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NONLINEAR ISOMETRIES BETWEEN FUNCTION SPACES

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Abstract. We demonstrate that any surjective isometry $T\colon \mathcal{A} \to \mathcal{B}$ not assumed to be linear between unital, completely regular subspaces of complex-valued, continuous functions on compact Hausdorff spaces is of the form

$$T(f) = T(0) + \mathrm{Re}\big[\mu \cdot (f \circ \tau)\big] + i\,\mathrm{Im}\big[\nu \cdot (f \circ \rho)\big],$$

where μ and ν are continuous and unimodular, there exists a clopen set K with $\nu = \mu$ on K and $\nu = -\mu$ on K^c, and τ and ρ are homeomorphisms.

1. Introduction

When investigating a mathematical object, it is worthwhile to study mappings that leave the relevant structures undisturbed. For example, the collection $C(X)$ of complex-valued, continuous functions on a compact Hausdorff space X is a normed vector space under the uniform norm $\|\cdot\|$, and so it is of interest to characterize the surjective, *complex-linear* isometries $T\colon C(X) \to C(Y)$. This was done by both Banach [2] and Stone [10], and such mappings are of the form

$$T(f) = \mu \cdot (f \circ \tau), \tag{1.1}$$

where $|\mu(y)| = 1$ for all $y \in Y$ and where $\tau\colon Y \to X$ is a homeomorphism.

This classic result has been extended to mappings between subspaces of $C(X)$ and $C(Y)$, and a general survey of such results can be found in [4]. We note one in

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particular: Myers [9] analyzed linear isometries between *completely regular subspaces*; that is, for subspaces $\mathcal{A}$ such that, given $x \in X$ and an open neighborhood U of x, there is an $f \in \mathcal{A}$ with $1 = |f(x)| = \|f\|$ and $|f| < 1$ on $X \setminus U$.

For a general surjective isometry T between subspaces of continuous functions, the Mazur–Ulam theorem [7, théorème] ensures that $T - T(0)$ is *real-linear*, and so it is a natural extension to characterize such mappings. There has been recent interest in this problem (see [8]), and the typical conclusion is that there is a clopen set K such that $T(f)|_K$ satisfies (1.1) and $T(f)$ is its conjugate on the complement K^c. However, there are other possibilities; for example, define $\mathcal{A}$ and $T\colon \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{A} = \bigl\{f(z) = az + b\colon a, b \in \mathbb{C}, |z| = 1\bigr\} \qquad \text{and} \qquad T(az+b) = az + \overline{b}.$$

It is known (see [6, Example 6.2]) that T is an isometry that cannot be of this form; however, note that $\mathcal{A}$ is completely regular and that T satisfies

$$T(az+b) = \operatorname{Re}[az+b] + i\operatorname{Im}\bigl[-\bigl(a(-z)+b\bigr)\bigr],$$

which suggests a possibility for the general isometries between such spaces.

The goal of this work is to give a complete characterization of surjective isometries $T\colon \mathcal{A} \to \mathcal{B}$ between completely regular subspaces. It is worth noting that a similar problem was recently investigated by Jamshidi and Sady [5]; however, our approach is significantly different. Instead of using the mapping T to induce a mapping $T^*\colon \mathcal{B}^* \to \mathcal{A}^*$ between the dual spaces and then investigating the extreme points of the unit ball thereof, we adapt Eilenberg's [3, Theorem 7.2] proof of the Banach–Stone theorem, whose arguments hinge on the fact that the maximal convex subsets of the unit sphere of $C(X)$ are essentially in a one-to-one correspondence with X.

We begin in Section 2 by demonstrating that this correspondence still holds for completely regular subspaces; in fact, it is shown that this is a necessary and sufficient condition for a subspace to be completely regular. Then we prove the following in Section 3.

Main Theorem. *Let $\mathcal{A} \subset C(X)$ and $\mathcal{B} \subset C(Y)$ be unital, completely regular spaces, and let $T\colon \mathcal{A} \to \mathcal{B}$ be a surjective mapping such that*

$$\bigl\|T(f) - T(g)\bigr\| = \|f - g\|$$

holds for all $f \in \mathcal{A}$. Then there exist continuous functions $\mu, \nu\colon Y \to \mathbb{C}$ with $|\mu(y)| = |\nu(y)| = 1$ for all $y \in Y$, a (potentially empty) clopen set $K \subset Y$ such that $\nu(y) = \mu(y)$ for $y \in K$ and $\nu(y) = -\mu(y)$ for $y \in Y \setminus K$, and (possibly distinct) homeomorphisms $\tau, \rho\colon Y \to X$ such that

$$T(f) = T(0) + \operatorname{Re}\bigl[\mu \cdot (f \circ \tau)\bigr] + i\operatorname{Im}\bigl[\nu \cdot (f \circ \rho)\bigr]$$

for all $f \in \mathcal{A}$.

2. Maximal convex sets of the unit sphere

Throughout this section, X is a compact Hausdorff space, $C(X)$ is the Banach space of complex-valued and continuous functions on X, and $\mathcal{A} \subset C(X)$ is a subspace. Specifically, $\mathcal{A}$ is nonempty and $\alpha f + \beta g \in \mathcal{A}$ for any $f, g \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$. Given $f \in \mathcal{A}$, we denote the maximizing set of f by

$$M(f) = \big\{x \in X \colon \big|f(x)\big| = \|f\|\big\},$$

and we note that $M(f)$ is nonempty since X is compact. Similarly, for any subset $\mathcal{F} \subset \mathcal{A}$, we define its maximizing set as

$$M(\mathcal{F}) = \bigcap_{f \in \mathcal{F}} M(f),$$

which is potentially empty.

Denote the unit sphere of $\mathcal{A}$ by

$$S_{\mathcal{A}} = \big\{f \in \mathcal{A} \colon \|f\| = 1\big\},$$

and denote the unit circle of $\mathbb{C}$ by

$$\mathbb{T} = \big\{z \in \mathbb{C} \colon |z| = 1\big\}.$$

The following lemmas regarding convex combinations in the unit sphere are straightforward to verify; however, they are essential for characterizing the maximal convex subsets of $S_{\mathcal{A}}$, and so we include their proofs for completeness.

Lemma 2.1. *Let $f_1, \dots, f_n \in S_{\mathcal{A}}$, and let $f = \sum_{k=1}^{n} f_k/n \in S_{\mathcal{A}}$. Then $M(f) \subset \bigcap_{k=1}^{n} M(f_j)$.*

Proof. Let $z \in M(f)$. Then $|f(z)| = \|f\| = 1$ since $f \in S_{\mathcal{A}}$, and we have

$$1 = \big|f(z)\big| = \Big|\sum_{k=1}^{n} \frac{f_k(z)}{n}\Big| \leq \frac{1}{n}\sum_{k=1}^{n}\big|f_k(z)\big|.$$

As $f_1, \dots, f_n \in S_{\mathcal{A}}$, it must be that $|f_k(z_0)| \leq 1$ holds for each $1 \leq k \leq n$. Suppose that $|f_k(z)| < 1$ for some k; then

$$1 \leq \frac{1}{n}\big(\big|f_1(z)\big| + \cdots + \big|f_n(z)\big|\big) < 1,$$

which is a contradiction. Therefore, $\|f_k\| = 1 = |f_k(z)|$ holds for all $1 \leq k \leq n$, and so $z \in \bigcap_{k=1}^{n} M(f_k)$. □

Lemma 2.2. *Let $f, g \in S_{\mathcal{A}}$ be such that $(1/2)[f+g] \in S_{\mathcal{A}}$. Then $f(x) = g(x)$ for any $x \in M((1/2)[f+g])$.*

Proof. Let $h = (1/2)[f+g]$, and let $x \in M(h)$. Then Lemma 2.1 implies that $x \in M(f) \cap M(g)$, and so $|f(x)| = |g(x)| = 1$. Since $h(x)$ is a convex combination of $f(x)$ and $g(x)$ and $h(x) \in \mathbb{T}$, it follows that $g(x) = f(x)$. □

Given an $x \in X$, we denote the collection of $f \in S_{\mathcal{A}}$ that maximize at x by

$$S_{\mathcal{A}}(x) = \{f \in S_{\mathcal{A}} : |f(x)| = 1\}$$

and with value $\alpha \in \mathbb{T}$ by

$$S_{\mathcal{A}}(x, \alpha) = \{f \in S_{\mathcal{A}} : f(x) = \alpha\}.$$

Note that $S_{\mathcal{A}}(x, \alpha)$ is a convex subset of $S_{\mathcal{A}}$, and this fact yields the following.

Lemma 2.3. *Let $x, y \in X$, and let $\alpha \in \mathbb{T}$ be such that $S_{\mathcal{A}}(x, \alpha) \subset S_{\mathcal{A}}(y)$. Then there exists a $\beta \in \mathbb{T}$ such that $S_{\mathcal{A}}(x, \alpha) \subset S_{\mathcal{A}}(y, \beta)$.*

Proof. First, we note that $y \in M(f)$ must hold for all $f \in S_{\mathcal{A}}(x, \alpha)$. Now, fix $f_0 \in S_{\mathcal{A}}(x, \alpha)$, and set $\beta = f_0(y) \in \mathbb{T}$. Given $f \in S_{\mathcal{A}}(x, \alpha)$, the convexity of this set implies that $(1/2)(f_0 + f) \in S_{\mathcal{A}}(x, \alpha) \subset S_{\mathcal{A}}(y)$. Therefore, $y \in M((1/2)[f_0 + f])$, and Lemma 2.2 implies that $f(y) = f_0(y) = \beta$. □

Furthermore, by the next lemma, any convex subset of $S_{\mathcal{A}}$ is contained in a set of the form $S_{\mathcal{A}}(x, \alpha)$.

Lemma 2.4. *Let $\mathcal{C} \subset S_{\mathcal{A}}$ be convex. Then there exists an $x \in X$ and an $\alpha \in \mathbb{T}$ such that $\mathcal{C} \subset S_{\mathcal{A}}(x, \alpha)$.*

Proof. For any $f_1, \ldots, f_n \in \mathcal{C}$, the convexity of $\mathcal{C}$ yields that $\sum_{k=1}^{n} f_k/n \in \mathcal{C}$, and so Lemma 2.1 implies that $\bigcap_{k=1}^{n} M(f_k)$ is nonempty. By the finite intersection property, we then have $M(\mathcal{C}) \neq \varnothing$. Fix $x \in M(\mathcal{C})$, $f_0 \in \mathcal{C}$, and set $\alpha = f_0(x)$. Given any $f \in \mathcal{C}$, we have $(1/2)[f_0 + f] \in \mathcal{C}$, and so $x \in M((1/2)[f_0 + f])$. Therefore, as $\mathcal{C} \subset S_{\mathcal{A}}$, Lemma 2.2 implies that $f(x) = f_0(x) = \alpha$, and thus $\mathcal{C} \subset S_{\mathcal{A}}(x, \alpha)$. □

In light of this, any maximal (with respect to inclusion) convex subset of $S_{\mathcal{A}}$ is of the form $S_{\mathcal{A}}(x, \alpha)$ for some $x \in X$ and $\alpha \in \mathbb{T}$. We say X is in *correspondence with the maximal convex subsets of* $S_{\mathcal{A}}$ if, given $x, y \in X$ and $\alpha, \beta \in \mathbb{T}$ with

$$S_{\mathcal{A}}(x, \alpha) \subset S_{\mathcal{A}}(y, \beta),$$

it holds that $x = y$. Note that $\alpha = \beta$ follows, and so $S_{\mathcal{A}}(x, \alpha) = S_{\mathcal{A}}(y, \beta)$; furthermore, this condition yields that $S_{\mathcal{A}}(x, \alpha)$ is maximal for each $x \in X$ and $\alpha \in \mathbb{T}$.

Lemma 2.5. *Let X be in correspondence with the maximal convex subsets of $S_{\mathcal{A}}$, and let $x \in X$ and $\alpha \in \mathbb{T}$. Then $S_{\mathcal{A}}(x, \alpha)$ is a maximal convex subset of $S_{\mathcal{A}}$.*

Proof. Let $\mathcal{C}$ be a convex subset of $S_{\mathcal{A}}$ with $S_{\mathcal{A}}(x, \alpha) \subset \mathcal{C}$. Lemma 2.4 implies that $\mathcal{C} \subset S_{\mathcal{A}}(y, \beta)$ for some $y \in Y$ and $\beta \in \mathbb{T}$, and so $S_{\mathcal{A}}(x, \alpha) = S_{\mathcal{A}}(y, \beta)$ must hold. Therefore, $S_{\mathcal{A}}(x, \alpha) = \mathcal{C}$, and so $S_{\mathcal{A}}(x, \alpha)$ must be maximal. □

Furthermore, this condition is equivalent to requiring that $\mathcal{A}$ be completely regular.

Lemma 2.6. *The subspace $\mathcal{A}$ is completely regular if and only if X is in correspondence with the maximal convex subsets of $S_{\mathcal{A}}$.*

Proof. Suppose that $\mathcal{A}$ is completely regular. Let $x, y \in X$ and $\alpha, \beta \in \mathbb{T}$ be such that $S_{\mathcal{A}}(x, \alpha) \subset S_{\mathcal{A}}(y, \beta)$. If $x \neq y$, then there exists an open neighborhood U of x with $y \notin U$. As $\mathcal{A}$ is completely regular, there is an $f \in \mathcal{A}$ such that $1 = |f(x)| = \|f\|$ and $|f(z)| < 1$ for all $z \in X \setminus U$. We may assume that $f(x) = \alpha$; otherwise, f is replaced with $\alpha\overline{f(x)}f$. Thus $f \in S_{\mathcal{A}}(x, \alpha) \subset S_{\mathcal{A}}(y, \beta)$, which yields the contradictory fact that $1 = |f(y)| < 1$.

Now, suppose that $\mathcal{A}$ is not completely regular. Then there exists an $x \in X$ and an open neighborhood U such that $M(f) \cap (X \setminus U) \neq \varnothing$ holds for all $f \in S_{\mathcal{A}}(x)$. We claim that the collection

$$\mathcal{F} = \big\{M(f) \cap (X \setminus U) \colon f \in S_{\mathcal{A}}(x)\big\}$$

of closed sets has the finite intersection property. Indeed, let $f_1, \ldots, f_n \in S_{\mathcal{A}}(x)$. Set

$$f = \sum_{k=1}^{n} \frac{\overline{f_k(x)} f_k}{n}.$$

Then $f \in S_{\mathcal{A}}(x)$ holds. Consequently, Lemma 2.1 implies that

$$\varnothing \neq M(f) \cap (X \setminus U) \subset \Big(\bigcap_{k=1}^{n} M\big(\overline{f_k(x)} f_k\big)\Big) \cap (X \setminus U) = \bigcap_{k=1}^{n} \big[M(f_k) \cap (X \setminus U)\big].$$

Consequently, there exists a $y \in M(f) \cap (X \setminus U)$ for all $f \in S_{\mathcal{A}}(x)$, and we note that $x \neq y$ and $S_{\mathcal{A}}(x, 1) \subset S_{\mathcal{A}}(y)$ hold. Therefore, Lemma 2.3 implies that there exists a $\beta \in \mathbb{T}$ such that $S_{\mathcal{A}}(x, 1) \subset S_{\mathcal{A}}(y, \beta)$, and so X fails to be in correspondence with the maximal convex subsets of $S_{\mathcal{A}}$. □

We conclude this section with a result that we will repeatedly use, which is inspired by arguments made by Araujo and Font in [1, Lemma 2.3].

Lemma 2.7. *Let $\mathcal{A}$ be completely regular, $x_0 \in X$, $f \in \mathcal{A}$, $\alpha \in \mathbb{T}$, and let $\varepsilon > 0$ be such that $|f(x_0)| < \varepsilon$. Then there exist an $h \in S_{\mathcal{A}}(x_0, \alpha)$ and an $M > 0$ such that $\|f + Mh\| < \varepsilon + M$.*

Proof. Let $U = \{x \in X \colon |f(x)| < \varepsilon\}$. Since $\mathcal{A}$ is completely regular, there exists an $h \in \mathcal{A}$ such that $1 = |h(x_0)| = \|h\|$ and $M(h) \subset U$. We can assume that $h \in S_{\mathcal{A}}(x_0, \alpha)$. As $X \setminus U$ is compact, there is an $s < 1$ with

$$s = \sup\big\{\big|h(x)\big| \colon x \in X \setminus U\big\}.$$

Choose $M > 0$ such that $\|f\| < \varepsilon + M(1 - s)$. Then $\|f\| + Ms < \varepsilon + M$. For $x \in U$, we have

$$\big|f(x) + Mh(x)\big| < \varepsilon + M,$$

and for $x \in X \setminus U$, it must be that

$$\big|f(x) + Mh(x)\big| < \|f\| + Ms < \varepsilon + M. \qquad \square$$

3. Nonlinear isometries between completely regular subspaces

In this section, X and Y are compact Hausdorff spaces, and $\mathcal{A} \subset C(X)$ and $\mathcal{B} \subset C(Y)$ are unital (the constant function 1 belongs to both $\mathcal{A}$ and $\mathcal{B}$), completely regular subspaces. Moreover, $T\colon \mathcal{A} \to \mathcal{B}$ is a surjective isometry that is real-linear, which is to say that

$$T(rf + sg) = rT(f) + sT(g) \qquad \text{and} \qquad \big\|T(f) - T(g)\big\| = \|f - g\|$$

hold for all $r, s \in \mathbb{R}$ and $f, g \in \mathcal{A}$. We will prove the following result regarding such mappings.

Theorem 3.1. *There exist continuous functions $\mu, \nu\colon Y \to \mathbb{T}$ and a clopen set $K \subset Y$ with $\nu(y) = \mu(y)$ for $y \in K$ and $\nu(y) = -\mu(y)$ for $y \in Y \setminus K$, and there exist homeomorphisms $\tau, \rho\colon Y \to X$ such that*

$$T(f) = \operatorname{Re}\big[\mu \cdot (f \circ \tau)\big] + i \operatorname{Im}\big[\nu \cdot (f \circ \rho)\big] \tag{3.1}$$

for all $f \in \mathcal{A}$.

The main theorem is thus a corollary of this theorem combined with the Mazur–Ulam theorem, and we will prove Theorem 3.1 via a sequence of lemmas.

As T is a surjective, real-linear isometry, it must be bijective and its inverse T^{-1} is also a real-linear isometry. As $\mathcal{B}$ is completely regular, Lemma 2.5 yields that $T^{-1}[S_{\mathcal{B}}(y, \lambda)]$ is a maximal convex subset of $S_{\mathcal{A}}$, where $y \in Y$ and $\lambda \in \mathbb{T}$. Moreover, Lemma 2.4 implies that there exist $x \in X$ and $\alpha \in \mathbb{T}$ with $T^{-1}[S_{\mathcal{B}}(y, \lambda)] = S_{\mathcal{A}}(x, \alpha)$, and Lemma 2.6 yields that these must be unique.

For each $\lambda \in \mathbb{T}$, we define mappings $\psi_\lambda\colon Y \to X$ and $\varphi_\lambda\colon Y \to \mathbb{T}$ by

$$T^{-1}\big[S_{\mathcal{B}}(y, \lambda)\big] = S_{\mathcal{A}}\big(\psi_\lambda(y), \varphi_\lambda(y)\big). \tag{3.2}$$

We begin by demonstrating that each $\psi_\lambda\colon Y \to X$ is a continuous bijection. As Y is compact and X is Hausdorff, it then follows that ψ_λ is a homeomorphism.

Lemma 3.2. *Let $\lambda \in \mathbb{T}$. Then the mapping $\psi_\lambda\colon Y \to X$ is injective.*

Proof. Let $y, z \in Y$ be such that $\psi_\lambda(z) = \psi_\lambda(y)$. The constant function λ belongs to both $S_{\mathcal{B}}(z, \lambda)$ and $S_{\mathcal{B}}(y, \lambda)$, and so (3.2) implies that

$$T^{-1}(\lambda) \in S_{\mathcal{A}}\big(\psi_\lambda(z), \varphi_\lambda(z)\big) \qquad \text{and} \qquad T^{-1}(\lambda) \in S_{\mathcal{A}}\big(\psi_\lambda(y), \varphi_\lambda(y)\big),$$

which implies that

$$\varphi_\lambda(z) = T^{-1}(\lambda)\big(\psi_\lambda(z)\big) = T^{-1}(\lambda)\big(\psi_\lambda(y)\big) = \varphi_\lambda(y)$$

must hold. Now, if $z \neq y$, then the complete regularity of $\mathcal{B}$ yields the existence of a $k \in S_{\mathcal{B}}(z, \lambda)$ such that $|k(y)| < 1$. By (3.2), we have $T^{-1}(k) \in S_{\mathcal{A}}(\psi_\lambda(z), \varphi_\lambda(z)) = S_{\mathcal{A}}(\psi_\lambda(y), \varphi_\lambda(y))$, which yields the contradictory $k \in S_{\mathcal{B}}(y, \lambda)$. Therefore, we must have $z = y$. ☐

Lemma 3.3. *Let $\lambda \in \mathbb{T}$. Then the mapping $\psi_\lambda\colon Y \to X$ is surjective.*

Proof. Let $x \in X$. Since T is a real-linear isometry, Lemmas 2.4, 2.5, and 2.6 yield the existence of $y \in Y$ and $\alpha \in \mathbb{T}$ with

$$T\big[S_{\mathcal{A}}(x,1)\big] = S_{\mathcal{B}}(y,\alpha).$$

Similarly,

$$T\big[S_{\mathcal{A}}(x,\overline{\alpha}\lambda)\big] = S_{\mathcal{B}}(w,\beta) \qquad \text{and} \qquad T\big[S_{\mathcal{A}}(x,\alpha\overline{\lambda})\big] = S_{\mathcal{B}}(z,\gamma)$$

for some $w, z \in Y$ and $\beta, \gamma \in \mathbb{T}$. Set $f = T^{-1}(\alpha)$, and note that $\overline{\alpha}\lambda f \in S_{\mathcal{A}}(x,\overline{\alpha}\lambda)$ and $\alpha\overline{\lambda} f \in S_{\mathcal{A}}(x,\alpha\overline{\lambda})$. As such, we arrive at the following inequalities:

$$\begin{aligned}
|1+\overline{\alpha}\beta| &= |\alpha+\beta| = \big|\alpha + T(\overline{\alpha}\lambda f)(w)\big| \le \big\|\alpha + T(\overline{\alpha}\lambda f)\big\| \\
&= \|f + \overline{\alpha}\lambda f\| = |1+\overline{\alpha}\lambda|, \\
|1-\overline{\alpha}\beta| &= |\alpha-\beta| = \big|\alpha - T(\overline{\alpha}\lambda f)(w)\big| \le \big\|\alpha - T(\overline{\alpha}\lambda f)\big\| \\
&= \|f - \overline{\alpha}\lambda f\| = |1-\overline{\alpha}\lambda|, \\
|1+\overline{\alpha}\gamma| &= |\alpha+\gamma| = \big|\alpha + T(\alpha\overline{\lambda} f)(z)\big| \le \big\|\alpha + T(\alpha\overline{\lambda} f)\big\| \\
&= \|f + \alpha\overline{\lambda} f\| = |1+\alpha\overline{\lambda}|, \\
|1-\overline{\alpha}\gamma| &= |\alpha-\gamma| = \big|\alpha - T(\alpha\overline{\lambda} f)(z)\big| \le \big\|\alpha - T(\alpha\overline{\lambda} f)\big\| \\
&= \|f - \alpha\overline{\lambda} f\| = |1-\alpha\overline{\lambda}|.
\end{aligned}$$

These inequalities force

$$\mathrm{Re}(\overline{\alpha}\beta) = \mathrm{Re}(\overline{\alpha}\lambda) = \mathrm{Re}(\overline{\alpha}\gamma).$$

And since $\{\overline{\alpha}\beta, \overline{\alpha}\lambda, \overline{\alpha}\gamma\} \subset \mathbb{T}$, it follows via the pigeonhole principle that at least two of these complex numbers must be equal. As such, there are three cases to consider.

If $\overline{\alpha}\beta = \overline{\alpha}\lambda$, then $\beta = \lambda$, and so (3.2) implies that

$$S_{\mathcal{A}}(x,\overline{\alpha}\lambda) = T^{-1}\big[S_{\mathcal{B}}(w,\beta)\big] = T^{-1}\big[S_{\mathcal{B}}(w,\lambda)\big] = S_{\mathcal{A}}\big(\psi_\lambda(w),\varphi_\lambda(w)\big),$$

$x = \psi_\lambda(w)$.

Similarly, if $\overline{\alpha}\lambda = \overline{\alpha}\gamma$, then $\lambda = \gamma$, and so

$$S_{\mathcal{A}}(x,\alpha\overline{\lambda}) = T^{-1}\big[S_{\mathcal{B}}(z,\gamma)\big] = T^{-1}\big[S_{\mathcal{B}}(z,\lambda)\big] = S_{\mathcal{A}}\big(\psi_\lambda(z),\varphi_\lambda(z)\big),$$

which gives $x = \psi_\lambda(z)$.

Finally, suppose that $\overline{\alpha}\beta = \overline{\alpha}\gamma$. Then $\beta = \gamma$, and so

$$\begin{aligned}
\beta &\in S_{\mathcal{B}}(w,\beta) = T\big[S_{\mathcal{A}}(x,\overline{\alpha}\lambda)\big] \qquad \text{and} \\
\beta &\in S_{\mathcal{B}}(z,\beta) = S_{\mathcal{B}}(z,\gamma) = T\big[S_{\mathcal{A}}(x,\alpha\overline{\lambda})\big]
\end{aligned}$$

hold. It follows that $\overline{\alpha}\lambda = \alpha\overline{\lambda} = \overline{\overline{\alpha}\lambda}$ holds, and thus $\overline{\alpha}\lambda \in \mathbb{R}$. As $|\overline{\alpha}\lambda| = 1$, we either have $\lambda = \alpha$ or $\lambda = -\alpha$. In the former case, we have

$$S_{\mathcal{A}}(x,1) = T^{-1}\big[S_{\mathcal{B}}(y,\alpha)\big] = T^{-1}\big[S_{\mathcal{B}}(y,\lambda)\big] = S_{\mathcal{A}}\big(\psi_\lambda(y),\varphi_\lambda(y)\big),$$

and so $x = \psi_\lambda(y)$. For the latter case, note that for any $f \in S_{\mathcal{A}}(x,-1)$, we have $-f \in S_{\mathcal{A}}(x,1)$, and so $T(-f) \in S_{\mathcal{B}}(y,\alpha)$, which implies that $T(f) \in S_{\mathcal{B}}(y,-\alpha) = S_{\mathcal{B}}(y,\lambda)$. Therefore, $S_{\mathcal{A}}(x,-1) \subset T^{-1}[S_{\mathcal{B}}(y,\lambda)] = S_{\mathcal{A}}(\psi_\lambda(y),\varphi_\lambda(y))$, and so $x = \psi_\lambda(y)$ follows from Lemma 2.6. $\square$

Lemma 3.4. *Let $\lambda \in \mathbb{T}$. Then the mapping $\psi_\lambda \colon Y \to X$ is continuous.*

Proof. Let $U \subset X$ be open, and fix $y_0 \in \psi_\lambda^{-1}[U]$. As $\psi_\lambda(y_0) \in U$, the complete regularity of $\mathcal{A}$ yields the existence of an $h \in S_{\mathcal{A}}(\psi_\lambda(y_0), \varphi_\lambda(y_0))$ with $M(h) \subset U$. Set

$$\varepsilon = \sup\{|h(x)| : x \in X \setminus U\},$$

and define

$$W = \{y \in Y : \varepsilon < \operatorname{Re}[\overline{\lambda} T(h)(y)]\}.$$

Note that $\varepsilon < 1 = \operatorname{Re}[\overline{\lambda} T(h)(y_0)]$ holds, and so W is an open neighborhood of y_0. We claim that $W \subset \psi_\lambda^{-1}[U]$, and thus $\psi_\lambda^{-1}[U]$ must be open. Indeed, let $z \in W$. If $\psi_\lambda(z) \in X \setminus U$, then $|h(\psi_\lambda(z))| < \varepsilon$, and so Lemma 2.7 implies that there exists an $M > 0$ and a $k \in S_{\mathcal{A}}(\psi_\lambda(z), \varphi_\lambda(z))$ such that $\|h + Mk\| < \varepsilon + M$. Since $z \in W$, it follows that

$$\begin{aligned}
\varepsilon + M &< \operatorname{Re}[\overline{\lambda} T(h)(z) + M] \\
&\le |\overline{\lambda} T(h)(z) + M| \\
&= |T(h)(z) + M\lambda| \\
&= |T(h)(z) + MT(k)(z)| \\
&\le \|T(h) + MT(k)\| \\
&= \|h + Mk\| < \varepsilon + M,
\end{aligned}$$

which is contradictory. Therefore, it must be that $\psi_\lambda(z) \in U$, and so $z \in \psi_\lambda^{-1}[U]$. □

Let us now prove a zero preservation property.

Lemma 3.5. *Let $y \in Y$, $\lambda \in \mathbb{T}$, and $f \in \mathcal{A}$ be such that $f(\psi_\lambda(y)) = 0$. Then $\operatorname{Re}[\overline{\lambda} T(f)(y)] = 0$.*

Proof. Suppose that $\operatorname{Re}[\overline{\lambda} T(f)(y)] \neq 0$. We can assume that $\operatorname{Re}[\overline{\lambda} T(f)(y)] = 1$; if not, then f is adjusted by an appropriate real scalar.

Since $|f(\psi_\lambda(y))| < 1$, Lemma 2.7 yields the existence of an $h \in S_{\mathcal{A}}(\psi_\lambda(y), \varphi_\lambda(y))$ and an $M > 0$ with $\|f + Mh\| < 1 + M$. As (3.2) gives $T(h) \in S_{\mathcal{B}}(y, \lambda)$, we have

$$\begin{aligned}
1 + M &= \operatorname{Re}[\overline{\lambda} T(f)(y) + M] \\
&\le |\overline{\lambda} T(f)(y) + M| \\
&= |T(f)(y) + M\lambda| \\
&= |T(f)(y) + MT(h)(y)| \\
&\le \|T(f) + MT(h)\| \\
&= \|f + Mh\| < 1 + M,
\end{aligned}$$

which is a contradiction. □

Next, we verify that φ_1 and φ_i differ by a scaling of $\pm i$.

Lemma 3.6. *Let $y \in Y$. Then $\varphi_i(y) = \pm i\varphi_1(y)$.*

Proof. First, we demonstrate that the function

$$T\Big(\frac{\varphi_1(y)+\varphi_i(y)}{\sqrt{2}}\Big)$$

has norm less than or equal to 1. Indeed, the constant function $\varphi_1(y)$ belongs to $S_{\mathcal{A}}(\psi_1(y),\varphi_1(y))$, and thus (3.2) implies that $T(\varphi_1(y)) \in S_{\mathcal{B}}(y,1)$. Let $g = T^{-1}(iT(\varphi_1(y)))$. Then $T(g) \in S_{\mathcal{B}}(y,i)$, and so $g \in S_{\mathcal{A}}(\psi_i(y),\varphi_i(y))$. The fact that T is a real-linear isometry then yields

$$\begin{aligned}\big|\varphi_1(y)+\varphi_i(y)\big| &= \big|\varphi_1(y)+g\big(\psi_i(y)\big)\big| \\ &\le \big\|\varphi_1(y)+g\big\| \\ &= \big\|T\big(\varphi_1(y)\big)+T(g)\big\| \\ &= \big\|(1+i)T\big(\varphi_1(y)\big)\big\| = \sqrt{2},\end{aligned}$$

and so

$$\Big\|T\Big(\frac{\varphi_1(y)+\varphi_i(y)}{\sqrt{2}}\Big)\Big\| = \Big\|\frac{\varphi_1(y)+\varphi_i(y)}{\sqrt{2}}\Big\| \le 1.$$

Now, let $\alpha = (1/\sqrt{2})[1+i]$. By (3.2), we have that the constant function $\varphi_i(y)$ satisfies $T(\varphi_i(y)) \in S_{\mathcal{B}}(y,i)$, and thus the real linearity of T then implies that

$$T\Big(\frac{\varphi_1(y)+\varphi_i(y)}{\sqrt{2}}\Big)(y) = \frac{T(\varphi_1(y))(y)+T(\varphi_i(y))(y)}{\sqrt{2}} = \frac{1+i}{\sqrt{2}},$$

which forces

$$T\Big(\frac{\varphi_1(y)+\varphi_i(y)}{\sqrt{2}}\Big) \in S_{\mathcal{B}}(y,\alpha)$$

to hold. Appealing to (3.2) again implies that $(1/\sqrt{2})[\varphi_1(y)+\varphi_i(y)]$ belongs to $S_{\mathcal{A}}(\psi_\alpha(y),\varphi_\alpha(y))$. This yields

$$\sqrt{2}\varphi_\alpha(y) = \varphi_1(y)+\varphi_i(y).$$

Therefore, $|\varphi_1(y)+\varphi_i(y)| = \sqrt{2}$, and thus $\varphi_i(y) = \pm i\varphi_1(y)$ follows. ☐

Define the set

$$K = \big\{y \in Y\colon \varphi_i(y) = i\varphi_1(y)\big\}. \tag{3.3}$$

Note that Lemma 3.6 implies that

$$Y \setminus K = \big\{y \in Y\colon \varphi_i(y) = -i\varphi_1(y)\big\}.$$

Our next task is to prove that K is clopen. To do so, we need some auxiliary results.

Lemma 3.7. *Let $y \in Y$, let $\lambda \in \mathbb{T}$, and let $f \in \mathcal{A}$. Then*

$$\operatorname{Re}\big[\overline{\lambda}T(f)(y)\big] = \operatorname{Re}\big[\overline{\lambda}T(1)(y)\big]\cdot\operatorname{Re}\big[f\big(\psi_\lambda(y)\big)\big] + \operatorname{Re}\big[\overline{\lambda}T(i)(y)\big]\cdot\operatorname{Im}\big[f\big(\psi_\lambda(y)\big)\big].$$

In particular,

$$\begin{aligned}\operatorname{Re}T(f)(y) &= \operatorname{Re}T(1)(y)\operatorname{Re}f\big(\psi_1(y)\big)+\operatorname{Re}T(i)(y)\operatorname{Im}f\big(\psi_1(y)\big),\\ \operatorname{Im}T(f)(y) &= \operatorname{Im}T(1)(y)\operatorname{Re}f\big(\psi_i(y)\big)+\operatorname{Im}T(i)(y)\operatorname{Im}f\big(\psi_i(y)\big)\end{aligned}$$

and

$$\begin{aligned}1 &= \operatorname{Re} T(1)(y) \operatorname{Re} \varphi_1(y) + \operatorname{Re} T(i)(y) \operatorname{Im} \varphi_1(y),\\ 1 &= \operatorname{Im} T(1)(y) \operatorname{Re} \varphi_i(y) + \operatorname{Im} T(i)(y) \operatorname{Im} \varphi_i(y).\end{aligned}$$

Proof. Let $g = f - f(\psi_\lambda(y))$, and denote $f(\psi_\lambda(y)) = a+bi$. Then the real linearity of T implies that

$$T(g) = T(f) - T(a+bi) = T(f) - aT(1) - bT(i).$$

Since $g(\psi_\lambda(y)) = 0$, Lemma 3.5 implies that

$$0 = \operatorname{Re}\big(\overline{\lambda} T(g)(y)\big) = \operatorname{Re}\big(\overline{\lambda}\big[T(f)(y) - aT(1)(y) - bT(i)(y)\big]\big),$$

and so

$$\begin{aligned}\operatorname{Re}\big[\overline{\lambda} T(f)(y)\big] &= a \operatorname{Re}\big[\overline{\lambda} T(1)(y)\big] + b \operatorname{Re}\big[\overline{\lambda} T(i)(y)\big]\\ &= \operatorname{Re}\big[\overline{\lambda} T(1)(y)\big] \cdot \operatorname{Re}\big[f\big(\psi_\lambda(y)\big)\big]\\ &\quad + \operatorname{Re}\big[\overline{\lambda} T(i)(y)\big] \cdot \operatorname{Im}\big[f\big(\psi_\lambda(y)\big)\big].\end{aligned}$$ □

Lemma 3.8. *Let $y \in Y$.*

(i) *Let $y \in K$. Then $T(1)(y) = \overline{\varphi_1(y)}$ and $T(i)(y) = iT(1)(y)$.*
(ii) *Let $y \in Y \setminus K$. Then $T(1)(y) = \varphi_1(y)$ and $T(i)(y) = -iT(1)(y)$.*

Proof. (i) As $y \in K$, (3.3) yields that $\varphi_i(y) = i\varphi_1(y)$, and so

$$\operatorname{Re} \varphi_i(y) = -\operatorname{Im} \varphi_1(y) \qquad \text{and} \qquad \operatorname{Im} \varphi_i(y) = \operatorname{Re} \varphi_1(y).$$

By Lemma 3.7, it must be that

$$\begin{aligned}1 &= \operatorname{Re} T(1)(y) \operatorname{Re} \varphi_1(y) + \operatorname{Re} T(i)(y) \operatorname{Im} \varphi_1(y),\\ 1 &= \operatorname{Im} T(i)(y) \operatorname{Re} \varphi_1(y) - \operatorname{Im} T(1)(y) \operatorname{Im} \varphi_1(y).\end{aligned}$$

Adding these produces

$$\begin{aligned}2 &= \operatorname{Re} T(1)(y) \operatorname{Re} \varphi_1(y) - \operatorname{Im} T(1)(y) \operatorname{Im} \varphi_1(y)\\ &\quad + \operatorname{Re} T(i)(y) \operatorname{Im} \varphi_1(y) + \operatorname{Im} T(i)(y) \operatorname{Re} \varphi_1(y)\\ &= \operatorname{Re}\big[T(1)(y)\varphi_1(y)\big] + \operatorname{Im}\big[T(i)(y)\varphi_1(y)\big].\end{aligned}$$

As T is norm-preserving, we have that $|T(1)(y)\varphi_1(y)|$ and $|T(i)(y)\varphi_1(y)|$ are less than 1, and so it follows that

$$1 = T(1)(y)\varphi_1(y) \qquad \text{and} \qquad i = T(i)(y)\varphi_1(y).$$

Therefore,

$$T(1)(y) = \overline{\varphi_1(y)} \qquad \text{and} \qquad T(i)(y) = i\overline{\varphi_1(y)}.$$

(ii) Since $y \in Y \setminus K$, we have that $\varphi_i(y) = -i\varphi_1(y)$, and so

$$\operatorname{Re} \varphi_i(y) = \operatorname{Im} \varphi_1(y) \qquad \text{and} \qquad \operatorname{Im} \varphi_i(y) = -\operatorname{Re} \varphi_1(y).$$

Appealing to Lemma 3.7 again gives

$$\begin{aligned}1 &= \operatorname{Re} T(1)(y) \operatorname{Re} \varphi_1(y) + \operatorname{Re} T(i)(y) \operatorname{Im} \varphi_1(y),\\ 1 &= \operatorname{Im} T(1)(y) \operatorname{Im} \varphi_1(y) - \operatorname{Im} T(i)(y) \operatorname{Re} \varphi_1(y).\end{aligned}$$

Adding the above two equations yields

$$\begin{aligned} 2 &= \operatorname{Re} T(1)(y) \operatorname{Re} \varphi_1(y) + \operatorname{Im} T(1)(y) \operatorname{Im} \varphi_1(y) \\ &\quad + \operatorname{Re} T(i)(y) \operatorname{Im} \varphi_1(y) - \operatorname{Im} T(i)(y) \operatorname{Re} \varphi_1(y) \\ &= \operatorname{Re}\bigl[T(1)(y)\overline{\varphi_1(y)}\bigr] + \operatorname{Im}\bigl[T(i)(y) \cdot \bigl(-\overline{\varphi_1(y)}\bigr)\bigr]. \end{aligned}$$

From this, we have that

$$1 = T(1)(y)\overline{\varphi_1(y)} \qquad \text{and} \qquad i = T(i)(y) \cdot \bigl(-\overline{\varphi_1(y)}\bigr),$$

and so

$$T(1)(y) = \varphi_1(y) \qquad \text{and} \qquad T(i)(y) = -i\varphi_1(y). \qquad \square$$

Using these, we can now demonstrate that the set K defined by (3.3) is clopen.

Lemma 3.9. *The set K satisfies*

$$\begin{aligned} K &= \bigl\{y \in Y : T(i)(y) = iT(1)(y)\bigr\}, \\ Y \setminus K &= \bigl\{y \in Y : T(i)(y) = -iT(1)(y)\bigr\}. \end{aligned}$$

Consequently, K is clopen and the mapping $\varphi_1 : Y \to \mathbb{T}$ is continuous.

Proof. In light of Lemma 3.8, we have the following inclusions:

$$\begin{aligned} K &\subset \bigl\{y \in Y : T(i)(y) = iT(1)(y)\bigr\} \qquad \text{and} \\ Y \setminus K &\subset \bigl\{y \in Y : T(i)(y) = -iT(1)(y)\bigr\}. \end{aligned}$$

Thus we need only prove the reverse inclusions.

Indeed, let $y \in Y$ satisfy $T(i)(y) = iT(1)(y)$. Then Lemma 3.7 implies that

$$\begin{aligned} 1 &= \operatorname{Re} T(1)(y) \operatorname{Re} \varphi_1(y) + \operatorname{Re} T(i)(y) \operatorname{Im} \varphi_1(y) \\ &= \operatorname{Re} T(1)(y) \operatorname{Re} \varphi_1(y) - \operatorname{Im} T(1)(y) \operatorname{Im} \varphi_1(y) = \operatorname{Re}\bigl[T(1)(y)\varphi_1(y)\bigr], \\ 1 &= \operatorname{Im} T(1)(y) \operatorname{Re} \varphi_i(y) + \operatorname{Im} T(i)(y) \operatorname{Im} \varphi_i(y) \\ &= \operatorname{Im} T(1)(y) \operatorname{Re} \varphi_i(y) + \operatorname{Re} T(1)(y) \operatorname{Im} \varphi_i(y) = \operatorname{Im}\bigl[T(1)(y)\varphi_i(y)\bigr]. \end{aligned}$$

This yields

$$1 = T(1)(y)\varphi_1(y) \qquad \text{and} \qquad i = T(1)(y)\varphi_i(y),$$

and so

$$\varphi_i(y) = i\overline{T(1)(y)} = i\varphi_1(y).$$

Consequently, $y \in K$.

Now, let $y \in Y$ be such that $T(i)(y) = -iT(1)(y)$. Lemma 3.7 then gives

$$\begin{aligned} 1 &= \operatorname{Re} T(1)(y) \operatorname{Re} \varphi_1(y) + \operatorname{Re} T(i)(y) \operatorname{Im} \varphi_1(y) \\ &= \operatorname{Re} T(1)(y) \operatorname{Re} \varphi_1(y) + \operatorname{Im} T(1)(y) \operatorname{Im} \varphi_1(y) = \operatorname{Re}\bigl[T(1)(y)\overline{\varphi_1(y)}\bigr] \end{aligned}$$

and

$$\begin{aligned} 1 &= \operatorname{Im} T(1)(y) \operatorname{Re} \varphi_i(y) + \operatorname{Im} T(i)(y) \operatorname{Im} \varphi_i(y) \\ &= \operatorname{Im} T(1)(y) \operatorname{Re} \varphi_i(y) - \operatorname{Re} T(1)(y) \operatorname{Im} \varphi_i(y) = \operatorname{Im}\bigl[T(1)(y)\overline{\varphi_i(y)}\bigr], \end{aligned}$$

and thus $1 = T(1)(y)\overline{\varphi_1(y)}$ and $i = T(1)(y)\overline{\varphi_i(y)}$. In light of this, we have

$$\varphi_i(y) = -iT(1)(y) = -i\varphi_1(y),$$

which yields $y \in Y \setminus K$. Finally, we note that

$$K = \{y \in Y\colon \big(T(i) - iT(1)\big)(y) = 0\},$$
$$Y \setminus K = \{y \in Y\colon \big(T(i) + iT(1)\big)(y) = 0\}.$$

Thus both K and $Y \setminus K$ are closed; consequently, K is clopen. Note that Lemma 3.8 yields that $\varphi_1|_K = \overline{T(1)}|_K$ and $\varphi_1|_{Y\setminus K} = T(1)|_{Y\setminus K}$. Since K and $Y \setminus K$ are disjoint closed sets and $T(1)$ is continuous, it follows that φ_1 is continuous. □

Define the mappings $\tau, \rho\colon Y \to X$, and $\mu, \nu\colon Y \to \mathbb{T}$ as follows:

$$\tau = \psi_1, \qquad \rho = \psi_i, \qquad \mu = \overline{\varphi_1}, \qquad \nu(y) = \begin{cases} \mu(y), & y \in K, \\ -\mu(y), & y \in Y \setminus K. \end{cases}$$

By Lemma 3.9, we have that μ and ν are continuous, and Lemmas 3.2, 3.3, and 3.4 imply that τ and ρ are homeomorphisms. To complete the proof of Theorem 3.1, it is only left to demonstrate that these mappings satisfy (3.1).

Lemma 3.10. *Let $y \in Y$, and let $f \in \mathcal{A}$. Then*

$$T(f)(y) = \operatorname{Re}\big[\mu(y)f\big(\tau(y)\big)\big] + i\operatorname{Im}\big[\nu(y)f\big(\rho(y)\big)\big].$$

Proof. Suppose that $y \in K$. Lemma 3.8 implies that both $T(1)(y) = \overline{\varphi_1(y)} = \mu(y) = \nu(y)$ and $T(i)(y) = i\mu(y) = i\nu(y)$ hold. From Lemma 3.7, we know that

$$\begin{aligned}\operatorname{Re} T(f)(y) &= \operatorname{Re} T(1)(y)\operatorname{Re} f\big(\tau(y)\big) + \operatorname{Re} T(i)(y)\operatorname{Im} f\big(\tau(y)\big)\\ &= \operatorname{Re}\mu(y)\operatorname{Re} f\big(\tau(y)\big) + \operatorname{Re}\big[i\mu(y)\big]\operatorname{Im} f\big(\tau(z)\big)\\ &= \operatorname{Re}\mu(y)\operatorname{Re} f\big(\tau(y)\big) - \operatorname{Im}\mu(y)\operatorname{Im} f\big(\tau(y)\big) = \operatorname{Re}\big[\mu(y)f\big(\tau(y)\big)\big]\end{aligned}$$

and that

$$\begin{aligned}\operatorname{Im} T(f)(y) &= \operatorname{Im} T(1)(y)\operatorname{Re} f\big(\rho(y)\big) + \operatorname{Im} T(i)(y)\operatorname{Im} f\big(\rho(y)\big)\\ &= \operatorname{Im}\nu(y)\operatorname{Re} f\big(\rho(y)\big) + \operatorname{Im}\big[i\nu(y)\big]\operatorname{Im} f\big(\rho(y)\big)\\ &= \operatorname{Im}\nu(y)\operatorname{Re} f\big(\rho(y)\big) + \operatorname{Re}\nu(y)\operatorname{Im} f\big(\rho(y)\big)\\ &= \operatorname{Im}\big[\nu(y)f\big(\rho(y)\big)\big].\end{aligned}$$

Consequently,

$$T(f)(y) = \operatorname{Re} T(f)(y) + i\operatorname{Im} T(f)(y) = \operatorname{Re}\big[\mu(y)f\big(\tau(y)\big)\big] + i\operatorname{Im}\big[\nu(y)f\big(\rho(y)\big)\big].$$

Now, let $y \in Y \setminus K$. Then $T(1)(y) = \varphi_1(y) = \overline{\mu(y)} = -\overline{\nu(y)}$ and $T(i)(y) = -i\overline{\mu(y)} = i\overline{\nu(y)}$. As such, Lemma 3.7 gives

$$\begin{aligned}\operatorname{Re} T(f)(y) &= \operatorname{Re} T(1)(y)\operatorname{Re} f\big(\tau(y)\big) + \operatorname{Re} T(i)(y)\operatorname{Im} f\big(\tau(y)\big)\\ &= \operatorname{Re}\overline{\mu(y)}\operatorname{Re} f\big(\tau(y)\big) + \operatorname{Re}\big[-i\overline{\mu(y)}\big]\operatorname{Im} f\big(\tau(z)\big)\\ &= \operatorname{Re}\mu(y)\operatorname{Re} f\big(\tau(y)\big) - \operatorname{Im}\mu(y)\operatorname{Im} f\big(\tau(y)\big) = \operatorname{Re}\big[\mu(y)f\big(\tau(y)\big)\big]\end{aligned}$$

and

$$\begin{aligned}\operatorname{Im} T(f)(y) &= \operatorname{Im} T(1)(y) \operatorname{Re} f\big(\rho(y)\big) + \operatorname{Im} T(i)(y) \operatorname{Im} f\big(\rho(y)\big)\\ &= \operatorname{Im}\big[-\overline{\nu(y)}\big] \operatorname{Re} f\big(\rho(y)\big) + \operatorname{Im}\big[i\overline{\nu(y)}\big] \operatorname{Im} f\big(\rho(y)\big)\\ &= \operatorname{Im} \nu(y) \operatorname{Re} f\big(\rho(y)\big) + \operatorname{Re} \nu(y) \operatorname{Im} f\big(\rho(y)\big)\\ &= \operatorname{Im}\big[\nu(y) f\big(\rho(y)\big)\big].\end{aligned}$$

Therefore,

$$\begin{aligned}T(f)(y) &= \operatorname{Re} T(f)(y) + i \operatorname{Im} T(f)(y)\\ &= \operatorname{Re}\big[\mu(y) f\big(\tau(y)\big)\big] + i \operatorname{Im}\big[\nu(y) f\big(\rho(y)\big)\big]. \qquad \square\end{aligned}$$

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