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ON MULTIPLIERS BETWEEN BOUNDED VARIATION SPACES

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Abstract. Wiener-type variation spaces, also known as BV_p-*spaces* ($1 \leq p < \infty$), are complete normed linear spaces. A function g is called a *multiplier* from BV_p to BV_q if the pointwise multiplication fg belongs to BV_q for each $f \in \mathrm{BV}_p$. In this article, we characterize the multipliers from BV_p to BV_q for the cases $1 \leq q < p$ and $1 \leq p \leq q$.

1. Introduction

Let E and F be spaces of real- (or complex-) valued functions defined on a set X. A real- (or complex-) valued function g defined on X is called a *multiplier* from E to F if the pointwise multiplication fg belongs to F for every $f \in E$. The set of all multipliers from E to F is denoted as $M(E \to F)$. When E and F are normed spaces, then it is natural to consider the operator $M_g : E \to F$ defined as

$$M_g(f) = fg.$$

The operator M_g is called a *multiplication operator* induced by g, and the function g is usually called the *symbol* of the multiplication operator.

It is then of interest to characterize the set $M(E \to F)$ as well as some properties of M_g (such as boundedness, compactness, closed range, etc.) in terms of conditions on the symbol g. For example, Takagi and Yokouchi [11] characterized the set $M(L_p \to L_q)$, where L_p stands for the usual Lebesgue space. Nakai [9] studied the set of multipliers between Lorentz spaces. The author, Castillo, and Ramos-Fernández [5] studied multiplication operators defined on Orlicz–Lorentz

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spaces; the author and Castillo [4] also studied multiplication operators defined on multidimensional Lorentz spaces. (We refer the reader to [10] for more information about these topics.)

In order to introduce the bounded variation spaces, we recall that a partition P of $[0,1]$ is a finite set $P=\{t_0,t_1,\dots,t_m\}$ such that

$$0=t_0<t_1<t_2<\cdots<t_m=1.$$

For a function $f:[0,1]\to\mathbb{R}$, we say that f has *bounded p-variation* (BV_p) if

$$\mathrm{Var}_p(f)=\sup_P\Big(\sum_{j=1}^m\big|f(t_j)-f(t_{j-1})\big|^p\Big)^{1/p}<\infty,$$

where the supremum is taken over all partitions P of $[0,1]$. The set of all functions $f:[0,1]\to\mathbb{R}$ with bounded p-variation will be denoted as $\mathrm{BV}_p([0,1])$. Bounded variation spaces were introduced by Jordan [6] in 1881. Since then, the concept of bounded variation has been generalized in many ways. The one we discuss here was introduced by Wiener [12] in 1924.

There are some special features that distinguish $\mathrm{BV}_p([0,1])$-spaces from other spaces such as Lebesgue spaces L_p (and their generalizations, e.g., Lorentz spaces, Orlicz spaces, etc.). For example, for functions f and g in L_p, if $f=g$ almost everywhere, then their L_p-norms are the same. This is not true for functions in $\mathrm{BV}_p([0,1])$. Even if $f,g\in\mathrm{BV}_p([0,1])$ differ only on one single point, their norms can be very different. So, in the context of $\mathrm{BV}_p([0,1])$, $f=g$ means $f(t)=g(t)$ for all $t\in[0,1]$.

Another important difference between $\mathrm{BV}_p([0,1])$- and L_p-spaces is the lack of the so-called *lattice property*: for f, g in L_p, if $|f|\le|g|$ almost everywhere, then $\|f\|_{L_p}\le\|g\|_{L_p}$. This property does not hold in $\mathrm{BV}_p([0,1])$, as is easily shown by defining on $[0,1]$ the functions

$$f(t)=\begin{cases}0 & \text{if } t\neq 1/2\\ 1 & \text{if } t=1/2\end{cases}\qquad\text{and}\qquad g(t)=1.$$

(For more details about bounded variation spaces and different types of variations, see [1].) There has been relatively little study of multipliers and multiplication operators on bounded variation spaces. One of the few examples we can cite is [2], where the authors obtained results about the multiplication operator $M_u:\mathrm{BV}_1([0,1])\to\mathrm{BV}_1([0,1])$.

In this article, we completely characterize the set

$$M\big(\mathrm{BV}_p\big([0,1]\big)\to\mathrm{BV}_q\big([0,1]\big)\big).$$

To describe our result more precisely, we divide the argument into two cases: CASE I: $1\le q<p$, CASE II: $1\le p\le q$. In Section 2, we give some auxiliary results and definitions. In Section 3, we state some theorems regarding the characterization described above.

2. Auxiliary results

We present some auxiliary results that will be useful later. Let us denote by $B([0,1])$ the set of all bounded functions $f:[0,1]\to\mathbb{R}$ with the norm

$$\|f\|_\infty := \sup_{0\le t\le 1}\big|f(t)\big|.$$

It is a well-known fact that $\mathrm{BV}_p([0,1])$ is a subspace of $B([0,1])$ (see [1, p. 85]). Moreover, if we set

$$\|f\|_{\mathrm{BV}_p} := \|f\|_\infty + \mathrm{Var}_p(f),$$

then $(\mathrm{BV}_p([0,1]), \|\cdot\|_{\mathrm{BV}_p})$ becomes a Banach space. With this norm, $\mathrm{BV}_p([0,1])$ is a normalized Banach algebra; that is,

$$\|fg\|_{\mathrm{BV}_p} \le \|f\|_{\mathrm{BV}_p}\|g\|_{\mathrm{BV}_p}. \tag{2.1}$$

(See [7, p. 171] for a proof of the above inequality.) Besides, since the inequality

$$\mathrm{Var}_p(f)^{1/p} \le \mathrm{Var}_q(f)^{1/q}, \quad 1\le q\le p<\infty, \tag{2.2}$$

holds, one concludes that

$$\mathrm{BV}_q\big([0,1]\big) \subset \mathrm{BV}_p\big([0,1]\big), \quad 1\le q\le p<\infty.$$

In the next lemma, we show that the above inclusion is strict. This fact will be useful later.

Lemma 2.1. *Given any strictly increasing sequence $\{t_j\}_{j\in\mathbb{N}}\subset[0,1]$, there exists a function f such that*

(1) *$f\in\mathrm{BV}_p([0,1])$ but $f\notin\mathrm{BV}_q([0,1])$ if $1\le q<p$,*
(2) *$\sup_{t\in(t_j,t_{j+1})} f(t) = f(t_j)$,*
(3) *$\inf_{t\in(t_j,t_{j+1})} f(t) = 0$.*

Proof. (1) For some $\theta>0$, consider the zigzag function Z_θ defined on $[0,1]$ as

$$Z_\theta(t) = \begin{cases} 0 & \text{if } t<t_0 \text{ or } t=(t_j+t_{j+1})/2,\ j=0,1,2,\dots,\\ \frac{1}{2^{1/p}(j+1)^\theta} & \text{if } t=t_j,\ j=0,1,2,\dots,\\ \text{linear} & \text{otherwise.}\end{cases} \tag{2.3}$$

It follows that

$$\mathrm{Var}_p(Z_\theta)^p = \sum_{j=1}^\infty \frac{1}{j^{p\theta}} \quad (1\le p<\infty).$$

This means that Z_θ belongs to $\mathrm{BV}_p([0,1])$ only if $p>1/\theta$. In particular, for $1\le q<p$, $Z_{1/q}(t)\in\mathrm{BV}_p([0,1])$ and $Z_{1/q}(t)\notin\mathrm{BV}_q([0,1])$ (see [1, p. 89] for a similar discussion). It is clear that $Z_{1/q}$ also satisfies conditions (2) and (3). $\square$

For any function $f:[0,1]\to\mathbb{R}$ and any set $E\subseteq[0,1]$, we call

$$\mathrm{osc}_E(f) = \sup_{t\in E} f(t) - \inf_{t\in E} f(t)$$

the *oscillation* of f on E. For $1 \le p < \infty$, we define

$$v_p(f) = \sup\Big(\sum_{k=1}^{m} \mathrm{osc}_{I_k}(f)^p\Big)^{1/p},$$

where the supremum is taken over all collections $\{I_k\}$ of disjoint intervals contained in $[0,1]$.

For the proof of Theorem 3.1, which is the main result of this paper, it will be convenient to use $v_p(f)$ instead of $\mathrm{Var}_p(f)$. We show that they are the same in the following lemma.

Lemma 2.2. *For any function* $f \in \mathrm{BV}_p([0,1])$,

$$\mathrm{Var}_p(f) = v_p(f).$$

Proof. Given any partition $P = \{0 = t_0, t_1, \ldots, t_m = 1\}$ of $[0,1]$, we construct a sequence of disjoint intervals

$$I_1 = (t_0, t_1), \qquad I_2 = (t_1, t_2), \qquad \ldots, \qquad I_m = (t_{m-1}, t_m).$$

It is clear that

$$\big|f(t_j) - f(t_{j-1})\big| \le \sup_{t \in I_j} f(t) - \inf_{t \in I_j} f(t) = \mathrm{osc}_{I_j}(f), \quad j = 1, 2, \ldots, m.$$

Then

$$\sum_{j=1}^{m}\big|f(t_j) - f(t_{j-1})\big|^p \le \sum_{j=1}^{m} \mathrm{osc}_{I_j}(f)^p,$$

from which one concludes that

$$\mathrm{Var}_p(f) \le v_p(f). \tag{2.4}$$

Now we will obtain the reverse inequality. Fix a sequence $X_1, X_2, \ldots, X_m$ of disjoint subintervals of $[0,1]$. Then, for any $\varepsilon > 0$, there exist $x_j \in X_j$ and $x_{j-1} \in X_j$ $(j = 1, \ldots, m)$ such that

$$f(x_j) > \sup_{x \in X_j} f(x) - \varepsilon \qquad \text{and} \qquad f(x_{j-1}) < \inf_{x \in X_j} f(x) + \varepsilon.$$

Therefore,

$$\big|f(x_j) - f(x_{j-1})\big| \ge f(x_j) - f(x_{j-1}) > \sup_{x \in X_j} f(x) - \inf_{x \in X_j} f(x) - 2\varepsilon.$$

And then we have

$$\mathrm{Var}_p(f)^p \ge \sum_{j=1}^{m}\big|f(x_j) - f(x_{j-1})\big|^p > \sum_{j=1}^{m}\big(\mathrm{osc}_{X_j}(f) - 2\varepsilon\big)^p.$$

From the above inequality, a standard argument shows that

$$v_p(f) \le \mathrm{Var}_p(f). \tag{2.5}$$

Combining (2.4) and (2.5), we obtain the desired result. ∎

3. Multipliers from $\mathrm{BV}_p([0,1])$ to $\mathrm{BV}_q([0,1])$

To facilitate our study of multipliers between $\mathrm{BV}_p([0,1])$- and $\mathrm{BV}_q([0,1])$-spaces, we separate it into two cases.

CASE I: $1 \le q < p$. Lemma 2.1 and the inequality

$$\mathrm{Var}_p(f) \le \mathrm{Var}_q(f) \quad (1 \le q < p),$$

show us that, for $1 \le q < p$, $\mathrm{BV}_q([0,1])$ is a proper subset of $\mathrm{BV}_p([0,1])$. If we take a function u belonging to $\mathrm{BV}_p([0,1]) \setminus \mathrm{BV}_q([0,1])$, then we cannot induce a multiplier from $\mathrm{BV}_p([0,1])$ into $\mathrm{BV}_q([0,1])$. For the constant function $f(t) = 1 \in \mathrm{BV}_p([0,1])$,

$$M_u(f) = u \cdot f = u \cdot 1 = u \notin \mathrm{BV}_q\big([0,1]\big).$$

Because of this, it is natural to restrict ourselves only to symbols u such that $u \in \mathrm{BV}_q([0,1])$.

For any subset $A \subset [0,1]$, we denote by $\#(A)$ the counting measure on A, that is,

$$\#(A) = \begin{cases} \text{number of elements in } A & \text{if } A \text{ is a finite set,} \\ \infty & \text{if } A \text{ is an infinite set.} \end{cases}$$

Moreover, for a function $u : [0,1] \to \mathbb{R}$, we define

$$\varphi_u(r) = \#\big(\{t \in [0,1] : |u(t)| \ge r\}\big).$$

In the next theorem, we will see that the function φ_u allows us to characterize the set $M(\mathrm{BV}_p([0,1]) \to \mathrm{BV}_q([0,1]))$.

Theorem 3.1. *Suppose that $1 \le q < p$, and let u be a function in $\mathrm{BV}_q([0,1])$. Then $u \in M(\mathrm{BV}_p([0,1]) \to \mathrm{BV}_q([0,1]))$ if and only if $\varphi_u(r) < \infty$ for all $r > 0$.*

Proof. Fix $u \in \mathrm{BV}_p([0,1])$. Assume that $\varphi_u(r) < \infty$ for all $r > 0$, and take arbitrary $f \in \mathrm{BV}_q([0,1])$. We will prove that $uf \in \mathrm{BV}_p([0,1])$.

There is no loss of generality in assuming that both u and f are positive. Otherwise, we decompose u and f as

$$u = u^+ - u^-, \qquad f = f^+ - f^-,$$

where the superscripts $^+$ and $^-$ stand for the positive and negative parts of the functions; that is,

$$f^+(t) = \max\{f(t), 0\}, \qquad f^-(t) = \max\{-f(t), 0\}.$$

Note that

$$uf = u^+f^+ - u^+f^- - u^-f^+ + u^+f^-$$

and that

$$\mathrm{Var}_p(uf)^{1/p} \le \mathrm{Var}_p(u^+f^+)^{1/p} + \mathrm{Var}_p(u^+f^-)^{1/p} + \mathrm{Var}_p(u^-f^+)^{1/p} + \mathrm{Var}_p(u^+f^-)^{1/p}.$$

Then, it is sufficient to estimate each term separately.

We prove the converse first. If $\varphi(r) < \infty$ for all $r > 0$, then for any interval $I_k \subset [0,1]$ there exists $t_k \in I_k$ such that $u(t_k) = 0$. Since u and f are positive functions, it follows that

$$\inf_{t\in I_k}(uf)(t) = 0 \qquad \text{and} \qquad \inf_{t\in I_k} u(t) = 0. \tag{3.1}$$

We know also that

$$\sup_{t\in I_k}(uf)(t) \le \|f\|_\infty \sup_{t\in I_k} u(t). \tag{3.2}$$

From (3.1) and (3.2), we conclude that

$$\operatorname{osc}_{I_k}(uf) \le \|f\|_\infty \operatorname{osc}_{I_k}(u).$$

And then, adding over disjoint intervals I_k,

$$\sum_{k=1}^{m} \operatorname{osc}_{I_k}(uf)^q \le \|f\|_\infty^q \sum_{k=1}^{m} \operatorname{osc}_{I_k}(u)^q.$$

Therefore,

$$\operatorname{Var}_q(uf) \le \|f\|_\infty \operatorname{Var}_q(u) \le \operatorname{Var}_p(f) \operatorname{Var}_q(u).$$

In order to prove the direct implication, by way of contradiction, assume that there exists a number $r_0 > 0$ such that $\varphi(r_0) = \infty$. Then we can find an increasing sequence $\{t_n\}_{n\in\mathbb{N}} \subset [0,1]$ such that $u(t_n) \ge r_0$. Recall the function $Z_{1/q}$ defined in the proof of Lemma 2.1. We know that $v_p(Z_{1/q}) < \infty$ and $v_q(Z_{1/q}) = \infty$. For this function, it is true that

$$\inf_{(t_j,t_{j+1})} Z_{1/q}(t) = 0, \qquad \text{and also} \qquad \inf_{(t_j,t_{j+1})} u(t)\cdot Z_{1/q}(t) = 0.$$

Then

$$\begin{aligned}
\operatorname{osc}_{(t_j,t_{j+1})}(u\cdot Z_{1/q}) &= \sup_{(t_j,t_{j+1})}(u\cdot Z_{1/q})(t) - \inf_{(t_j,t_{j+1})}(u\cdot Z_{1/q})(t)\\
&= \sup_{(t_j,t_{j+1})}(u\cdot Z_{1/q})(t)\\
&\ge u(t_j)\cdot Z_{1/q}(t_j)\\
&\ge r_0\cdot Z_{1/q}(t_j)\\
&= r_0\cdot \sup_{(t_j,t_{j+1})} Z_{1/q}(t)\\
&= r_0\cdot\big(\sup_{(t_j,t_{j+1})} Z_{1/q}(t) - \inf_{(t_j,t_{j+1})} Z_{1/q}(t)\big)\\
&= r_0\cdot \operatorname{osc}_{(t_j,t_{j+1})}(Z_{1/q}).
\end{aligned}$$

From this one concludes that

$$\begin{aligned}
\operatorname{Var}_q(u\cdot Z_{1/q})^q &\ge r_0^q \operatorname{Var}_q(Z_{1/q})^q\\
&= r_0^q \sum_{k=1}^{\infty}\frac{1}{k}\\
&= \infty.
\end{aligned}$$

□

CASE II: $1 \leq p \leq q$. The study of $M(\mathrm{BV}_p([0,1]) \to \mathrm{BV}_q([0,1]))$ for the case $1 \leq p \leq q$ is quite easy. We provide it here for the sake of completeness.

The following result relies on the fact that $\mathrm{BV}_q([0,1])$ is a normalized Banach algebra, and also on the fact that, for $1 \leq p \leq q$, we have the continuous embedding $\mathrm{BV}_p([0,1]) \hookrightarrow \mathrm{BV}_q([0,1])$.

Theorem 3.2. *Suppose that $1 \leq p \leq q$. Then $u \in M(\mathrm{BV}_p([0,1]) \to \mathrm{BV}_q([0,1]))$ if and only if $u \in \mathrm{BV}_q([0,1])$. In this case, M_u, the multiplication operator induced by u, is a bounded linear operator from $\mathrm{BV}_p([0,1])$ into $\mathrm{BV}_q([0,1])$, and its norm is given by $\|M_u\| = \|u\|_{\mathrm{BV}_q}$.*

Proof. If $u \in \mathrm{BV}_q([0,1])$, then from (2.1) and (2.2) we get

$$\|uf\|_{\mathrm{BV}_q} \leq \|u\|_{\mathrm{BV}_q}\|f\|_{\mathrm{BV}_q} \leq \|u\|_{\mathrm{BV}_q}\|f\|_{\mathrm{BV}_p} < \infty. \tag{3.3}$$

Then $uf \in \mathrm{BV}_q([0,1])$. Conversely, if $u \in M(\mathrm{BV}_p([0,1]) \to \mathrm{BV}_q([0,1]))$, then, since the constant function $h(t) = 1$ belongs to $\mathrm{BV}_p([0,1])$, we have

$$M_u h(x) = u(x) \cdot h(x) = u(x) \cdot 1 = u(x), \tag{3.4}$$

and so $u \in \mathrm{BV}_q([0,1])$. Finally, from (3.3) and (3.4) we can also conclude that $\|M_u\| = \|u\|_{\mathrm{BV}_q}$. □

Remark 3.3. The results we have obtained in this article can be performed, with some modifications, for functions of several variables. For more information about bounded variation in this setting, we refer the reader to [1, p. 91], [3], and [8].

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References

1. J. Appell, J. Banaś, and N. Merentes, *Bounded Variation and Around*, de Gruyter Ser. Nonlinear Anal. Appl. **17**, de Gruyter, Berlin, 2014. Zbl 1282.26001. MR3156940. 377, 378, 382
2. F. R. Astudillo-Villalba and J. C. Ramos-Fernández, *Multiplication operators on the space of functions of bounded variation*, Demonstratio Math. **50** (2017), no. 1, 105–115. Zbl 1367.26030. MR3657865. DOI 10.1515/dema-2017-0012. 377
3. S. Barza and M. Lind, *A new variational characterization of Sobolev spaces*, J. Geom. Anal. **25** (2015), no. 4, 2185–2195. Zbl 1348.46034. MR3427120. DOI 10.1007/s12220-014-9508-z. 382
4. R. E. Castillo and H. C. Chaparro, *Weighted composition operator on two-dimensional Lorentz spaces*, Math. Inequal. Appl. **20** (2017), no. 3, 773–799. Zbl 06802019. MR3653918. DOI 10.7153/mia-20-49. 377
5. R. E. Castillo, H. C. Chaparro, and J. C. Ramos-Fernández, *Orlicz-Lorentz spaces and their multiplication operators*, Hacet. J. Math. Stat. **44** (2015), no. 5, 991–1009. Zbl 06632060. MR3469921. DOI 10.15672/HJMS.2015449663. 376
6. C. Jordan, *Sur la série de Fourier*, C. R. Math. Acad. Sci. Paris **92** (1881), no. 5, 228–230. JFM 13.0184.01. 377
7. R. Kantrowitz, *Submultiplicativity of norms for spaces of generalized BV-functions*, Real Anal. Exchange **36** (2010/2011), no. 1, 169–175. Zbl 1251.26008. MR3016410. 378

8. J. Malý, *Absolutely continuous functions of several variables*, J. Math. Anal. Appl. **231** (1999), no. 2, 492–508. Zbl 0924.26008. MR1669167. DOI 10.1006/jmaa.1998.6246. 382
9. E. Nakai, *Pointwise multipliers on the Lorentz spaces*, Mem. Osaka Kyoiku Univ. Ser. III Nat. Sci. Appl. Sci. **45** (1996), no. 1, 1–7. MR1448884. 376
10. J. C. Ramos-Fernández, *Some properties of multiplication operators acting on Banach spaces of measurable functions*, Bol. Mat. (N.S.) **23** (2016), no. 2, 221–237. MR3608457. 377
11. H. Takagi and K. Yokouchi, "Multiplication and composition operators between two L^p-spaces" in *Function Spaces (Edwardsville, Ill., 1998)*, Contemp. Math. **232**, Amer. Math. Soc., Providence, 1999, 321–338. Zbl 0939.47028. MR1678344. DOI 10.1090/conm/232/03408. 376
12. N. Wiener, *The quadratic variation of a function and its Fourier coefficients*, Stud. Appl. Math. **3** (1924), no. 2, 72–94. JFM 50.0203.01. 377

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