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EQUIVALENT PROPERTIES OF A HILBERT-TYPE INTEGRAL INEQUALITY WITH THE BEST CONSTANT FACTOR RELATED TO THE HURWITZ ZETA FUNCTION

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ABSTRACT. By the use of methods of real analysis and weight functions, we study the equivalent properties of a Hilbert-type integral inequality with the nonhomogeneous kernel. The constant factor related to the Hurwitz zeta function is proved to be the best possible. As a corollary, a few equivalent conditions of a Hilbert-type integral inequality with the homogeneous kernel are deduced. We also consider their operator expressions.

1. INTRODUCTION

In 1925, by introducing one pair of conjugate exponents (p, q), Hardy in [2] proved the following integral inequality. For

$$p > 1, \qquad \frac{1}{p} + \frac{1}{q} = 1, \qquad f(x), g(y) \geq 0,$$

$$0 < \int_0^\infty f^p(x)\, dx < \infty \qquad \text{and} \qquad 0 < \int_0^\infty g^q(y)\, dy < \infty,$$

we have the following Hardy–Hilbert inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\, dx\, dy < \frac{\pi}{\sin(\pi/p)} \Big(\int_0^\infty f^p(x)\, dx \Big)^{\frac{1}{p}} \Big(\int_0^\infty g^q(y)\, dy \Big)^{\frac{1}{q}}, \qquad (1.1)$$

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where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. For $p = q = 2$, (1.1) reduces to the well-known Hilbert integral inequality. Inequality (1.1) as well as the Hilbert integral inequality are important in mathematical analysis and its applications (see [3], [10]).

In 1934, Hardy et al. presented an extension of (1.1) as follows. If $k_1(x, y)$ is a nonnegative homogeneous function of degree -1,

$$k_p = \int_0^\infty k_1(u, 1)u^{\frac{-1}{p}}\, du \in \mathbf{R}_+ = (0, \infty),$$

then we have the following Hardy–Hilbert-type integral inequality:

$$\int_0^\infty \int_0^\infty k_1(x, y)f(x)g(y)\, dx\, dy < k_p\Big(\int_0^\infty f^p(x)\, dx\Big)^{\frac{1}{p}}\Big(\int_0^\infty g^q(y)\, dy\Big)^{\frac{1}{q}}, \quad (1.2)$$

where the constant factor k_p is the best possible (see [3, Theorem 319]). Additionally, the following Hilbert-type integral inequality with the nonhomogeneous kernel was proved: if $h(u) > 0, \phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}\, du \in \mathbf{R}_+$, then

$$\begin{aligned}\int_0^\infty \int_0^\infty & h(xy)f(x)g(y)\, dx\, dy \\ &< \phi\Big(\frac{1}{p}\Big)\Big(\int_0^\infty x^{p-2}f^p(x)\, dx\Big)^{\frac{1}{p}}\Big(\int_0^\infty g^q(y)\, dy\Big)^{\frac{1}{q}}, \end{aligned} \quad (1.3)$$

where the constant factor $\phi(\frac{1}{p})$ is the best possible (see [3, Theorem 350]). In 1998, by introducing an independent parameter $\lambda > 0$, Yang [13], [14] extended the Hilbert integral inequality, proving that

$$\begin{aligned}\int_0^\infty \int_0^\infty & \frac{f(x)g(y)}{(x+y)^\lambda}\, dx\, dy \\ &< B\Big(\frac{\lambda}{2}, \frac{\lambda}{2}\Big)\Big(\int_0^\infty x^{1-\lambda}f^2(x)\, dx \int_0^\infty y^{1-\lambda}g^2(y)\, dy\Big)^{\frac{1}{2}}, \end{aligned} \quad (1.4)$$

where the constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ is the best possible ($B(u, v)$ is the beta function). In 2004, by introducing another pair of conjugate exponents (r, s), Yang in [18] presented the following extension of (1.1): if $\lambda > 0, r > 1, f(x), g(y) \geq 0$,

$$\frac{1}{r} + \frac{1}{s} = 1,$$

$$0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)\, dx < \infty \qquad \text{and} \qquad 0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)\, dy < \infty,$$

then

$$\begin{aligned}\int_0^\infty \int_0^\infty & \frac{f(x)g(y)}{x^\lambda + y^\lambda}\, dx\, dy \\ &< \frac{\pi}{\lambda \sin(\pi/r)}\Big[\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)\, dx\Big]^{\frac{1}{p}}\Big[\int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)\, dy\Big]^{\frac{1}{q}}, \end{aligned} \quad (1.5)$$

where the constant factor $\frac{\pi}{\lambda\sin(\pi/r)}$ is the best possible. For $\lambda = 1, r = q$, and $s = p$, (1.5) reduces to (1.1); for $\lambda = 1, r = p$, and $s = q$, (1.5) reduces to the dual form of (1.1); namely,

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\Big(\int_0^\infty x^{p-2}f^p(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_0^\infty y^{q-2}g^q(y)\,dy\Big)^{\frac{1}{q}}, \tag{1.6}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is still the best possible.

In 2005, Yang et al. [19] proved an extension of (1.1) and (1.4) with the kernel $\frac{1}{(x+y)^\lambda}$ and two pairs of conjugate exponents. Various authors (see [1], [4], [6], [9], [12], [20]) provided some extensions and particular cases of (1.1), (1.2), and (1.3) with parameters. In 2009, Yang [15], [16] gave an extension of (1.2), (1.4), and (1.5) as follows. If $\lambda_1 + \lambda_2 = \lambda \in \mathbf{R} = (-\infty, \infty)$, $k_\lambda(x, y)$, is a nonnegative homogeneous function of degree $-\lambda$ satisfying

$$k_\lambda(ux, uy) = u^{-\lambda}k_\lambda(x, y) \quad (u, x, y > 0)$$

and

$$k(\lambda_1) = \int_0^\infty k_\lambda(u, 1)u^{\lambda_1-1}\,du \in \mathbf{R}_+ = (0, \infty),$$

then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x, y)f(x)g(y)\,dx\,dy < k(\lambda_1)\Big(\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_0^\infty y^{q(1-\lambda_2)-1}g^q(y)\,dy\Big)^{\frac{1}{q}}, \tag{1.7}$$

where the constant factor $k(\lambda_1)$ is the best possible. For $\lambda = 1, \lambda_1 = \frac{1}{q}$, and $\lambda_2 = \frac{1}{p}$, we have that (1.7) reduces to (1.2); for

$$p = q = 2, \qquad \lambda_1 = \lambda_2 = \frac{\lambda}{2} > 0, \qquad k_\lambda(x, y) = \frac{1}{(x+y)^\lambda},$$

(1.7) reduces to (1.4); for

$$\lambda > 0, \qquad \lambda_1 = \frac{\lambda}{r}, \qquad \lambda_2 = \frac{\lambda}{s}, \qquad k_\lambda(x, y) = \frac{1}{x^\lambda + y^\lambda},$$

(1.7) reduces to (1.5). Moreover, the following extension of (1.3) was proved (see [17]):

$$\int_0^\infty \int_0^\infty h(xy)f(x)g(y)\,dx\,dy < \phi(\sigma)\Big(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)\,dx\Big)^{\frac{1}{p}}\Big(\int_0^\infty y^{q(1-\sigma)-1}g^q(y)\,dy\Big)^{\frac{1}{q}}, \tag{1.8}$$

where the constant factor $\phi(\sigma)$ is the best possible. For $\sigma = \frac{1}{p}$, (1.8) reduces to (1.3). Some equivalent inequalities of (1.7) and (1.8) were established in [15]. In 2013, Yang [17] also studied the equivalency between (1.7) and (1.8) under the additional condition $h(u) = k_\lambda(u, 1)$. In 2017, Hong [5] considered an equivalent condition between (1.7) and a few parameters.

In the present paper, by the use of methods of real analysis and weight functions, we study the equivalent properties of a Hilbert-type integral inequality with the nonhomogeneous kernel

$$e^{\alpha xy} \operatorname{csc} h(xy) \quad (\alpha < 1).$$

The constant factor related to the Hurwitz zeta function is proved to be the best possible. As a corollary, a few equivalent conditions of a Hilbert-type integral inequality with the homogeneous kernel are deduced. We also consider their operator expressions.

2. An example and a lemma

Example 2.1. For $\alpha < 1$, we set

$$h(u) := e^{\alpha u} \operatorname{csc} h(u) = \frac{2e^{\alpha u}}{e^u - e^{-u}} \quad (u > 0),$$

where

$$\operatorname{csc} h(u) = \frac{2}{e^u - e^{-u}}$$

is the hyperbolic cosecant function (see [21]). For $\sigma > 1$, by the Lebesgue term by the term-integration theorem (see [8]), we derive that

$$\begin{aligned} k(\sigma, \alpha) &:= \int_0^\infty e^{\alpha u} \operatorname{csc} h(u) u^{\sigma-1}\, du \\ &= \int_0^\infty \frac{2e^{\alpha u} u^{\sigma-1}}{e^u - e^{-u}}\, du = \int_0^\infty \frac{2u^{\sigma-1} e^{(\alpha-1)u}}{1 - e^{-2u}}\, du \\ &= 2\int_0^\infty u^{\sigma-1} \sum_{k=0}^\infty e^{-(2k-\alpha+1)u}\, du = 2\sum_{k=0}^\infty \int_0^\infty u^{\sigma-1} e^{-(2k-\alpha+1)u}\, du. \end{aligned}$$

Setting $v = (2k - \alpha + 1)u$ in the above integral, we obtain

$$\begin{aligned} k(\sigma, \alpha) &= 2\int_0^\infty v^{\sigma-1} e^{-v}\, dv \sum_{k=0}^\infty \frac{1}{(2k - \alpha + 1)^\sigma} \\ &= 2^{1-\sigma} \Gamma(\sigma) \zeta\Big(\sigma, \frac{1-\alpha}{2}\Big) \in \mathbf{R}_+, \end{aligned} \tag{2.1}$$

where

$$\Gamma(\eta) := \int_0^\infty v^{\eta-1} e^{-v}\, dv \quad (\eta > 0)$$

is the gamma function and where

$$\zeta(\sigma, a) := \sum_{k=0}^\infty \frac{1}{(k+a)^\sigma} \quad (a > 0, \sigma > 1)$$

is the Hurwitz zeta function (in particular, $\zeta(\sigma) := \zeta(\sigma, 1) = \sum_{k=1}^\infty \frac{1}{k^\sigma}$ is the Riemann zeta function) (see [11]).

For $\alpha = 0$, we have

$$
\begin{aligned}
k(\sigma, 0) &= 2^{1-\sigma}\Gamma(\sigma)\zeta\Big(\sigma, \frac{1}{2}\Big) = 2\Gamma(\sigma)\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{\sigma}} \\
&= 2\Gamma(\sigma)\Big[\sum_{k=1}^{\infty}\frac{1}{k^{\sigma}} - \sum_{k=1}^{\infty}\frac{1}{(2k)^{\sigma}}\Big] \\
&= 2\Gamma(\sigma)\Big(1 - \frac{1}{2^{\sigma}}\Big)\zeta(\sigma);
\end{aligned}
$$

for $\alpha = -1$, we get

$$
k(\sigma, -1) = 2^{1-\sigma}\Gamma(\sigma)\zeta(\sigma, 1) = 2^{1-\sigma}\Gamma(\sigma)\zeta(\sigma).
$$

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha < 1, \sigma > 1, \sigma_1 \in \mathbf{R}$, for $n \in \mathbf{N} = \{1, 2, \dots\}$, then we define the following two expressions:

$$
I_1 := \int_1^{\infty}\Big(\int_0^1 e^{\alpha xy}\csc h(xy)x^{\sigma+\frac{1}{pn}-1}\,dx\Big)y^{\sigma_1-\frac{1}{qn}-1}\,dy, \tag{2.2}
$$

$$
I_2 := \int_0^{1}\Big(\int_1^{\infty} e^{\alpha xy}\csc h(xy)x^{\sigma-\frac{1}{pn}-1}\,dx\Big)y^{\sigma_1+\frac{1}{qn}-1}\,dy. \tag{2.3}
$$

Setting $u = xy$ in (2.2) and (2.3), by Fubini's theorem (see [8]), we obtain

$$
\begin{aligned}
I_1 &= \int_1^{\infty} y^{(\sigma_1-\sigma)-\frac{1}{n}-1}\Big(\int_0^y e^{\alpha u}\csc h(u)u^{\sigma+\frac{1}{pn}-1}\,du\Big)\,dy \\
&= \int_1^{\infty} y^{(\sigma_1-\sigma)-\frac{1}{n}-1}\,dy\int_0^1 e^{\alpha u}\csc h(u)u^{\sigma+\frac{1}{pn}-1}\,du \\
&\quad + \int_1^{\infty}\Big[\int_u^{\infty} y^{(\sigma_1-\sigma)-\frac{1}{n}-1}\,dy\Big]e^{\alpha u}\csc h(u)u^{\sigma+\frac{1}{pn}-1}\,du;
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
I_2 &= \int_0^{1} y^{(\sigma_1-\sigma)+\frac{1}{n}-1}\Big(\int_y^{\infty} e^{\alpha u}\csc h(u)u^{\sigma-\frac{1}{pn}-1}\,du\Big)\,dy \\
&= \int_0^{1}\Big[\int_0^{u} y^{(\sigma_1-\sigma)+\frac{1}{n}-1}\,dy\Big]e^{\alpha u}\csc h(u)u^{\sigma-\frac{1}{pn}-1}\,du \\
&\quad + \int_0^{1} y^{(\sigma_1-\sigma)+\frac{1}{n}-1}\,dy\int_1^{\infty} e^{\alpha u}\csc h(u)u^{\sigma-\frac{1}{pn}-1}\,du.
\end{aligned} \tag{2.5}
$$

Lemma 2.2. *If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha < 1, \sigma > 1, \sigma_1 \in \mathbf{R}$, and there exists a constant M, such that for any nonnegative measurable functions $f(x)$ and $g(y)$ $(x, y \in (0, \infty))$, the inequality*

$$
\begin{aligned}
I &:= \int_0^{\infty}\int_0^{\infty} e^{\alpha xy}\csc h(xy)f(x)g(y)\,dx\,dy \\
&\le M\Big[\int_0^{\infty} x^{p(1-\sigma)-1}f^p(x)\,dx\Big]^{\frac{1}{p}}\Big[\int_0^{\infty} y^{q(1-\sigma_1)-1}g^q(y)\,dy\Big]^{\frac{1}{q}}
\end{aligned} \tag{2.6}
$$

holds true, then we have $\sigma_1 = \sigma$. For $\sigma_1 = \sigma$, we still have $M \ge k(\sigma, \alpha)$.

Proof. If $\sigma_1 < \sigma$, then for

$$n > \frac{1}{\sigma - \sigma_1} \quad (n \in \mathbf{N}),$$

we set the following two functions:

$$f_n(x) := \begin{cases} 0, & 0 < x < 1, \\ x^{\sigma - \frac{1}{pn} - 1}, & x \geq 1, \end{cases} \qquad g_n(y) := \begin{cases} y^{\sigma_1 + \frac{1}{qn} - 1}, & 0 < y \leq 1, \\ 0, & y > 1. \end{cases}$$

Hence, we obtain

$$\begin{aligned} J_2 &:= \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x)\, dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y)\, dy\right]^{\frac{1}{q}} \\ &= \left(\int_1^\infty x^{-\frac{1}{n}-1}\, dx\right)^{\frac{1}{p}} \left(\int_0^1 y^{\frac{1}{n}-1}\, dy\right)^{\frac{1}{q}} = n. \end{aligned}$$

By (2.5) and (2.6), we have

$$\begin{aligned} &\int_0^1 \left[\int_0^u y^{(\sigma_1-\sigma)+\frac{1}{n}-1}\, dy\right] e^{\alpha u} \csc h(u) u^{\sigma - \frac{1}{pn} - 1}\, du \\ &\quad \leq I_2 = \int_0^\infty \int_0^\infty e^{\alpha xy} \csc h(xy) f_n(x) g_n(y)\, dx\, dy \leq M J_2 = Mn. \end{aligned} \tag{2.7}$$

Since

$$(\sigma_1 - \sigma) + \frac{1}{n} < 0,$$

it follows that for any $u \in (0,1)$,

$$\int_0^u y^{(\sigma_1-\sigma)+\frac{1}{n}-1}\, dy = \infty.$$

By (2.7), in view of the fact that

$$e^{\alpha u} \csc h(u) u^{\sigma - \frac{1}{pn} - 1} > 0, \quad u \in (0,1),$$

we derive that $\infty < \infty$, which is a contradiction. If $\sigma_1 > \sigma$, then for

$$n > \frac{1}{\sigma_1 - \sigma} \quad (n \in \mathbf{N}),$$

we set

$$\widetilde{f}_n(x) := \begin{cases} x^{\sigma + \frac{1}{pn} - 1}, & 0 < x \leq 1, \\ 0, & x > 1, \end{cases} \qquad \widetilde{g}_n(y) := \begin{cases} 0, & 0 < y < 1, \\ y^{\sigma_1 - \frac{1}{qn} - 1}, & y \geq 1. \end{cases}$$

Hence, we find that

$$\begin{aligned} \widetilde{J}_2 &:= \left[\int_0^\infty x^{p(1-\sigma)-1} \widetilde{f}_n^p(x)\, dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} \widetilde{g}_n^q(y)\, dy\right]^{\frac{1}{q}} \\ &= \left(\int_0^1 x^{\frac{1}{n}-1}\, dx\right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\frac{1}{n}-1}\, dy\right)^{\frac{1}{q}} = n. \end{aligned}$$

By (2.4) and (2.6), we have

$$\begin{aligned}&\int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1}\,dy\int_0^1 e^{\alpha u}\csc h(u)u^{\sigma+\frac{1}{pn}-1}\,du\\&\quad\le I_1=\int_0^\infty\int_0^\infty e^{\alpha xy}\csc h(xy)\widetilde{f}_n(x)\widetilde{g}_n(y)\,dx\,dy\le M\widetilde{J}_2=Mn. \qquad (2.8)\end{aligned}$$

Since

$$(\sigma_1-\sigma)-\frac{1}{n}>0,$$

it follows that

$$\int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1}\,dy=\infty.$$

By (2.8), in view of the fact that

$$\int_0^1 e^{\alpha u}\csc h(u)u^{\sigma+\frac{1}{pn}-1}\,du>0,$$

we have $\infty<\infty$, which is a contradiction. Hence, we conclude that $\sigma_1=\sigma$.

For $\sigma_1=\sigma$, we reduce (2.4) and then apply (2.8) as follows:

$$\begin{aligned}\frac{1}{n}I_1&=\frac{1}{n}\Big[\int_1^\infty y^{-\frac{1}{n}-1}\,dy\int_0^1 e^{\alpha u}\csc h(u)u^{\sigma+\frac{1}{pn}-1}\,du\\&\qquad+\int_1^\infty\Big(\int_u^\infty y^{-\frac{1}{n}-1}\,dy\Big)e^{\alpha u}\csc h(u)u^{\sigma+\frac{1}{pn}-1}\,du\Big]\\&=\int_0^1 e^{\alpha u}\csc h(u)u^{\sigma+\frac{1}{pn}-1}\,du+\int_1^\infty e^{\alpha u}\csc h(u)u^{\sigma-\frac{1}{qn}-1}\,du\\&\le\frac{1}{n}M\widetilde{J}_2=M. \qquad (2.9)\end{aligned}$$

Since

$$\big\{e^{\alpha u}\csc h(u)u^{\sigma+\frac{1}{pn}-1}\big\}_{n=1}^\infty \qquad \big(\big\{e^{\alpha u}\csc h(u)u^{\sigma-\frac{1}{qn}-1}\big\}_{n=1}^\infty\big)$$

is nonnegative and increasing in $(0,1)$ $((1,\infty))$, by Levi's theorem (see [8]) we obtain

$$\begin{aligned}k(\sigma,\alpha)&=\int_0^1\lim_{n\to\infty}e^{\alpha u}\csc h(u)u^{\sigma+\frac{1}{pn}-1}\,du+\int_1^\infty\lim_{n\to\infty}e^{\alpha u}\csc h(u)u^{\sigma-\frac{1}{qn}-1}\,du\\&=\lim_{n\to\infty}\Big[\int_0^1 e^{\alpha u}\csc h(u)u^{\sigma+\frac{1}{pn}-1}\,du+\int_1^\infty e^{\alpha u}\csc h(u)u^{\sigma-\frac{1}{qn}-1}\,du\Big]\\&\le M. \qquad (2.10)\end{aligned}$$

The lemma is proved. □

3. Main results

Theorem 3.1. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha < 1, \sigma > 1,$ *and* $\sigma_1 \in \mathbf{R}$, *then the following conditions are equivalent.*

(i) *There exists a constant* M *such that for any* $f(x) \geq 0$ *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\, dx < \infty,$$

we have the following inequality:

$$\begin{aligned} J &:= \Big[\int_0^\infty y^{p\sigma_1 - 1}\Big(\int_0^\infty e^{\alpha xy} \csc h(xy) f(x)\, dx\Big)^p dy\Big]^{\frac{1}{p}} \\ &< M\Big[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\, dx\Big]^{\frac{1}{p}}. \end{aligned} \tag{3.1}$$

(ii) *There exists a constant* M*, such that for any* $f(x), g(y) \geq 0$ *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\, dx < \infty$$

and

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)\, dy < \infty,$$

we have the following Hilbert-type integral inequality:

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty e^{\alpha xy} \csc h(xy) f(x) g(y)\, dx\, dy \\ &< M\Big[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\, dx\Big]^{\frac{1}{p}} \Big[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)\, dy\Big]^{\frac{1}{q}}. \end{aligned} \tag{3.2}$$

(iii) $\sigma_1 = \sigma$.

If condition (iii) follows, then $M \geq k(\sigma, \alpha)$ *and the constant factor*

$$M = k(\sigma, \alpha) = 2^{1-\sigma}\Gamma(\sigma)\zeta\Big(\sigma, \frac{1-\alpha}{2}\Big)$$

in (3.1) and (3.2) is the best possible.

Proof. (i) $\Longrightarrow$ (ii). By Hölder's inequality (see [7]), we have

$$\begin{aligned} I &= \int_0^\infty \Big(y^{\sigma_1 - \frac{1}{p}} \int_0^\infty e^{\alpha xy} \csc h(xy) f(x)\, dx\Big)\big(y^{\frac{1}{p} - \sigma_1} g(y)\big)\, dy \\ &\leq J\Big[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)\, dy\Big]^{\frac{1}{q}}. \end{aligned} \tag{3.3}$$

Then by (3.1), we get (3.2).

(ii) $\Longrightarrow$ (iii). By Lemma 2.2, we have $\sigma_1 = \sigma$.

(iii) $\Longrightarrow$ (i). Setting $u = xy$, we obtain the following weight function:

$$\omega(\sigma, y) := y^{\sigma} \int_0^{\infty} e^{\alpha xy} \csc h(xy) x^{\sigma-1}\, dx$$
$$= \int_0^{\infty} e^{\alpha u} \csc h(u) u^{\sigma-1}\, du = k(\sigma, \alpha) \quad (y > 0). \tag{3.4}$$

By Hölder's inequality with weight and (3.4), we have

$$\Big(\int_0^{\infty} e^{\alpha xy} \csc h(xy) f(x)\, dx\Big)^p$$
$$= \Big\{\int_0^{\infty} e^{\alpha xy} \csc h(xy) \Big[\frac{y^{(\sigma-1)/p}}{x^{(\sigma-1)/q}} f(x)\Big] \Big[\frac{x^{(\sigma-1)/q}}{y^{(\sigma-1)/p}}\Big]\, dx\Big\}^p$$
$$\leq \int_0^{\infty} e^{\alpha xy} \csc h(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x)\, dx$$
$$\times \Big[\int_0^{\infty} e^{\alpha xy} \csc h(xy) \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}}\, dx\Big]^{p/q}$$
$$= \big[\omega(\sigma, y) y^{q(1-\sigma)-1}\big]^{p-1} \int_0^{\infty} e^{\alpha xy} \csc h(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x)\, dx$$
$$= \big(k(\sigma, \alpha)\big)^{p-1} y^{-p\sigma+1} \int_0^{\infty} e^{\alpha xy} \csc h(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x)\, dx. \tag{3.5}$$

If (3.5) takes the form of an equality for some $y \in (0, \infty)$, then (see [7]) there exist constants A and B, such that they are not all zero, and

$$A \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) = B \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} \quad \text{a.e. in } \mathbf{R}_+.$$

We assume that $A \neq 0$ (otherwise, $B = A = 0$). Then it follows that

$$x^{p(1-\sigma)-1} f^p(x) = y^{q(1-\sigma)} \frac{B}{Ax} \quad \text{a.e. in } \mathbf{R}_+,$$

which contradicts the fact that

$$0 < \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x)\, dx < \infty.$$

Hence, (3.5) takes the form of strict inequality. For $\sigma_1 = \sigma$, by Fubini's theorem, we have

$$J < \big(k(\sigma, \alpha)\big)^{\frac{1}{q}} \Big[\int_0^{\infty} \int_0^{\infty} e^{\alpha xy} \csc h(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x)\, dx\, dy\Big]^{\frac{1}{p}}$$
$$= \big(k(\sigma, \alpha)\big)^{\frac{1}{q}} \Big\{\int_0^{\infty} \Big[\int_0^{\infty} e^{\alpha xy} \csc h(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)(p-1)}}\, dy\Big] f^p(x)\, dx\Big\}^{\frac{1}{p}}$$
$$= \big(k(\sigma, \alpha)\big)^{\frac{1}{q}} \Big[\int_0^{\infty} \omega(\sigma, x) x^{p(1-\sigma)-1} f^p(x)\, dx\Big]^{\frac{1}{p}}$$
$$= k(\sigma, \alpha) \Big[\int_0^{\infty} x^{p(1-\sigma)-1} f^p(x)\, dx\Big]^{\frac{1}{p}}.$$

Setting $M \geq k(\sigma, \alpha)$, inequality (3.1) follows. Therefore, the conditions (i), (ii), and (iii) are equivalent.

In the case when condition (iii) follows, if there exists a constant $M < k(\sigma, \alpha)$, such that (3.2) is valid, then by Lemma 2.2 we have $M \geq k(\sigma, \alpha)$. From this contradiction it follows that the constant factor $M = k(\sigma, \alpha)$ in (3.2) is the best possible. The constant factor $M = k(\sigma, \alpha)$ in (3.1) is still the best possible. Otherwise, by (3.3) (for $\sigma_1 = \sigma$), we can conclude that the constant factor $M = k(\sigma, \alpha)$ in (3.2) is not the best possible. □

Setting $y = \frac{1}{Y}$, $G(Y) = \frac{1}{Y^2} g(\frac{1}{Y})$ in Theorem 3.1, then replacing Y $(G(Y))$ by y $(g(y))$, we obtain the following corollary.

Corollary 3.2. *If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha < 1, \sigma > 1$, and $\sigma_1 \in \mathbf{R}$, then the following conditions are equivalent.*

(i) *There exists a constant M, such that for any $f(x) \geq 0$, satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\, dx < \infty,$$

we have the following inequality:

$$\begin{aligned}\Big\{\int_0^\infty y^{-p\sigma_1 - 1}\Big[\int_0^\infty e^{\alpha x/y} \csc h(x/y) f(x)\, dx\Big]^p dy\Big\}^{\frac{1}{p}} \\ < M\Big[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\, dx\Big]^{\frac{1}{p}}. \qquad (3.6)\end{aligned}$$

(ii) *There exists a constant M, such that for any $f(x), g(y) \geq 0$, satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\, dx < \infty$$

and

$$0 < \int_0^\infty y^{q(1+\sigma_1)-1} g^q(y)\, dy < \infty,$$

we have the following Hilbert-type integral inequality with the homogeneous kernel:

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{\alpha x/y} \csc h(x/y) f(x) g(y)\, dx\, dy \\ < M\Big[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)\, dx\Big]^{\frac{1}{p}} \Big[\int_0^\infty y^{q(1+\sigma_1)-1} g^q(y)\, dy\Big]^{\frac{1}{q}}. \qquad (3.7)\end{aligned}$$

(iii) $\sigma_1 = \sigma$.

In the case when condition (iii) holds, we get $M \geq k(\sigma, \alpha)$, and the constant $M = k(\sigma, \alpha)$ in (3.6) and (3.7) is the best possible.

Remark 3.3. On the other hand, setting $y = \frac{1}{Y}$, $G(Y) = \frac{1}{Y^2} g(\frac{1}{Y})$, in Corollary 3.2, then replacing Y $(G(Y))$ by y $(g(y))$, we obtain Theorem 3.1. Hence, Theorem 3.1 and Corollary 3.2 are equivalent.

4. Operator expressions

We set the functions

$$\varphi(x) := x^{p(1-\sigma)-1}, \qquad \psi(y) := y^{q(1-\sigma)-1}, \qquad \phi(y) := y^{q(1+\sigma)-1},$$

wherefrom

$$\psi^{1-p}(y) = y^{p\sigma-1}, \qquad \phi^{1-p}(y) = y^{-p\sigma-1} \quad (x, y \in \mathbf{R}_+),$$

and we define the following real normed linear spaces,

$$L_{p,\varphi}(\mathbf{R}_+) := \Big\{ f : \|f\|_{p,\varphi} := \Big(\int_0^\infty \varphi(x) \big|f(x)\big|^p \, dx \Big)^{\frac{1}{p}} < \infty \Big\},$$

wherefrom

$$L_{q,\psi}(\mathbf{R}_+) = \Big\{ g : \|g\|_{q,\psi} := \Big(\int_0^\infty \psi(y) \big|g(y)\big|^q \, dy \Big)^{\frac{1}{q}} < \infty \Big\},$$

$$L_{q,\phi}(\mathbf{R}_+) = \Big\{ g : \|g\|_{q,\phi} := \Big(\int_0^\infty \phi(y) \big|g(y)\big|^q \, dy \Big)^{\frac{1}{q}} < \infty \Big\},$$

$$L_{p,\psi^{1-p}}(\mathbf{R}_+) = \Big\{ h : \|h\|_{p,\psi^{1-p}} = \Big(\int_0^\infty \psi^{1-p}(y) \big|h(y)\big|^p \, dy \Big)^{\frac{1}{p}} < \infty \Big\},$$

$$L_{q,\phi^{1-p}}(\mathbf{R}_+) = \Big\{ h : \|h\|_{p,\phi^{1-p}} = \Big(\int_0^\infty \phi^{1-p}(y) \big|h(y)\big|^p \, dy \Big)^{\frac{1}{p}} < \infty \Big\}.$$

In view of Theorem 3.1 (when $\sigma_1 = \sigma$), for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$h_1(y) := \int_0^\infty e^{\alpha xy} \csc h(xy) f(x) \, dx \quad (y \in \mathbf{R}_+),$$

by (3.1) we have

$$\|h_1\|_{p,\psi^{1-p}} = \Big[\int_0^\infty \psi^{1-p}(y) h_1^p(y) \, dy \Big]^{\frac{1}{p}} < M \|f\|_{p,\varphi} < \infty. \tag{4.1}$$

Definition 4.1. Define a Hilbert-type integral operator with the nonhomogeneous kernel $T^{(1)}$: $L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\psi^{1-p}}(\mathbf{R}_+)$ as follows. For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $T^{(1)} f = h_1 \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$ satisfying $T^{(1)} f(y) = h_1(y)$ for any $y \in \mathbf{R}_+$.

In view of (4.1), it follows that

$$\|T^{(1)} f\|_{p,\psi^{1-p}} = \|h_1\|_{p,\psi^{1-p}} \leq M \|f\|_{p,\varphi},$$

and then the operator $T^{(1)}$ is bounded satisfying

$$\|T^{(1)}\| = \sup_{f(\neq\theta) \in L_{p,\varphi}(\mathbf{R}_+)} \frac{\|T^{(1)} f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq M.$$

If we define the formal inner product of $T^{(1)} f$ and g by

$$(T^{(1)} f, g) := \int_0^\infty \Big(\int_0^\infty e^{\alpha xy} \csc h(xy) f(x) \, dx \Big) g(y) \, dy,$$

then we can rewrite Theorem 3.1 as follows.

Theorem 4.2. *If* $p > 1, \frac{1}{p}+\frac{1}{q} = 1, \alpha < 1$, *and* $\sigma > 1$, *then the following conditions are equivalent.*

(i) *There exists a constant* M, *such that for any*

$$f(x) \geq 0, \qquad f \in L_{p,\varphi}(\mathbf{R}_+), \qquad \|f\|_{p,\varphi} > 0,$$

the following inequality holds true:

$$\|T^{(1)} f\|_{p,\psi^{1-p}} < M\|f\|_{p,\varphi}. \tag{4.2}$$

(ii) *There exists a constant* M, *such that for any*

$$f(x), g(y) \geq 0, \qquad f \in L_{p,\varphi}(\mathbf{R}_+), \qquad g \in L_{q,\psi}(\mathbf{R}_+), \qquad \|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0,$$

the following inequality holds true:

$$(T^{(1)} f, g) < M\|f\|_{p,\varphi}\|g\|_{q,\psi}. \tag{4.3}$$

We still have $\|T^{(1)}\| = k(\sigma, \alpha) \leq M$.

In view of Corollary 3.2 (when $\sigma_1 = \sigma$), for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$h_2(y) := \int_0^\infty e^{\alpha x/y} \operatorname{csc} h(x/y) f(x)\, dx \quad (y \in \mathbf{R}_+),$$

by (3.6) we have

$$\|h_2\|_{p,\phi^{1-p}} = \Big[\int_0^\infty \phi^{1-p}(y) h_2^p(y)\, dy\Big]^{\frac{1}{p}} < M\|f\|_{p,\varphi} < \infty. \tag{4.4}$$

Definition 4.3. Define a Hilbert-type integral operator with the homogeneous kernel $T^{(2)}$: $L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\phi^{1-p}}(\mathbf{R}_+)$ as follows. For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $T^{(2)} f = h_2 \in L_{p,\phi^{1-p}}(\mathbf{R}_+)$ satisfying $T^{(2)} f(y) = h_2(y)$ for any $y \in \mathbf{R}_+$.

In view of (4.4), it follows that

$$\|T^{(2)} f\|_{p,\phi^{1-p}} = \|h_2\|_{p,\phi^{1-p}} \leq M\|f\|_{p,\varphi},$$

and thus the operator $T^{(2)}$ is bounded satisfying

$$\|T^{(2)}\| = \sup_{f(\neq\theta) \in L_{p,\varphi}(\mathbf{R}_+)} \frac{\|T^{(2)} f\|_{p,\phi^{1-p}}}{\|f\|_{p,\varphi}} \leq M.$$

If we define the formal inner product of $T^{(2)} f$ and g by

$$(T^{(2)} f, g) := \int_0^\infty \Big(\int_0^\infty e^{\alpha x/y} \operatorname{csc} h(x/y) f(x)\, dx\Big) g(y)\, dy,$$

then we can rewrite Corollary 3.2 as follows:

Corollary 4.4. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha < 1$, *and* $\sigma > 1$, *then the following conditions are equivalent.*

(i) *There exists a constant M, such that for any*

$$f(x) \geq 0, \qquad f \in L_{p,\varphi}(\mathbf{R}_+), \qquad \|f\|_{p,\varphi} > 0,$$

the following inequality holds true:

$$\|T^{(2)} f\|_{p,\phi^{1-p}} < M \|f\|_{p,\varphi}. \tag{4.5}$$

(ii) *There exists a constant M, such that for any*

$$f(x), g(y) \geq 0, \qquad f \in L_{p,\varphi}(\mathbf{R}_+), \qquad g \in L_{q,\phi}(\mathbf{R}_+), \qquad \|f\|_{p,\varphi}, \|g\|_{q,\phi} > 0,$$

the following inequality holds true:

$$(T^{(2)} f, g) < M \|f\|_{p,\varphi} \|g\|_{q,\phi}. \tag{4.6}$$

We still have $\|T^{(2)}\| = k(\sigma, \alpha) \leq M$.

Remark 4.5. Theorem 4.2 and Corollary 4.4 are equivalent.

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