



Ann. Funct. Anal. 9 (2018), no. 3, 322–333  
<https://doi.org/10.1215/20088752-2017-0039>  
ISSN: 2008-8752 (electronic)  
<http://projecteuclid.org/afa>

## A NOTE ON THE HYPERCYCLICITY OF OPERATOR-WEIGHTED SHIFTS

YA WANG and ZE-HUA ZHOU\*

Communicated by I. M. Spitkovsky

ABSTRACT. In this article, we give equivalent conditions for the hypercyclicity of bilateral operator-weighted shifts on  $L^2(\mathcal{K})$  with weight sequence  $\{A_n\}_{n=-\infty}^{\infty}$  of positive invertible diagonal operators on a separable complex Hilbert space  $\mathcal{K}$ , as well as for hereditarily hypercyclicity and supercyclicity.

### 1. Introduction and preliminaries

Let  $L(X)$  denote the space of continuous linear operators on a separable, infinite-dimensional complex Fréchet space  $X$ . An operator  $T \in L(X)$  is said to be *hypercyclic* if there is a vector  $x \in X$  such that the orbit  $\text{orb}(T, x) = \{T^n x : n \geq 0\}$  is dense in  $X$ . In such a case,  $x$  is called a *hypercyclic vector* for  $T$ . Accordingly,  $T \in L(X)$  is called *supercyclic* if there exists a vector  $x \in X$  such that the projective orbit  $\{\lambda T^n x : n \geq 0, \lambda \in \mathbb{C}\}$  is dense in  $X$ . It is well known that if  $T$  is hypercyclic, then the set of hypercyclic vectors of  $T$  is a dense  $G_\delta$  subset of  $X$  (see [7]). Many fundamental results regarding the theory of hypercyclic operators were established by Kitai in [10]. The excellent books by Bayart and Matheron [2] and by Grosse-Erdmann and Peris [8] also provide a solid foundation and give an overview of the dynamics of linear operators.

The first example of a hypercyclic operator was offered by Rolewicz [12], who showed that if  $B$  is the unweighted unilateral backward shift on  $\ell^2(\mathbb{N})$ , then the

---

Copyright 2018 by the Tusi Mathematical Research Group.

Received Apr. 27, 2017; Accepted Jul. 6, 2017.

First published online Jan. 29, 2018.

\*Corresponding author.

2010 *Mathematics Subject Classification*. Primary 47A16; Secondary 47B37.

*Keywords*. hypercyclic operator, supercyclic operator, operator-weighted shifts.

scaled shift  $\lambda B$  is hypercyclic if and only if  $|\lambda| > 1$ . After that, Salas [13] characterized the hypercyclic unilateral weighted backward shifts and hypercyclic bilateral weighted shifts on  $\ell^p(\mathbb{Z})$  ( $1 \leq p < \infty$ ) in terms of their weight sequences. In recent years, the dynamics of weighted shifts have been studied by many authors. For example, Bès, Martin, Peris, and Shkarin [4] verified that for any mixing bilateral weighted shift  $T$  on  $\ell^p(\mathbb{Z})$  ( $1 \leq p < \infty$ ) and any positive integer  $r$ ,  $(T, T^2, \dots, T^r)$  is  $d$ -mixing. Bès, Martin, and Sanders [5] studied the disjoint hypercyclicity of unilateral and bilateral weighted shift operators. In addition, Hazarika and Arora [9] characterized the hypercyclicity of the bilateral operator-weighted shift  $T$  on  $L^2(\mathcal{K})$  with weight sequence  $\{A_n\}_{n=-\infty}^\infty$  of positive invertible diagonal operators on a separable complex Hilbert space  $\mathcal{K}$ . Inspired by Hazarika and Arora’s work, Liang and Zhou [11] discussed the supercyclicity and hereditarily hypercyclicity of operator-weighted shifts. In the present article, we provide a counterexample to show that the characterization in [9] is not a necessary condition for hypercyclicity. Moreover, we give new equivalent conditions for the bilateral operator-weighted shifts to qualify as hypercyclic, hereditarily hypercyclic, and supercyclic.

The following criterion, due to Bès [3], will be used several times in this article.

**Theorem 1.1** (Hypercyclicity criterion [2, Definition 1.5]). *Let  $X$  be a separable  $F$ -space, and let  $T \in L(X)$ . If there are dense subsets  $X_0, Y_0 \subset X$ , an increasing sequence  $(n_k)_k$  of positive integers, and maps  $S_{n_k} : Y_0 \rightarrow X, k \geq 1$  such that*

- (1)  $T^{n_k}(x) \rightarrow 0$  for any  $x \in X_0$ ,
- (2)  $S_{n_k}(y) \rightarrow 0$  for any  $y \in Y_0$ ,
- (3)  $T^{n_k}S_{n_k}(y) \rightarrow y$  for each  $y \in Y_0$ ,

*then we say that  $T$  satisfies the hypercyclicity criterion with respect to  $(n_k)_k$ ; in particular,  $T$  is hypercyclic.*

We note that the hypercyclicity criterion is not necessary for hypercyclicity (see [1]). Moreover, the above criterion has been shown in [6] to be equivalent to *hereditarily hypercyclic*; that is, there exists some increasing sequence of positive integers  $(n_k)_k$  such that for each subsequence  $(m)_k$  of  $(n)_k$ ,  $\{T^{m_k}\}_k$  is hypercyclic.

Before proceeding further, let us specify some terminology about operator-weighted shifts on the space  $L^2(\mathcal{K})$ . Let  $\mathbb{Z}$  be the set of integers, and let  $\mathcal{K}$  be a separable complex Hilbert space with an orthonormal basis  $\{f_k\}_{k=0}^\infty$ . Then  $L^2(\mathcal{K})$  is the separable Hilbert space

$$L^2(\mathcal{K}) = \left\{ x = (\dots, x_{-1}, [x_0], x_1, \dots) : x_i \in \mathcal{K} \text{ and } \sum_{i \in \mathbb{Z}} \|x_i\|^2 < \infty \right\}$$

under the inner product  $\langle x, y \rangle = \sum_{i \in \mathbb{Z}} \langle x_i, y_i \rangle$  for  $x = (x_i)_{i \in \mathbb{Z}}$  and  $y = (y_i)_{i \in \mathbb{Z}}$  in  $L^2(\mathcal{K})$ .

Let  $\{A_n\}_{n=-\infty}^\infty$  be a uniformly bounded sequence of invertible operators on  $\mathcal{K}$ , where the operators  $\{A_n\}_{n=-\infty}^\infty$  are all positive diagonal with respect to the basis  $\{f_k\}_{k=0}^\infty$ . We define the bilateral forward and backward operator-weighted shift on  $L^2(\mathcal{K})$  as follows.

(i) For  $x = (x_i) \in L^2(\mathcal{K})$ , the bilateral forward operator-weighted shift  $T$  on  $L^2(\mathcal{K})$  is defined by

$$T(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, A_{-2}x_{-2}, [A_{-1}x_{-1}], A_0x_0, \dots).$$

Since  $\{A_n\}_{n=-\infty}^{\infty}$  is uniformly bounded,  $T$  is bounded and  $\|T\| = \sup_{i \in \mathbb{Z}} \|A_i\| < \infty$ . For  $n > 0$ ,

$$T^n(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, y_{-1}, [y_0], y_1, \dots),$$

where

$$y_j = A_{j-1}A_{j-2} \cdots A_{j-n}x_{j-n} \quad \text{or} \quad y_{n+j} = A_{j+n-1}A_{j+n-2} \cdots A_jx_j; \quad (1.1)$$

also, we have

$$y_j = \prod_{s=0}^{n-1} A_{j+s-n}x_{j-n} \quad \text{or} \quad y_{n+j} = \prod_{s=0}^{n-1} A_{j+s}x_j, \quad (1.2)$$

since  $A_nA_m = A_mA_n$  for any  $n, m \in \mathbb{Z}$ .

Hence

$$\|T^n\| = \sup_j \left\| \prod_{s=0}^{n-1} A_{j+s} \right\|.$$

(ii) For  $x = (x_i) \in L^2(\mathcal{K})$ , the bilateral backward operator weighted shift  $T$  on  $L^2(\mathcal{K})$  is defined by

$$T(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, A_0x_0, [A_1x_1], A_2x_2, \dots).$$

Then

$$T^n(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, y_{-1}, [y_0], y_1, \dots),$$

where

$$y_j = \prod_{s=1}^n A_{j+s}x_{j+n} \quad \text{or} \quad y_{j-n} = \prod_{s=1}^n A_{j+s-n}x_j.$$

Then we have

$$\|T^n\| = \sup_j \left\| \prod_{s=1}^n A_{j+s} \right\|.$$

Since each  $A_n$  is an invertible diagonal operator on  $\mathcal{K}$ , we conclude that

$$\|A_n\| = \sup_k \|A_n f_k\|, \quad \|A_n^{-1}\| = \sup_k \|A_n^{-1} f_k\|, \quad \sup_k \|A_n f_k\| = \frac{1}{\inf_k \|A_n^{-1} f_k\|}.$$

### 2. Hypercyclic operator-weighted shift

The following statement is the core theorem in [9], which provides a characterization for the hypercyclicity of bilateral operator-weighted shifts on  $L^2(\mathcal{K})$ .

**Theorem 2.1** ([9, Theorem 3.1]). *Let  $T$  be a bilateral forward operator-weighted shift on  $L^2(\mathcal{K})$  with weight sequence  $\{A_n\}_{n=-\infty}^\infty$ , where  $\{A_n\}_n$  is a uniformly bounded sequence of positive invertible diagonal operators on  $\mathcal{K}$ . Then the following are equivalent:*

- (a)  $T$  is hypercyclic;
- (b) for  $\varepsilon > 0$  and  $q \in \mathbb{N}$ , there exists  $n$  arbitrarily large such that for all  $|j| \leq q$ ,

$$\left\| \prod_{s=0}^{n-1} A_{s+j} \right\| < \varepsilon \quad \text{and} \quad \left\| \prod_{s=1}^n A_{j-s}^{-1} \right\| < \varepsilon.$$

However, we found a simple counterexample, which is a hypercyclic operator-weighted shift on  $L^2(\mathcal{K})$  that does not satisfy Theorem 2.1(b).

*Example 2.2.* Let  $\{A_n\}_{n=-\infty}^\infty$  be a uniformly bounded sequence of positive invertible diagonal operators on  $\mathcal{K}$ , defined as follows:

$$\begin{aligned} \text{if } n > 0: \quad A_n(f_k) &= \begin{cases} \frac{1}{2}f_k, & 0 \leq k < n, \\ f_k, & k = n, \\ 2f_k, & k > n, \end{cases} \\ \text{if } n = 0: \quad A_0(f_k) &= \begin{cases} f_k, & k = 0, \\ 2f_k, & k > 0, \end{cases} \\ \text{if } n < 0: \quad A_n(f_k) &= 2f_k \quad \text{for all } k \geq 0, \end{aligned}$$

where  $\{f_k\}_{k=0}^\infty$  is the orthonormal basis of  $\mathcal{K}$ . Let  $T$  be the bilateral forward operator-weighted shift on  $L^2(\mathcal{K})$  with weight sequence  $\{A_n\}_{n=-\infty}^\infty$ . Then  $T$  is hypercyclic but does not satisfy Theorem 2.1(b).

*Proof.* For each  $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$ , set  $e_{i_0, j_0} := (\dots, z_{-1}, [z_0], z_1, \dots)$ , where  $z_{j_0} = f_{i_0}$  and  $z_j = 0$  for all  $j \neq j_0$ . It is obvious that the set  $\text{span}\{e_{i,j} : (i, j) \in \mathbb{N} \times \mathbb{Z}\}$  is dense in  $L^2(\mathcal{K})$ . To prove that  $T$  is hypercyclic, we apply the hypercyclicity criterion to the whole sequence of integers  $(n_k) = (k)$ , the same dense set  $X_0 = Y_0 = \text{span}\{e_{i,j} : (i, j) \in \mathbb{N} \times \mathbb{Z}\}$ , and the maps  $S_k = S^k$ , where  $S$  is the backward operator-weighted shift defined by

$$S(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, B_0x_0, [B_1x_1], B_2x_2, \dots), \tag{2.1}$$

where  $B_j = A_{j-1}^{-1}$  for each  $j \in \mathbb{Z}$ . Now we check that conditions (1), (2), and (3) of Theorem 1.1 are satisfied. Indeed, (3) holds because  $TS = I$  on  $Y_0$ . To prove (1) and (2), it is enough to check that for any  $(i, j) \in \mathbb{N} \times \mathbb{Z}$ , both  $T^k(e_{i,j})$  and  $S^k(e_{i,j})$  tend to zero as  $k \rightarrow \infty$  (then we conclude by using linearity). But this is clear, since for any  $k > 2i + 2|j| + 1$ , we have

$$\|T^k(e_{i,j})\| = \left\| \prod_{s=0}^{k-1} A_{s+j} f_i \right\|$$

$$\begin{aligned}
 &= \|A_j A_{j+1} \cdots A_{j+k-1} f_i\| \\
 &= \begin{cases} \|A_j \cdots A_i \cdots A_{j+k-1} f_i\| & \text{if } j \leq i, \\ \|A_j A_{j+1} \cdots A_{j+k-1} f_i\| & \text{if } j > i \end{cases} \\
 &= \begin{cases} \|A_j \cdots A_i (\frac{1}{2^{j+k-1-i}} f_i)\| & \text{if } j \leq i, \\ \|\frac{1}{2^k} f_i\| & \text{if } j > i \end{cases} \\
 &\leq \begin{cases} 2^{i-j} \cdot 1 \cdot \frac{1}{2^{j+k-1-i}} & \text{if } j \leq i, \\ \frac{1}{2^k} & \text{if } j > i \end{cases} \\
 &\leq 2^{|j|} \cdot 2^i \cdot 1 \cdot \frac{1}{2^{k-|j|-i-1}} = \frac{1}{2^{k-2|j|-2i-1}},
 \end{aligned}$$

and

$$\begin{aligned}
 \|S^k(e_{i,j})\| &= \left\| \prod_{s=1}^k B_{j+s-k} f_i \right\| \\
 &= \left\| \prod_{s=1}^k A_{j-s}^{-1} f_i \right\| \\
 &= \begin{cases} \|A_{j-1}^{-1} \cdots A_{j-|j|}^{-1} A_{j-(|j|+1)}^{-1} \cdots A_{j-k}^{-1} f_i\| & \text{if } j \neq 0, \\ \|A_{-1}^{-1} A_{-2}^{-1} \cdots A_{-k}^{-1} f_i\| & \text{if } j = 0 \end{cases} \\
 &= \begin{cases} \|A_{j-1}^{-1} \cdots A_{j-|j|}^{-1} (\frac{1}{2^{k-|j|}} f_i)\| & \text{if } j \neq 0, \\ \|\frac{1}{2^k} f_i\| & \text{if } j = 0 \end{cases} \\
 &\leq 2^{|j|} \cdot \frac{1}{2^{k-|j|}} = \frac{1}{2^{k-2|j|}}.
 \end{aligned}$$

However, for any  $n \in \mathbb{N}$ ,  $j \in \mathbb{Z}$

$$\left\| \prod_{s=0}^{n-1} A_{s+j} \right\| = \sup_{k \in \mathbb{N}} \left\| \prod_{s=0}^{n-1} A_{s+j} f_k \right\| = 2^n \geq 1.$$

Therefore,  $T$  does not satisfy Theorem 2.1(b). □

Now let us briefly recall the proof of “(a)  $\Rightarrow$  (b)” in article [9] and see what leads to the contradiction. Suppose that  $T$  is hypercyclic, let  $\varepsilon > 0$ ,  $q \in \mathbb{N}$  be given, and choose  $\delta > 0$  such that  $\frac{\delta}{1-\delta} < \varepsilon$ . For an arbitrarily fixed nonnegative integer  $i$ , consider the vector  $f = (\dots, a_{-1}, [a_0], a_1, \dots)$  in  $L^2(\mathcal{K})$ , where  $a_j = f_i$  for all  $|j| \leq q$  and  $a_j = 0$  if  $|j| > q$ . By the density of hypercyclic vectors, there exists a hypercyclic vector  $x = (\dots, x_{-1}, [x_0], x_1, \dots)$  such that

$$\|x - f\| < \delta. \tag{2.2}$$

Also, since  $\text{orb}(T, x)$  is dense in  $L^2(\mathcal{K})$ , there exists an arbitrarily large integer  $n$  (choose  $n > 2q$ ) such that

$$\|T^n x - f\| < \delta. \tag{2.3}$$

From (2.2), (2.3), and some details (which we omit here; interested readers are referred to [9]), we get that, for all  $|j| \leq q$ ,

$$\begin{cases} \|\prod_{s=0}^{n-1} A_{j+s} f_i\| < \frac{\delta}{1-\delta}, \\ \|\prod_{s=1}^n A_{j-s} f_i\| > \frac{1-\delta}{\delta}. \end{cases} \tag{2.4}$$

Then the authors of [9] claim that, for all  $|j| \leq q$ ,

$$\begin{cases} \|\prod_{s=0}^{n-1} A_{j+s}\| = \sup_i \|\prod_{s=0}^{n-1} A_{j+s} f_i\| < \frac{\delta}{1-\delta}, \\ \inf_i \|\prod_{s=1}^n A_{j-s} f_i\| > \frac{1-\delta}{\delta}, \end{cases} \tag{2.5}$$

since  $i$  is arbitrarily fixed. The contradiction lies in the claims that

$$\sup_i \left\| \prod_{s=0}^{n-1} A_{j+s} f_i \right\| < \frac{\delta}{1-\delta} \quad \text{and} \quad \inf_i \left\| \prod_{s=1}^n A_{j-s} f_i \right\| > \frac{1-\delta}{\delta}.$$

Since the selection of the integer  $n$  depends on  $f_i$ , we cannot obtain (2.5) directly from (2.4).

We improve Theorem 2.1 with the following result.

**Theorem 2.3.** *Let  $T$  be a bilateral forward operator-weighted shift on  $L^2(\mathcal{K})$  with weight sequence  $\{A_n\}_{n=-\infty}^{\infty}$ , where  $\{A_n\}_{n=-\infty}^{\infty}$  is a uniformly bounded sequence of positive invertible diagonal operators on  $\mathcal{K}$ . Then  $T$  is hypercyclic if and only if, given  $\varepsilon > 0$ ,  $q \in \mathbb{N}$ , and  $K \in \mathbb{N}$ , there exists  $n$  arbitrarily large such that, for all  $|j| \leq q$ ,*

$$\left\| \prod_{s=0}^{n-1} A_{s+j}|_{X_K} \right\| < \varepsilon \quad \text{and} \quad \left\| \prod_{s=1}^n A_{j-s}^{-1}|_{X_K} \right\| < \varepsilon, \tag{*}$$

where  $X_K = \text{span}\{f_0, f_1, f_2, \dots, f_K\}$  and  $\{f_k\}_{k=0}^{\infty}$  is the orthonormal basis of  $\mathcal{K}$ .

*Proof.* Suppose that  $T$  is hypercyclic. Let  $\varepsilon > 0$ ,  $q \in \mathbb{N}$ , and  $K \in \mathbb{N}$  be given, and choose  $0 < \delta < 1$  such that  $\frac{\delta}{1-\delta} < \varepsilon$ . Consider the vector  $f = f_0 + f_1 + \dots + f_K$  in  $X_K$ . For any  $i \in \mathbb{Z}$ , set  $f(i) := (\dots, y_{-1}, [y_0], y_1, \dots)$ , where  $y_i = f$  and  $y_j = 0$  for all  $j \neq i$ . Since the hypercyclic vectors are dense in  $L^2(\mathcal{K})$ , one may find a hypercyclic vector  $x = (\dots, x_{-1}, [x_0], x_1, \dots)$  and an integer  $n > 2q$  such that

$$\left\| x - \sum_{|j| \leq q} f(j) \right\| < \delta \quad \text{and} \quad \left\| T^n x - \sum_{|j| \leq q} f(j) \right\| < \delta. \tag{2.6}$$

Looking at the first inequality of (2.6), we get

$$\begin{cases} \text{(i) } \|x_j\| < \delta & \text{for } |j| > q, \\ \text{(ii) } \|x_j - f\| < \delta & \text{for } |j| \leq q. \end{cases} \tag{2.7}$$

Since each  $x_j$  is in  $\mathcal{K}$ ,  $x_j$  can be written as  $x_j = \sum_{k=0}^{\infty} \alpha_k^{(j)} f_k$ , where  $\alpha_k^{(j)} = \langle x_j, f_k \rangle$ . It follows from (2.7) that

$$\begin{cases} \text{if } |j| > q: & |\alpha_k^{(j)}| < \delta \quad \text{for all } k, \\ \text{if } |j| \leq q: & \begin{cases} |\alpha_k^{(j)}| < \delta & \text{for } k > K, \\ |\alpha_k^{(j)}| > 1 - \delta & \text{for } k \leq K. \end{cases} \end{cases} \tag{2.8}$$

Now we denote

$$T^n x = (\dots, y_{-1}, [y_0], y_1, \dots),$$

where

$$y_j = \prod_{s=0}^{n-1} A_{j+s-n} x_{j-n} \quad \text{or} \quad y_{n+j} = \prod_{s=0}^{n-1} A_{j+s} x_j = \sum_{k=0}^{\infty} \alpha_k^{(j)} \prod_{s=0}^{n-1} A_{j+s} f_k.$$

The second inequality of (2.6) gives

$$\begin{cases} \text{(i)} \ \|y_j\| < \delta & \text{for } |j| > q, \\ \text{(ii)} \ \|y_j - f\| < \delta & \text{for } |j| \leq q. \end{cases} \quad (2.9)$$

By the hypothesis  $n > 2q$ , we have  $j+n > q$  for all  $|j| \leq q$ , and hence  $\|y_{j+n}\| < \delta$  for all  $|j| \leq q$ . Moreover, as  $y_{j+n} = \prod_{s=0}^{n-1} A_{j+s} x_j = \sum_{k=0}^{\infty} \alpha_k^{(j)} \prod_{s=0}^{n-1} A_{j+s} f_k$ , it follows that  $|\alpha_k^{(j)}| \|\prod_{s=0}^{n-1} A_{j+s} f_k\| < \delta$  for all  $k \in \mathbb{N}$  and  $|j| \leq q$ . Then from (2.8), we have  $\|\prod_{s=0}^{n-1} A_{j+s} f_k\| < \frac{\delta}{1-\delta}$  for  $k = 0, 1, \dots, K$  and  $|j| \leq q$ . Since for each  $m \in \mathbb{Z}$ ,  $A_m$  is diagonal on  $\mathcal{K}$ ,  $X_K$  is an invariant subspace of operator  $\prod_{s=0}^{n-1} A_{j+s}$ . Thus, we have

$$\left\| \prod_{s=0}^{n-1} A_{s+j}|_{X_K} \right\| = \sup_{0 \leq i \leq K} \left\| \prod_{s=0}^{n-1} A_{j+s} f_i \right\| < \frac{\delta}{1-\delta} < \varepsilon \quad \text{for all } |j| \leq q.$$

Again from  $x_{j-n} = \sum_{k=0}^{\infty} \alpha_k^{(j-n)} f_k$ , we can get

$$y_j = \prod_{s=0}^{n-1} A_{j+s-n} x_{j-n} = \sum_{k=0}^{\infty} \alpha_k^{(j-n)} \prod_{s=0}^{n-1} A_{j+s-n} f_k.$$

Therefore, (ii) of (2.9) gives

$$\begin{cases} \text{(i)} \ \|\alpha_k^{(j-n)} \prod_{s=0}^{n-1} A_{j+s-n} f_k - f_k\| < \delta & \text{for } k \leq K, |j| \leq q, \\ \text{(ii)} \ \|\alpha_k^{(j-n)} \prod_{s=0}^{n-1} A_{j+s-n} f_k\| < \delta & \text{for } k > K, |j| \leq q. \end{cases} \quad (2.10)$$

The hypothesis  $n > 2q$  implies that  $|j-n| > q$  for all  $|j| \leq q$ ; hence by (2.8)  $|\alpha_k^{(j-n)}| < \delta$  for all  $k \in \mathbb{N}$  and  $|j| \leq q$ . So from (i) of (2.10),

$$\left\| \prod_{s=0}^{n-1} A_{j+s-n} f_k \right\| > \frac{1-\delta}{\delta} \quad \text{when } k \leq K, |j| \leq q.$$

Then it follows that

$$\inf_{0 \leq i \leq K} \left\| \prod_{s=0}^{n-1} A_{j+s-n} f_i \right\| > \frac{1-\delta}{\delta} \quad \text{for all } |j| \leq q,$$

which implies that for  $|j| \leq q$ ,

$$\sup_{0 \leq i \leq K} \left\| \prod_{s=1}^n A_{j-s}^{-1} f_i \right\| = \frac{1}{\inf_{0 \leq i \leq K} \left\| \prod_{s=1}^n A_{j-s} f_i \right\|}$$

$$\begin{aligned}
 &= \frac{1}{\inf_{0 \leq i \leq K} \left\| \prod_{s=0}^{n-1} A_{j+s-n} f_i \right\|} \\
 &< \frac{\delta}{1-\delta}.
 \end{aligned}$$

Hence for all  $|j| \leq q$ ,

$$\left\| \prod_{s=1}^n A_{j-s}^{-1} |_{X_K} \right\| = \sup_{0 \leq i \leq K} \left\| \prod_{s=1}^n A_{j-s}^{-1} f_i \right\| < \frac{\delta}{1-\delta} < \varepsilon.$$

Conversely, suppose that  $(*)$  holds. Let us show that  $T$  satisfies the hypercyclicity criterion. For each  $k \in \mathbb{N}$  with  $k \geq 1$ , let

$$\varepsilon = \frac{1}{k}, \quad q = k, \quad K = k \quad \text{and} \quad X_k = \text{span}\{f_0, f_1, f_2, \dots, f_k\}.$$

By  $(*)$ , one can find an increasing sequence  $(n_k)_k$  of positive integers such that, for all  $|j| \leq k$ ,

$$\left\| \prod_{s=0}^{n_k-1} A_{s+j} |_{X_k} \right\| < \frac{1}{k} \quad \text{and} \quad \left\| \prod_{s=1}^{n_k} A_{j-s}^{-1} |_{X_k} \right\| < \frac{1}{k}.$$

For any  $m, n \in \mathbb{N}$ , we set

$$H_{m,n} := \{x = (x_i) \in L^2(\mathcal{K}) : x_i \in X_n \text{ when } |i| \leq m, \text{ and } x_i = 0 \text{ when } |i| > m\}.$$

Now let  $X_0 = Y_0 = \bigcup_{m,n \in \mathbb{N}} H_{m,n}$ , which is a dense subset of  $L^2(\mathcal{K})$ , and let  $S : Y_0 \rightarrow L^2(\mathcal{K})$  be the mapping defined by

$$S(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, A_{-1}^{-1}x_0, [A_0^{-1}x_1], A_1^{-1}x_2, \dots).$$

Note that  $TS = I$  on  $Y_0$ . We just need to prove that for any  $g \in X_0 = Y_0$ , both  $T^{n_k}(g)$  and  $S^{n_k}(g)$  tend to zero as  $k \rightarrow \infty$ . Now fix  $g \in X_0 = Y_0$ . By definition, we can find  $N \in \mathbb{N}$  such that  $g \in H_{N,N}$ . Thus there exist vectors  $g_{-N}, g_{-N+1}, \dots, g_{-1}, g_0, g_1, \dots, g_N$  in  $X_N$  such that

$$g = (\dots, 0, 0, g_{-N}, g_{-N+1}, \dots, g_{-1}, [g_0], g_1, \dots, g_N, 0, 0, \dots).$$

So for each  $k \geq 1$ ,

$$T^{n_k}g = (\dots, y_{-1}, [y_0], y_1, \dots),$$

where

$$y_{n_k+j} = \prod_{s=0}^{n_k-1} A_{s+j} g_j \quad \text{or} \quad y_j = \prod_{s=0}^{n_k-1} A_{s+j-n_k} g_{j-n_k},$$

and for each  $k \geq 1$ ,

$$S^{n_k}g = (\dots, z_{-1}, [z_0], z_1, \dots),$$

where

$$z_{j-n_k} = \prod_{s=1}^{n_k} A_{j-s}^{-1} g_j \quad \text{or} \quad z_j = \prod_{s=1}^{n_k} A_{j-s+n_k}^{-1} g_{j+n_k}.$$

Since  $g \in H_{N,N}$ , for each  $k > N$  we have

$$\begin{aligned} \|T^{n_k}g\| &\leq \sup_{|j|\leq N} \left\| \prod_{s=0}^{n_k-1} A_{s+j}|_{X_N} \right\| \|g\| \\ &\leq \sup_{|j|\leq k} \left\| \prod_{s=0}^{n_k-1} A_{s+j}|_{X_k} \right\| \|g\| \\ &\leq \frac{1}{k} \|g\| \xrightarrow{k \rightarrow \infty} 0 \end{aligned}$$

and

$$\begin{aligned} \|S^{n_k}g\| &\leq \sup_{|j|\leq N} \left\| \prod_{s=1}^{n_k} A_{j-s}^{-1}|_{X_N} \right\| \|g\| \\ &\leq \sup_{|j|\leq k} \left\| \prod_{s=1}^{n_k} A_{j-s}^{-1}|_{X_k} \right\| \|g\| \\ &\leq \frac{1}{k} \|g\| \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

Thus  $T$  is hypercyclic. □

By the process of the above proof, we can also deduce the following two equivalent conditions for hypercyclicity of operator-weighted shifts.

**Theorem 2.4.** *Let  $T$  be a bilateral forward operator-weighted shift on  $L^2(\mathcal{K})$  with weight sequence  $\{A_n\}_{n=-\infty}^\infty$ , where  $\{A_n\}_{n=-\infty}^\infty$  is a uniformly bounded sequence of positive invertible diagonal operators on  $\mathcal{K}$ . Then  $T$  is hypercyclic if and only if there exists an increasing sequence of positive integers  $(n_k)_k$  such that*

$$\lim_{k \rightarrow \infty} \max \left\{ \left\| \prod_{s=0}^{n_k-1} A_{s+j}|_{X_k} \right\|, |j| \leq k \right\} = 0$$

and

$$\lim_{k \rightarrow \infty} \max \left\{ \left\| \prod_{s=1}^{n_k} A_{j-s}^{-1}|_{X_k} \right\|, |j| \leq k \right\} = 0,$$

where  $X_k = \text{span}\{f_0, f_1, f_2, \dots, f_k\}$ .

**Theorem 2.5.** *Let  $T$  be a bilateral forward operator-weighted shift on  $L^2(\mathcal{K})$  with weight sequence  $\{A_n\}_{n=-\infty}^\infty$ , where  $\{A_n\}_{n=-\infty}^\infty$  is a uniformly bounded sequence of positive invertible diagonal operators on  $\mathcal{K}$ . Let  $(n_k)_{k \geq 1}$  be an increasing sequence of positive integers. Then  $\{T^{n_k}\}_{k \geq 1}$  is hypercyclic if and only if, given  $\varepsilon > 0$ ,  $q \in \mathbb{N}$ , and  $K \in \mathbb{N}$ , there exists  $n \in (n_k)_{k \geq 1}$  such that for all  $|j| \leq q$ ,*

$$\left\| \prod_{s=0}^{n-1} A_{s+j}|_{X_K} \right\| < \varepsilon \quad \text{and} \quad \left\| \prod_{s=1}^n A_{j-s}^{-1}|_{X_K} \right\| < \varepsilon,$$

where  $X_K = \text{span}\{f_0, f_1, f_2, \dots, f_K\}$ .

To illustrate Theorem 2.3, let us see how it applies to Example 2.2.

*Example 2.6.* Let  $\{A_n\}_{n=-\infty}^\infty$  be the uniformly bounded sequence of positive invertible diagonal operators on  $\mathcal{K}$  given in Example 2.2. Let  $T$  be the bilateral (forward) operator-weighted shift on  $L^2(\mathcal{K})$  with weight sequence  $\{A_n\}_{n=-\infty}^\infty$ . Then  $T$  is hypercyclic.

*Proof.* Let  $\varepsilon > 0$ ,  $q \in \mathbb{N}$ , and  $K \in \mathbb{N}$  be given. Set  $X_K = \text{span}\{f_0, f_1, f_2, \dots, f_K\}$ . Then for all  $|j| \leq q$  and  $n \in \mathbb{N}$  with  $n > 2q + 2K + 1$ , by the definition of  $\{A_n\}_{n=-\infty}^\infty$  we have

$$\left\| \prod_{s=0}^{n-1} A_{s+j}|_{X_K} \right\| = \sup_{0 \leq k \leq K} \left\| \prod_{s=0}^{n-1} A_{s+j} f_k \right\| \leq 2^q \cdot 2^K \cdot 1 \cdot \frac{1}{2^{n-q-K-1}} = \frac{1}{2^{n-2q-2K-1}}$$

and

$$\left\| \prod_{s=1}^n A_{j-s}^{-1}|_{X_K} \right\| = \sup_{0 \leq k \leq K} \left\| \prod_{s=1}^n A_{j-s}^{-1} f_k \right\| \leq 2^q \cdot \frac{1}{2^{n-q}} = \frac{1}{2^{n-2q}}.$$

Since  $\lim_{n \rightarrow \infty} \frac{1}{2^{n-2q-2K-1}} = 0$  and  $\lim_{n \rightarrow \infty} \frac{1}{2^{n-2q}} = 0$ , there exists  $n \in \mathbb{N}$  such that

$$\left\| \prod_{s=0}^{n-1} A_{s+j}|_{X_K} \right\| < \varepsilon \quad \text{and} \quad \left\| \prod_{s=1}^n A_{j-s}^{-1}|_{X_K} \right\| < \varepsilon.$$

By Theorem 2.3,  $T$  is hypercyclic. □

### 3. Supercyclic and hereditarily hypercyclic operator-weighted shift

Liang and Zhou [11] investigated the hereditarily hypercyclic and supercyclic operator-weighted shifts on  $L^2(\mathcal{K})$ , respectively. Their theorems are listed below.

**Theorem 3.1** ([11, Theorem 3.2]). *Let  $T$  be a forward bilateral operator-weighted shift on  $X = L^2(\mathcal{K})$  with weight sequence  $\{A_n\}_{n=-\infty}^\infty$ , where  $\{A_n\}_{n=-\infty}^\infty$  is a uniformly bounded sequence of positive invertible diagonal operators on  $\mathcal{K}$ . Then  $T$  is supercyclic if and only if for every  $q \in \mathbb{N}$ ,*

$$\liminf_{n \rightarrow \infty} \max \left\{ \left\| \prod_{k=j}^{j+n-1} A_k \right\| \left\| \prod_{k=h-n}^{h-1} A_k^{-1} \right\| : |j|, |h| \leq q \right\} = 0.$$

**Theorem 3.2** ([11, Theorem 2.6]). *Let  $T$  be a forward bilateral operator-weighted shift on  $L^2(\mathcal{K})$  with weight sequence  $\{A_n\}_{n=-\infty}^\infty$ , where  $\{A_n\}_{n=-\infty}^\infty$  is a uniformly bounded sequence of positive invertible diagonal operators on  $\mathcal{K}$  and  $\{A_n^{-1}\}_{n=-\infty}^\infty$  is also a uniformly bounded sequence. Also let  $(n_k)_k \subset \mathbb{N}$ . Then the following are equivalent.*

- (1)  $T$  is hereditarily hypercyclic with respect to  $(n_k)_k$ .
- (2) For all  $\varepsilon > 0$  and  $q \in \mathbb{N}$ , there exists  $k_0 \in \mathbb{N}$  satisfying: for all  $k \geq k_0$  and all  $|j| \leq q$ ,

$$\begin{cases} \text{(i)} \quad \left\| \prod_{s=0}^{n_k-1} A_{s+j} \right\| < \varepsilon, \\ \text{(ii)} \quad \left\| \prod_{s=1}^{n_k} A_{j-s}^{-1} \right\| < \varepsilon. \end{cases}$$

(3) For all  $j \in \mathbb{Z}$ ,

$$\lim_{k \rightarrow \infty} \left\| \prod_{s=0}^{j+n_k-1} A_s \right\| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \left\| \prod_{s=1}^{j+n_k} A_{-s}^{-1} \right\| = 0.$$

By the Bès–Peris theorem [6, Theorem 2.3],  $T$  is hereditarily hypercyclic if and only if  $T$  satisfies the hypercyclicity criterion. It follows that the operator in Example 2.2 is also hereditarily hypercyclic and supercyclic, but fails to satisfy the characterizations in the two preceding theorems. Therefore, the conditions in Theorem 3.1 and Theorem 3.2 are not equivalent to supercyclicity and hereditary hypercyclicity, respectively.

Referencing the proof of Theorem 2.3, the improvements of the above two theorems are listed below.

**Theorem 3.3.** *Let  $T$  be a forward bilateral operator-weighted shift on  $X = L^2(\mathcal{K})$  with weight sequence  $\{A_n\}_{n=-\infty}^\infty$ , where  $\{A_n\}_{n=-\infty}^\infty$  is a uniformly bounded sequence of positive invertible diagonal operators on  $\mathcal{K}$ . Then  $T$  is supercyclic if and only if, for every  $q \in \mathbb{N}$  and  $K \in \mathbb{N}$ ,*

$$\liminf_{n \rightarrow \infty} \max \left\{ \left\| \prod_{k=j}^{j+n-1} A_k|_{X_K} \right\|, \left\| \prod_{k=h-n}^{h-1} A_k^{-1}|_{X_K} \right\| : |j|, |h| \leq q \right\} = 0,$$

where  $X_K = \text{span}\{f_0, f_1, f_2, \dots, f_K\}$ .

**Theorem 3.4.** *Let  $T$  be a forward bilateral operator-weighted shift on  $L^2(\mathcal{K})$  with weight sequence  $\{A_n\}_{n=-\infty}^\infty$ , where  $\{A_n\}_{n=-\infty}^\infty$  is a uniformly bounded sequence of positive invertible diagonal operators on  $\mathcal{K}$  and  $\{A_n^{-1}\}_{n=-\infty}^\infty$  is also a uniformly bounded sequence. Also let  $(n_k)_k \subset \mathbb{N}$ . Then the following are equivalent.*

- (1)  $T$  is hereditarily hypercyclic with respect to  $(n_k)_k$ .
- (2) For any  $\varepsilon > 0$ ,  $q \in \mathbb{N}$ , and  $K \in \mathbb{N}$ , there exists  $k_0 \in \mathbb{N}$  satisfying, for all  $k \geq k_0$  and all  $|j| \leq q$ ,

$$\begin{cases} \text{(i)} \quad \left\| \prod_{s=0}^{n_k-1} A_{s+j}|_{X_K} \right\| < \varepsilon, \\ \text{(ii)} \quad \left\| \prod_{s=1}^{n_k} A_{j-s}^{-1}|_{X_K} \right\| < \varepsilon, \end{cases}$$

where  $X_K := \text{span}\{f_0, f_1, f_2, \dots, f_K\}$ .

- (3) For any  $j \in \mathbb{Z}$  and  $K \in \mathbb{N}$ ,

$$\lim_{k \rightarrow \infty} \left\| \prod_{s=0}^{j+n_k-1} A_s|_{X_K} \right\| = 0$$

and

$$\lim_{k \rightarrow \infty} \left\| \prod_{s=1}^{j+n_k} A_{-s}^{-1}|_{X_K} \right\| = 0,$$

where  $X_K = \text{span}\{f_0, f_1, f_2, \dots, f_K\}$ .

**Acknowledgments.** We would like to thank the referee whose thoughtful comments led to an improvement of exposition.

Wang's work was partially supported by National Natural Science Foundation of China (NSFC) grant 11771323. Zhou's work was partially supported by NSFC grant 11371276.

## References

1. F. Bayart and É. Matheron, *Hypercyclic operators failing the hypercyclicity criterion on classical Banach spaces*, J. Funct. Anal. **250** (2007), no. 2, 426–441. [Zbl 1131.47006](#). [MR2352487](#). [DOI 10.1016/j.jfa.2007.05.001](#). [323](#)
2. F. Bayart and É. Matheron, *Dynamics of Linear Operators*, Cambridge Tracts in Math. **179**, Cambridge Univ. Press, Cambridge, 2009. [Zbl 1187.47001](#). [MR2533318](#). [DOI 10.1017/CBO9780511581113](#). [322](#), [323](#)
3. J. Bès, *Three problems on hypercyclic operators*, Ph.D. dissertation, Kent State University, Kent, OH, 1998. [MR2697588](#). [323](#)
4. J. Bès, Ö. Martin, A. Peris, and S. Shkarin, *Disjoint mixing operators*, J. Funct. Anal. **263** (2012), no. 5, 1283–1322. [Zbl 1266.47013](#). [MR2943730](#). [DOI 10.1016/j.jfa.2012.05.018](#). [323](#)
5. J. Bès, Ö. Martin, and R. Sanders, *Weighted shifts and disjoint hypercyclicity*, J. Operator Theory **72** (2014), no. 1, 15–40. [Zbl 1318.47010](#). [MR3246979](#). [DOI 10.7900/jot.2012aug20.1970](#). [323](#)
6. J. Bès and A. Peris, *Hereditarily hypercyclic operators*, J. Funct. Anal. **167** (1999), no. 1, 94–112. [Zbl 0941.47002](#). [MR1710637](#). [DOI 10.1006/jfan.1999.3437](#). [323](#), [332](#)
7. G. D. Birkhoff, *Surface transformations and their dynamical applications*, Acta Math. **43** (1922), no. 1, 1–119. [JFM 47.0985.03](#). [MR1555175](#). [DOI 10.1007/BF02401754](#). [322](#)
8. K.-G. Grosse-Erdmann and A. Peris Manguillot, *Linear Chaos*, Universitext, Springer, London, 2011. [Zbl 1246.47004](#). [MR2919812](#). [DOI 10.1007/978-1-4471-2170-1](#). [322](#)
9. M. Hazarika and S. C. Arora, *Hypercyclic operator-weighted shifts*, Bull. Korean Math. Soc. **41** (2004), no. 4, 589–598. [Zbl 1079.47011](#). [MR2103108](#). [DOI 10.4134/BKMS.2004.41.4.589](#). [323](#), [325](#), [326](#), [327](#)
10. C. Kitai, *Invariant closed sets for linear operators*, Ph.D. dissertation, University of Toronto, Ontario, Canada, 1982. [MR2632793](#). [322](#)
11. Y. Liang and Z. Zhou, *Hereditarily hypercyclicity and supercyclicity of weighted shifts*, J. Korean Math. Soc. **51** (2014), no. 2, 363–382. [Zbl 1292.47007](#). [MR3178589](#). [DOI 10.4134/JKMS.2014.51.2.363](#). [323](#), [331](#)
12. S. Rolewicz, *On orbits of elements*, Studia Math. **32** (1969), 17–22. [Zbl 0174.44203](#). [MR0241956](#). [322](#)
13. H. N. Salas, *Hypercyclic weighted shifts*, Trans. Amer. Math. Soc. **347**, no. 3 (1995), 993–1004. [Zbl 0822.47030](#). [MR1249890](#). [DOI 10.2307/2154883](#). [323](#)

SCHOOL OF MATHEMATICS, TIANJIN UNIVERSITY, TIANJIN 300354, PEOPLE'S REPUBLIC OF CHINA

*E-mail address:* [wangyasjxsy0802@163.com](mailto:wangyasjxsy0802@163.com); [zehuazhoumath@aliyun.com](mailto:zehuazhoumath@aliyun.com); [zhzhou@tju.edu.cn](mailto:zhzhou@tju.edu.cn)