Ann. Funct. Anal. 8 (2017), no. 3, 411–423
http://dx.doi.org/10.1215/20088752-2017-0006
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

FUNCTIONAL EQUATIONS ON DOUBLE COSET HYPERGROUPS

ŻYWILLA FECHNER[1*] and LÁSZLÓ SZÉKELYHIDI[2,3]

Communicated by J. Chmieliński

Abstract. In this paper we describe the complex-valued solutions defined on a double coset hypergroup of the exponential, additive, and quadratic functional equations. Moreover, the m-sine functions on a double coset hypergroup are discussed. The double coset hypergroup we consider is closely related to affine groups and spherical functions on them.

1. Introduction and preliminaries

In this article we consider functional equations and we describe some basic function classes on some hypergroups called *double coset hypergroups* (see [2, Section 1.5]). Throughout, $\mathbb{C}$ denotes the set of complex numbers. By a *hypergroup*, we mean a locally compact hypergroup. (For basics about hypergroups, see the monograph [2]; a comprehensive monograph on the subject is [8].)

Let K be a hypergroup with identity o and involution $^\vee$. We call K *Hermitian* if involution is the identity mapping. The nonidentically zero continuous function m is called an *exponential* on K if $m : K \to \mathbb{C}$ satisfies $m(x * y) = m(x)m(y)$ for each x, y in K. The continuous function $a : K \to \mathbb{C}$ is called an *additive function* if it satisfies $a(x * y) = a(x) + a(y)$. The description of exponentials and additive functions on some types of commutative hypergroups can be found in [8]. The continuous function $f : K \to \mathbb{C}$ will be called an *m-sine function* if it

Received Aug. 10, 2016; Accepted Nov. 15, 2016.
First published online May 13, 2017.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 43A62; Secondary 39B99, 20N20.
Keywords. hypergroup, functional equation, affine group.

satisfies

$$f(x * y) = f(x)m(y) + f(y)m(x)$$

for each x, y in K. The function f is called a *sine function* if it is an m-sine function for some exponential m. Clearly, every sine function f satisfies $f(o) = 0$. For a given exponential m all m-sine functions form a linear space. Obviously, $m \equiv 1$ is an exponential on any hypergroup, and 1-sine functions are exactly the additive functions. The description of sine functions on some types of hypergroups can be found in [4]. We note that if K is a commutative group, then every m-sine function is the product of m and an additive function (see [4, Theorem 1]). The continuous function $q : K \to \mathbb{C}$ is called a *quadratic function* if it satisfies

$$q(x * y) + q(x * y^\vee) = 2q(x) + 2q(y)$$

for each x, y in K. Obviously, if K is Hermitian, then every quadratic function on K is additive.

Let G be a locally compact group with identity e, and let K be a compact subgroup with normed Haar measure ω: $\int_K d\omega(k) = 1$. As K is unimodular, ω is left and right invariant, and also inversion invariant. For each x in G we define the *double coset* of x as the set

$$KxK = \{kxl : k, l \in K\}.$$

We introduce a hypergroup structure on the set $L = G//K$ of all double cosets: the topology of L is the factor topology, which is locally compact. The identity o is the coset $K = KeK$ itself, and the involution is defined by

$$(KxK)^\vee = Kx^{-1}K.$$

Finally, the convolution of δ_{KxK} and δ_{KyK} is defined by

$$\delta_{KxK} * \delta_{KyK} = \int_K \delta_{KxkyK}\, d\omega(k).$$

It is known that this gives a hypergroup structure on L (see [2], page 12), which is noncommutative in general. If K is a normal subgroup, then L is isomorphic to the hypergroup arising from the factor group G/K.

In [5] representations of double coset hypergroups are investigated. Such representations can be canonically obtained from those of the group in question. Nevertheless, not every representation arises in this way. However, our results show that, on some affine groups, basic representing functions, like exponentials, additive functions, and quadratic functions, are closely related to the corresponding functions on the original group.

We note that continuous functions on L can be identified with those continuous functions on G which are K-invariant: $f(x) = f(kxl)$ for each x in G and k, l in K. Hence, for a continuous function $f : L \to \mathbb{C}$, the simplified—and somewhat loose—notation $f(x)$ can be used for the function value $f(KxK)$. Using this convention, we can write, for each continuous function $f : L \to \mathbb{C}$ and for each x, y in G,

$$f(x * y) = \int_K f(xky)\, d\omega(k).$$

The following theorem exhibits a close connection between exponentials on double coset hypergroups and spherical functions on locally compact groups. Following the terminology of [2] (see also [3]), we recall the concept of spherical functions. Let G be a locally compact group, and let K be a compact subgroup with Haar measure ω. The continuous bounded K-invariant function $f : G \to \mathbb{C}$ is called a *K-spherical function* if $f(e) = 1$ and

$$\int_K f(xky)\, d\omega(k) = f(x)f(y) \tag{1.1}$$

holds for each x, y in G. A *generalized K spherical function* on G is the same as above without the boundedness hypothesis. For the sake of simplicity, in this paper we use the term *spherical function* for continuous functions satisfying (1.1) without the boundedness assumption. The following theorem, which is an immediate consequence of the previous considerations, gives the link between spherical functions and exponentials of double coset hypergroups.

Theorem 1.1. *Let G be a locally compact group, and let $K \subseteq G$ be a compact subgroup. Then the nonzero continuous complex-valued function m is a K-spherical function on G if and only if it is an exponential on the double coset hypergroup $G//K$. In particular, K-spherical functions on G can be identified with the characters of $G//K$.*

It is obvious that if G is a locally compact Abelian group, and K is a compact subgroup of G, then K-spherical functions are exactly the exponentials of the (locally compact) Abelian group G/K.

2. Affine groups

An important type of double coset hypergroup arises form the concept of the affine group. Let V be an n-dimensional vector space over the field $\mathbb{K}$, and let $GL(V)$ denote the *general linear group* of V, the invertible linear transformations on V. For each subgroup H of $GL(V)$ we form the *semidirect product*

$$\mathrm{Aff}(H) = H \ltimes V$$

in the following way: we equip the set $H \times V$ with the following multiplication:

$$(x, u) \cdot (y, v) = (x \cdot y, x \cdot v + u)$$

for each x, y in H and u, v in V. Here $x \cdot v$ is the image of v under the linear mapping x. Then $\mathrm{Aff}(H)$ is a group with identity $(\mathrm{id}, 0)$, where id is the identity operator and 0 is the zero of the vector space V. The inverse of (x, u) is

$$(x, u)^{-1} = (x^{-1}, -x^{-1} \cdot u)$$

for each x in H and u in V. We note that, in general, $\mathrm{Aff}(H)$ is noncommutative even if H is commutative. In any case, V—as a commutative group—is isomorphic to the normal subgroup consisting of all elements of the form (id, u) with u in V the isomorphism provided by the mapping $u \mapsto (\mathrm{id}, u)$. Indeed, we have

$$(x,u)\cdot(\mathrm{id},v)\cdot(x,u)^{-1} = (\mathrm{id}, x\cdot v),$$

which proves that the image of V, which we identify with V, is normal. The set of all elements of the form $(x, 0)$ with x in H is a subgroup of $\mathrm{Aff}(H)$ isomorphic to H, and it will be identified with H. Affine groups play an important role in geometry and physics. For instance, the *Poincaré group* $\mathrm{Aff}(O(1,3))$ is the affine group of the *Lorentz group* $O(1,3)$: $O(1,3) \ltimes \mathbb{R}^{1,3}$, where $O(1,3)$ is the *isometry group* of the real vector space $\mathbb{R}^{1,3} = \mathbb{R} \oplus \mathbb{R}^3$ equipped with the quadratic form

$$\langle v, w\rangle = \sum_{t=1}^{p} v_t w_t - \sum_{t=p+1}^{p+q} v_t w_t,$$

where $v = (v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q})$ and $w = (w_1, \ldots, w_p, w_{p+1}, \ldots, w_{p+q})$. For more about affine groups and geometry, see, for example, [1] and [7].

In the case $V = \mathbb{K}^n$ we have $GL(V) = GL(\mathbb{K}, n)$, which is a matrix group. With the usual euclidean topology, V and $GL(V)$ are locally compact topological groups. Suppose that H is a closed subgroup and that K is a compact subgroup in H. Then K—identified with the set of all elements of the form (x, u) with x in K and u in V—is a compact subgroup in $H \ltimes V$. Let L denote the double coset hypergroup $\mathrm{Aff}(H)//K$. As we have seen above, K-invariant continuous functions on $\mathrm{Aff}(H)$ can be identified with the continuous functions on L; we use the notation $f(x, u)$ for $f(K(x, u)K)$. Using this notation, K-invariance means $f(kx, lu) = f(x, u)$ for each k, l in K and u in V. If $*$ denotes convolution in L, then we have

$$f\big((x,u)*(y,v)\big) = \int_K f(xky, xkv+u)\,d\omega(k)$$

for each x, y in H and u, v in V, where ω is the Haar measure on K.

In the following result we describe the exponential functions on L for commutative H. We recall that this does not mean that L is commutative.

Theorem 2.1. *Let H be a closed commutative subgroup of $GL(V)$, and let K be a compact subgroup in H with the normalized Haar measure ω. The nonzero K-invariant continuous function $m : \mathrm{Aff}(H) \to \mathbb{C}$ is an exponential on the double coset hypergroup L if and only if it has the form*

$$m(x,u) = e(x)\varphi(u) \tag{2.1}$$

for each x in H and u in V, where $e : H \to \mathbb{C}$ is a K-invariant exponential and $\varphi : V \to \mathbb{C}$ is a nonzero continuous H-invariant function satisfying

$$\varphi(u)\varphi(v) = \int_K \varphi(ku+v)\,d\omega(k) \tag{2.2}$$

for each u, v in V.

Proof. Suppose that $m \neq 0$ is an exponential on L. By the commutativity of H, we have

$$m(x,u)m(y,v) = m\big((x,u)*(y,v)\big) = \int_K m(xky, xkv+u)\,d\omega(k)$$
$$= \int_K m(xy, xkv+u)\,d\omega(k) \tag{2.3}$$

for each x, y in H and u, v in V. Substituting $v = 0, y = \mathrm{id}$, we obtain

$$m(x,u)m(\mathrm{id},0) = \int_K m(x,u)\,d\omega(k) = m(x,u)$$

for each x in H and u in V; that is, $m(\mathrm{id},0) = 1$. Then substituting $u = v = 0$, we have by the commutativity of H and by the K-invariance of m

$$m(x,0)m(y,0) = \int_K m(xky,0)\,d\omega(k) = m(xy,0)$$

for each x, y in H. Hence $m(x,0) = e(x)$, where $e : H \to \mathbb{C}$ is a nonzero K-invariant homomorphism of H into the multiplicative group of $\mathbb{C}\backslash\{0\}$; in other words, it is a K-invariant exponential on H. Now we substitute $u = 0, y = \mathrm{id}$ in (2.3) to get

$$e(x)m(\mathrm{id},v) = \int_K m(xk, xkv)\,d\omega(k) = m(x, xv)$$

for each x in H and v in V. Let $u = xv$. Then we have

$$m(x,u) = e(x)m(\mathrm{id}, x^{-1}u), \tag{2.4}$$

which we substitute into (2.3) to obtain

$$e(x)m(\mathrm{id},x^{-1}u)e(y)m(\mathrm{id},y^{-1}v) = \int_K e(xy)m(\mathrm{id}, y^{-1}kv + y^{-1}x^{-1}u)\,d\omega(k),$$

which implies that

$$m(\mathrm{id},x^{-1}u)m(\mathrm{id},y^{-1}v) = \int_K m(\mathrm{id}, y^{-1}kv + y^{-1}x^{-1}u)\,d\omega(k) \tag{2.5}$$

for each u, v in V. For $v = 0$ this yields $m(\mathrm{id}, x^{-1}u) = m(\mathrm{id}, y^{-1}x^{-1}u)$ for each x, y in H and u in V, and here we can replace u by xu and y by y^{-1} to conclude

$$m(\mathrm{id},u) = m(\mathrm{id},yu)$$

for each y in H and u in V. On the other hand, from (2.5) we derive

$$m(\mathrm{id},u)m(\mathrm{id},v) = \int_K m(\mathrm{id}, kv+u)\,d\omega(k)$$

for each u, v in V. Letting $\varphi(u) = m(\mathrm{id},u)$ for each u in V, we have that $\varphi : V \to \mathbb{C}$ is H-invariant: $\varphi(u) = \varphi(xu)$ holds for each x in H and u in V. Further, it satisfies (2.2), and, finally, by (2.4), we have (2.1).

To prove the converse, suppose that m satisfies (2.1) with the K-invariant exponential $e : H \to \mathbb{C}$ and the continuous, H-invariant nonzero $\varphi : V \to \mathbb{C}$, which satisfies equation (2.2). Then we have for each x, y in H and u, v in V

$$\begin{aligned} m(x,u)m(y,v) &= e(x)\varphi(u)e(y)\varphi(v) = e(xy)\varphi(xu)\varphi(v) \\ &= \int_K e(xy)\varphi(kxu+v)\,d\omega(k) = \int_K m(xy, xku+v)\,d\omega(k), \end{aligned}$$

which is equation (2.3), as we intended. □

The theorem says that in the given situation exponentials of the double coset hypergroup $\mathrm{Aff}(H)//K$ "split" into K-spherical functions on H and H-invariant K-spherical functions on V. "Split" means that they are tensor products of the form given in (2.1). We note that for obvious reasons we call the continuous function $\varphi : V \to \mathbb{C}$ a *K-spherical function* if φ satisfies (2.2). The question whether the same holds if H is noncommutative remains open.

3. Additive functions

Additive functions on the double coset hypergroup L can be described in a similar manner.

Theorem 3.1. *Let H be a closed commutative subgroup of $GL(V)$, and let K be a compact subgroup in H with the normalized Haar measure ω. The K-invariant continuous function $a : \mathrm{Aff}(H) \to \mathbb{C}$ is an additive function on the double coset hypergroup L if and only if it has the form*

$$a(x,u) = l(x) + \varphi(u) \tag{3.1}$$

for each x in H and u in V, where $l : H \to \mathbb{C}$ is a K-invariant additive function, that is, a function satisfying

$$l(xy) = l(x) + l(y),$$

and $\varphi : V \to \mathbb{C}$ is a H-invariant K-spherical function, that is, a function satisfying

$$\varphi(u) + \varphi(v) = \int_K \varphi(kv+u)\,d\omega(k). \tag{3.2}$$

Proof. Let a be an additive function on L. By the commutativity of H, we have

$$\begin{aligned} a(x,u) + a(y,v) = a\big((x,u) * (y,v)\big) &= \int_K a(xky, xkv+u)\,d\omega(k) \\ &= \int_K a(xy, xkv+u)\,d\omega(k) \end{aligned} \tag{3.3}$$

for each x, y in H and u, v in V. Substituting $v = 0, y = \mathrm{id}$, we obtain

$$a(x,u) + a(\mathrm{id}, 0) = \int_K a(x,u)\,d\omega(k) = a(x,u)$$

for each x in H and u in V; that is, $a(\mathrm{id},0)=0$. Then substituting $u=v=0$, we have by the commutativity of H and by the K-invariance of m

$$a(x,0)+a(y,0)=\int_K a(xky,0)\,d\omega(k)=a(xy,0)$$

for each x,y in H. Hence $a(x,0)=l(x)$, where $l:H\to\mathbb{C}$ is a nonzero K-invariant homomorphism of H into the additive group of $\mathbb{C}$; in other words, it is a K-invariant additive function on H. Now we substitute $u=0, y=\mathrm{id}$ in (3.3) to get by the commutativity of K

$$a(x,0)+a(\mathrm{id},v)=\int_K a(xk,xkv)\,d\omega(k)=a(x,xv)$$

for each x in H and v in V. Let $u=xv$. Then we have

$$a(x,u)=l(x)+a(\mathrm{id},x^{-1}u), \tag{3.4}$$

which we substitute into (3.3) to obtain

$$l(x)+a(\mathrm{id},x^{-1}u)+l(y)+a(\mathrm{id},y^{-1}v)=\int_K \big(l(xy)+a(\mathrm{id},y^{-1}kv+y^{-1}x^{-1}u)\big)\,d\omega(k),$$

which implies that

$$a(\mathrm{id},x^{-1}u)+a(\mathrm{id},y^{-1}v)=\int_K a(\mathrm{id},y^{-1}kv+y^{-1}x^{-1}u)\,d\omega(k) \tag{3.5}$$

for each u,v in V. For $v=0$ this yields $a(\mathrm{id},x^{-1}u)=a(\mathrm{id},y^{-1}x^{-1}u)$ for each x,y in H and u in V, and here we can replace u by xu and y by y^{-1} to conclude

$$a(\mathrm{id},u)=a(\mathrm{id},yu)$$

for each y in H and u in V. On the other hand, from (3.5) we derive

$$a(\mathrm{id},u)+a(\mathrm{id},v)=\int_K a(\mathrm{id},kv+u)\,d\omega(k)$$

for each u,v in V. Letting $\varphi(u)=a(\mathrm{id},u)$ for each u in V, we have that $\varphi:V\to\mathbb{C}$ is H-invariant: $\varphi(u)=\varphi(xu)$ holds for each x in H and u in V. Further, it satisfies (3.2), and finally, by (3.4), we have (3.1).

To prove the converse, suppose that a satisfies (3.1) with the K-invariant additive function $l:H\to\mathbb{C}$ and the continuous, H-invariant K-spherical function $\varphi:V\to\mathbb{C}$, which satisfies equation (3.2). Then we have for each x,y in H and u,v in V

$$\begin{aligned}a(x,u)+a(y,v)&=l(x)+\varphi(u)+l(y)+\varphi(v)=l(xy)+\varphi(xu)+\varphi(v)\\&=\int_K\big(l(xy)+\varphi(kxu+v)\big)\,d\omega(k)=\int_K a(xy,xku+v)\,d\omega(k),\end{aligned}$$

which is equation (3.3), as we intended to prove. $\square$

4. Sine functions

In this paragraph we describe sine functions on the double coset hypergroup L.

Theorem 4.1. *Let H be a closed commutative subgroup of $GL(V)$, and let K be a compact subgroup in H with the normalized Haar measure ω, and let $m : \mathrm{Aff}(H) \to \mathbb{C}$ be an exponential on the double coset hypergroup L. The K-invariant continuous function $f : \mathrm{Aff}(H) \to \mathbb{C}$ is an m-sine function on the double coset hypergroup L if and only if it has the form*

$$f(x,u) = \big[a(x)\varphi(u) + s(u)\big]e(x) \tag{4.1}$$

for each x in H and u in V, where $e : H \to \mathbb{C}$ is a K-invariant exponential, $a : H \to \mathbb{C}$ is a K-invariant additive function, $\varphi : V \to \mathbb{C}$ is a H-invariant K-spherical function on V, and $s : V \to \mathbb{C}$ is a continuous H-invariant function satisfying

$$\int_K s(ku+v)\, d\omega(k) = s(u)\varphi(v) + s(v)\varphi(u) \tag{4.2}$$

for each u, v in V.

Proof. Let $f : \mathrm{Aff}(H) \to \mathbb{C}$ be an m-sine function on the double coset hypergroup L. By Theorem 2.1, m has the form (2.1), where $e : H \to \mathbb{C}$ is an exponential, and $\varphi : V \to \mathbb{C}$ is a H-invariant K-spherical function. Then we have by definition that f is an m-sine function if and only if

$$\int_K f(xy, xkv + u)\, d\omega(k) = f(x,u)e(y)\varphi(v) + f(y,v)e(x)\varphi(u) \tag{4.3}$$

holds for each x, y in H and u, v in V. Substituting $x = y = \mathrm{id}$ and $u = v = 0$, we get $f(\mathrm{id}, 0) = 0$, and the substitution $u = v = 0$ in (4.3) gives

$$f(xy, 0) = f(x,0)e(y) + f(y,0)e(x)$$

for each x, y in H; hence $x \mapsto f(x,0)$ is a K-invariant e-sine function on H. It follows that we have

$$f(x,0) = a(x)e(x)$$

for each x in H, where $a : H \to \mathbb{C}$ is a K-invariant additive function.

Now we substitute $y = \mathrm{id}$ and $u = 0$ in (4.3) to obtain

$$f(x, xv) = f(x,0)\varphi(v) + f(\mathrm{id}, v)e(x)$$

for each x in H and v in V. Here, putting $x^{-1}v$ for v, we conclude that

$$f(x,v) = f(x,0)\varphi(x^{-1}v) + f(\mathrm{id}, x^{-1}v)e(x) \tag{4.4}$$

whenever x is in H and v is in V. We substitute this into (4.3), and after simplification using the H-invariance of φ and the additivity of the function $x \mapsto f(x,0)/e(x)$, we obtain the functional equation

$$\int_K f(\mathrm{id}, ky^{-1}v + y^{-1}x^{-1}u)\, d\omega(k) = f(\mathrm{id}, x^{-1}u)\varphi(v) + f(\mathrm{id}, y^{-1}v)\varphi(u)$$

for each x, y in H and u, v in V. We put yv for v and xyu for u to get

$$\int_K f(\mathrm{id}, kv+u)\, d\omega(k) = f(\mathrm{id}, yu)\varphi(v) + f(\mathrm{id}, v)\varphi(u)$$

for each y in H and u, v in V. If $\varphi \equiv 0$, then we have $f(\mathrm{id}, u) = 0$ for each u in V. However, if $\varphi \not\equiv 0$, then the above equation implies that $f(\mathrm{id}, yu) = f(\mathrm{id}, u)$ for each y in H and u in V; that is, the function $s(u) = f(\mathrm{id}, u)$ is H-invariant and satisfies (4.2) for each u, v in V. Obviously, this holds in the case $\varphi \equiv 0$, too. Finally, from (4.4), the H-invariance of φ, we infer (4.1).

Conversely, suppose that f has the form given by (4.1). Then we have for each x, y in H and u, v in V

$$\begin{aligned}
&\int_K f(xy, xkv+u)\, d\omega(k) \\
&\quad = a(xy)e(xy)\int_K \varphi(xkv+u)\, d\omega(k) + e(xy)\int_K s(xkv+u)\, d\omega(k) \\
&\quad = a(xy)e(xy)\varphi(xv)\varphi(u) + e(xy)\big[s(xv)\varphi(u) + s(u)\varphi(xv)\big] \\
&\quad = a(xy)e(xy)\varphi(v)\varphi(u) + e(xy)\big[s(v)\varphi(u) + s(u)\varphi(v)\big] \\
&\quad = \big[e(x)a(x)\varphi(u) + e(x)s(u)\big]e(y)\varphi(v) + \big[e(y)a(y)\varphi(v) + e(y)s(v)\big]e(x)\varphi(u) \\
&\quad = f(x,u)e(y)\varphi(v) + f(y,v)e(x)\varphi(u),
\end{aligned}$$

which is (4.3), as we intended to prove. The proof is complete. □

5. Quadratic functions

In this section we describe quadratic functions on the double coset hypergroup L.

Theorem 5.1. *Let H be a closed commutative subgroup of $GL(V)$, and let K be a compact subgroup in H with the normalized Haar measure ω. The K-invariant continuous function $f : \mathrm{Aff}(H) \to \mathbb{C}$ is a quadratic function on the double coset hypergroup L if and only if it has the form*

$$f(x, u) = k(x) + q(u) \tag{5.1}$$

for each x in H and u in V, where $k : H \to \mathbb{C}$ is a K-invariant quadratic function, and $q : V \to \mathbb{C}$ is a continuous H-invariant function satisfying

$$\int_K \big[q(kv+u) + q(kv-u)\big]\, d\omega(k) = 2q(u) + 2q(v) \tag{5.2}$$

for each u, v in V.

Proof. Suppose that $f : \mathrm{Aff}(H) \to \mathbb{C}$ is a K-invariant quadratic function on the double coset hypergroup L. Then it satisfies

$$\int_K \big[f(xy, xkv+u) + f(xy^{-1}, -xky^{-1}v+u)\big]\, d\omega(k) = 2f(x,u) + 2f(y,v) \tag{5.3}$$

for each x, y in H and u, v in V. Substituting $y = \mathrm{id}$ and $v = 0$, we have

$$2f(x,u) = 2f(x,u) + 2f(\mathrm{id}, 0);$$

hence $f(\mathrm{id},0)=0$. On the other hand, substituting $u=v=0$ in (5.3) gives

$$f(xy,0)+f(xy^{-1},0)=2f(x,0)+2f(y,0)$$

for each x,y in H; hence $k(x)=f(x,0)$ is a K-invariant quadratic function on H. Now we substitute $y=\mathrm{id}$ and $u=0$ in (5.3) to obtain

$$\int_K \big[f(x,xkv)+f(x,-xkv)\big]\,d\omega(k)=2f(x,0)+2f(\mathrm{id},v),$$

or, by the K-invariance of f and the commutativity of H,

$$f(x,xv)+f(x,-xv)=2f(x,0)+2f(\mathrm{id},v)$$

for each x in H and v in V. This implies that the function $q(u)=f(\mathrm{id},u)$ is even, and it satisfies

$$f(x,u)=k(x)+q(\mathrm{id},x^{-1}u)$$

for each x in H and u in V. We substitute this into (5.3) after simplification. Then we conclude that

$$\int_K \big[q(y^{-1}kv+y^{-1}x^{-1}u)+q(kv-yx^{-1}u)\big]\,d\omega(k)=2q(x^{-1}u)+2q(y^{-1}v)$$

for each x,y in H and u,v in V. Here the substitution $v=0$ and $x=\mathrm{id}$ yields $q(u)=q(yu)$ for each y in H and u in V; hence the function q is H-invariant, which implies, by the above equation applied for $x=\mathrm{id}$, that

$$\int_K \big[q(kv+u)+q(kv-yu)\big]\,d\omega(k)=2q(u)+2q(v)$$

for each y in H and u,v in V. Putting $y=\mathrm{id}$, we have (5.2).

Now suppose that f has the form given in (5.1). Obviously, (5.2) implies that q is an even function; further, by H-invariance, it satisfies

$$\int_K \big[q(xkv+u)+q(xkv-yu)\big]\,d\omega(k)=2q(u)+2q(v)$$

for each x,y in H and u,v in V. We can proceed as follows:

$$\begin{aligned}
&\int_K \big[f(xy,xkv+u)+f(xy^{-1},-xky^{-1}v+u)\big]\,d\omega(k)\\
&\quad=\int_K \big[k(xy)+q(xkv+u)+k(xy^{-1})+q(-xky^{-1}v+u)\big]\,d\omega(k)\\
&\quad=2k(x)+2k(y)+\int_K \big[q(xkv+u)+q(-xky^{-1}v+u)\big]\,d\omega(k)\\
&\quad=2k(x)+2k(y)+\int_K \big[q(xkv+u)+q(xkv-yu)\big]\,d\omega(k)\\
&\quad=2k(x)+2k(y)+2q(u)+2q(v)=2f(x,u)+2f(y,v),
\end{aligned}$$

as we intended. $\square$

6. An example

In this section we consider the following group G: it is the multiplicative group of matrices of the form

$$\begin{pmatrix} x & u \\ 0 & 1 \end{pmatrix},$$

where x, u are complex numbers, $x \neq 0$. All these matrices form a subgroup of $GL(2, \mathbb{C})$ which can be identified with a subset of $\mathbb{C}^2$, and it is a locally compact topological group G when equipped with the topology inherited from $\mathbb{C}^2$. As we have

$$\begin{pmatrix} x & u \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} xy & xu+v \\ 0 & 1 \end{pmatrix},$$

we can describe the group operation on the set $G = \{(x, u) : x, u \in \mathbb{C}, x \neq 0\}$ in the following way:

$$(x, u) \cdot (y, v) = (xy, xv + u).$$

Let H denote the set of all matrices of the form

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

where $x \neq 0$ is a complex number, and let K denote the set of all matrices of the form

$$\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix},$$

where z is a complex number with $|z| = 1$. Then H is topologically isomorphic to the multiplicative group $\mathbb{C}\backslash\{0\}$ of nonzero complex numbers, and K is topologically isomorphic to the complex unit circle group with multiplication. Clearly, K is a compact subgroup of the locally compact Abelian group H, which is a closed subgroup of $GL(2, \mathbb{C})$. Finally, G is topologically isomorphic to the affine group $\mathrm{Aff}(H) = H \ltimes \mathbb{C}$. For more about this group see, for example, [6], page 201 (see also [4]).

The Haar measure on K is given by

$$\int_K \varphi(z, 0)\, du = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}, 0)\, dt$$

for each continuous function $\varphi : K \to \mathbb{C}$. It is easy to check that K is not a normal subgroup; hence the hypergroup structure on the double coset space $G//K$ is not induced by a group operation. The function $f : G \to \mathbb{C}$ is K-invariant if and only if it satisfies the compatibility condition

$$f(x, u) = f(e^{it}x, e^{is}u)$$

for each (x, u) in G and t, s in $\mathbb{R}$. Moreover, the continuous function $\varphi : \mathbb{C} \to \mathbb{C}$ is H-invariant if and only if it is constant. Hence, by Theorem 2.1, we have the following result.

Theorem 6.1. *The nonzero continuous function $m : L \to \mathbb{C}$ is an exponential on L if and only if it has the form*

$$m(x, u) = |x|^\lambda$$

for each (x, u) in G with some complex number λ. In particular, the exponential family on L is of the form

$$\Phi(x, u, \lambda) = |x|^\lambda.$$

Proof. Clearly, the K invariant exponentials on the multiplicative group $\mathbb{C}\backslash\{0\}$ are the functions $x \mapsto |x|^\lambda$, where λ is any complex number. □

The following result follows in a similar manner by Theorem 3.1.

Theorem 6.2. *The continuous function $a : L \to \mathbb{C}$ is an additive function on L if and only if it has the form*

$$a(x, u) = c \ln |x|$$

for each (x, u) in G with some complex number c.

The following theorem, as a corollary of Theorem 4.1, characterizes sine functions on the double coset hypergroup L.

Theorem 6.3. *Let λ be a complex number, and let m_λ denote the exponential $(x, u) \mapsto |x|^\lambda$ on L. The continuous function $s : L \to \mathbb{C}$ is an m_λ-sine function if and only if it has the form*

$$s(x, u) = c|x|^\lambda \ln |x|$$

for each x, u in $\mathbb{C}$ with $x \neq 0$ with some complex number c.

Finally, our following theorem describes all quadratic functions on L using the result in Theorem 5.1.

Theorem 6.4. *The continuous function $q : L \to \mathbb{C}$ is a quadratic function on L if and only if it has the form*

$$q(x, u) = c \ln^2 |x|$$

for each (x, u) in G with some complex number c.

Acknowledgments. Székelyhidi's work was partly supported by the Hungarian National Foundation for Scientific Research (OTKA) grant K111651.

References

1. R. Artzy, *Linear Geometry*, Addison-Wesley, Reading, MA, 1966. Zbl 0166.16005. MR0188842. 414
2. W. R. Bloom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, De Gruyter Stud. Math. **20**, de Gruyter, Berlin, 1995. Zbl 0828.43005. MR1312826. 411, 412, 413
3. J. Dieudonné, *Treatise on Analysis, VI*, Pure Appl. Math. **10-VI**, Academic Press, New York, 1978. Zbl 0435.43001. MR0621691. 413
4. Ż. Fechner and L. Székelyhidi, *Sine functions on hypergroups*, Arch. Math. (Basel) **106** (2016), no. 4, 371–382. Zbl 1336.43008. MR3475630. DOI 10.1007/s00013-016-0884-4. 412, 421

5. P. Hermann, *Representations of double coset hypergroups and induced representations*, Manuscripta Math. **88** (1995), no. 1, 1–24. Zbl 0844.43007. MR1348786. DOI 10.1007/BF02567801. 412
6. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, I*, Springer, Berlin, 1979. Zbl 0416.43001. MR0551496. 421
7. R. C. Lyndon, *Groups and Geometry*, London Math. Soc. Lecture Note Ser. **101**, Cambridge Univ. Press, Cambridge, 1985. Zbl 0561.51014. MR0794131. DOI 10.1017/CBO9781107325685. 414
8. L. Székelyhidi, *Functional Equations on Hypergroups*, World Scientific, Hackensack, NJ, 2013. Zbl 1270.39016. MR2978690. DOI 10.1142/8481. 411

[1]Institute of Mathematics, University of Silesia in Katowice, Bankowa 14, PL-40-007 Katowice, Poland.

E-mail address: zfechner@gmail.com

[2]Institute of Mathematics, University of Debrecen, 4032 Egyetem tér 1, Debrecen, Hungary.

E-mail address: lszekelyhidi@gmail.com

[3]Department of Mathematics, University of Botswana, 4775 Notwane Road, Gaborone, Botswana.

E-mail address: lszekelyhidi@gmail.com