 2, \ldots, n\}$.

Proposition 2.1. *If $\mathcal{A}$, $\mathcal{B}$ are commutative Banach algebras and $\theta \in \Delta(\mathcal{B})$, then the functions f, $\widetilde{f}$ and g are homeomorphisms.*

Proof. Clearly f, $\widetilde{f}$, and g are bijections. Let $U(\widetilde{\varphi}, (a_1, \lambda_1), \ldots, (a_n, \lambda_n), \epsilon)$ be a basic open set containing $\widetilde{\varphi} \in \Delta(\mathcal{A}_e)$. Let $b_i \in \mathcal{B}$ be such that $\theta(b_i) = \lambda_i$ for all

$i = 1, 2, \dots, n$. Then

$$\begin{aligned}&\widetilde{f}\big(U\big(\widetilde{\varphi}, (a_1, \lambda_1), \dots, (a_n, \lambda_n), \epsilon\big)\big)\\&\quad = U\big((\varphi, \theta), (a_1, b_1), \dots, (a_n, b_n), \epsilon\big) \cap \big(E \cup \big\{(0, \theta)\big\}\big),\end{aligned}$$

which is a basic open set in $E \cup \{(0, \theta)\}$ containing (φ, θ).

If $U((\varphi, \theta), (a_1, b_1), \dots, (a_n, b_n), \epsilon) \cap (E \cup \{(0, \theta)\})$ is a basic open set containing $(\varphi, \theta) \in E \cup \{(0, \theta)\}$, then

$$\begin{aligned}&(\widetilde{f})^{-1}\big(U\big((\varphi, \theta), (a_1, b_1), \dots, (a_n, b_n), \epsilon\big) \cap \big(E \cup \big\{(0, \theta)\big\}\big)\big)\\&\quad = U\big(\widetilde{\varphi}, \big(a_1, \theta(b_1)\big), \dots, \big(a_n, \theta(b_n)\big), \epsilon\big)\end{aligned}$$

is a basic open set in $\Delta(\mathcal{A}_e)$ containing $\widetilde{\varphi}$.

If $U(\varphi_\infty, (a_1, \lambda_1), \dots, (a_n, \lambda_n), \epsilon)$ is a basic open set containing $\varphi_\infty \in \Delta(\mathcal{A}_e)$, then

$$\begin{aligned}&\widetilde{f}\big(U\big(\varphi_\infty, (a_1, \lambda_1), \dots, (a_n, \lambda_n), \epsilon\big)\big)\\&\quad = U\big((0, \theta), (a_1, b_1), \dots, (a_n, b_n), \epsilon\big) \cap \big(E \cup \big\{(0, \theta)\big\}\big)\end{aligned}$$

is a basic open set in $E \cup \{(0, \theta)\}$ containing $(0, \theta)$, where $\theta(b_i) = \lambda_i$ for all i. In addition, if $U((0, \theta), (a_1, b_1), \dots, (a_n, b_n), \epsilon) \cap (E \cup \{(0, \theta)\})$ is a basic open set containing $(0, \theta) \in E \cup \{(0, \theta)\}$, then

$$\begin{aligned}&(\widetilde{f})^{-1}\big(U\big((0, \theta), (a_1, b_1), \dots, (a_n, b_n), \epsilon\big) \cap \big(E \cup \big\{(0, \theta)\big\}\big)\big)\\&\quad = U\big(\varphi_\infty, \big(a_1, \theta(b_1)\big), \dots, \big(a_n, \theta(b_n)\big), \epsilon\big)\end{aligned}$$

is a basic open set in $\Delta(\mathcal{A}_e)$ containing φ_∞. Thus $\widetilde{f}$ is a homeomorphism.

Since $\widetilde{f}|_{\Delta(\mathcal{A})} = f$, we have that f is a homeomorphism. If $U(\psi, b_1, \dots, b_n, \epsilon)$ is a basic open set in $\Delta(\mathcal{B})$ containing ψ, then

$$g\big(U(\psi, b_1, \dots, b_n, \epsilon)\big) = U\big((0, \psi), (0, b_1), \dots, (0, b_n), \epsilon\big) \cap F$$

is a basic open set in F containing $(0, \psi)$. If $U((0, \psi), (a_1, b_1), \dots, (a_n, b_n), \epsilon)$ is a basic open set in F containing $(0, \psi)$, then

$$g^{-1}\big(U\big((0, \psi), (a_1, b_1), \dots, (a_n, b_n), \epsilon\big)\big) = U(\psi, b_1, \dots, b_n, \epsilon)$$

is a basic open set in $\Delta(\mathcal{B})$ containing ψ. Hence g is a homeomorphism. □

The last result implies that $E \cup \{(0, \theta)\}$ is homeomorphic to $\Delta(\mathcal{A}_e)$ and that F is homeomorphic to $\Delta(\mathcal{B})$.

Let X be a locally compact Hausdorff space, and let $\mathcal{A}$ be a subalgebra of $C_0(X)$ which strongly separates the points of X. A subset M of X is a *boundary* of $\mathcal{A}$ if, for every $f \in \mathcal{A}$, there is $y \in M$ such that $|f(y)| = \sup\{|f(x)| : x \in X\} = \|f\|_\infty$. The intersection of all closed boundaries of $\mathcal{A}$, which is a boundary [8, Theorem 3.3.2], is called the *Shilov boundary* of $\mathcal{A}$. It is denoted by $\partial(\mathcal{A})$.

Let $\mathcal{A}$ be a commutative Banach algebra, and let $\Gamma : \mathcal{A} \to C_0(\Delta(\mathcal{A}))$, $\Gamma(a) = \widehat{a}$, be the Gelfand representation of $\mathcal{A}$. The set $\partial(\Gamma(\mathcal{A}))$ is considered the Shilov boundary of $\mathcal{A}$ and is denoted by $\partial(\mathcal{A})$. We now see the relation between the Shilov boundaries of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A} \times_\theta \mathcal{B}$.

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras and let $\theta \in \Delta(\mathcal{B})$. Then $\partial(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0,\theta)\} = (f(\partial(\mathcal{A})) \cup g(\partial(\mathcal{B})) \setminus \{(0,\theta)\}$.*

Proof. We may identify an element $\varphi \in \Delta(\mathcal{A})$ with the element $\widetilde{\varphi}$ of $\Delta(\mathcal{A}_e)$. Let $\varphi \notin \partial(\mathcal{A})$. Since $\partial(\mathcal{A}) = \Delta(\mathcal{A}) \cap \partial(\mathcal{A}_e)$ (see [8, Remark 3.3.8]), there is an open set $U \subset \Delta(\mathcal{A}_e)$ containing φ such that $\|\widehat{(a,\lambda)}|_U\|_\infty \le \|\widehat{(a,\lambda)}|_{U^c}\|_\infty$ for all $(a,\lambda) \in \mathcal{A}_e$, where U^c is the complement of U in $\Delta(\mathcal{A}_e)$. Consequently $\|\widehat{(a,b)}|_{f(U)}\|_\infty \le \|\widehat{(a,b)}|_{f(U^c)}\|_\infty \le \|\widehat{(a,b)}|_{f(U)^c}\|_\infty$ for all $(a,b) \in \mathcal{A} \times_\theta \mathcal{B}$. Hence $(\varphi,\theta) \notin \partial(\mathcal{A} \times_\theta \mathcal{B})$. Thus if $(\varphi,\theta) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$, then $\varphi \in \partial(\mathcal{A})$.

If we let $\varphi \in \partial(\mathcal{A})$, then, for any neighborhood U of (φ,θ) not containing $(0,\theta)$, there exists $a \in \mathcal{A}$ such that

$$\|\widehat{a}|_{\Delta(\mathcal{A})\setminus f^{-1}(U)}\|_\infty < \|\widehat{a}|_{f^{-1}(U)}\|_\infty.$$

Thus $\sup\{|(\varphi,\theta)(a,0)| : (\varphi,\theta) \in \Delta(\mathcal{A} \times_\theta \mathcal{B}) \setminus U\} < \sup\{|(\varphi,\theta)(a,0)| : (\varphi,\theta) \in U\}$. It is the same as $\|\widehat{(a,0)}|_{\Delta(\mathcal{A}\times_\theta\mathcal{B})\setminus U}\|_\infty < \|\widehat{(a,0)}|_U\|_\infty$; therefore $(\varphi,\theta) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$.

Let $(0,\theta) \neq (0,\psi) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$. If U is a neighborhood of ψ in F not containing θ, then there exists $(a,b) \in \mathcal{A} \times_\theta \mathcal{B}$ such that

$$\left\|\widehat{(a,b)}|_{\Delta(\mathcal{A}\times_\theta\mathcal{B})\setminus g(U)}\right\|_\infty < \left\|\widehat{(a,b)}|_{g(U)}\right\|_\infty;$$

that is, $\max\{\sup\{|\varphi(a) + \theta(b)| : \varphi \in \Delta(\mathcal{A})\}, \sup\{|\psi(b)| : \psi \in \Delta(\mathcal{B}) \setminus U\} < \sup\{|\psi(b)| : \psi \in U\}$. Hence $\sup\{|\psi(b)| : \psi \in \Delta(\mathcal{B}) \setminus U\} < \sup\{|\psi(b)| : \psi \in U\}$, which means that $\psi \in \partial(\mathcal{B})$.

Let $\theta \neq \psi \in \partial(\mathcal{B})$. If U is a neighborhood of ψ not containing θ, then there exists $b \in \mathcal{B}$ such that

$$\|\widehat{b}|_{\Delta(\mathcal{B})\setminus g^{-1}(U)}\|_\infty < \|\widehat{b}|_{g^{-1}(U)}\|_\infty;$$

that is, $\sup\{|\psi(b)| : \psi \in \Delta(\mathcal{B})\setminus g^{-1}(U)\} < \sup\{|\psi(b)| : \psi \in g^{-1}(U)\}$, which means that $\sup\{|(0,\psi)(0,b)| : (0,\psi) \in \Delta(\mathcal{B}) \setminus U\} < \sup\{|(0,\psi)(0,b)| : (0,\psi) \in U\}$. Hence $\|\widehat{(0,b)}|_{\Delta(\mathcal{A}\times_\theta\mathcal{B})\setminus U}\|_\infty < \|\widehat{(0,b)}|_U\|_\infty$, and $(0,\psi) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$. ☐

Proposition 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras, let $\theta \in \Delta(\mathcal{B})$, and let $(0,\theta) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$. Then $\varphi_\infty \in \partial(\mathcal{A}_e)$, or $(0,\theta) \in \overline{\partial(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0,\theta)\}}$.*

Proof. Suppose that $(0,\theta) \notin \overline{\partial(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0,\theta)\}}$. Let U be a neighborhood of φ_∞ in $\Delta(\mathcal{A}_e)$. Since g is a homeomorphism, we have $\theta \notin \overline{\partial(\mathcal{B}) \setminus \{\theta\}}$. Consider $\Delta(\mathcal{B})$ as a part of $\Delta(\mathcal{B}_e)$. Since $\Delta(\mathcal{B}_e)$ is compact, $\overline{\partial(\mathcal{B}) \setminus \{\theta\}}$ is a compact subset of $\Delta(\mathcal{B}_e)$. Since topology on $\Delta(\mathcal{B}_e)$ is Hausdorff, we have open sets V and V' such that $\theta \in V$, and $(\overline{\partial(\mathcal{B}) \setminus \{\theta\}}) \cup \{\phi_\infty\} \subset V'$, and $V' \cap V = \emptyset$. Since $\phi_\infty \notin \overline{V}$, we have that V is a precompact subset of $\Delta(\mathcal{B})$. Since V^c is closed, it is a compact subset of $\Delta(\mathcal{B}_e)$. By similar arguments we have open sets S and S' such that $\theta \in S$, and $V^c \subset S'$ and $S \cap S' = \emptyset$. Hence $\overline{S} \subset V$, where V is a precompact subset of $\Delta(\mathcal{B})$, and $(\partial(\mathcal{B}) \setminus \{\theta\}) \cap \overline{S} = \emptyset$. Note that the set $W = f(U) \cup g(S) \cup \{(0,\theta)\}$ is a neighborhood of $(0,\theta)$ and that $\overline{W}$ is compact. Since $(0,\theta) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$, there exists $(a,b) \in \mathcal{A} \times_\theta \mathcal{B}$ such that $\|\widehat{(a,b)}|_W\|_\infty > \|\widehat{(a,b)}|_{W^c}\|_\infty$. Also, since W is precompact, $\|\widehat{(a,b)}|_W\|_\infty$ is attained at some point in $\overline{W}$. If $\|\widehat{(a,b)}|_W\|_\infty$

is not attained at any point of $\overline{W} \cap (f(\Delta(\mathcal{A})) \cup \{(0,\theta)\})$, then there is $\varphi \in \overline{W} \setminus (f(\Delta(\mathcal{A})) \cup \{(0,\theta)\}) = \overline{g(S)} \setminus \{(0,\theta)\}$, and $\varphi \in \partial(\mathcal{A} \times_\theta \mathcal{B})$. This contradicts the fact that $(\partial(\mathcal{B}) \setminus \{\theta\}) \cap \overline{S} = \emptyset$. Thus $\|\widehat{(a,b)}|_W\|_\infty$ is attained at some point of $\overline{W} \cap (f(\Delta(\mathcal{A})) \cup \{(0,\theta)\})$, which shows that $\|\widehat{a+\theta(b)}|_U\|_\infty > \|\widehat{a+\theta(b)}|_{U^c}\|_\infty$. Hence $\varphi_\infty \in \partial(\mathcal{A}_e)$. □

Remark 2.4. Note that $\theta \in \partial(\mathcal{B})$ and $(0,\theta) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$ are not related to each other. This can be seen from the following examples.

(1) Let $\mathcal{A} = C_0[0,1)$, and let $\mathcal{B} = A(\mathbb{D})$. Let $\theta = \phi_0 \in \Delta(A(\mathbb{D}))$, where $\phi_x(f) = f(x)$ for all $f \in A(\mathbb{D})$. It follows from Theorem 2.2 that $(\phi_x, \theta) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$ for every $\phi_x \in \Delta(\mathcal{A})$ and that $(0, \phi_y) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$ for every $\phi_y \in \Delta(\mathcal{B})$ with $|y| = 1$. If we let U be any neighborhood of $(0,\theta)$ in $\Delta(\mathcal{A} \times_\theta \mathcal{B})$, then $f^{-1}(U) \cup \{\phi_\infty\}$ is a neighborhood of $\phi_\infty \in \Delta(\mathcal{A}_e)$. If we now consider $h \in C_0[0,1)$ such that $\|h|_{f^{-1}(U)\cup\{\phi_\infty\}}\|_\infty > 0$, and $h|_{f^{-1}(U^c)} = 0$, then $\|h|_{f^{-1}(U)}\|_\infty = \|(h,0)|_U\|_\infty > 0 = \|(h,0)|_{U^c}\|_\infty$. Hence $(0,\theta) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$, but $\theta \notin \partial(\mathcal{B}) = \partial(A(\mathbb{D}))$.

(2) Let $\mathcal{A} = \{f \in A(\mathbb{D}) : f(0) = 0\}$, $\mathcal{B} = \mathbb{C}$ and $\theta(\lambda) = \lambda$. Then the Shilov boundary of $\mathcal{A}$ is $\mathbb{T}$, and the Shilov boundary of $\mathcal{A} \times_\theta \mathcal{B} = \mathcal{A} \times_\theta \mathbb{C} = \mathcal{A}_e = A(\mathbb{D})$ is $\mathbb{T}$. Hence $\theta \in \partial(\mathcal{B}) = \partial(\mathbb{C}) = \{\theta\}$, but $(0,\theta) \notin \partial(\mathcal{A} \times_\theta \mathcal{B})$.

(3) Let $\mathcal{A}$ be a commutative Banach algebra with identity e. Let B be a commutative Banach algebra, and let $\theta \in \Delta(\mathcal{B})$. Then $\mathcal{A} \times_\theta \mathcal{B}$ is isometrically isomorphic to the perturbed product $\mathcal{A} \times_T \mathcal{B}$, where $T : \mathcal{B} \to \mathcal{A}$ is defined by $T(b) = \theta(b)e$ ($b \in \mathcal{B}$). By [6, Lemma 2.6], $\partial(\mathcal{A}) \cup \partial(\mathcal{B}) \cong \partial(\mathcal{A} \times_T \mathcal{B})$.

3. Uniqueness of uniform norm and spectral extension property

Let $(\mathcal{A}, \|\cdot\|)$ be a commutative Banach algebra. A norm $|\cdot|$ (not necessarily complete) on $\mathcal{A}$ is a *uniform norm* if $|a^2| = |a|^2$ ($a \in \mathcal{A}$). The Banach algebra $\mathcal{A}$ has *unique uniform norm property* (UUNP) if it has exactly one uniform norm (see [5], [4], [6]). A subset F of $\Delta(\mathcal{A})$ is a *set of uniqueness* if $|a|_F = \sup\{|\varphi(a)| : \varphi \in F\}$ is a norm on $\mathcal{A}$. Let $r(a)$ be the spectral radius of an element a of a Banach algebra. We will prove the following.

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be semisimple commutative Banach algebras, and let $\theta \in \Delta(\mathcal{B})$. Then we have the following.*

1. *If $\mathcal{A}$ and $\mathcal{B}$ have UUNP, then $\mathcal{A} \times_\theta \mathcal{B}$ has UUNP.*
2. *If we suppose that $\mathcal{A} \times_\theta \mathcal{B}$ has UUNP, then $\mathcal{A}$ has UUNP. Moreover, if θ is continuous with respect to every uniform norm on $\mathcal{B}$, then $\mathcal{B}$ also has UUNP.*

The proof of Theorem 3.1 will follow Lemma 3.5. The following gives a relation between sets of uniqueness of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A} \times_\theta \mathcal{B}$.

Lemma 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras, and let $\theta \in \Delta(\mathcal{B})$.*

1. *If M is a set of uniqueness for $\mathcal{A} \times_\theta \mathcal{B}$, then $f^{-1}(M)$ is a set of uniqueness for $\mathcal{A}$, and $g^{-1}(M) \cup \{\theta\}$ is a set of uniqueness for $\mathcal{B}$.*
2. *If M and M' are sets of uniqueness for $\mathcal{A}$ and $\mathcal{B}$, respectively, then $f(M) \cup g(M')$ is a set of uniqueness for $\mathcal{A} \times_\theta \mathcal{B}$.*

Proof. (1) Let $a \in \mathcal{A}$, and let $\widehat{a}(f^{-1}(M)) = \{0\}$. Thus $\widehat{(a,0)}(M) = \{0\}$. Since M is a set of uniqueness for $\mathcal{A} \times_\theta \mathcal{B}$, we have $a = 0$. Hence $f^{-1}(M)$ is a set of uniqueness for $\mathcal{A}$. If $b \in \mathcal{B}$, and $\widehat{b}(g^{-1}(M) \cup \{\theta\}) = \{0\}$, then $\widehat{(0,b)}(M) = \{0\}$. Since M is a set of uniqueness for $\mathcal{A} \times_\theta \mathcal{B}$, $b = 0$, we have that $g^{-1}(M) \cup \{\theta\}$ is a set of uniqueness for $\mathcal{B}$.

(2) If $(a,b) \in \mathcal{A} \times_\theta \mathcal{B}$, and $\widehat{(a,b)}(f(M) \cup g(M')) = \{0\}$, then $\widehat{b}(M') = \{0\}$. Since M' is a set of uniqueness for $\mathcal{B}$, we have $b = 0$, and $\widehat{(a,0)}(f(M) \cup g(M')) = \{0\}$ implies that $\widehat{a}(M) = \{0\}$. Since M is a set of uniqueness for $\mathcal{A}$, we have $a = 0$. Hence $f(M) \cup g(M')$ is a set of uniqueness for $\mathcal{A} \times_\theta \mathcal{B}$. □

We use the following result to show that UUNP is stable with respect to the Lau product.

Theorem 3.3 ([4, Theorem 2.3])**.** *If we let $\mathcal{A}$ be a semisimple commutative Banach algebra, then the following are equivalent.*

1. *$\mathcal{A}$ has UUNP.*
2. *∂A is the smallest closed set of uniqueness.*
3. *If $F \subset \Delta(\mathcal{A})$ is closed and not containing ∂A, then there exists $a \in \mathcal{A}$ such that $r(a) > 0$, and $\widehat{a}|_F = 0$.*

Lemma 3.4. *Let $\mathcal{A}$ be a semisimple commutative Banach algebra such that $\mathcal{A}$ has UUNP, and $\partial(\mathcal{A})$ is a compact subset of $\Delta(\mathcal{A})$. Then $\varphi_\infty \in \partial(\mathcal{A}_e)$ if and only if $\mathcal{A}$ has identity.*

Proof. Assume that $\varphi_\infty \in \partial(\mathcal{A}_e)$. The set $K = \partial(\mathcal{A})$ is a closed subset of $\Delta(\mathcal{A}_e)$, and it does not contain φ_∞. Since $\mathcal{A}$ has UUNP, it holds that $\mathcal{A}_e$ has UUNP (see [3, Theorem 3.1]). Thus by Theorem 3.3 there exists $(a,\lambda) \in \mathcal{A}_e$ such that $\widehat{(a,\lambda)}|_K = 0$, and $r(a,\lambda) \neq 0$. Hence $\varphi(a) = -\lambda$ for all $\varphi \in \Delta(\mathcal{A})$. The case $\lambda = 0$ is not possible if $\lambda = 0$. In that case, $a = 0$; hence $(a,\lambda) = (0,0)$. Since $\mathcal{A}$ is semisimple and $\varphi(b + \frac{1}{\lambda}ab) = \varphi(b) + \varphi(\frac{1}{\lambda}ab) = \varphi(b) + \varphi(\frac{1}{\lambda}a)\varphi(b) = \varphi(b) - \varphi(b) = 0$ for all $b \in \mathcal{A}$, and $\varphi \in \Delta(\mathcal{A})$, the element $\frac{-1}{\lambda}a$ is the identity for $\mathcal{A}$.

If $\mathcal{A}$ has identity e, then for any neighborhood U of ϕ_∞ we have $(-e,1) \in \mathcal{A}_e$ such that $\|\widehat{(-e,1)}|_U\|_\infty = 1$, and $\|\widehat{(-e,1)}|_{U^c}\|_\infty = 0$. Hence $\phi_\infty \in \partial(\mathcal{A}_e)$. □

Lemma 3.5. *Let $\mathcal{A}$ be a semisimple commutative Banach algebra, and let $|\cdot|$ be a uniform norm on $\mathcal{A}$. For $a \in \mathcal{A}$, let $|a|_{\mathrm{op}} = \sup\{|ab| : b \in \mathcal{A}, |b| \leq 1\}$. Then $|\cdot|_{\mathrm{op}}$ is a norm on $\mathcal{A}$, and $|\cdot|_{\mathrm{op}} = |\cdot|$.*

Proof. Since $\mathcal{A}$ is semisimple, $|\cdot|_{\mathrm{op}}$ is a norm on $\mathcal{A}$, and $|\cdot|_{\mathrm{op}} \leq |\cdot|$. If $a \in \mathcal{A}$, then $|a|^2 = |a^2| \leq |a|_{\mathrm{op}}|a|$; therefore $|\cdot| \leq |\cdot|_{\mathrm{op}}$. □

Proof of Theorem 3.1. (1) Let $\mathcal{A}$ and $\mathcal{B}$ have UUNP. Let K be a closed subset of $\Delta(\mathcal{A} \times_\theta \mathcal{B})$ not containing $\partial(\mathcal{A} \times_\theta \mathcal{B})$.

If K does not contain at least one element (φ,θ) of $\partial(\mathcal{A} \times_\theta \mathcal{B})$, then $f^{-1}(K)$ is a closed subset of $\Delta(\mathcal{A})$ not containing $\varphi \in \partial(\mathcal{A})$. Therefore by Theorem 3.3 there exists a nonzero $a \in \mathcal{A}$ such that $\widehat{a}|_{f^{-1}(K)} = 0$. Consequently $\widehat{(a,0)}|_K = 0$, and $(a,0)$ is nonzero.

If K does not contain at least one element $(0,\psi)$ of $\partial(\mathcal{A}\times_\theta\mathcal{B})$ with $(0,\theta)\neq(0,\psi)$, then $g^{-1}(K)\cup\{\theta\}$ is a closed subset of $\Delta(\mathcal{B})$ not containing $\psi\in\partial(\mathcal{B})$. Thus there exists a nonzero $b\in\mathcal{B}$ such that $\widehat{b}|_{g^{-1}(K)\cup\{\theta\}}=0$; hence $\widehat{(0,b)}|_K=0$, and $(0,b)$ is nonzero.

If we suppose that K does not contain only $(0,\theta)\in\partial(\mathcal{A}\times_\theta\mathcal{B})$, then $\varphi_\infty\in\partial(\mathcal{A}_e)$ by Proposition 2.3, and $\mathcal{A}$ has identity by Lemma 3.4. Thus $\mathcal{A}\times_\theta\mathcal{B}\cong\mathcal{A}\times_T\mathcal{B}$, and, by [6, Theorem 2.9], there is a nonzero $(a,b)\in\mathcal{A}\times_\theta\mathcal{B}$ such that $\widehat{(a,b)}|_K=0$; hence $\mathcal{A}\times_\theta\mathcal{B}$ has UUNP.

(2) Let $|\cdot|$ be a uniform norm on $\mathcal{A}$. For $(a,b)\in\mathcal{A}\times_\theta\mathcal{B}$, let

$$\big\|(a,b)\big\|=\max\big\{\big|a+\theta(b)\big|_{\mathrm{op}},r(0,b)\big\},$$

where

$$\big|a+\theta(b)\big|_{\mathrm{op}}=\sup\big\{\big|\big(a+\theta(b)\big)x\big|:x\in\mathcal{A},|x|\le 1\big\}.$$

Then $\|\cdot\|$ is a norm on $\mathcal{A}\times_\theta\mathcal{B}$. If $(a,b)\in\mathcal{A}\times_\theta\mathcal{B}$, then

$$\begin{aligned}\big|\big(a+\theta(b)\big)^2\big|_{\mathrm{op}}&=\sup\big\{\big|\big(a+\theta(b)\big)^2x\big|:|x|\le 1\big\}\\&\ge\sup\big\{\big|\big(a+\theta(b)\big)^2x^2\big|:|x^2|\le 1\big\}\\&=\sup\big\{\big|\big(a+\theta(b)\big)x\big|^2:|x|^2\le 1\big\}\\&=\big|\big(a+\theta(b)\big)\big|_{\mathrm{op}}^2.\end{aligned}$$

Clearly, $|(a+\theta(b))^2|_{\mathrm{op}}\le|a+\theta(b)|_{\mathrm{op}}^2$. Thus $\|(a,b)^2\|=\|(a,b)\|^2$ for all $(a,b)\in\mathcal{A}\times_\theta\mathcal{B}$; that is, $\|\cdot\|$ is a uniform norm on $\mathcal{A}\times_\theta\mathcal{B}$. Since $\mathcal{A}\times_\theta\mathcal{B}$ has UUNP, the spectral radius is the only uniform norm on $\mathcal{A}\times_\theta\mathcal{B}$; therefore $\|(a,b)\|=r(a,b)$ for all $(a,b)\in\mathcal{A}\times_\theta\mathcal{B}$. If $a\in\mathcal{A}$, then $r(a,0)=\max\{|a|_{\mathrm{op}},r(0,0)\}=|a|_{\mathrm{op}}=|a|$. Thus any uniform norm on $\mathcal{A}$ is the restriction of the spectral radius of $\mathcal{A}\times_\theta\mathcal{B}$, and consequently $\mathcal{A}$ has UUNP.

Let $|\cdot|$ be a uniform norm on $\mathcal{B}$. Since θ is continuous with respect to any uniform norm on $\mathcal{B}$, we have $|\theta(b)|\le|b|$ for all $b\in\mathcal{B}$. For $(a,b)\in\mathcal{A}\times_\theta\mathcal{B}$, let

$$\big\|(a,b)\big\|=\max\big\{r_{\mathrm{op}}\big(a+\theta(b)\big),|b|\big\},$$

where

$$r_{\mathrm{op}}\big(a+\theta(b)\big)=\sup\big\{r\big(\big(a+\theta(b)\big)x,0\big):x\in\mathcal{A},r(x,0)\le 1\big\}.$$

If $(a,b)\in\mathcal{A}\times_\theta\mathcal{B}$, then

$$\begin{aligned}r_{\mathrm{op}}\big(\big(a+\theta(b)\big)^2\big)&=\sup\big\{r\big(\big(a+\theta(b)\big)^2x,0\big):r(x,0)\le 1\big\}\\&\ge\sup\big\{r\big(\big(a+\theta(b)\big)^2x^2,0\big):r(x^2,0)\le 1\big\}\\&=\sup\big\{r\big(\big(a+\theta(b)\big)x,0\big)^2:r(x,0)^2\le 1\big\}\\&=r_{\mathrm{op}}\big(a+\theta(b)\big)^2.\end{aligned}$$

Moreover,

$$\begin{aligned}r_{\mathrm{op}}\big(\big(a+\theta(b)\big)^2\big)&=\sup\big\{r\big(\big(a+\theta(b)\big)^2x,0\big):r(x,0)\le 1\big\}\\&\le\sup\big\{r_{\mathrm{op}}\big(a+\theta(b)\big)r\big(\big(a+\theta(b)\big)x,0\big):r(x,0)\le 1\big\}\end{aligned}$$

$$= r_{\mathrm{op}}\big(a+\theta(b)\big)\sup\big\{r\big(a+\theta(b)x,0\big) : r(x,0)\le 1\big\}$$
$$= r_{\mathrm{op}}\big(a+\theta(b)\big)^2.$$

Hence $\|\cdot\|$ is a uniform norm on $\mathcal{A}\times_\theta\mathcal{B}$. Since $\mathcal{A}\times_\theta\mathcal{B}$ has UUNP, we have $\|(a,b)\| = r(a,b)$ for all $(a,b)\in\mathcal{A}\times_\theta\mathcal{B}$. This gives $r(0,b)=\max\{|\theta(b)|,|b|\}=|b|$ for all $b\in\mathcal{B}$; hence $\mathcal{B}$ has UUNP. □

An *extension* of a Banach algebra $\mathcal{A}$ is a Banach algebra $\mathcal{B}$ such that $\mathcal{A}$ is algebraically, but not necessarily continuously, embedded in $\mathcal{B}$. A commutative Banach algebra $\mathcal{A}$ has the *spectral extension property (SEP)* if $r_{\mathcal{B}}(a)=r_{\mathcal{A}}(a)$ for every extension $\mathcal{B}$ of $\mathcal{A}$ and for every $a\in\mathcal{A}$ (see [11]). Equivalently, a commutative Banach algebra $\mathcal{A}$ has SEP if $r(a)\le|a|$ $(a\in\mathcal{A})$ for every norm $|\cdot|$ on $\mathcal{A}$. The SEP has precedents in [16]; it was formally introduced in [11], [12]; and it has been investigated by several authors (see [3]–[5], [11], [12]). Clearly, SEP implies UUNP, and it is not known whether UUNP implies SEP (see [5]). In what follows we discuss the stability of SEP with respect to the Lau product.

The *permanent spectral radius* [11] of an element a of $\mathcal{A}$ is

$$r_p(a)=\inf\big\{|a| : |\cdot| \text{ is a norm on } \mathcal{A}\big\}.$$

For the convenience of our readers we recall the following result.

Theorem 3.6 ([8, Theorem 4.5.3])**.** *If we let $\mathcal{A}$ be a semisimple commutative Banach algebra, then the following are equivalent.*

(1) *$\mathcal{A}$ has SEP.*
(2) *If E is a closed subset of $\Delta(\mathcal{A})$ that does not contain the Shilov boundary of $\mathcal{A}$, then there exists an element $a\in\mathcal{A}$ such that $\widehat{a}=0$ on E, and $r_p(a)>0$.*
(3) *Whenever $\mathcal{B}$ is a commutative extension of $\mathcal{A}$, every $\varphi\in\partial(\mathcal{A})$ extends to some element of $\Delta(\mathcal{B})$.*

Theorem 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be semisimple commutative Banach algebras, and let $\theta\in\Delta(\mathcal{B})$.*

(1) *If $\mathcal{A}$ and $\mathcal{B}$ have SEP, then $\mathcal{A}\times_\theta\mathcal{B}$ has SEP.*
(2) *If we suppose that $\mathcal{A}\times_\theta\mathcal{B}$ has SEP, then $\mathcal{A}$ has SEP. Moreover, if θ is continuous with respect to every norm on $\mathcal{B}$, then $\mathcal{B}$ has SEP.*

The proof of the Theorem 3.7 will follow Lemma 3.9.

Lemma 3.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be semisimple commutative Banach algebras, and let $(0,\theta)\notin\partial(\mathcal{A}\times_\theta\mathcal{B})$. If $\mathcal{A}$ and $\mathcal{B}$ have SEP, then $\mathcal{A}\times_\theta\mathcal{B}$ has SEP.*

Proof. Let $\mathcal{D}$ be a Banach algebra containing $\mathcal{A}\times_\theta\mathcal{B}$. If $(\varphi,\theta)\in\partial(\mathcal{A}\times_\theta\mathcal{B})$, then $\varphi\in\partial(\mathcal{A})$, by Theorem 2.2. Since $\mathcal{D}$ is an extension of $\mathcal{A}$, by Theorem 3.6, we have ϕ in $\Delta(\mathcal{D})$ such that $\phi|_{\mathcal{A}}=\varphi$; hence $\phi|_{\mathcal{A}\times_\theta\mathcal{B}}=(\varphi,\theta)$. Similarly, if $(0,\psi)\in\partial(\mathcal{A}\times_\theta\mathcal{B})$, then $\psi\in\partial(\mathcal{B})$, by Theorem 2.2. As $\mathcal{D}$ is an extension of $\mathcal{B}$, by Theorem 3.6, we have Ψ in $\Delta(\mathcal{D})$ such that $\Psi|_{\mathcal{B}}=\psi$, but then we also see that $\Psi|_{\mathcal{A}\times_\theta\mathcal{B}}=(0,\psi)$. This proves that $\mathcal{A}\times_\theta\mathcal{B}$ has SEP. □

Lemma 3.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be semisimple commutative Banach algebras, and let $(0,\theta)\in\partial(\mathcal{A}\times_\theta\mathcal{B})$. If $\mathcal{A}$ and $\mathcal{B}$ have SEP, then $\mathcal{A}\times_\theta\mathcal{B}$ has SEP.*

Proof. Let M be a closed subset of $\Delta(\mathcal{A} \times_\theta \mathcal{B})$ containing $\partial(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0,\theta)\}$ and $(0,\theta) \notin M$. Then, by Proposition 2.3, we have $\varphi_\infty \in \partial(\mathcal{A}_e)$, and $\partial(\mathcal{A})$ is a compact subset of $\Delta(\mathcal{A})$. Thus $\mathcal{A}$ has identity, by Theorem 3.4. If we let $T(b) = \theta(b)e$, then $\mathcal{A} \times_T \mathcal{B} \cong \mathcal{A} \times_\theta \mathcal{B}$; thus $\mathcal{A} \times_\theta \mathcal{B}$ has SEP by [6, Theorem 2.30].

If, for any closed subset M of $\Delta(\mathcal{A} \times_\theta \mathcal{B})$ containing $\partial(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0,\theta)\}$, we let $(0,\theta)$ be in M, and we let M be a closed subset of $\Delta(\mathcal{A} \times_\theta \mathcal{B})$ not containing $(\varphi,\theta) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$, then $U = f^{-1}(M^c)$ is an open subset of $\Delta(\mathcal{A})$ containing $\varphi \in \partial(\mathcal{A})$. Hence there exists $a \in \mathcal{A}$ such that $\widehat{a}|_{U^c} = 0$, and $r_p(a) \neq 0$. Consequently $\widehat{(a,0)}|_M = 0$, and $r_p(a,0) \neq 0$.

If we let M be a closed subset of $\Delta(\mathcal{A} \times_\theta \mathcal{B})$ not containing $(0,\psi) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$, and we let $(0,\theta) \neq (0,\psi)$, then $U = g^{-1}(M^c) \cup \{\theta\}$ is an open subset of $\Delta(\mathcal{B})$ containing $\psi \in \partial(\mathcal{B})$. Thus there exists $b \in \mathcal{B}$ such that $\widehat{b}|_{U^c} = 0$, and $r_p(b) \neq 0$; hence $\widehat{(0,b)}|_E = 0$, and $r_p(0,b) \neq 0$, and therefore $\mathcal{A} \times_\theta \mathcal{B}$ has SEP. □

Proof of Theorem 3.7. (1) This follows from Lemma 3.8 and Lemma 3.9.

(2) Let $|\cdot|$ be a norm on $\mathcal{A}$. For $(a,b) \in \mathcal{A} \times_\theta \mathcal{B}$, let $|(a,b)|_1 = |a| + \|b\|$. Consequently $|\cdot|_1$ is a norm. Since $\mathcal{A} \times_\theta \mathcal{B}$ has SEP, $r(a) = r(a,0) \leq |(a,0)|_1 = |a|$ for all $a \in \mathcal{A}$; hence $\mathcal{A}$ has SEP. Let $|\cdot|$ be a norm on $\mathcal{B}$. Define $|(a,b)|_2 = \|a\| + |b|$ for all $(a,b) \in \mathcal{A} \times_\theta \mathcal{B}$. Since θ is continuous with respect to every norm on $\mathcal{B}$, we have $|\theta(b)| \leq |b|$ $(b \in \mathcal{B})$, and $|\cdot|_2$ is a norm on $\mathcal{A} \times_\theta \mathcal{B}$. Since $\mathcal{A} \times_\theta \mathcal{B}$ has SEP, $r(b) = r(0,b) \leq |(0,b)|_2 = |b|$ for all $b \in \mathcal{B}$; hence $\mathcal{B}$ has SEP. □

A commutative Banach algebra $\mathcal{A}$ has the *multiplicative Hahn–Banach property (MHBP)* if, given any commutative extension $\mathcal{B}$ of $\mathcal{A}$, every $\varphi \in \Delta(\mathcal{A})$ extends to some element of $\Delta(\mathcal{B})$ (see [11]). We show that MHBP is stable with respect to the Lau product.

Theorem 3.10 ([8, Theorem 4.5.7])**.** *If we let $\mathcal{A}$ be a semisimple commutative Banach algebra, then $\mathcal{A}$ has MHBP if and only if $\mathcal{A}$ has SEP, and $\partial(\mathcal{A}) = \Delta(\mathcal{A})$.*

Theorem 3.11. *If we let $\mathcal{A}$ and $\mathcal{B}$ be semisimple commutative Banach algebras, and we let $\theta \in \Delta(\mathcal{B})$, then we see the following.*

(1) *If both $\mathcal{A}$ and $\mathcal{B}$ have MHBP, then $\mathcal{A} \times_\theta \mathcal{B}$ has MHBP.*
(2) *If we suppose that $\mathcal{A} \times_\theta \mathcal{B}$ has MHBP, then $\mathcal{A}$ has MHBP. Moreover, if θ is continuous with respect to every norm on $\mathcal{B}$, then $\mathcal{B}$ has MHBP.*

Proof. (1) Since $\mathcal{A}$ and $\mathcal{B}$ have MHBP, we have $\partial(\mathcal{A}) = \Delta(\mathcal{A})$, $\partial(\mathcal{B}) = \Delta(\mathcal{B})$, and $\mathcal{A}$ and $\mathcal{B}$ have SEP. By Theorem 3.7, $\mathcal{A} \times_\theta \mathcal{B}$ has SEP, and by Theorem 2.2 we have $\Delta(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0,\theta)\} = \partial(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0,\theta)\}$. If $(0,\theta) \in \overline{\Delta(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0,\theta)\}}$, then we have $\partial(\mathcal{A} \times_\theta \mathcal{B}) = \Delta(\mathcal{A} \times_\theta \mathcal{B})$ because the Shilov boundary is a closed subset of Gelfand space. If we let $(0,\theta) \notin \overline{\Delta(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0,\theta)\}}$, then $\{(0,\theta)\}$ is a compact open subset of $\Delta(\mathcal{A} \times_\theta \mathcal{B})$. By Shilov's idempotent theorem [8, Theorem 3.5.1] there exists $(a,b) \in \mathcal{A} \times_\theta \mathcal{B}$ such that $\widehat{(a,b)}|_{\{(0,\theta)\}} = \{1\}$, and $\widehat{(a,b)}_{\{\Delta(\mathcal{A} \times_\theta \mathcal{B}) \setminus \{(0,\theta)\}\}} = \{0\}$; that is, $(0,\theta) \in \partial(\mathcal{A} \times_\theta \mathcal{B})$. Thus $\mathcal{A} \times_\theta \mathcal{B}$ has MHBP.

(2) If we let $\mathcal{A} \times_\theta \mathcal{B}$ have MHBP, then $\mathcal{A} \times_\theta \mathcal{B}$ has SEP, and $\partial(\mathcal{A} \times_\theta \mathcal{B}) = \Delta(\mathcal{A} \times_\theta \mathcal{B})$. Consequently $\mathcal{A}$ and $\mathcal{B}$ have SEP by Theorem 3.7, and $\partial(\mathcal{A}) = \Delta(\mathcal{A})$, and $\partial(\mathcal{B}) \setminus \{\theta\} = \Delta(\mathcal{B}) \setminus \{\theta\}$ by Theorem 2.2. If $\theta \in \overline{\Delta(\mathcal{B}) \setminus \{\theta\}}$, then, since $\partial(\mathcal{B})$ is closed subset of $\Delta(\mathcal{B})$, we have $\partial(\mathcal{B}) = \Delta(\mathcal{B})$. If $\theta \notin \overline{\Delta(\mathcal{B}) \setminus \{\theta\}}$, it holds that $\{\theta\}$ is an open and compact subset of $\Delta(\mathcal{B})$. By Shilov's idempotent theorem [8, Theorem 3.5.1], we have $b \in \mathcal{B}$ such that $\widehat{b}|_{\{\theta\}} = 1$, and $\widehat{b}|_{\Delta(\mathcal{B}) \setminus \{\theta\}} = 0$; that is, $\theta \in \partial(\mathcal{B})$. Hence $\mathcal{A}$ and $\mathcal{B}$ have MHBP. □

A commutative Banach algebra $\mathcal{A}$ has *unique semisimple norm property (USNP)* if any two semisimple norms on $\mathcal{A}$ are equivalent (see [4]). We will use the following to show that USNP is stable with respect to the Lau product.

Theorem 3.12. *[4, Proposition 2.2] A commutative Banach algebra $(\mathcal{A}, \|\cdot\|_\mathcal{A})$ has USNP if and only if $\mathcal{A}$ has UUNP, and $\|\cdot\|_\mathcal{A}$ is equivalent to a uniform norm on $\mathcal{A}$.*

Theorem 3.13. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras, and let $\theta \in \Delta(\mathcal{B})$.*

(1) *If $\mathcal{A}$ and $\mathcal{B}$ have USNP, then $\mathcal{A} \times_\theta \mathcal{B}$ has USNP.*
(2) *If we suppose that $\mathcal{A} \times_\theta \mathcal{B}$ has USNP, then $\mathcal{A}$ has USNP. Moreover, if θ is continuous with respect to every uniform norm on $\mathcal{B}$, then $\mathcal{B}$ has USNP.*

Proof. (1) Since $(\mathcal{A}, \|\cdot\|_\mathcal{A})$ and $(\mathcal{B}, \|\cdot\|_\mathcal{B})$ have USNP, they have UUNP and the norms $\|\cdot\|_\mathcal{A}$ and $\|\cdot\|_\mathcal{B}$ are equivalent to uniform norms $\|\cdot\|'$ and $\|\cdot\|''$ on $\mathcal{A}$ and $\mathcal{B}$ respectively. Since $\mathcal{A}$ and $\mathcal{B}$ have UUNP, $\mathcal{A} \times_\theta \mathcal{B}$ has UUNP. Define

$$\left|(a,b)\right| = \max\left\{\left\|a + \theta(b)\right\|'_{\mathrm{op}}, \|b\|''\right\} \quad \left((a,b) \in \mathcal{A} \times_\theta \mathcal{B}\right),$$

where $\|a + \theta(b)\|'_{\mathrm{op}} = \sup\{\|(a + \theta(b))x\|' : x \in \mathcal{A}, \|x\|' \leq 1\}$. Then $\|\cdot\|'_{\mathrm{op}}$ is a uniform norm on the unitization $\mathcal{A}_e$ of $\mathcal{A}$; hence $|\cdot|$ is a uniform norm on $\mathcal{A} \times_\theta \mathcal{B}$. Since $\|\cdot\|''$ and $\|\cdot\|_\mathcal{B}$ are equivalent, θ is continuous with respect to $\|\cdot\|''$. This gives $|\theta(b)| \leq \|b\|''$ for all $b \in \mathcal{B}$. If $(a,b) \in \mathcal{A} \times_\theta \mathcal{B}$, and $x \in \mathcal{A}$ with $\|x\|' \leq 1$, then $\|(a + \theta(b))x\|' \leq \|a\|' + \|b\|'' \leq K(\|a\|_\mathcal{A} + \|b\|_\mathcal{B})$ for some constant $K > 0$ as the norms $\|\cdot\|_\mathcal{A}$ and $\|\cdot\|'$ are equivalent on $\mathcal{A}$, and the norms $\|\cdot\|_\mathcal{B}$ and $\|\cdot\|''$ are equivalent on $\mathcal{B}$. Thus $|(a,b)| \leq K(\|a\|_\mathcal{A} + \|b\|_\mathcal{B})$ for all $(a,b) \in \mathcal{A} \times_\theta \mathcal{B}$. To show that the norms $|\cdot|$ and $\|\cdot\|_\mathcal{A} + \|\cdot\|_\mathcal{B}$ are equivalent, we need to prove that $|\cdot|$ is a complete norm on $\mathcal{A} \times_\theta \mathcal{B}$. Let $((a_n, b_n))$ be a Cauchy sequence in $\mathcal{A} \times_\theta \mathcal{B}$ in the norm $|\cdot|$. Since $\|\cdot\|''$ is a complete norm on $\mathcal{B}$, there is $b \in \mathcal{B}$ such that $\|b_n - b\|'' \to 0$ as $n \to \infty$. If we let $m, n \in \mathbb{N}$, then $\|a_n - a_m\|'_{\mathrm{op}} \leq \|a_n + \theta(b) - a_m - \theta(b_m)\|'_{\mathrm{op}} + |\theta(b_n) - \theta(b_m)|$. Since $\|\cdot\|'_{\mathrm{op}} = \|\cdot\|'$ on $\mathcal{A}$, by Lemma 3.5, and $\|\cdot\|'$ is a complete norm on $\mathcal{A}$, there is $a \in \mathcal{A}$ such that $\|a_n - a\|'_{\mathrm{op}} \to 0$ as $n \to \infty$. It follows that $|(a_n, b_n) - (a,b)| \to 0$ as $n \to \infty$. Hence $|\cdot|$ is a complete norm on $\mathcal{A} \times_\theta \mathcal{B}$. This proves that the norm on $\mathcal{A} \times_\theta \mathcal{B}$ is equivalent to a uniform norm on $\mathcal{A} \times_\theta \mathcal{B}$; therefore $\mathcal{A} \times_\theta \mathcal{B}$ has USNP.

(2) Assume that $\mathcal{A} \times_\theta \mathcal{B}$ has USNP. Since θ is continuous with respect to every uniform norm $\mathcal{B}$, it holds that both $\mathcal{A}$ and $\mathcal{B}$ have UUNP, by Theorem 3.1. Since $\mathcal{A} \times_\theta \mathcal{B}$ has USNP, the norm $\|\cdot\|$ on $\mathcal{A} \times_\theta \mathcal{B}$ is equivalent to a uniform norm on $\mathcal{A} \times_\theta \mathcal{B}$. However the restrictions of the uniform norm on $\mathcal{A}$ and $\mathcal{B}$ are uniform

norms and these uniform norms are equivalent to norms of $\mathcal{A}$ and $\mathcal{B}$ respectively. Therefore A and $\mathcal{B}$ have USNP. □

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