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STABILITY OF THE LYAPUNOV EXPONENTS UNDER PERTURBATIONS

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Abstract. For a linear-delay equation on an arbitrary Banach space, we describe a condition so that the Lyapunov exponents of the equation persist under sufficiently small linear as well as nonlinear perturbations. We consider both cases of discrete and continuous time with the study of delay-difference equations and delay equations, respectively. The delay can be any number from zero to infinity.

1. Introduction

For a linear-delay equation on an arbitrary Banach space, we describe a condition so that the Lyapunov exponents of the equation persist under sufficiently small linear and nonlinear perturbations. In a certain sense this condition is optimal (as illustrated later on in the Introduction). It is motivated by the abstract theory of Lyapunov exponents (more precisely, by the regularity theory) and can be expressed in terms of a regularity coefficient. However, since we consider dynamics on an arbitrary Banach space, the usual regularity theory on a finite-dimensional space cannot be applied, and part of the difficulty in our work is precisely how to formulate appropriate assumptions for the persistence of the Lyapunov exponents that reduce to the usual conditions on a Euclidean space.

Moreover, we consider both cases of discrete and continuous time with the study of delay-difference equations and delay equations. The delay can be any number from zero to infinity. A zero delay corresponds to the absence of delays, and thus

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amounts to considering ordinary differential equations in the case of continuous time. On the other hand, in the case of discrete time an infinite delay means that we cannot reduce the delay-difference equation to a difference equation (without delay) on some higher-dimensional space.

More precisely, in the case of continuous time we consider a linear-delay equation

$$x' = L(t)x_t, \tag{1.1}$$

where $x_t(\tau) = x(t+\tau)$ for some linear operators $L(t)$ (the case of discrete time is analogous, and we leave it for the main text). We assume that there exists an invariant splitting

$$\mathcal{B} = F^1(t) \oplus \cdots \oplus F^p(t), \quad t \geq 0, \tag{1.2}$$

of the appropriate phase space $\mathcal{B}$ such that

(1) the Lyapunov exponents

$$\lambda_i = \limsup_{t\to+\infty} \frac{1}{t} \log\bigl\|T(t,0) \mid F^i(0)\bigr\|$$

of the bundles F^i satisfy $\lambda_1 < \cdots < \lambda_p$;

(2) the projections $P^i(t)\colon \mathcal{B} \to F^i(t)$ associated to (1.2) satisfy

$$\limsup_{t\to+\infty} \frac{1}{t} \log\bigl\|P^i(t)\bigr\| \leq 0. \tag{1.3}$$

On a finite-dimensional space condition, (1.3) corresponds to requiring that the angles between any two bundles $F^i(t)$ and $F^j(t)$ with $i \neq j$ cannot decrease exponentially with t.

In particular, we consider linear perturbations of the linear-delay equation (1.1), and we give a condition so that its Lyapunov exponents persist under sufficiently small linear perturbations. Namely, given linear operators $L(t)$ and $M(t)$ for $t \geq 0$, consider the equation

$$y' = \bigl(L(t) + M(t)\bigr)y_t. \tag{1.4}$$

We show in Theorem 3.1 that if

$$\limsup_{t\to+\infty} \frac{1}{t} \log\bigl\|M(t)\bigr\| < -\max_{i=1,\ldots,p}(\lambda_i + \mu_i), \tag{1.5}$$

where

$$\mu_i = \limsup_{t\to+\infty} \frac{1}{t} \log\Bigl\|\bigl(T(t,0)^*\bigr)^{-1} \mid \bigl(F^i(0)\bigr)^*\Bigr\|$$

for $i = 1, \ldots, p$, then any Lyapunov exponent for equation (1.4) is a Lyapunov exponent for equation (1.1). In Theorem 3.3 we obtain a corresponding result for a class of nonlinear perturbations.

The proofs are inspired by arguments of Czornik and Nawrat in [5], where they consider the particular case of linear perturbations of a difference equation (without delay) on a finite-dimensional space. Our work is thus a generalization of their corresponding result simultaneously for discrete and continuous time, infinite-dimensional spaces, and nonlinear perturbations. Our work can also be seen as a generalization of work of Pituk [13], [14] (for values in a finite-dimensional space and finite delay) and Matsui, Matsunaga, and Murakami [11] (for values in a Banach space and infinite delay), although those authors considered instead perturbations of an autonomous equation, which corresponds to having a zero regularity coefficient. For nonlinear perturbations of an arbitrary nonautonomous linear equation, corresponding results for the regular case were obtained in [3] and [4] for delay equations, respectively, with discrete and continuous time.

We note that Theorem 3.1 (as well as the other results in the paper) cannot be extended to the case when inequality (1.5) is replaced by an equality. For example, consider an autonomous ordinary differential equation on a finite-dimensional space. Then the right-hand side of (1.5) vanishes while any autonomous linear perturbation of the equation clearly changes the Lyapunov exponents (which are simply the logarithms of the absolute values of the eigenvalues).

Besides equations with constant and periodic coefficients, for which our assumptions are automatically satisfied, it turns out that from the point of view of ergodic theory almost all trajectories satisfy our assumptions. Namely, for almost all trajectories the evolution operator of the corresponding linear variational equation is in block form up to a tempered coordinate change, with each block corresponding to one Lyapunov exponent. In fact, for almost all trajectories each Lyapunov exponent is a limit, and the angles between subspaces corresponding to different Lyapunov exponents cannot decrease exponentially (we refer the reader to the book [1] for a detailed exposition of the theory). In addition, there exist corresponding results on infinite-dimensional spaces, namely by Ruelle [15] in Hilbert spaces and by Mañé [10] in Banach spaces. Finally, there exists also a variant of the regularity theory in Hilbert spaces (see [2, Section 2]). All these results are a major motivation for our approach. Notice also that a tempered coordinate change keeps the values of the Lyapunov exponents unchanged together with their multiplicities, and so the problem does not change if we consider from the beginning the dynamics already in block form, corresponding precisely to the existence of an invariant splitting as in (1.2).

2. Discrete time

2.1. **Preliminaries.** Let $X = (X, |\cdot|)$ be a Banach space. Given a function $x\colon (-\infty, m] \cap \mathbb{N} \to X$ and an integer $\ell \leq m$, we define $x_\ell\colon \mathbb{Z}_0^- \to X$ by

$$x_\ell(j) = x(\ell + j) \quad \text{for } j \in \mathbb{Z}_0^-.$$

Following the approach in [12] (as proposed by Hale in [6] in the case of continuous time), we consider a Banach space $\mathcal{B} = (\mathcal{B}, \|\cdot\|)$ of functions $\phi\colon \mathbb{Z}_0^- \to X$ with the property that there exist $N_0 > 0$ and functions $K_1, K_2\colon \mathbb{N} \to \mathbb{R}_0^+$ such that, if $x\colon \mathbb{Z} \to X$ is a function with $x_0 \in \mathcal{B}$, then for all $n \in \mathbb{N}$ we have $x_n \in \mathcal{B}$ and

$$N_0\big|x(n)\big| \leq \|x_n\| \leq K_1(n) \sup_{0\leq m\leq n} \big|x(m)\big| + K_2(n)\|x_0\|.$$

An example of such a space $\mathcal{B}$ is the following. Given $\gamma > 0$, let $\mathcal{B}$ be the set of all functions $\phi\colon \mathbb{Z}_0^- \to X$ such that

$$\|\phi\| := \sup_{j\in\mathbb{N}}\big(\big|\phi(j)\big|e^{\gamma j}\big) < +\infty$$

(one can take $N_0 = 1$ and $K_1(n) = K_2(n) = 1$ for all $n \in \mathbb{N}$). (We refer the reader to the book [8] for many other examples of appropriate Banach spaces.)

Given linear operators $L_m \colon \mathcal{B} \to X$, for $m \in \mathbb{N}$, we consider the linear-delay equation

$$x(m+1) = L_m x_m. \tag{2.1}$$

For each $\ell \in \mathbb{N}$ and $\phi \in \mathcal{B}$, there exists a unique function $x = x(\cdot, \ell, \phi) \colon \mathbb{Z} \to X$ with $x_\ell = \phi$ satisfying (2.1) for all $m \geq \ell$. For each $m, \ell \in \mathbb{Z}$ with $m \geq \ell$, we define a linear operator $T(m, \ell)$ on $\mathcal{B}$ by

$$T(m,\ell)\phi = x_m(\cdot, \ell, \phi), \quad \phi \in \mathcal{B}.$$

Then $T(m,m) = \mathrm{Id}$ and

$$T(n,m)T(m,\ell) = T(n,\ell)$$

for $n \geq m \geq \ell$. The Lyapunov exponent of a function $\phi \in \mathcal{B}$ for equation (2.1) is defined by

$$\lambda_L(\phi) = \limsup_{m \to +\infty} \frac{1}{m} \log \big\| T(m,0)\phi \big\|.$$

We always assume that the evolution operators $T(m,\ell)$ are invertible and that there exist decompositions

$$\mathcal{B} = F_m^1 \oplus F_m^2 \oplus \cdots \oplus F_m^p, \quad m \in \mathbb{N}, \tag{2.2}$$

and numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_p$ such that

(1) for each $m, \ell \in \mathbb{N}$ and $i = 1, \ldots, p$ we have

$$T(m,\ell)F_\ell^i = F_m^i$$

and

$$\lambda_i = \limsup_{m \to +\infty} \frac{1}{m} \log \big\| T(m,0) \mid F_0^i \big\|;$$

(2) for each $i = 1, \ldots, p$ the projection $P_m^i \colon \mathcal{B} \to F_m^i$ associated to the decomposition in (2.2) satisfies

$$\limsup_{m \to +\infty} \frac{1}{m} \log \|P_m^i\| \leq 0. \tag{2.3}$$

Note that $\lambda_L(\phi) \leq \lambda_i$ for $\phi \in F_0^i$. Finally, let $G_0^i = (F_0^i)^*$ be the topological dual of F_0^i and define

$$\mu_i = \limsup_{m \to +\infty} \frac{1}{m} \log \big\| \big(T(m,0)^*\big)^{-1} \mid G_0^i \big\|.$$

2.2. **Behavior under linear perturbations.** In this section, we consider linear perturbations of the linear-delay equation (2.1). Namely, given linear operators $L_m, M_m \colon \mathcal{B} \to X$, for $m \in \mathbb{N}$, we consider the equation

$$y(m+1) = L_m y_m + M_m y_m. \tag{2.4}$$

We define

$$\Lambda_M = \limsup_{m\to+\infty} \frac{1}{m} \log \|M_m\|.$$

Moreover, given a sequence $\mathbf{v} = (v^m)_{m\in\mathbb{N}} \subset \mathcal{B}$, let

$$\Lambda(\mathbf{v}) = \limsup_{m\to+\infty} \frac{1}{m} \log \|v^m\|.$$

Theorem 2.1. *If*

$$\Lambda_M < -\max_{i=1,\dots,p} (\lambda_i + \mu_i), \tag{2.5}$$

then any Lyapunov exponent for equation (2.4) is a Lyapunov exponent for equation (2.1).

Proof. Let $y(m)$ be a solution of equation (2.4) with $y_0 \in \mathcal{B}$. Then

$$y_m = T(m,0)y_0 + T(m,0)\sum_{l=0}^{m-1} T(l+1,0)^{-1}(\Gamma M_l y_l) \tag{2.6}$$

for $m \in \mathbb{N}$, where

$$\Gamma(j) = \begin{cases} \mathrm{Id} & \text{if } j = 0, \\ 0 & \text{if } j < 0. \end{cases}$$

Now let $v^l = (v^{1l}, \dots, v^{pl}) \in \mathcal{B}$, where

$$v^{il} = P_0^i T(l+1,0)^{-1}(\Gamma M_l y_l) \tag{2.7}$$

for $l \geq 0$ and $i = 1, \dots, p$. We define

$$u^m = -c + \sum_{l=0}^{m-1} v^l, \tag{2.8}$$

where the function $c = (c^1, \dots, c^p) \in \mathcal{B}$ has components

$$c^i = \begin{cases} 0 & \text{if } \Lambda(\mathbf{v}^i) \geq 0, \\ \sum_{l=0}^\infty v^{il} & \text{if } \Lambda(\mathbf{v}^i) < 0, \end{cases} \tag{2.9}$$

writing $\mathbf{v}^i = (v^{im})_{m\in\mathbb{N}}$. We show that the former series is well defined. Given $\varepsilon > 0$, there exists $C > 0$ such that

$$\|v^{il}\| \leq Ce^{(\Lambda(\mathbf{v}^i)+\varepsilon)l} \quad \text{for } l \geq 0 \tag{2.10}$$

and $i = 1, \dots, p$. When $\Lambda(\mathbf{v}^i) < 0$ and ε is so small that $\Lambda(\mathbf{v}^i) + \varepsilon < 0$, we have

$$\sum_{l=0}^\infty \|v^{il}\| \leq C\sum_{l=0}^\infty e^{(\Lambda(\mathbf{v}^i)+\varepsilon)l} = \frac{C}{1 - e^{\Lambda(\mathbf{v}^i)+\varepsilon}}.$$

Hence the series in (2.9) is well defined and

$$u^{im} := P_0^i u^m = -\sum_{l=m}^{\infty} v^{il} \quad \text{whenever } \Lambda(\mathbf{v}^i) < 0.$$

We want to show that $\Lambda(\mathbf{u}^i) \le \Lambda(\mathbf{v}^i)$ for $i = 1, \dots, p$, where $\mathbf{u}^i = (u^{im})_{m\in\mathbb{N}}$. When $\Lambda(\mathbf{v}^i) \ge 0$, using (2.10), we obtain

$$\sum_{l=0}^{m-1} \|v^{il}\| \le C \sum_{l=0}^{m-1} e^{(\Lambda(\mathbf{v}^i)+\varepsilon)l} \le \frac{C}{e^{\Lambda(\mathbf{v}^i)+\varepsilon} - 1} e^{(\Lambda(\mathbf{v}^i)+\varepsilon)m},$$

and so

$$\Lambda(\mathbf{u}^i) = \limsup_{m\to+\infty} \frac{1}{m} \log \Big\| \sum_{l=0}^{m-1} v^{il} \Big\| \le \Lambda(\mathbf{v}^i) + \varepsilon.$$

Letting $\varepsilon \to 0$, we conclude that

$$\Lambda(\mathbf{u}^i) \le \Lambda(\mathbf{v}^i). \tag{2.11}$$

On the other hand, when $\Lambda(\mathbf{v}^i) < 0$, taking $\varepsilon > 0$ such that $\Lambda(\mathbf{v}^i) + \varepsilon < 0$, we obtain

$$\sum_{l=m}^{\infty} \|v^{il}\| \le C \sum_{l=m}^{\infty} e^{(\Lambda(\mathbf{v}^i)+\varepsilon)l} = \frac{C}{1 - e^{\Lambda(\mathbf{v}^i)+\varepsilon}} e^{(\Lambda(\mathbf{v}^i)+\varepsilon)m},$$

and so

$$\Lambda(\mathbf{u}^i) = \limsup_{m\to+\infty} \frac{1}{m} \log \Big\| \sum_{l=m}^{\infty} v^{il} \Big\| \le \Lambda(\mathbf{v}^i) + \varepsilon.$$

Again letting $\varepsilon \to 0$, we conclude that (2.11) holds.

We proceed with the proof of the theorem. Since

$$\big\| \big(T(m,0)^*\big)^{-1} \mid G_0^i \big\| = \big\| T(m,0)^{-1} \mid F_m^i \big\|,$$

we obtain

$$\begin{aligned} \big\| P_0^i T(m+1,0)^{-1} \big\| &= \big\| T(m+1,0)^{-1} P_{m+1}^i \big\| \\ &\le \big\| T(m+1,0)^{-1} \mid F_{m+1}^i \big\| \cdot \|P_{m+1}^i\| \\ &\le \big\| \big(T(m+1,0)^*\big)^{-1} \mid G_0^i \big\| \cdot \|P_{m+1}^i\|, \end{aligned} \tag{2.12}$$

and it follows from (2.3) and (2.7) that

$$\Lambda(\mathbf{v}^i) \le \mu_i + \Lambda_M + \lambda_{L+M}(y_0). \tag{2.13}$$

Now we rewrite identity (2.6) in the form

$$y_m = T(m,0)(y_0 + c) + w^m, \tag{2.14}$$

where $w^m = T(m,0)u^m$. Let $w^m = (w^{1m}, \dots, w^{pm})$, where

$$w^{im} = P_m^i w^m = T(m,0) u^{im}.$$

By (2.5) and (2.13) we obtain

$$\begin{aligned}\Lambda(\mathbf{w}^i) &\le \lambda_i + \Lambda(\mathbf{u}^i) \le \lambda_i + \Lambda(\mathbf{v}^i)\\ &\le \lambda_i + \mu_i + \Lambda_M + \lambda_{L+M}(y_0).\end{aligned} \tag{2.15}$$

Therefore,

$$\Lambda(\mathbf{w}) \le \max_{i=1,\dots,p} (\lambda_i + \mu_i) + \Lambda_M + \lambda_{L+M}(y_0) < \lambda_{L+M}(y_0). \tag{2.16}$$

Finally, since

$$\lambda_{L+M}(y_0) = \limsup_{m\to+\infty} \frac{1}{m} \log \|y_m\|,$$

it follows from (2.14) that

$$\lambda_L(y_0 + c) = \lambda_{L+M}(y_0).$$

This shows that any Lyapunov exponent for equation (2.4) is a Lyapunov exponent for equation (2.1). ☐

The following example gives an application of Theorem 2.1.

Example 2.2. Take $p = 1$. Given $\omega < 0$ and $\delta > 0$, we consider linear operators $L_m\colon \mathcal{B} \to X$ for $m \in \mathbb{N}$ defined by

$$L_m y = e^{\omega + \delta(m+1)\sin(m+1) - \delta m \sin m} y.$$

For each $m \ge 0$, we have

$$\big\|T(m, 0)\big\| = e^{\omega m + \delta m \sin m},$$

and so $\lambda_1 = \omega + \delta$. On the other hand,

$$\Big\|\big(T(m, 0)^*\big)^{-1}\Big\| = e^{-\omega m - \delta m \sin m},$$

and so $\mu_1 = -\omega + \delta$. Hence, by (2.5), one can apply Theorem 2.1 to any linear operators $M_m\colon \mathcal{B} \to X$ for $m \in \mathbb{N}$ such that

$$\Lambda_M = \limsup_{m\to+\infty} \frac{1}{m} \log \|M_m\| < -2\delta.$$

2.3. **Behavior under nonlinear perturbations.** In this section we consider nonlinear perturbations of the linear-delay equation (2.1). Namely, we consider the equation

$$y(m + 1) = L_m y_m + f_m(y_m) \tag{2.17}$$

for some functions $f_m\colon \mathcal{B} \to X$ for $m \in \mathbb{N}$ such that

$$\big|f_m(\phi)\big| \le \theta_m \|\phi\|$$

for $m \in \mathbb{N}$ and $\phi \in \mathcal{B}$. We define

$$\Lambda_f = \limsup_{m\to+\infty} \frac{1}{m} \log |\theta_m|.$$

Theorem 2.3. *If*

$$\Lambda_f < -\max_{i=1,\ldots,p}(\lambda_i + \mu_i), \tag{2.18}$$

then for each solution $y(m)$ of equation (2.17) there exists $\phi \in \mathcal{B}$ such that

$$\limsup_{m\to+\infty} \frac{1}{m} \log \|y_m\| = \lambda_L(\phi).$$

Proof. We follow the proof of Theorem 2.1. Let $y(m)$ be a solution of equation (2.17). Then

$$y_m = T(m,0)y_0 + T(m,0)\sum_{l=0}^{m-1} T(l+1,0)^{-1}\big(\Gamma f_l(y_l)\big).$$

Again, let $v^l = (v^{1l}, \ldots, v^{pl})$, where

$$v^{il} = P_0^i T(l+1,0)^{-1}\big(\Gamma f_l(y_l)\big), \tag{2.19}$$

and define u^m as in (2.8) and (2.9). Proceeding as in the proof of Theorem 2.1, we can show that $\Lambda(\mathbf{u}^i) \le \Lambda(\mathbf{v}^i)$ for $i = 1, \ldots, p$. Moreover, it follows from (2.12) together with (2.3) and (2.19) that

$$\Lambda(\mathbf{v}^i) \le \mu_i + \Lambda_f + \limsup_{m\to+\infty} \frac{1}{m} \log \|y_m\|,$$

which allows us to conclude that

$$\limsup_{m\to+\infty} \frac{1}{m} \log \|y_m\| = \lambda_L(y_0 + c).$$

This completes the proof of the theorem. □

3. Continuous time

3.1. **Preliminaries.** Again, let $X = (X, |\cdot|)$ be a Banach space. Given a function $x\colon (-\infty, \tau] \to X$ and $s \le \tau$, we define $x_s\colon (-\infty, 0] \to X$ by

$$x_s(t) = x(s+t) \quad \text{for } t \le 0.$$

As proposed by Hale in [6] (see also [9]), we consider a Banach space $\mathcal{B} = (\mathcal{B}, \|\cdot\|)$ of functions $\phi\colon (-\infty, 0] \to X$ with the property that there exist $N_0 > 0$ and continuous functions $K_1, K_2\colon \mathbb{R}_0^- \to \mathbb{R}_0^+$ such that, if $x\colon \mathbb{R} \to X$ is continuous on $[0, +\infty)$ and $x_0 \in \mathcal{B}$, then we have a continuous map $[0, +\infty) \ni t \mapsto x_t \in \mathcal{B}$ and

$$N_0\big|x(t)\big| \le \|x_t\| \le K_1(t) \sup_{s\le t}\big|x(s)\big| + K_2(t)\|x_0\|.$$

Given linear operators $L(t)\colon \mathcal{B} \to X$, for $t \ge 0$ such that $(t, x) \mapsto L(t)x$ is continuous, we consider the linear equation

$$x' = L(t)x_t, \tag{3.1}$$

where x' denotes the right-sided derivative. We always assume that, for each $s \ge 0$ and $\phi \in \mathcal{B}$, there exists a unique function $x(\cdot, s, \phi)\colon \mathbb{R} \to X$ with $x_s = \phi$ satisfying (3.1) for all $t \ge s$ (we refer the reader, e.g., to [7, Section 2.2] for a related discussion). We emphasize that no specific conditions for the existence and uniqueness of solutions are used in the remainder of the section. We define linear evolution operators $T(t,s)\colon \mathcal{B} \to \mathcal{B}$ for $t \geq s$ by

$$T(t,s)\phi = x_t(\cdot, s, \phi), \quad \phi \in \mathcal{B}.$$

Clearly, $T(t,t) = \mathrm{Id}$ and

$$T(t,s)T(s,r) = T(t,r)$$

for $t \geq s \geq r$. The Lyapunov exponent of a function $\phi \in \mathcal{B}$ for equation (3.1) is defined by

$$\lambda_L(\phi) = \limsup_{t\to+\infty} \frac{1}{t} \log\bigl\|T(t,0)\phi\bigr\|.$$

Moreover, we assume that the evolution operators $T(t,s)$ are invertible and that there exist decompositions

$$\mathcal{B} = F^1(t) \oplus F^2(t) \oplus \cdots \oplus F^p(t), \quad t \geq 0, \tag{3.2}$$

and numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_p$ such that

(1) for each $t, s \geq 0$ and $i = 1, \ldots, p$ we have

$$T(t,s)F^i(s) = F^i(t)$$

and

$$\lambda_i = \limsup_{t\to+\infty} \frac{1}{t} \log\bigl\|T(t,0) \mid F^i(0)\bigr\|;$$

(2) for each $i = 1, \ldots, p$ the projection $P^i(t)\colon \mathcal{B} \to F^i(t)$ associated to the decomposition in (3.2) satisfies

$$\limsup_{t\to+\infty} \frac{1}{t} \log\bigl\|P^i(t)\bigr\| \leq 0. \tag{3.3}$$

Finally, let $G^i(0) = (F^i(0))^*$ be the topological dual of $F^i(0)$, and define

$$\mu_i = \limsup_{t\to+\infty} \frac{1}{t} \log\Bigl\|\bigl(T(t,0)^*\bigr)^{-1} \mid G^i(0)\Bigr\|.$$

3.2. **Behavior under linear perturbations.** In this section, we consider linear perturbations of the linear-delay equation (3.1). Given linear operators $L(t)$, $M(t)\colon \mathcal{B} \to X$, for $t \geq 0$ such that $(t,x) \mapsto L(t)x$ and $(t,x) \mapsto M(t)x$ are continuous, we consider the equation

$$y' = \bigl(L(t) + M(t)\bigr)y_t, \tag{3.4}$$

where y' denotes the right-sided derivative. We define

$$\Lambda_M = \limsup_{t\to+\infty} \frac{1}{t} \log\bigl\|M(t)\bigr\|.$$

Moreover, given a family $\mathbf{v} = (v^t)_{t\geq 0} \subset \mathcal{B}$, let

$$\Lambda(\mathbf{v}) = \limsup_{t\to+\infty} \frac{1}{t} \log \|v^t\|.$$

Theorem 3.1. *If*

$$\Lambda_M < -\max_{i=1,\ldots,p}(\lambda_i + \mu_i), \tag{3.5}$$

then any Lyapunov exponent for equation (3.4) is a Lyapunov exponent for equation (3.1).

Proof. To the extent possible, we follow the proof of Theorem 2.1. Let $y(t)$ be a solution of equation (3.4) with $y_0 \in \mathcal{B}$. Then

$$y_t = T(t,0)y_0 + T(t,0)\int_0^t T(\tau,0)^{-1}\big(X_0 M(\tau) y_\tau\big)\,d\tau, \tag{3.6}$$

where

$$(X_0 u)(\theta) = \begin{cases} u & \text{if } \theta = 0, \\ 0 & \text{if } \theta < 0. \end{cases} \tag{3.7}$$

Let $v^\tau = (v^{1\tau}, \ldots, v^{p\tau}) \in \mathcal{B}$, where

$$v^{i\tau} = P^i(0)T(\tau,0)^{-1}\big(X_0 M(\tau) y_\tau\big). \tag{3.8}$$

We define

$$u^t = \int_0^t v^\tau\,d\tau - c,$$

where the function $c = (c^1, \ldots, c^p) \in \mathcal{B}$ has components

$$c^i = \begin{cases} 0 & \text{if } \Lambda(\mathbf{v}^i) \ge 0, \\ \int_0^\infty v^{i\tau}\,d\tau & \text{if } \Lambda(\mathbf{v}^i) < 0; \end{cases}$$

that is, writing $u^{it} = P^i(t)u^t$, we have

$$u^{it} = \int_0^t v^{i\tau}\,d\tau \quad \text{for } \Lambda(\mathbf{v}^i) \ge 0 \tag{3.9}$$

and

$$u^{it} = -\int_t^\infty v^{i\tau}\,d\tau \quad \text{for } \Lambda(\mathbf{v}^i) < 0. \tag{3.10}$$

Proceeding as in the proof of Theorem 2.1, one can show that the latter integral converges.

Now we show that

$$\Lambda(\mathbf{u}^i) \le \Lambda(\mathbf{v}^i), \quad i = 1, \ldots, p. \tag{3.11}$$

Given $\varepsilon > 0$, there exists C such that

$$\|v^{i\tau}\| \le Ce^{(\Lambda(\mathbf{v}^i)+\varepsilon)\tau} \quad \text{for } \tau \ge 0 \tag{3.12}$$

and $i = 1, \ldots, p$. When $\Lambda(\mathbf{v}^i) \ge 0$, using (3.12), we obtain

$$\int_0^t \|v^{i\tau}\|\,d\tau \le C\int_0^t e^{(\Lambda(\mathbf{v}^i)+\varepsilon)\tau}\,d\tau \le \frac{C}{\Lambda(\mathbf{v}^i)+\varepsilon} e^{(\Lambda(\mathbf{v}^i)+\varepsilon)t},$$

and by (3.9) we get

$$\Lambda(\mathbf{u}^i) \le \Lambda(\mathbf{v}^i) + \varepsilon. \tag{3.13}$$

Letting $\varepsilon \to 0$, we conclude that $\Lambda(\mathbf{u}^i) \le \Lambda(\mathbf{v}^i)$. On the other hand, when $\Lambda(\mathbf{v}^i) < 0$, taking $\varepsilon > 0$ so small that $\Lambda(\mathbf{v}^i) + \varepsilon < 0$ and using (3.12), we obtain

$$\int_t^\infty \|v^{i\tau}\|\,d\tau \le C \int_t^\infty e^{(\Lambda(\mathbf{v}^i)+\varepsilon)\tau}\,d\tau \le \frac{C}{|\Lambda(\mathbf{v}^i)+\varepsilon|} e^{(\Lambda(\mathbf{v}^i)+\varepsilon)t},$$

and by (3.10) we get inequality (3.13). Again letting $\varepsilon \to 0$, we conclude that $\Lambda(\mathbf{u}^i) \le \Lambda(\mathbf{v}^i)$. This establishes property (3.11).

Since

$$\big\|\big(T(t,0)^*\big)^{-1} \mid G^i(0)\big\| = \big\|T(t,0)^{-1} \mid F^i(t)\big\|,$$

proceeding as in (2.12), we obtain

$$\big\|P^i(0)T(t,0)^{-1}\big\| \le \big\|\big(T(t,0)^*\big)^{-1} \mid G^i(0)\big\| \cdot \big\|P^i(t)\big\|,$$

and it follows from (3.3) and (3.8) that

$$\Lambda(\mathbf{v}^i) \le \mu_i + \Lambda_M + \lambda_{L+M}(y_0). \tag{3.14}$$

Now we rewrite identity (3.6) in the form

$$y_t = T(t,0)(y_0 + c) + w^t,$$

where $w^t = T(t,0)u^t$. Let $w^t = (w^{1t}, \dots, w^{pt})$, where

$$w^{it} = P^i(t)w^t = T(t,0)u^{it}.$$

By (3.5) and (3.14), proceeding as in (2.15) and (2.16), we obtain

$$\Lambda(\mathbf{w}) \le \max_{i=1,\dots,p}(\lambda_i + \mu_i) + \Lambda_M + \lambda_{L+M}(y_0) < \lambda_{L+M}(y_0).$$

Therefore, $\lambda_L(y_0 + c) = \lambda_{L+M}(y_0)$. □

The following example gives an application of Theorem 3.1.

Example 3.2. Take $p = 1$. Given $\omega < 0$ and $\delta > 0$, we consider the equation

$$y' = (\omega + \delta t \sin t)y.$$

For each $t \ge 0$ we have

$$\big\|T(t,0)\big\| = e^{\omega t - \delta t \cos t + \delta \sin t},$$

and so $\lambda_1 = \omega + \delta$. On the other hand,

$$\big\|\big(T(t,0)^*\big)^{-1}\big\| = e^{-\omega t + \delta t \cos t - \delta \sin t},$$

and so $\mu_1 = -\omega + \delta$. Hence, by (3.5), one can apply Theorem 3.1 to any linear operators $M(t)\colon \mathcal{B} \to X$ for $t \ge 0$ such that

$$\Lambda_M = \limsup_{t\to+\infty} \frac{1}{t} \log\big\|M(t)\big\| < -2\delta.$$

3.3. **Behavior under nonlinear perturbations.** We can also consider nonlinear perturbations of a linear delay equation. Namely, we consider the equation

$$y' = L(t)y_t + f(t, y_t) \tag{3.15}$$

for some functions $f\colon \mathbb{R}_0^+ \times \mathcal{B} \to X$ such that

$$\bigl|f(t,\phi)\bigr| \le \theta(t)\|\phi\|$$

for $t \ge 0$ and $\phi \in \mathcal{B}$. We define

$$\Lambda_f = \limsup_{t\to+\infty} \frac{1}{t}\log\bigl|\theta(t)\bigr|.$$

Theorem 3.3. *If (2.18) holds, then for each solution $y(t)$ of equation (3.15) there exists $\phi \in \mathcal{B}$ such that*

$$\limsup_{t\to+\infty} \frac{1}{t}\log\|y_t\| = \lambda_L(\phi).$$

Proof. The argument is entirely analogous to the one in the proof of Theorem 2.3 using the variation of parameters formula

$$y_t = T(t,0)y_0 + T(t,0)\int_0^t T(\tau,0)^{-1}\bigl(X_0 f_\tau(y_\tau)\bigr)\,d\tau,$$

with X_0 as in (3.7). We avoid repeating all the details. □

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