Ann. Funct. Anal. 9 (2018), no. 1, 72–86
https://doi.org/10.1215/20088752-2017-0025
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

INHOMOGENEOUS LIPSCHITZ SPACES OF VARIABLE ORDER AND THEIR APPLICATIONS

JIAN TAN and JIMAN ZHAO*

Communicated by K. Guerlebeck

Abstract. In this article, the authors first give a Littlewood–Paley characterization for inhomogeneous Lipschitz spaces of variable order with the help of inhomogeneous Calderón identity and almost-orthogonality estimates. As applications, the boundedness of inhomogeneous Calderón–Zygmund singular integral operators of order (ϵ, σ) on these spaces has been presented. Finally, we note that a class of pseudodifferential operators $T_a \in \mathcal{O}pS^0_{1,1}$ are continuous on the inhomogeneous Lipschitz spaces of variable order as a corollary. We may observe that those operators are not, in general, continuous in L^2.

1. Introduction and statement of main results

The classical Lipschitz spaces $\dot{\mathcal{C}}^\eta$ play an important role in harmonic analysis and partial differential equations. It is well known that the spaces $\dot{\mathcal{C}}^\eta$ can be characterized via Littlewood–Paley decomposition (see [7] and [18]). Much research has been carried out on Lipschitz spaces and their applications. One direction is variable-exponent Lipschitz spaces (see [1], [2], [15]). Another direction (see [8]) is the study of multiparameter Lipschitz spaces. (For more about the Lipschitz spaces or so called *Hölder–Zygmund spaces*, see also [3], [11], [12], [14], [16].)

In many applications, as we know, use of the homogeneous spaces $\dot{\mathcal{C}}^s$ rather than the inhomogeneous Hölder spaces $\mathcal{C}^s = \dot{\mathcal{C}}^s \cap L^\infty$ is not successful. For instance, the continuity property of pseudodifferential operators $T \in \mathcal{O}pS^m_{1,0}$

Received Oct. 16, 2016; Accepted Feb. 20, 2017.
First published online Jun. 29, 2017.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B25, 46E35.
Keywords. inhomogeneous Lipschitz space, variable exponent, almost-orthogonal estimate, Littlewood–Paley theory, pseudodifferential operator.

(whose symbols fulfill $|\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)| \leq C(\alpha,\beta)(1+|\xi|_e)^{m-|\alpha|}$) in the the inhomogeneous Hölder spaces $\mathcal{C}^s$ is considered in [16]. Also, $T \in \mathcal{O}pS^0_{1,1}$ (whose symbols satisfy $|\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)| \leq C(\alpha,\beta)(1+|\xi|_e)^{|\beta|-|\alpha|}$) is continuous on inhomogeneous Hölder–Zygmund spaces $\mathcal{C}^s$ (see [13]). Moreover, Stein and Yung in [17] showed that a class of pseudodifferential operators preserve the isotropic and nonisotropic Lipschitz spaces.

On the other hand, due to its application to partial differential equations and the calculus of variations, *variable-exponent function space theory* has attracted much attention (see ([4], [6]). In many applications, a crucial step has been to show that the classical operators of harmonic analysis, such as maximal operators, singular integrals, and fractional integrals, are bounded on variable-exponent function spaces. So we will mainly focus on the boundedness of a class of Calderón–Zygmund singular integral operators on inhomogeneous Hölder–Zygmund spaces of variable order.

The purpose of this work is to characterize inhomogeneous Hölder–Zygmund spaces via the Littlewood–Paley theory and to prove that inhomogeneous Calderón–Zygmund singular integral operators are bounded on these spaces. If these results are established at once, we will see that pseudodifferential operators $T_a \in \mathcal{O}pS^0_{1,1}$ are continuous on the inhomogeneous Hölder–Zygmund spaces. We also observe that those operators are not, in general, continuous in L^2.

Before we state our results, we first recall some notions concerning variable-exponent and Hölder–Zygmund spaces. For a measurable subset $E \subset \mathbb{R}^n$, we denote $p^-(E) = \inf_{x\in E} p(x)$ and $p^+(E) = \sup_{x\in E} p(x)$. Especially, we denote $p^- = p^-(\mathbb{R}^n)$ and $p^+ = p^+(\mathbb{R}^n)$. Let $p(\cdot)\colon \mathbb{R}^n \to (0,\infty)$ be a measurable function with $0 < p^- \leq p^+ < \infty$ and let $\mathcal{P}^0$ be the set of all these $p(\cdot)$.

We say that $p(\cdot) \in LH_0$ if $p(\cdot)$ satisfies

$$\big|p(x)-p(y)\big| \leq \frac{C}{-\log(|x-y|)}, \quad |x-y| \leq \frac{1}{2}.$$

Throughout this article we use C to denote positive constants, whose value may vary from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. We denote by $f \sim g$ the fact that there exists a constant $C > 0$ independent of the main parameters such that $C^{-1}g < f < Cg$. We also denote that

$$\Delta_u f(x) = f(x+u) - f(x), \qquad \Delta_u^2 f(x) = \Delta_u(\Delta_u) = f(x+2u) + f(x) - 2f(x+u).$$

Now we recall the definition of inhomogeneous Hölder–Zygmund space of variable order. In [1], Almeida and Hästö generalized the definition of Hölder–Zygmund spaces to the variable-order setting for $0 < \alpha^- \leq \alpha^+ \leq 1$ (see also [2], [15]).

Definition 1.1. Let $\alpha(\cdot) : \mathbb{R}^n \to (0,\infty)$. The inhomogeneous Hölder space of variable order $H^{\alpha(\cdot)}$ is defined to be the space of all bounded uniformly continuous f defined on $\mathbb{R}^n$ in what follows. When $0 < \alpha^- \leq \alpha^+ < 1$,

$$\|f\|_{H_0^{\alpha(\cdot)}} := \|f\|_\infty + \sup_{x\in\mathbb{R}^n, u\neq 0} \frac{|f(x-u)-f(x)|}{|u|^{\alpha(x-u)}} < \infty.$$

When $m < \alpha^- \le \alpha^+ < m+1$, we write $\alpha(x) = m + r(x)$, where m is an integer and $0 < r^- \le r^- < 1$. Here $f \in H^{\alpha(\cdot)}$ means that f is a $\mathcal{C}^m$ function such that

$$\|f\|_{H_m^{\alpha(\cdot)}} := \sum_{|\beta|\le m} \|\partial^\beta f\|_\infty + \sum_{|\beta|=m} \sup_{x\in\mathbb{R}^n, u\neq 0} \frac{|\partial^\beta f(x-u) - \partial^\beta f(x)|}{|u|^{r(x-u)}} < \infty.$$

When $0 < \alpha^- \le \alpha^+ < \infty$ and $\alpha(x) \neq$ integer, we have $\alpha(x) = \sum_{i=[\alpha^-]}^{[\alpha^+]} \alpha_i(x)$, where $\alpha_i = \alpha\chi_i$ and $\chi_i(x) = 1$ for $\alpha(x) \in (i, i+1)$; otherwise $\chi_i(x) = 0$. $f \in H^{\alpha(\cdot)}$ means that f is a $\mathcal{C}^{[\alpha^+]}$ function such that

$$\|f\|_{H^{\alpha(\cdot)}} := \sum_{m=[\alpha^-]}^{[\alpha^+]} \|f\|_{H_m^{\alpha(\cdot)}} < \infty.$$

The inhomogeneous Zygmund space of variable order $\Lambda^{\alpha(\cdot)}$ is defined analogously but with the norm given as follows. When $0 < \alpha^- \le \alpha^+ \le 1$,

$$\|f\|_{\Lambda_0^{\alpha(\cdot)}} := \|f\|_\infty + \sup_{x\in\mathbb{R}^n, u\neq 0} \frac{|f(x+u) + f(x-u) - 2f(x)|}{|u|^{\alpha(x-u)}};$$

When $m < \alpha^- \le \alpha^+ \le m+1$, we write $\alpha(x) = m + r(x)$, where m is integer and $0 < r^- \le r^+ \le 1$:

$$\|f\|_{\Lambda_m^{\alpha(\cdot)}} := \sum_{|\beta|\le m} \|\partial^\beta f\|_\infty + \sum_{|\beta|=m} \sup_{x\in\mathbb{R}^n, u\neq 0} \frac{|\partial^\beta f(x+u) + \partial^\beta f(x-u) - 2\partial^\beta f(x)|}{|u|^{r(x-u)}}.$$

When $0 < \alpha^- \le \alpha^+ < \infty$, we have $\alpha(x) = \sum_{i=[\alpha^-]}^{[\alpha^+]} \alpha_i(x)$, where $\alpha_i = \alpha\chi_i$ and $\chi_i(x) = 1$ for $\alpha(x) \in (i, i+1]$; otherwise $\chi_i(x) = 0$:

$$\|f\|_{\Lambda^{\alpha(\cdot)}} := \sum_{m=[\alpha^-]}^{[\alpha^+]} \|f\|_{\Lambda_m^{\alpha(\cdot)}}.$$

Next we give the Littlewood–Paley characterization for $H^{\alpha(\cdot)}$ and $\Lambda^{\alpha(\cdot)}$. Let $\hat{\psi}$ be the Fourier transform of $\psi \in \mathcal{S}$. For this purpose, let $\psi, \Psi \in \mathcal{S}(\mathbb{R}^n)$ with

$$\operatorname{supp} \widehat{\psi}(\xi) \subset \big\{\xi : 1/2 < |\xi| \le 2\big\},$$

and Ψ with

$$\big|\widehat{\Psi}(\xi)\big| \ge c > 0, \qquad \operatorname{supp} \widehat{\Psi} \subset \big\{|\xi| \le 2\big\}$$

satisfying

$$\big|\widehat{\Psi}(\xi)\big|^2 + \sum_{j=1}^{\infty} \big|\widehat{\psi}(2^{-j}\xi)\big|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}^n.$$

We set $\psi_j(x) = 2^{jn}\psi(2^j x)$ and $\Psi(x) =: \psi_0(x)$.

For $f \in L^2$, we have the inhomogeneous continuous Calderón identity

$$f = \sum_{j=0}^{\infty} \psi_j * \psi_j * f$$

via taking the Fourier transform, where the series converges in its $L^2(\mathbb{R}^n)$ norm. Before we state the result, we note that in [1], Almeida and Hästö have proved that $B^{\alpha(\cdot)}_{\infty,\infty} = H^{\alpha(\cdot)}(\alpha < 1)$ and $B^{\alpha(\cdot)}_{\infty,\infty} = \Lambda^{\alpha(\cdot)}(\alpha^+ \leq 1)$ with the help of the so-called *Peetre maximal function*.

Theorem 1.2. *Suppose that* $\alpha(\cdot) \in LH_0 \cap \mathcal{P}^0$. *Note that* $f \in H^{\alpha(\cdot)}$ *if and only if* $f \in \mathcal{S}'$ *and*

$$\big|\psi_j * f(x)\big| \leq C2^{-j\alpha(x)}$$

for any x *such that* $\alpha(x) \neq \text{integer}$; $f \in \Lambda^{\alpha(\cdot)}$ *if and only if* $f \in \mathcal{S}'$ *and*

$$\big|\psi_j * f(x)\big| \leq C2^{-j\alpha(x)}$$

for any $x \in \mathbb{R}^n$. *Furthermore,*

$$\|f\|_{H^{\alpha(\cdot)}} \sim \sup_{j\geq 0, x\in\mathbb{R}^n} 2^{j\alpha(x)}\big|\psi_j * f(x)\big|, \qquad \|f\|_{\Lambda^{\alpha(\cdot)}} \sim \sup_{j\geq 0, x\in\mathbb{R}^n} 2^{j\alpha(x)}\big|\psi_j * f(x)\big|.$$

We will state that inhomogeneous Calderón–Zygmund singular integral operators of order (ϵ, σ) are bounded operators on the new inhomogeneous Hölder–Zygmund spaces.

First, we recall some definitions. For $\eta \in (0, 1]$, let $\dot{\mathcal{C}}^\eta$ be the set of all continuous functions f on $\mathbb{R}^n$ having compact support such that

$$\|f\|_{\dot{\mathcal{C}}^\eta} = \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\eta} < \infty.$$

Endow $\dot{\mathcal{C}}^\eta$ with the natural topology and let $(\dot{\mathcal{C}}^\eta)'$ be its dual space.

The following definition is the classical inhomogeneous Calderón–Zygmund singular integral kernel which was first introduced by Meyer and Coifman in [13]. For the framework of this kernel on spaces of homogeneous type, the reader is referred to [9].

Definition 1.3. A continuous complex-valued function K on $\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ is called an *inhomogeneous Calderón–Zygmund kernel of type* (ϵ, σ) if there exist constants $\epsilon \in (0, 1], \sigma > 0$ and $C_1 > 0$ such that

(i) $|K(x, y)| \leq C_1 \frac{1}{|x-y|^n}$,

(ii) $|K(x, y)| \leq C_1 \frac{1}{|x-y|^{n+\delta}}$ for $|x - y| \geq 1$,

(iii) $|K(x, y) - K(x', y)| \leq C_1 \frac{|x-x'|^\epsilon}{|x-y|^{n+\epsilon}}$ for $|x - x'| \leq \frac{1}{2}|x - y|$.

We now recall inhomogeneous Calderón–Zygmund singular integral operators.

Definition 1.4. A continuous linear operator $T\colon \dot{C}^\eta \to (\dot{C}^\eta)'$ is an inhomogeneous Calderón–Zygmund singular integral operator if there exists an inhomogeneous kernel K such that

$$\langle Tf, g\rangle = \int\int K(x, y)f(y)g(x)\,dx\,dy$$

for all $f, g \in \dot{C}^\eta$ with disjoint supports.

The following definition is the classical weak boundedness property.

Definition 1.5 ([5, p.5]). A Calderón–Zygmund singular integral operator T is said to have the weak boundedness property, if there exist constants $C_2 > 0$ and $\eta \in (0,1]$ such that for all $x_0 \in \mathbb{R}^n$ and $r > 0$,

$$\big|\langle Tf, g\rangle\big| \le C_2 r^{n+2\eta} \|g\|_{\dot{C}^\eta} \|f\|_{\dot{C}^\eta},$$

where $f, g \in \dot{C}^\eta$ with $\operatorname{supp} f, g \subset \{x : |x - x_0| \le r\}$, $\|f\|_\infty \le 1$, $\|g\|_\infty \le 1$, $\|f\|_{\dot{C}^\eta} \le r^{-\eta}$, and $\|g\|_{\dot{C}^\eta} \le r^{-\eta}$, and we denote this by $T \in WBP$.

Theorem 1.6. *Suppose that T is the inhomogeneous Calderón–Zygmund singular integral operator and the kernel satisfying Definition 1.3. Also assume that $T(1) = 0, T \in WBP, \alpha(\cdot) \in LH_0$, and $0 < \alpha^- \le \alpha^+ < \epsilon \le 1$. Then T can be extended to a bounded linear operator on $H^{\alpha(\cdot)}$ and $\Lambda^{\alpha(\cdot)}$.*

2. Proof of Theorem 1.2

Proof. We only give the proof for $\Lambda^{\alpha(\cdot)}$; the proof for $H^{\alpha(\cdot)}$ is similar. First, it is easy to see that $f \in \mathcal{S}'$, when $f \in \Lambda_0^{\alpha(\cdot)}$ with $0 < \alpha^- \le \alpha^+ \le 1$. Next we will estimate the term $|\psi_j * f(x)|$. Now we consider the following two cases.

When $j = 0$, we have

$$\big|\psi_0 * f(x)\big| = \int \big|\psi_0(u)\big|\big|f(x-u)\big|\,du \le C\|f\|_\infty \le C\|f\|_{\Lambda_0^\alpha(\cdot)}.$$

Applying LH_0 condition of $\alpha(\cdot)$ yields $|u|^{\alpha(x-u)} \le C|u|^{\alpha(x)}$ for $|u| < 1$ (see [1]).

When $j \ge 1$, we may assume that ψ_j is a radial function, and then applying the cancellation conditions on ψ_j, we have

$$\begin{aligned}
\big|\psi_j * f(x)\big| &= \Big|\int \psi_j(u)\big[f(x-u) - f(x)\big]\,du\Big| \\
&= \frac{1}{2}\Big|\int \psi_j(u)\big[f(x+u) + f(x-u) - 2f(x)\big]\,du\Big| \\
&\le C\int_{|u|<1} \big|\Delta_u^2 f(x-u)\big|\big|\psi_j(u)\big|\,du + C\int_{|u|\ge 1} \big|\Delta_u^2 f(x-u)\big|\big|\psi_j(u)\big|\,du \\
&\le C\|f\|_{\Lambda_0^\alpha(\cdot)}\Big\{\int_{|u|<1} |u|^{\alpha(x-u)}\big|\psi_j(u)\big|\,du + \int_{|u|\ge 1} |u|^{\alpha(x-u)}\big|\psi_j(u)\big|\,du\Big\} \\
&\le C2^{-j\alpha(x)}\|f\|_{\Lambda_0^\alpha(\cdot)}\Big\{\int \big[|u|^{\alpha^-} + |u|^{\alpha^+}\big]\big|\psi(u)\big|\,du\Big\} \\
&\le C2^{-j\alpha(x)}\|f\|_{\Lambda_0^\alpha(\cdot)}.
\end{aligned}$$

Thus, we have obtained

$$\sup_{j\ge 0, x\in\mathbb{R}^n} 2^{j\alpha(x)}\big|\psi_j * f(x)\big| \le C\|f\|_{\Lambda_0^\alpha(\cdot)}.$$

Next we will consider the case where $m < \alpha^- \le \alpha^+ \le m+1, m \in \mathbb{Z}_+$. First, we consider the case $j = 0$,

$$\big|\psi_0 * f(x)\big| = \int \big|\psi_0(u)\big|\big|f(x-u)\big|\,du \le C\|f\|_\infty \le C\sum_{|\beta|\le m} \|\partial^\beta f\|_\infty \le C\|f\|_{\Lambda_m^\alpha(\cdot)}.$$

For the case $j > 0$, we now write $|\beta| = m$, $\widehat{\tilde{\psi}_j}(\xi) = \frac{(2\pi i\xi)^\beta \widehat{\psi_j}(\xi)}{(4\pi^2|\xi|^2)^m}$. Then $\psi_j * f = \partial^\beta \tilde{\psi}_j * f = (-1)^m \tilde{\psi}_j * \partial^\beta f$. Notice that every $2^{jm}\tilde{\psi}_j$ satisfies the similar smoothness, size and cancellation conditions as ψ_j. Therefore, the similar argument yields that for any $j > 0$, $|\beta| = m$, and $x \in \mathbb{R}^n$:

$$\begin{aligned}\big|\psi_j * f(x)\big| &= \big|2^{-jm}\big(2^{jm}\widetilde{\psi}_j * \partial^\beta f(x)\big)\big| \\ &\leq C2^{-jm}2^{-jr(x)}\|\partial^\beta f\|_{\Lambda_0^{r(\cdot)}}.\end{aligned}$$

That is,

$$\sup_{j\geq 0, x\in\mathbb{R}^n} 2^{j\alpha(x)}\big|\psi_j * f(x)\big| \leq C\|f\|_{\Lambda_m^{\alpha(\cdot)}}.$$

Note that $\alpha(\cdot) \in LH_0$ implies that $\alpha(\cdot)$ is uniformly continuous. Let Ω_i be the domain of $\alpha_i(x) \neq 0$. Then we can get $\bigcup_j I_{i,j} = \Omega_i$ and $\alpha_i(\cdot)$ is continuous on every $I_{i,j}$.

When $0 < \alpha^- \leq \alpha^+ \leq \infty$, since $\alpha(\cdot) \in LH_0$ implies that all $\alpha_i(\cdot) \in LH_0(I_{i,j})$ for $[\alpha^-] \leq i \leq [\alpha^+]$,

$$\|f\|_{\Lambda^{\alpha(\cdot)}} := \sum_{m=[\alpha^-]}^{[\alpha^+]} \|f\|_{\Lambda_m^{\alpha(\cdot)}} = \sum_{m=[\alpha^-]}^{[\alpha^+]} \sum_{|\beta|=m} \|\partial^\beta f\|_{\Lambda_0^{r(\cdot)}}$$

is a finite sum, so we are done.

To prove the converse statement, we first show that every distribution $f \in \mathcal{S}'$ that fulfills

$$\sup_{j\geq 0, x\in\mathbb{R}^n} 2^{j\alpha(x)}\big|\psi_j * f(x)\big| \leq C$$

coincides with a bounded continuous function in $\mathbb{R}^n$. As mentioned, $f(x) = \sum_{j\geq 0} \psi_j * \psi_j * f(x)$ in $\mathcal{S}'$. Observe that

$$\big|\psi_j * \psi_j * f(x)\big| \leq \|\psi_j * f\|_\infty \|\psi_j\|_{L^1} \leq C\big(\sup_{j\geq 0, x\in\mathbb{R}^n} 2^{j\alpha(x)}\big|\psi_j * f(x)\big|\big)2^{-j\alpha^-}.$$

Thus, the series $\sum_{j\geq 0} \psi_j * \psi_j * f$ converges uniformly in x. Since $\psi_j * \psi_j * f$ is continuous in $\mathbb{R}^n$, the sum function f is also continuous in $\mathbb{R}^n$. Moreover, we can get that

$$\|f\|_\infty \leq C\big(\sup_{j\geq 0, x\in\mathbb{R}^n} 2^{j\alpha(x)}\big|\psi_j * f(x)\big|\big).$$

Now we estimate $\|f\|_{\Lambda_0^{\alpha(\cdot)}}$, as follows. When $0 < \alpha^- \leq \alpha^+ \leq 1$, to prove this, we only need to estimate that, for any $u \neq 0$,

$$\begin{aligned}\big|\Delta_u^2 f(x-u)\big| &= \big|f(x+u) + f(x-u) - 2f(x)\big| \\ &\leq C\big(\sup_{j\geq 0, x\in\mathbb{R}^n} 2^{j\alpha(x)}\big|\psi_j * f(x)\big|\big)|u|^{\alpha(x-u)}.\end{aligned}$$

Observe that

$$\begin{aligned} &f(x-u)+f(x+u)-2f(x)\\ &\quad=\sum_{j\geq 0}\int\big[\psi_j(x-u-w)+\psi_j(x+u-w)-2\psi_j(x-w)\big](\psi_j*f)(w)\,dw. \end{aligned}$$

When $|u|\geq 1$, we only need to apply the size condition of ψ_j. Hence we can obtain

$$\begin{aligned} \big|\Delta_u^2 f(x)\big| &\leq C\big(\sup_{j\geq 0,x\in\mathbb{R}^n}2^{j\alpha(x)}\big|\psi_j*f(x)\big|\big)\sum_{j\geq 0}2^{-j\alpha^-}\\ &\leq C\big(\sup_{j\geq 0,x\in\mathbb{R}^n}2^{j\alpha(x)}\big|\psi_j*f(x)\big|\big)|u|^{\alpha(x)}. \end{aligned}$$

When $|u|\leq 1$, we need to apply the smoothness condition and size conditon on ψ_j. Let l be the unique nonnegative integer such that $2^{-l-1}\leq|u|<2^{-l}$. Hence we can obtain

$$\begin{aligned} &\big|\Delta_u^2 f(x)\big|\\ &\quad\leq\big(\sup_{j\geq 0,x\in\mathbb{R}^n}2^{j\alpha(x)}\big|\psi_j*f(x)\big|\big)\\ &\qquad\times\sum_{j\geq 0}\int 2^{-j\alpha(x)}\big[\psi_j(x-u-w)+\psi_j(x+u-w)-2\psi_j(x-w)\big]\,dw\\ &\quad\leq C\big(\sup_{j\geq 0,x\in\mathbb{R}^n}2^{j\alpha(x)}\big|\psi_j*f(x)\big|\big)\Big(\sum_{j=0}^{l}2^{-j\alpha(x)}|2^j u|^2+\sum_{j=l}^{\infty}2^{-j\alpha(x)}\Big)\\ &\quad\sim\big(\sup_{j\geq 0,x\in\mathbb{R}^n}2^{j\alpha(x)}\big|\psi_j*f(x)\big|\big)|u|^{\alpha(x)}. \end{aligned}$$

When $m<\alpha^-\leq\alpha^+\leq m+1$, we have $\partial^\beta f(x)=\sum_{j\geq 0}\partial^\beta\psi_j*\psi_j*f(x)$ in $\mathcal{S}'$. Since $\psi\in\mathcal{S}$, then

$$\big|\partial^\beta\psi_j*\psi_j*f(x)\big|\leq\|\psi_j*f\|_\infty\|\partial^\beta\psi_j\|_{L^1}\leq C\big(\sup_{j\geq 0,x\in\mathbb{R}^n}2^{j\alpha(x)}\big|\psi_j*f(x)\big|\big)2^{-j(\alpha^- -|\beta|)}.$$

Thus,

$$\sum_{|\beta|\leq m}\|\partial^\beta f\|_\infty\leq C\sum_{|\beta|\leq m}\sum_{j\geq 0}2^{-j(\alpha^- -|\beta|)}\big(\sup_{j\geq 0,x\in\mathbb{R}^n}2^{j\alpha(x)}\big|\psi_j*f(x)\big|\big)\leq C.$$

On the other hand, observe that $|\beta|=m,\alpha(x)=m+r(x)$ and that

$$\begin{aligned} &\partial^\beta f(x-u)+\partial^\beta f(x+u)-2\partial^\beta f(x)\\ &\quad=\sum_{j\geq 0}\int\big[\partial^\beta\psi_j(x-u-w)+\partial^\beta\psi_j(x+u-w)-2\partial^\beta\psi_j(x-w)\big]\\ &\qquad\times(\psi_j*f)(w)\,dw. \end{aligned}$$

Here we note that the properties of $\partial^\beta\psi_j$ are similar to $2^{jm}\psi_j$. Hence the estimate for this case is the same as the proof for the case above. When $0<\alpha^-\leq\alpha^+\leq\infty$, by Definition 1.1 we split $\alpha=\sum_{[\alpha^-]}^{[\alpha^+]}\alpha_i$, where the decomposition is finite sum. So this case can be handled similarly. With this, we have proved Theorem 1.2. □

3. Proof of Theorem 1.6

In order to prove Theorem 1.6, we need an inhomogeneous Calderón-type identity on $H^{\alpha(\cdot)}$ and $\Lambda^{\alpha(\cdot)}$. To do this, let $\phi \in \mathcal{S}$ with $\operatorname{supp} \phi \subseteq B(0,1)$ and $\Phi \in \mathcal{S}$ with

$$\big|\widehat{\Phi}(\xi)\big| \geq C > 0, \qquad \operatorname{supp} \Phi \subset \big\{|\xi| \leq 2\big\}$$

satisfying

$$\big|\widehat{\Phi}(\xi)\big|^2 + \sum_{j\geq 1} \big|\widehat{\phi}(2^{-j}\xi)\big|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}^n,$$

and

$$\int_{\mathbb{R}^n} \phi(x) x^\alpha \, dx = 0 \quad \text{for all } |\alpha| \leq 10M,$$

where M is a fixed large positive integer depending on α. We denote $\Phi =: \phi_0$ and $\phi_j(x) = 2^{jn}\phi(2^j x)$.

The inhomogeneous Calderón-type identity is given by the following.

Proposition 3.1. *Suppose that $\alpha(\cdot) \in LH_0 \cap \mathcal{P}^0$. Let $\phi \in \mathcal{S}$ satisfy conditions above. Then for any $f \in H^{\alpha(\cdot)}$ or $f \in \Lambda^{\alpha(\cdot)}$, we have*

$$f = \sum_{j\geq 0} \phi_j * \phi_j * f \tag{3.1}$$

in the distribution sense. Moreover, if we denote

$$\|f\|^{\phi}_{H^{\alpha(\cdot)}} = \sup_{j\geq 0} 2^{j\alpha(x)} \big|\phi_j * f(x)\big| (\alpha^+ < 1);$$
$$\|f\|^{\phi}_{\Lambda^{\alpha(\cdot)}} = \sup_{j\geq 0} 2^{j\alpha(x)} \big|\phi_j * f(x)\big| (\alpha^+ \leq 1),$$

then

$$\|f\|^{\phi}_{H^{\alpha(\cdot)}} \sim \|f\|^{\psi}_{H^{\alpha(\cdot)}}; \qquad \|f\|^{\phi}_{\Lambda^{\alpha(\cdot)}} \sim \|f\|^{\psi}_{\Lambda^{\alpha(\cdot)}}.$$

Proof. By taking the Fourier transform, we have, for any $f \in L^2$,

$$f(x) = \sum_{j\geq 0} \phi_j * \phi_j * f(x).$$

Now we prove that the series in (3.1) converges in $\mathcal{S}$. To do this, it suffices to show that, for any fixed $L > 0$ and any given integer $M \geq 0$, $|\alpha| \geq 0$,

$$\big|D^\alpha(\phi_j * \phi_j * f)(x)\big| \leq C 2^{-jL}\big(1 + |x|\big)^{-M}. \tag{3.2}$$

Here and below, we will apply the almost-orthogonal estimate which can be found in many monographs (see [7] for more details). To be more precise, for any given positive integers L, M and $\psi, \varphi \in \mathcal{S}$ satisfying cancellation conditions, then

$$\big|\psi_j * \varphi_k(x)\big| \leq C \frac{2^{-|j-k|L} 2^{(j\wedge k)n}}{(1 + 2^{(j\wedge k)}|x|)^{(n+M)}}.$$

Using the almost-orthogonal estimate in [7, p.595] with the case one function has cancellation, we get that

$$\left|\psi_j * g(x)\right| \leq C2^{-jL}\frac{1}{(1+|x|)^{n+M}}$$

for any $L, M \geq 0$, where $j \in \mathbb{Z}_+$.

To prove (3.2), we need to apply the classical almost-orthogonality argument. On one hand, from the size conditions of the functions ϕ, we have, for any given large M,

$$\left|D^\alpha \phi_j(u)\right| \leq C2^{j(n+|\alpha|)}\frac{1}{(1+|2^j u|)^M}.$$

On the other hand, for any $L > 0$, we have

$$\left|\phi_j * f(u)\right| \leq C2^{-jL}\frac{1}{(1+|u|)^M}.$$

Set $L > n+|\alpha|$, and we get the desired result. By the duality argument, we obtain the series in (3.1) converges in $\mathcal{S}'$. Next we will show that

$$\|f\|^\phi_{H^{\alpha(\cdot)}} \sim \|f\|^\psi_{H^{\alpha(\cdot)}};$$

the proof for $\Lambda^{\alpha(\cdot)}$ is similar.

To conclude the proof, applying the Calderón identity, the classical almost-orthogonality argument, and Theorem 1.2, we get that for any $j \geq 0$,

$$\begin{aligned} 2^{j\alpha(x)}\left|\phi_j * f(x)\right| &= \sup_{j\geq 0, x\in\mathbb{R}^n} 2^{j\alpha(x)}\Big|\sum_{j'\geq 0}\phi_j * \psi_{j'} * \psi_{j'} * f(x)\Big| \\ &\leq \sup_{j'\geq 0, x\in\mathbb{R}^n} 2^{j'\alpha(x)}\left|\psi_{j',k'} * f(x)\right| \leq C\|f\|^\psi_{H^{\alpha(\cdot)}}. \end{aligned}$$

It follows that

$$\|f\|^\phi_{H^{\alpha(\cdot)}} \leq C\|f\|^\psi_{H^{\alpha(\cdot)}}.$$

Similarly, by (3.1), the classical almost-orthogonality argument, and Theorem 1.2, we get

$$\|f\|^\psi_{H^{\alpha(\cdot)}} \leq C\|f\|^\phi_{H^{\alpha(\cdot)}}.$$

Therefore, the proof of Proposition 3.1 is concluded. □

The following proposition plays a key role in the proof of Theorem 1.6.

Proposition 3.2. *Let $\alpha(\cdot) \in LH_0 \cap \mathcal{P}^0$. If $f \in H^{\alpha(\cdot)}$ or $\Lambda^{\alpha(\cdot)}$, then there exists a sequence $\{f_n\} \in \mathcal{B}^{\alpha^+}_{2,2} \cap H^{\alpha(\cdot)}$ or $\mathcal{B}^{\alpha^+}_{2,2} \cap \Lambda^{\alpha(\cdot)}$ such that f_n converges to f in the distribution sense, where $\mathcal{B}^{\alpha^+}_{2,2}$ is the classical Besov space. Furthermore,*

$$\|f_n\|_{H^{\alpha(\cdot)}} \leq \|f\|_{H^{\alpha(\cdot)}}, \qquad \|f_n\|_{\Lambda^{\alpha(\cdot)}} \leq \|f\|_{\Lambda^{\alpha(\cdot)}}.$$

Proof. Suppose that $f \in H^{\alpha(\cdot)}$; then we have the inhomogeneous Calderón's identity

$$f(x) = \sum_{j\geq 0} \psi_j * \psi_j * f(x) \quad \text{in } \mathcal{S}'. \tag{3.3}$$

The partial sum of the above series will be denoted by f_n and is given by

$$f_n(x) = \sum_{0\leq j\leq n} \psi_j * \psi_j * f(x).$$

Then we get that

$$\|f_n\|_{\mathcal{B}^{\alpha^+}_{2,2}} < \infty.$$

In fact, applying the fact that $|\psi_j * f(x)| \leq C2^{-j\alpha(x)}$ proved in Theorem 1.2 yields $\|\psi_j * \psi_j * f\|_{\mathcal{B}^{\alpha^+}_{2,2}} \leq C$.

For any $g \in \mathcal{S}$, choosing $m \geq n > 0$, we obtain

$$\begin{aligned} \big|\langle f - f_n, g\rangle\big| &\leq \liminf_{m\to\infty} \big|\langle f_m - f_n, g\rangle\big| \\ &\leq \liminf_{m\to\infty} \Big|\Big\langle \sum_{n<j\leq m} \psi_j * \psi_j * f, g\Big\rangle\Big| \to 0, \quad \text{as } n \to 0, \end{aligned}$$

where the last inequality follows from the fact that the series in (3.3) converges in $\mathcal{S}'$. Thus, $f_n \in \mathcal{B}^{\alpha^+}_{2,2}$ and converges to f in the distribution sense.

To conclude the proof, note that

$$\psi_j * f_n(x) = \sum_{0\leq j'\leq n} \psi_j * \psi_{j'} * \psi_{j'} * f(x),$$

and by Theorem 1.2, it follows that

$$\|f_n\|_{H^{\alpha(\cdot)}} \leq C \sup_{j\geq 0, x\in\mathbb{R}^n} 2^{j\alpha(x)}\big|\psi_j * f_n(x)\big|.$$

Again applying the almost-orthogonal estimate and Theorem 1.2, for any $j \geq 0$ we have

$$2^{j\alpha(x)}\big|\psi_j * f_n(x)\big| \leq C \sup_{j'\geq 0} 2^{j'\alpha(x)}\big|\psi_{j'} * f(x)\big| \leq C\|f\|_{H^{\alpha(\cdot)}}.$$

By similar argument, we can prove $\|f_n\|_{\Lambda^{\alpha(\cdot)}} \leq \|f\|_{\Lambda^{\alpha(\cdot)}}$. Therefore, the proof of Proposition 3.2 is completed. □

Now we prove Theorem 1.6.

Proof of Theorem 1.6. We will prove that T is a bounded operator on $H^{\alpha(\cdot)}$ with $\alpha^+ < \epsilon$ for any $f \in \mathcal{B}^{\alpha^+}_{2,2} \cap H^{\alpha(\cdot)}$. In fact, by Theorem 1.2 and Proposition 3.1, it follows that

$$\|Tf\|_{H^{\alpha(\cdot)}} \leq C \sup_{j\geq 0, x\in\mathbb{R}^n} 2^{j\alpha(x)}\big|\phi_j * Tf(x)\big|. \tag{3.4}$$

First we claim that

$$\begin{aligned}\big|\phi_j T\phi_{j'}(x,y)\big| &:= \int\int \phi_j(x-u)K(u,v)\phi_{j'}(v-y)\,du\,dv\\ &\le C(2^{(j'-j)\epsilon}\wedge 1)\frac{1+(j-j\wedge j')}{[2^{-(j\wedge j')}+|x-y|]^{n+\sigma}},\end{aligned}\tag{3.5}$$

where $\sigma=\delta$ when $j=0$ or $j'=0$, otherwise $\sigma=\epsilon$. By the fact that T is bounded on $\mathcal{B}^{\alpha^+}_{2,2}$ for $0<\alpha^+<\epsilon$ given in [13], we then get that

$$\phi_j * Tf(x) = \sum_{j'\ge 0}(\phi_j T\phi_{j'}) * \phi_{j'} * f(x).\tag{3.6}$$

To prove the claim, we will consider the cases where $j,j'>0$, $j=0,j'>0$ and $j'=0,j>0$. (The idea here comes from [10].) When $j,j'>0$ we consider the following four cases:

Case 1: $j>j'$ and $|x-y|\le 5\;2^{-j'}$. Since $T(1)=0$, we have

$$\begin{aligned}\phi_j T\phi_{j'}(x,y) &= \int\int \phi_j(x-u)K(u,v)\phi_{j'}(v-y)\,du\,dv\\ &= \int\int \phi_j(x-u)K(u,v)\big(\phi_{j'}(v-y)-\phi_{j'}(x-y)\big)\,du\,dv.\end{aligned}$$

Choose a smooth function η_0 such that $\operatorname{supp}\eta_0\subset\{x:|x|\le 6\}$, and let $\eta_0=1$ when $|x|\le 2$. Set $\eta_1=1-\eta_0$. Then we get

$$\begin{aligned}&\big|\phi_j T\phi_{j'}(x,y)\big|\\ &\quad= \Big|\int\int \phi_j(x-u)K(u,v)\big(\phi_{j'}(v-y)-\phi_{j'}(x-y)\big)\eta_0\big(2^j(v-x)\big)\,du\,dv\Big|\\ &\qquad+\Big|\int\int \phi_j(x-u)K(u,v)\big(\phi_{j'}(v-y)-\phi_{j'}(x-y)\big)\eta_1\big(2^j(v-x)\big)\,du\,dv\Big|\\ &\quad= I+II.\end{aligned}$$

For I, we denote $\varphi(v)=(\psi_{j'}(v-y)-\psi_{j'}(x-y))\eta_0(2^j(v-x))$ and $\omega(u)=\phi_j(x-u)$. Since $T\in WBP$, we have

$$\begin{aligned}I = \big|\langle T\varphi,\omega\rangle\big| &\le C2^{-j(n+2\eta)}\|\varphi\|_{\dot{\mathcal{C}}_\eta}\|\omega\|_{\dot{\mathcal{C}}_\eta}\\ &\le C2^{-j(n+2\eta)}\{2^{-(j-j')}2^{j'n}2^{j\eta}\}\{2^{jn}2^{j\eta}\}\\ &\le C2^{-(j-j')}2^{j'n}.\end{aligned}$$

We now deal with the term II. By the cancellation condition of ϕ, we get

$$\begin{aligned}II &= \Big|\int\int \phi_j(x-u)\big[K(u,v)-K(x,v)\big]\\ &\qquad\times\big(\phi_{j'}(v-y)-\phi_{j'}(x-y)\big)\eta_1\big(2^j(v-x)\big)\,du\,dv\Big|\\ &\le C\big(1+(j-j')\big)2^{(j'-j)\epsilon}2^{j'n}.\end{aligned}$$

Case 2: $j > j'$ and $|x-y| \geq 5\ 2^{-j'}$. In this case, it is easy to see that $|x-y| \sim |u-v|$. Using the smoothness condition on the kernel $K(u,v)$, we have

$$\begin{aligned}\phi_j T\phi_{j'}(x,y) &= \int\int \phi_j(x-u)K(u,v)\phi_{j'}(v-y)\,du\,dv\\ &= \int\int \phi_j(x-u)\big[K(u,v)-K(x,v)\big]\big(\phi_{j'}(v-y)\big)\,du\,dv\\ &\leq C\frac{2^{-j\epsilon}}{|x-y|^{n+\epsilon}}.\end{aligned}$$

Case 3: $j \leq j'$ and $|x-y| \leq 5\ 2^{-j}$. In this case, we have

$$\begin{aligned}&\big|\phi_j T\phi_{j'}(x,y)\big|\\ &\quad= \Big|\int\int \phi_j(x-u)K(u,v)\phi_{j'}(v-y)\eta_0\big(2^{j'}(u-y)\big)\,du\,dv\Big|\\ &\qquad+\Big|\int\int \phi_j(x-u)K(u,v)\phi_{j'}(v-y)\eta_1\big(2^{j'}(u-y)\big)\,du\,dv\Big|\\ &\quad= I + II.\end{aligned}$$

For I, we denote $\tilde{\varphi}(u) = \phi_j(x-u)\eta_0(2^{j'}(u-y))$ and $\tilde{\phi}(u) = \phi_{j'}(v-y)$. Since $T \in WBP$, we have

$$\begin{aligned}I = \big|\langle T\tilde{\varphi},\tilde{\phi}\rangle\big| &\leq C2^{-j'(n+2\eta)}\|\tilde{\varphi}\|_{\dot{C}_\eta}\|\tilde{\phi}\|_{\dot{C}_\eta}\\ &\leq C2^{-j'(n+2\eta)}\{2^{jn}2^{j'\eta}\}\{2^{j'n}2^{j'\eta}\}\\ &\leq C2^{jn}.\end{aligned}$$

For II, observing that $|u-v| \geq C2^{-j}$ and the size condition of kernel K, we obtain

$$\begin{aligned}II &= \Big|\int\int \phi_j(x-u)K(u,v)\phi_{j'}(v-y)\eta_1\big(2^{j'}(u-y)\big)\,du\,dv\Big|\\ &\leq \Big|\int\int \phi_j(x-u)\frac{1}{|u-v|^n}\psi_{j'}(v-y)\eta_1\big(2^{j'}(u-y)\big)\,du\,dv\Big|\\ &\leq C2^{jn}.\end{aligned}$$

Case 4: $j \leq j'$ and $|x-y| \geq 5\ 2^{-j}$. Noting that $|x-y| \sim |u-v|$ and using the fact that $\phi_j 1 = 0$ and the smoothness condition on the kernel $K(u,v)$, we have

$$\begin{aligned}\big|\phi_j T\phi_{j'}(x,y)\big| &= \Big|\int\int \phi_j(x-u)K(u,v)\phi_{j'}(v-y)\,du\,dv\Big|\\ &= \Big|\int\int \phi_j(x-u)\big[K(u,v)-K(x,v)\big]\big(\phi_{j'}(v-y)\big)\,du\,dv\Big|\\ &\leq C\frac{2^{-j\epsilon}}{|x-y|^{n+\epsilon}}.\end{aligned}$$

The other cases are similar but simple. Thus, we prove the claim.

Observe that $0 < \alpha^+ < \epsilon$. Combining (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned} &\|Tf\|_{H^{\alpha(\cdot)}} \\ &\quad\le C \sup_{j\ge 0, x\in\mathbb{R}^n} 2^{j\alpha(x)} \sum_{j'\ge 0} (2^{(j'-j)\epsilon}\wedge 1)\big(1+(j-j\wedge j')\big)\big|\phi_j * f(x)\big| \\ &\quad\le C \sup_{j\ge 0, x\in\mathbb{R}^n} \sum_{j'\ge 0} 2^{j'\alpha(x)} 2^{(j-j')\alpha(x)} 2(2^{(j'-j)\epsilon}\wedge 1)\big(1+(j-j\wedge j')\big)\big|\phi_j * f(x)\big| \\ &\quad\le C \sup_{j'\ge 0, x\in\mathbb{R}^n} 2^{j'\alpha(x)}\big|\phi_{j'} * f(x)\big| \le C\|f\|_{H^{\alpha(\cdot)}}. \end{aligned} \tag{3.7}$$

Next we can extend T to $H^{\alpha(\cdot)}$ as follows. By Proposition 3.2, if $f \in H^{\alpha(\cdot)}$, then there exists a sequence $\{f_n\} \in \mathcal{B}^{\alpha^+}_{2,2} \cap H^{\alpha(\cdot)}$ such that f_n converges to f in the distribution sense. Furthermore,

$$\|f_n\|_{H^{\alpha(\cdot)}} \le \|f\|_{H^{\alpha(\cdot)}}.$$

Using (3.7) shows that

$$\big\|T(f_n - f_m)\big\|_{H^{\alpha(\cdot)}} \le \|f_n - f_m\|_{H^{\alpha(\cdot)}}.$$

On the other hand, by duality, for any $g \in \mathcal{S}$ we get that

$$\big\langle T(f_n - f_m), g\big\rangle = \langle f_n - f_m, T^* g\rangle \to 0, \quad \text{as } n, m \to \infty,$$

where T^* is the adjoint operator of T. Hence, Tf_n converges in the distribution sense and we can define

$$Tf = \lim_{n\to\infty} Tf_n \quad \text{in } \mathcal{S}'.$$

Applying Theorem 1.2 again and Fatou's lemma, we get

$$\begin{aligned} \|Tf\|_{H^{\alpha(\cdot)}} &\le C \sup_{j\ge 0, x\in\mathbb{R}^n} 2^{j\alpha(x)}\big|\lim_{n\to\infty} \psi_j * Tf_n(x)\big| \\ &\le C \liminf_{n\to\infty} \sup_{j\ge 0, x\in\mathbb{R}^n} 2^{j\alpha(x)}\big|\psi_j * Tf_n(x)\big| \\ &\le C \liminf_{n\to\infty} \|f_n\|_{H^{\alpha(\cdot)}} \le C\|f\|_{H^{\alpha(\cdot)}}. \end{aligned}$$

Therefore, we conclude the proof of Theorem 1.6. □

4. Remark

In this last section, we remark that a class of the pseudodifferential operators is continuous on the inhomogeneous Hölder–Zygmund spaces of variable order, although these operators are not, in general, continuous on L^2 (see [13]).

Repeating the analogous argument in the proof of Theorem 1.6, we can obtain the following Proposition.

Proposition 4.1. *Let $\alpha(\cdot) \in LH_0 \cap \mathcal{P}^0$. Suppose that the kernel $K(x,y)$ of T satisfying the following estimates for $|x-y| \ge 1$,*

(i) $|\partial_x^\beta K(x,y)| \le C_2 \frac{1}{|x-y|^N}$ *for any* $|\beta| \le \gamma$ *and* $N > 1$;

(ii) $|\partial_x^\beta K(x,y) - \partial_x^\beta K(x',y)| \le C_2 \frac{|x-y|^\epsilon}{|x-y|^N}$

where $m \in \mathbb{N}$ and r are defined by $\gamma = m + \epsilon$ with $0 < \epsilon \leq 1$ and where $|\beta| = m$ and $|x - x'| \leq 1/2|x - y|$. Also, $T(x^\beta) = 0$ when $|\beta| \leq m$ and $T \in WBP$. Then T can be extended to a bounded linear operator on $H^{\alpha(\cdot)}$ and $\Lambda^{\alpha(\cdot)}$ for any $\alpha^+ \leq \gamma$.

Remark 4.2. An immediate result of the proposition is that the pseudodifferential operators $T \in \mathcal{O}pS^0_{1,1}$ (whose symbols fulfill that $|\partial^\alpha_\xi \partial^\beta_x \sigma(x, \xi)| \leq C(\alpha, \beta)(1 + |\xi|)^{|\beta| - |\alpha|}$) are continuous on the inhomogeneous Hölder–Zygmund spaces of variable order, since the corresponding kernel $K(x, y)$ of the symbol $\sigma(x, \xi)$ satisfies

$$\left|\partial^\alpha_x \partial^\beta_y K(x, y)\right| \leq C_3 \frac{1}{|x - y|^{(n+|\alpha|+|\beta|)}},$$

when $|x - y| \leq 1$, and

$$\left|\partial^\alpha_x \partial^\beta_y K(x, y)\right| \leq C_3 \frac{1}{|x - y|^N}$$

for all $N \geq 1$, where $|x - y| \geq 1$ ($\alpha, \beta \in \mathbb{N}^n$).

Acknowledgments. The authors wish to express their heartfelt thanks to the anonymous referee for many corrections and valuable suggestions that improved the paper significantly. We especially thank the referee for pointing out some shortcomings of the original proof in Theorem 1.2.

The authors' work was partially supported by National Natural Science Foundation of China (NSFC) grant 11471040 and by Fundamental Research Funds for the Central Universities grant 2014KJJCA10.

References

1. A. Almeida and P. Hästö, *Besov spaces with variable smoothness and integrability*, J. Funct. Anal. **258** (2010), no. 5, 1628–1655. Zbl 1194.46045. MR2566313. DOI 10.1016/j.jfa.2009.09.012. 72, 73, 75, 76
2. A. Almeida and S. Samko, *Embeddings of variable Hajłasz-Sobolev spaces into Hölder spaces of variable order*, J. Math. Anal. Appl. **353** (2009), no. 2, 489–496. Zbl 1176.46036. MR2508952. DOI 10.1016/j.jmaa.2008.12.034. 72, 73
3. J. Cheeger. *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **9** (1999), no. 3, 428–517. Zbl 0942.58018. MR1708448. DOI 10.1007/s000390050094. 72
4. D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue spaces: Foundations and Harmonic Analysis*, Birkhäuser, New York, 2013. Zbl 1268.46002. MR3026953. DOI 10.1007/978-3-0348-0548-3. 73
5. G. David, J. Journé, and S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation*, Rev. Mat. Iberoam. **1** (1985), no. 4, 1–56. Zbl 0604.42014. MR0850408. DOI 10.4171/RMI/17. 76
6. L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Math. **2017**, Springer, Heidelberg, 2011. Zbl 1222.46002. MR2790542. DOI 10.1007/978-3-642-18363-8. 73
7. L. Grafakos, *Modern Fourier Analysis*, 3rd ed., Grad. Texts in Math. **250**, Springer, New York, 2014. Zbl 1304.42002. MR3243741. DOI 10.1007/978-1-4939-1230-8. 72, 79, 80
8. Y. Han and Y. Han, *Boundedness of composition operators associated with different homogeneities on Lipschitz spaces*, Math. Res. Lett. **23** (2016), no. 5, 1387–1403. Zbl 06686300. MR3601071. DOI 10.4310/MRL.2016.v23.n5.a7. 72

9. Y. Han, S. Lu, and D. Yang, *Inhomogeneous Besov and Triebel-Lizorkin spaces on spaces of homogeneous type*, Approx. Theory Appl. (N.S.) **15** (1999), no. 3, 37–65. Zbl 0957.46025. MR1746473. 75
10. Y. Han and E. Sawyer, *Para-accretive functions, the weak boundedness property and the Tb Theorem*, Rev. Mat. Iberoam. **6** (1990), no. 1–2, 17–41. Zbl 0723.42005. MR1086149. DOI 10.4171/RMI/93. 82
11. E. Harboure, O. Salinas, and B. Viviani, *Boundedness of the fractional integral on weighted Lebesgue and Lipschitz spaces*, Trans. Amer. Math. Soc. **349** (1997), no. 1, 235–255. Zbl 0865.42017. MR1357395. DOI 10.1090/S0002-9947-97-01644-9. 72
12. S. G. Krantz. *Lipschitz spaces on stratified groups*, Trans. Amer. Math. Soc. **269** (1982), no. 1, 39–66. Zbl 0529.22006. MR0637028. DOI 10.2307/1998593. 72
13. Y. Meyer and R. Coifman, *Wavelets: Calderón-Zygmund and Multilinear Operators*, Cambridge Stud. Adv. Math. **48**, Cambridge Univ. Press, Cambridge, 1997. Zbl 0916.42023. MR1456993. 73, 75, 82, 84
14. J. Peetre, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces*, J. Funct. Anal. **4** (1969), 71–87. Zbl 0175.42601. MR0241965. 72
15. B. Ross and S. Samko, *Fractional integration operator of variable order in the Hölder spaces $H^{\lambda(x)}$*, Int. J. Math. Math. Sci. **18** (1995), no. 4, 777–788. Zbl 0838.26005. MR1347069. DOI 10.1155/S0161171295001001. 72, 73
16. E. M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. **43**, Princeton Univ. Press, Princeton, 1993. Zbl 0821.42001. MR1232192. 72, 73
17. E. M. Stein and P. Yung, *Pseudodifferential operators of mixed type adapted to distributions of k-planes*, Math. Res. Lett. **20** (2013), no. 6, 1183–1208. Zbl 1312.47060. MR3228630. DOI 10.4310/MRL.2013.v20.n6.a15. 73
18. H. Triebel, *Theory of Function Spaces*, Monogr. Math. **78**, Birkhäuser, Basel, 1983. Zbl 0546.46027. MR0781540. DOI 10.1007/978-3-0346-0416-1. 72

School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, People's Republic of China.
E-mail address: tanjian89@126.com; jzhao@bnu.edu.cn