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THE COMMUTANT OF A MULTIPLICATION OPERATOR WITH A FINITE BLASCHKE PRODUCT SYMBOL ON THE SOBOLEV DISK ALGEBRA

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Abstract. Let $R(\mathbb{D})$ be the algebra generated in the Sobolev space $W^{22}(\mathbb{D})$ by the rational functions with poles outside the unit disk $\overline{\mathbb{D}}$. This is called the *Sobolev disk algebra*. In this article, the commutant of the multiplication operator $M_{B(z)}$ on $R(\mathbb{D})$ is studied, where $B(z)$ is an n-Blaschke product. We prove that an operator $A \in \mathcal{L}(R(\mathbb{D}))$ is in $\mathcal{A}'(M_{B(z)})$ if and only if $A = \sum_{i=1}^{n} M_{h_i} M_{\Delta(z)}^{-1} T_i$, where $\{h_i\}_{i=1}^{n} \subset R(\mathbb{D})$, and $T_i \in \mathcal{L}(R(\mathbb{D}))$ is given by $(T_i g)(z) = \sum_{j=1}^{n} (-1)^{i+j} \Delta_{ij}(z) g(G_{j-1}(z))$, $i = 1, 2, \dots, n$, $G_0(z) \equiv z$.

1. Introduction

Let Ω be an analytic Cauchy domain in the complex plane $\mathbb{C}$, and let $W^{22}(\Omega)$ denote the Sobolev space

$$W^{22}(\Omega) = \left\{ f \in L^2(\Omega, dm) : \begin{array}{l} \text{the distributional partial derivatives of first} \\ \text{and second order of } f \text{ belong to } L^2(\Omega, dm) \end{array} \right\},$$

where dm denotes the planar Lebesgue measure. For $f, g \in W^{22}(\Omega)$, we define

$$\langle f, g \rangle = \sum_{|\alpha| \le 2} \int D^\alpha f \overline{D^\alpha g} \, dm.$$

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Then $W^{22}(\Omega)$ is a Hilbert space and a Banach algebra with identity under an equivalent norm. The space $W^{22}(\Omega)$ can be continuously embedded in the space $C(\overline{\Omega})$ of continuous functions on $\overline{\Omega}$ by the Sobolev embedding theorem, where $\overline{\Omega}$ is the closure of Ω.

Dixmier and Foiaş [3] constructed operator models based on the Sobolev space. Using these models, Herrero, Taylor, and Wang [5] discussed the variation of point spectrum under compact perturbation. Jiang and Wang in [6, Chapter 4.5] obtained some interesting results on strongly irreducible operators. If $\Omega = \mathbb{D}$, the unit disk in $\mathbb{C}$, then let $R(\mathbb{D})$ be the subalgebra of $W^{22}(\mathbb{D})$ generated by rational functions with poles outside $\overline{\mathbb{D}}$. This subalgebra is called the *Sobolev disk algebra* (see [6], [8]). In fact, $R(\mathbb{D})$ consists of all analytic functions in $W^{22}(\mathbb{D})$, and the Hilbert space $R(\mathbb{D})$ possesses an orthonormal basis $\{e_n\}_{n=0}^{\infty}$, $e_n = \beta_n z^n$, $\beta_n = [\frac{n+1}{(3n^4-n^2+2n+1)\pi}]^{1/2}$ $(n = 0, 1, 2, \ldots)$. A function $g(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic on $\mathbb{D}$ belongs to $R(\mathbb{D})$ if and only if $\sum_{n=0}^{\infty} |\frac{a_n}{\beta_n}|^2 < \infty$. Given $g \in R(\mathbb{D})$, the multiplication operator M_g is given by $M_g f = gf$, $f \in R(\mathbb{D})$. It was proved that $\mathcal{A}'(M_z) = \{M_g; g \in R(\mathbb{D})\}$, where $\mathcal{A}'(M_z)$ is the commutant of M_z (see [6, p. 95]). The commutant of a bounded linear operator A on the Hilbert space $\mathcal{H}$ is defined by

$$\mathcal{A}'(A) = \big\{B \in \mathcal{L}(\mathcal{H}); AB = BA\big\},$$

where $\mathcal{L}(\mathcal{H})$ denotes the set of all bounded linear operators on $\mathcal{H}$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is Fredholm if $\operatorname{ran} T$, the range of T, is closed and $\operatorname{ind} T = \dim\ker T - \dim\ker T^*$ is finite. If $g \in R(\mathbb{D})$, $\sigma(M_g) = g(\overline{\mathbb{D}})$, $\sigma_e(M_g) = \sigma_{lre}(M_g) = g(\mathbb{T})$, and if $z_0 \in \mathbb{D}$ and $g(z_0) \notin g(\mathbb{T})$, then $M_g - g(z_0)$ is a Fredholm operator and

$$\operatorname{ind}\big(M_g - g(z_0)\big) = -\dim\ker\big(M_g - g(z_0)\big)^* = -n,$$

where $\sigma(M_g)$, $\sigma_e(M_g)$, $\sigma_{lre}(M_g)$ denote, respectively, the spectrum, the essential spectrum, and the Wolf-spectrum of M_g. Here $g(\overline{\mathbb{D}})$ and $g(\mathbb{T})$ are the images of $\overline{\mathbb{D}}$ and, respectively, the unit circle $\mathbb{T}$ under g and n is the number of zeros (counting multiplicity) of $g(z)-g(z_0)$ in $\mathbb{D}$. (For more details about the Sobolev disk algebra $R(\mathbb{D})$, see [6].)

It is well known that the commutant of a bounded linear operator or operators on a complex, separable Hilbert space plays an important role in determining the structure of this operator or these operators. In [2], Cuckovic investigated the commutant of M_{z^n} on the Bergman space. The commutant of multiplication by a univalent function in the disk algebra was discussed in [7]. On the Sobolev disk algebra, Jiang and Wang [6] and Liu and Wang [8] investigated the commutant $\mathcal{A}'(M_{z^n})$. Wang, Zhao, and Jin [10, Theorem 1] proved that M_g is similar to M_{z^n} if and only if g is an n-Blaschke product. The main point of the main theorem of the present article, Theorem 2.8, is to characterize the commutant $\mathcal{A}'(M_B)$ of an n-Blaschke product $B(z) = \alpha\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}$, $a_i \in \mathbb{D}$, $|\alpha| = 1$.

2. The commutant of a multiplication operator

Lemma 2.1 ([10, Lemma 1])**.** *Let $B(z) = \alpha\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}$ be an n-Blaschke product. Then the derivative $B'(z)$ of $B(z)$ has no zeros on the unit circle $\mathbb{T}$.*

Lemma 2.2 ([10, Proposition 1]). *Let $B(z)$ be a finite Blaschke product, $f \in R(\mathbb{D})$. Then $f[B(z)] \in R(\mathbb{D})$, and the operator C_B defined by $C_B(f)(z) = f[B(z)]$ is bounded.*

Without loss of generality, we can assume that $\alpha = 1$. By Lemma 4.5.9 of [6], for each $z_0 \in \mathbb{D}$, we have

$$B(z) - B(z_0) = (z - z_0)(z - z_1) \cdots (z - z_{n-1}) B_{z_0}(z),$$

where $B_{z_0}(z) \neq 0$ for $z \in \mathbb{D}$. Let $N_{z_0} = \{z_i\}_{i=0}^{n-1}$, and let

$$\Gamma = \bigcup \bigl\{ N_{z_0}; \text{there is at least one } z_i \ (0 \leq i \leq n-1) \text{ such that } B'(z_i) = 0 \bigr\}.$$

The set Γ is finite.

Lemma 2.3 ([8, Theorem 2.3]). *Let $f \in R(\mathbb{D})$, let M_f^* be a Cowen–Douglas operator of index n on Ω, and let $D_1 = f^{-1}(\Omega)$. Here Ω is the component of the semi-Fredholm domain $\rho_{s-F}(M_f)$ containing $f(z_0)$ for $z_0 \in \mathbb{D}$, and n is the number of zeros of $f(z) - f(z_0)$ in $\mathbb{D}$. If the set of $\{z \in D_1, f'(z) = 0\}$ is finite, then there exist analytic functions $\alpha_1(z), \ldots, \alpha_n(z)$ and $G_1(z), \ldots, G_{n-1}(z)$ on $D_1 \setminus \Gamma$ such that, for each $A \in \mathcal{A}'(M_f)$ and $g \in R(\mathbb{D})$,*

$$(Ag)(z) = \alpha_1(z) g(z) + \alpha_2(z) g\bigl(G_1(z)\bigr) + \cdots + \alpha_n(z) g\bigl(G_{n-1}(z)\bigr), \quad z \in D_1 \setminus \Gamma.$$

For this lemma, we need some explanation. In the proof of Theorem 2.3 of [8], the authors used an implicit holomorphic function theorem (see [4, Theorem 9.6]) to prove that, for every $\omega \in D_1 \setminus \Gamma$, there is a neighborhood $U(\omega, \delta_\omega)$ where δ_ω depends on ω, and there are $n-1$ functions $G_1(z), \ldots, G_{n-1}(z)$ analytic on $U(\omega, \delta_\omega)$ such that, for $v \in U(\omega, \delta_\omega)$,

$$f(z) - f(v) = (z - v)\bigl(z - G_1(v)\bigr) \cdots \bigl(z - G_{n-1}(v)\bigr) g_v(z), \quad g_v(z) \neq 0, z \in \mathbb{D}.$$

Obviously, there are at most countable such balls $U(\omega_k, \delta_{\omega_k})$ that can cover $D_1 \setminus \Gamma$. In these balls, suppose that $U(\omega_i, \delta_{\omega_i}) \cap U(\omega_j, \delta\omega_j) \neq \emptyset$ for $i \neq j$. For $U(\omega_i, \delta_{\omega_i})$, we have $n-1$ functions $G_1, \ldots, G_{n-1}$ analytic on $U(\omega_i, \delta_{\omega_i})$ such that, for $v \in U(\omega_i, \delta_{\omega_i})$,

$$f(z) - f(v) = (z - v)\bigl(z - G_1(v)\bigr) \cdots \bigl(z - G_{n-1}(v)\bigr) g_v(z), \quad g_v(z) \neq 0, z \in \mathbb{D}. \quad (2.1)$$

For $U(\omega_j, \delta_{\omega_j})$, we also have $n-1$ functions $Z_1, \ldots, Z_{n-1}$ analytic on $U(\omega_j, \delta_{\omega_j})$ such that, for $v \in U(\omega_j, \delta_{\omega_j})$, we have

$$f(z) - f(v) = (z - v)\bigl(z - G_1(v)\bigr) \cdots \bigl(z - G_{n-1}(v)\bigr) g_v(z), \quad g_v(z) \neq 0, z \in \mathbb{D}. \quad (2.2)$$

Then for $v \in U(\omega_i, \delta_{\omega_i}) \cap U(\omega_j, \delta_{\omega_j})$, by (2.1) and (2.2), we have $\{G_m(v)\}_{m=1}^{n-1} = \{Z_m(v)\}_{m=1}^{n-1}$, and so we can rearrange the index of $G_{m's}$ such that $G_m(v) = Z_m(v)$, $m = 1, 2, \ldots, n-1$. Since $\{G_m, Z_m\}_{m=1}^{n-1}$ are analytic functions on the open set $U(\omega_i, \delta_{\omega_i}) \cap U(\omega_j, \delta_{\omega_j})$, we have $G_m(v) = Z_m(v)$ for $v \in U(\omega_i, \delta_{\omega_i}) \cap U(\omega_j, \delta_{\omega_j})$. Thus the analytic continuation is possible.

By [8, Theorem 2.3] and the argument in its proof in [8, p. 68–69], there exist $n-1$ functions $G_1, \ldots, G_{n-1}$ analytic on $\overline{\mathbb{D}} \setminus \Gamma$ such that

$$B(z) - B(z_0) = (z - z_0)\big(z - G_1(z_0)\big) \cdots \big(z - G_{n-1}(z_0)\big) B_{z_0}(z).$$

For these $\{G_i\}_{i=1}^{n-1}$, we have the following.

Lemma 2.4.

(i) *Each* G_i $(i = 1, 2, \ldots, n-1)$ *is a Blaschke product of order* 1*; that is,* $G_i(z) = \alpha_i \frac{z - b_i}{1 - \overline{b_i} z}$, $|\alpha_i| = 1$, *and* $\alpha_i \neq 1$*;*

(ii) *Each* $G_i(z) - z$ $(i = 1, 2, \ldots, n-1)$ *and each* $G_i(z) - G_j(z)$ $(i \neq j;$ $i, j = 1, 2, \ldots, n-1)$ *has precisely one zero in* $\mathbb{D}$*;*

(iii) *The point* $z \in \mathbb{D}$ *is a zero of* $B'(z)$ *if and only if either* $z = z_0$ *for some* $z_0 \in \mathbb{D}$ *such that* $G_i(z_0) = z_0$ *for some* i, *or* $z = G_i(z_0)$ *for some* $z_0 \in \mathbb{D}$ *such that* $G_i(z_0) = G_j(z_0)$ *for some* $i \neq j$.

Proof. (i) For $z_0 \in \mathbb{D}$ and $i = 1, 2, \ldots, n-1$, $|B(G_i(z_0))| = |B(z_0)| < 1$. This implies that $|G_i(z_0)| < 1$. If $z_0 \in \mathbb{T}$, then $|B(G_i(z_0))| = |B(z_0)| = 1$. Thus $|G_i(z_0)| = 1$. Since G_i is bounded on $\overline{\mathbb{D}} \setminus \Gamma$ and Γ is a finite set, each point in Γ is a removable singularity (see [1]). We can assume that G_i is analytic on $\overline{\mathbb{D}}$. Thus G_i is an inner function. Since $|G_i(z)| = 1$ at each point of $\mathbb{T}$, $G_i(z)$ is not a singular inner function (see [9]); that is, $G_i(z)$ is a Blaschke product. Note that $B(z) - B(z_0) = 0$ has n roots $z_0, G_1(z_0), \ldots, G_{n-1}(z_0)$ when $z_0 \in \mathbb{D}$. Similarly, if $z \in \mathbb{D}$ is fixed and we solve for z_0, then

$$B(z) - B(z_0) = (z - z_0)\big(z - G_1(z_0)\big) \cdots \big(z - G_{n-1}(z_0)\big) B_{z_0}(z) = 0$$

has n roots. Thus $1 + k_1 + \cdots + k_{n-1} = n$, where k_i is the order of G_i, $i = 1, 2, \ldots, n-1$. This implies that each G_i is a Blashcke product of order 1. Suppose $G_i(z) = \alpha_i \frac{z - b_i}{1 - \overline{b_i} z}$ for some b_i with $|b_i| < 1$ and α_i with $|\alpha_i| = 1$. If $\alpha_i = 1$, then a computation shows that $G_i(z) - z$ has two zeros $z_0 = \pm e^{i\theta}$, where θ is the argument of b_i. Thus $B(z) - B(z_0) = (z - z_0)^2 f(z)$ for some f and $B'(z_0) = 0$, which contradicts Lemma 2.1.

(ii) Solve the equation $G_i(z) - z = 0$ or, equivalently,

$$\overline{b_i} z^2 - (1 - \alpha_i) z - \alpha_i b_i = 0.$$

Let z_1, z_2 be the two solutions in $\mathbb{C}$. Then

$$|z_1 z_2| = \left| \frac{-\alpha_i b_i}{\overline{b_i}} \right| = 1.$$

If both $z_1, z_2 \in \mathbb{T}$, then $B'(z_1) = B'(z_2) = 0$, which contradicts Lemma 2.1. Thus one of z_1, z_2 is located in $\mathbb{D}$ and the other is out of $\overline{\mathbb{D}}$.

Solve the equation $G_i(z) = G_j(z)$ $(i \neq j)$ or, equivalently,

$$\frac{cz - cb_i}{1 - \overline{b_i} z} = \frac{z - b_j}{1 - \overline{b_j} z},$$

where $c = \frac{\alpha_i}{\alpha_j}$. We get

$$(\overline{b_i} - c\overline{b_j}) z^2 + (c + cb_i\overline{b_j} - 1 - \overline{b_i} b_j) z + b_j - cb_i = 0.$$

If the solutions are the complex numbers z_1, z_2, then

$$|z_1 z_2| = \left|\frac{b_j - cb_i}{\overline{b_i} - c\overline{b_j}}\right| = 1.$$

For the same reason one and only one of z_1, z_2 is in $\mathbb{D}$.

(iii) If $B'(z_0) = 0$ and $z_0 \in \mathbb{D}$, then

$$\frac{B(z) - B(z_0)}{z - z_0} = \big(z - G_1(z_0)\big)\big(z - G_2(z_0)\big)\cdots\big(z - G_{n-1}(z_0)\big)B_{z_0}(z) \to 0$$

as $z \to z_0$. Since $B_{z_0}(z) \neq 0$ for all $z \in \overline{\mathbb{D}}, z - G_i(z_0) \to 0$ at least for one i; that is, $G_i(z_0) = z_0$. Conversely, if $G_i(z_0) = z_0$ for some i and $z_0 \in \mathbb{D}$, then

$$B(z) - B(z_0) = (z - z_0)^2 f(z)$$

for some f. Hence $B'(z_0) = 0$.

If $G_i(z_0) = G_j(z_0)$, then

$$B(z) - B\big(G_i(z_0)\big) = (z - z_0)\big(z - G_i(z_0)\big)^2 h(z)$$

for some h. Thus $B'(G_i(z_0)) = 0$. □

Example 2.5.

(i) Let $B(z) = \frac{(z-a)(z-b)}{(1-\overline{a}z)(1-\overline{b}z)}$. Calculations show that

$$G(z) = -\frac{z - c}{1 - \overline{c}z},$$

where $c = \frac{(a+b)-ab\overline{(a+b)}}{1-|ab|^2}$.

(ii) Let $B(z) = (\frac{z-a}{1-\overline{a}z})^4$. Calculations show that

$$G_i(z) = \alpha_i \frac{z - b_i}{1 - \overline{b_i}z} \quad (i = 1, 2, 3),$$

where $\alpha_1 = -1$, $b_1 = \frac{2a}{1+|a|^2}$; $\alpha_2 = \frac{i(1+i|a|^2)}{1-i|a|^2}$, $b_2 = \frac{(1+i)a}{1+i|a|^2}$; $\alpha_3 = \frac{i(|a|^2+i)}{|a|^2-i}$, $b_3 = \frac{(1+i)a}{|a|^2+i}$.

In what follows, $G_1, G_2, \ldots, G_{n-1}$ are the order one Blaschke products associated with the n-Blaschke product $B(z)$.

Let $\Delta(z)$ denote the Vandermonde determinant

$$\Delta(z) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z & G_1 & \cdots & G_{n-1} \\ z^2 & G_1^2 & \cdots & G_{n-1}^2 \\ \vdots & \vdots & \cdots & \vdots \\ z^{n-1} & G_1^{n-1} & \cdots & G_{n-1}^{n-1} \end{vmatrix}.$$

Let $\Delta_{kj}(z)$ be the k, j-algebra cofactor of $\Delta(z)$ $(k, j = 1, 2, \ldots, n-1)$. It is known that

$$\Delta(z) = (G_1 - z)(G_2 - z)\cdots(G_{n-1} - z)(G_2 - G_1)\cdots(G_{n-1} - G_{n-2}),$$

and that $\Delta(z)$ and $\Delta_{kj}(z)$ are in $R(\mathbb{D})$.

Lemma 2.6 (see [6, Proposition 4.5.3])**.** *A function* $f(z)=\sum_{n=0}^{\infty} f_n z^n$ *analytic on* $\mathbb{D}$ *belongs to* $R(\mathbb{D})$ *if and only if* $\sum_{n=0}^{\infty}|\frac{f_n}{\beta_n}|^2<\infty$*, where* $\beta_n=[\frac{n+1}{(3n^4-n^2+2n+1)\pi}]^{\frac{1}{2}}$ *and* $f_n\in\mathbb{C}$ *(*$n=0,1,2,\ldots$*).*

Lemma 2.7.

(i) $\operatorname{ran} M_{z-z_0}=\{f\in R(\mathbb{D}); f(z_0)=0\}$, $z_0\in\mathbb{D}$;

(ii) *Given* $g\in R(\mathbb{D})$ *and* $g(z)=(G_i(z)-z)f(z)$ *for* i *(*$i=1,2,\ldots,n-1$*) and some function* f*, then* $f\in R(\mathbb{D})$*,* $g\in\operatorname{ran} M_{G_i-z}$*, and* $f=M_{G_i-z}^{-1}g$*;*

(iii) *Given* $g\in R(\mathbb{D})$ *and* $g(z)=(G_i(z)-G_j(z))f(z)$ *for* $i\neq j$ *(*$i,j=1,2,\ldots,n-1$*) and some function f, then* $f\in R(\mathbb{D})$*,* $g\in\operatorname{ran} M_{G_i-G_j}$*, and* $f=M_{G_i-G_j}^{-1}g$*;*

(iv) *Given* $g\in R(\mathbb{D})$ *and* $g(z)=\Delta(z)f(z)$ *for some function* f*, then* $g\in\operatorname{ran} M_{\Delta(z)}$ *and* $f=M_{\Delta(z)}^{-1}g$*.*

Proof. (i) Note that M_{z-z_0} is Fredholm and that

$$\operatorname{ran} M_{z-z_0}=\big\{(z-z_0)g(z); g\in R(\mathbb{D})\big\}$$

is always closed. If $z_0=0$ and $f(0)=0$, then we can suppose that

$$f(z)=\sum_{k=1}^{\infty} f_k z^k = z\sum_{k=1}^{\infty} f_k z^{k-1}.$$

Then, by Lemma 2.6,

$$\sum_{k=1}^{\infty}\Big|\frac{f_k}{\beta_{k-1}}\Big|^2=\sum_{k=1}^{\infty}\Big|\frac{f_k}{\beta_k}\Big|^2\Big|\frac{\beta_k}{\beta_{k-1}}\Big|^2<\infty$$

implies that $h(z):=\sum_{k=1}^{\infty} f_k z^{k-1}\in R(\mathbb{D})$ and $f\in\operatorname{ran} M_z$. Thus $\operatorname{ran} M_z=\{f\in R(\mathbb{D}); f(0)=0\}$.

If $z_0\neq 0, z_0\in\mathbb{D}$, and $f(z_0)=0$, then set $g(z)=f(\frac{z+z_0}{1+\overline{z_0}z})$. By Lemma 2.2, $g\in R(\mathbb{D})$. Since $g(0)=0$, $g\in\operatorname{ran} M_z$, and $g(z)=zg_1(z)$, $g_1(z)\in R(\mathbb{D})$. Hence

$$f(z)=g\Big(\frac{z-z_0}{1-\overline{z_0}z}\Big)=(z-z_0)g_1\Big(\frac{z-z_0}{1-\overline{z_0}z}\Big)(1-\overline{z_0}z)^{-1}=(z-z_0)h(z),$$

where

$$h(z)=g_1\Big(\frac{z-z_0}{1-\overline{z_0}z}\Big)(1-\overline{z_0}z)^{-1}\in R(\mathbb{D}).$$

Thus $f\in\operatorname{ran} M_{z-z_0}$. The opposite inclusion is obvious.

(ii) Let $z_0\in\mathbb{D}$ be the only zero of $G_i(z)-z$ in $\overline{\mathbb{D}}$. Then

$$G_i(z)-z=(z-z_0)G_{z_0}(z),\quad G_{z_0}(z)\neq 0$$

for $z\in\mathbb{D}$, and $G_{z_0}\in R(\mathbb{D})$ by (i). For $0\notin G_{z_0}(\overline{\mathbb{D}})=\sigma(M_{G_{z_0}})$, $M_{G_{z_0}}$ is invertible and $G_{z_0}^{-1}(z)\in R(\mathbb{D})$. Since $g(z_0)=0$, we have $g(z)=(z-z_0)g_1(z)$, and $g_1\in R(\mathbb{D})$ by (i). Hence

$$g(z)=\big(G_i(z)-z\big)G_{z_0}^{-1}(z)g_1(z)$$

and

$$f(z)=G_{z_0}^{-1}(z)g_1(z)\in R(\mathbb{D});\quad \text{that is, } g\in\operatorname{ran} M_{G_i-z}.$$

Note that $G_i(z)-z \neq 0$ for all $z \in \mathbb{T}$, M_{G_i-z} is Fredholm, and ran M_{G_i-z} is closed. Also, M_{G_i-z} is injective. Therefore,

$$M_{G_i-z} : R(\mathbb{D}) \to \operatorname{ran} M_{G_i-z}$$

is invertible, and $f = M_{G_i-z}^{-1} g$.

(iii), (iv) By a similar argument to that used in (ii), we can easily prove (iii). Using (ii) and (iii), we can easily prove (iv). □

Theorem 2.8. *An operator $A \in \mathcal{L}(R(\mathbb{D}))$ is in $\mathcal{A}'(M_{B(z)})$ if and only if*

$$A = \sum_{i=1}^{n} M_{h_i} M_{\Delta(z)}^{-1} T_i,$$

where $\{h_i\}_{i=1}^n \subset R(\mathbb{D})$, $T_i \in \mathcal{L}(R(\mathbb{D}))$ is given by

$$(T_i g)(z) = \sum_{j=1}^{n} (-1)^{i+j} \Delta_{ij}(z) g\big(G_{j-1}(z)\big), \quad i = 1, 2, \ldots, n, G_0(z) \equiv z.$$

Proof. Let $A \in \mathcal{A}'(M_{B(z)})$. By Lemma 4.5.11 of [6] and Lemma 2.4, we have

$$(Ag)(z) = \alpha_1(z) g(z) + \alpha_2(z) g\big(G_1(z)\big) + \cdots + \alpha_n(z) g\big(G_{n-1}(z)\big)$$

for $g \in R(\mathbb{D})$ and $z \in \mathbb{D} \setminus \Gamma$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are analytic on $\mathbb{D} \setminus \Gamma$ and $G_1, G_2, \ldots, G_{n-1}$ are the order one Blaschke products associated with $B(z)$. Take $g = 1, z, z^2, \ldots, z^{n-1}$ sequentially. We get

$$\begin{aligned}
\alpha_1(z) + \alpha_2(z) + \cdots + \alpha_n(z) &= (A1)(z) = h_1(z), \\
z\alpha_1(z) + G_1(z)\alpha_2(z) + \cdots + G_{n-1}(z)\alpha_n(z) &= (Az)(z) = h_2(z), \\
&\vdots \\
z^{n-1}\alpha_1(z) + G_1^{n-1}(z)\alpha_2(z) + \cdots + G_{n-1}^{n-1}(z)\alpha_n(z) &= (Az^{n-1})(z) = h_n(z).
\end{aligned}$$

Solving for $\alpha_1(z), \alpha_2(z), \ldots, \alpha_n(z)$ by Cramer's rule, we get

$$\begin{aligned}
\alpha_1(z) &= \frac{1}{\Delta(z)} \begin{vmatrix} h_1 & 1 & \cdots & 1 \\ h_2 & G_1 & \cdots & G_{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ h_n & G_1^{n-1} & \cdots & G_{n-1}^{n-1} \end{vmatrix}, \\
\alpha_2(z) &= \frac{1}{\Delta(z)} \begin{vmatrix} 1 & h_1 & \cdots & 1 \\ z & h_2 & \cdots & G_{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ z^{n-1} & h_n & \cdots & G_{n-1}^{n-1} \end{vmatrix}, \\
&\cdots \\
\alpha_n(z) &= \frac{1}{\Delta(z)} \begin{vmatrix} 1 & 1 & \cdots & h_1 \\ z & G_1 & \cdots & h_2 \\ \vdots & \vdots & \cdots & \vdots \\ z^{n-1} & G_1^{n-1} & \cdots & h_n \end{vmatrix}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
(Ag)(z) = {} & \frac{h_1(z)}{\Delta(z)}\big[\Delta_{11}(z)g(z) - \Delta_{12}(z)g\big(G_1(z)\big) + \Delta_{13}(z)g\big(G_2(z)\big) + \cdots \\
& + (-1)^{1+n}\Delta_{1n}(z)g\big(G_{n-1}(z)\big)\big] \\
& + \frac{h_2(z)}{\Delta(z)}\big[-\Delta_{21}(z)g(z) + \Delta_{22}(z)g\big(G_1(z)\big) \\
& - \Delta_{23}(z)g\big(G_2(z)\big) + \cdots + (-1)^{2+n}\Delta_{2n}(z)g\big(G_{n-1}(z)\big)\big] + \cdots \\
& + \frac{h_n(z)}{\Delta(z)}\big[(-1)^{n+1}\Delta_{n1}(z)g(z) + (-1)^{n+2}\Delta_{n2}(z)g\big(G_1(z)\big) \\
& + (-1)^{n+3}\Delta_{n3}(z)g\big(G_2(z)\big) + \cdots + \Delta_{nn}(z)g\big(G_{n-1}(z)\big)\big] \\
= {} & \sum_{k=1}^{n} h_k(z)\frac{g_k(z)}{\Delta(z)},
\end{aligned}
$$

where $z \in \mathbb{D}\backslash\Gamma$ and $g_k(z) = \sum_{j=1}^{n}(-1)^{k+j}\Delta_{kj}(z)g(G_{j-1}(z))$. It is clear $g_k \in R(\mathbb{D})$.

Define $T_k g = g_k$ $(k = 1, 2, \ldots, n)$. Since $R(\mathbb{D})$ is a Banach algebra under an equivalent norm, applying Lemma 2.2, we have that T_k is a bounded linear operator in $\mathcal{L}(R(\mathbb{D}))$ and $\|T_k g\| \leq M\|g\|$ for $g \in R(\mathbb{D})$ and $k = 1, 2, \ldots, n$.

Consider the determinant expression of g_1,

$$
g_1(z) = \begin{vmatrix}
g(z) & g(G_1(z)) & \cdots & g(G_{n-1}(z)) \\
z & G_1(z) & \cdots & G_{n-1}(z) \\
z^2 & G_1^2(z) & \cdots & G_{n-1}^2(z) \\
\vdots & \vdots & \cdots & \vdots \\
z^{n-1} & G_1^{n-1}(z) & \cdots & G_{n-1}^{n-1}(z)
\end{vmatrix},
$$

where $g(z) = \sum_{k=0}^{\infty} a_k z^k$. For any $\varepsilon > 0$, let K be a positive integer such that

$$
\Big\|\sum_{i=k}^{\infty} a_i z^i\Big\| < \varepsilon, \qquad \Big\|\sum_{i=k}^{\infty} a_i G_1^i(z)\Big\| < \varepsilon, \qquad \ldots, \qquad \Big\|\sum_{i=k}^{\infty} a_i G_{n-1}^i(z)\Big\| < \varepsilon
$$

for any $k \geq K$. Thus

$$
\begin{aligned}
g_1(z) &= \begin{vmatrix}
\sum_{i=0}^{k-1} a_i z^i + \sum_{i=k}^{\infty} a_i z^i & \sum_{i=0}^{k-1} a_i G_1^i + \sum_{i=k}^{\infty} a_i G_1^i & \cdots & \sum_{i=0}^{k-1} a_i G_{n-1}^i + \sum_{i=k}^{\infty} a_i G_{n-1}^i \\
z & G_1 & \cdots & G_{n-1} \\
z^2 & G_1^2 & \cdots & G_{n-1}^2 \\
\vdots & \vdots & \cdots & \vdots \\
z^{n-1} & G_1^{n-1} & \cdots & G_{n-1}^{n-1}
\end{vmatrix} \\
&= \sum_{i=0}^{k-1} a_i Z_i + R_k,
\end{aligned}
$$

where

$$
Z_i = \begin{vmatrix}
z^i & G_1^i(z) & \cdots & G_{n-1}^i(z) \\
z & G_1(z) & \cdots & G_{n-1}(z) \\
z^2 & G_1^2(z) & \cdots & G_{n-1}^2(z) \\
\vdots & \vdots & \cdots & \vdots \\
z^{n-1} & G_1^{n-1}(z) & \cdots & G_{n-1}^{n-1}(z)
\end{vmatrix}
$$

and

$$R_k = \begin{vmatrix} \sum_{i=k}^{\infty} a_i z^i & \sum_{i=k}^{\infty} a_i G_1^i & \cdots & \sum_{i=k}^{\infty} a_i G_{n-1}^i \\ z & G_1 & \cdots & G_{n-1} \\ z^2 & G_1^2 & \cdots & G_{n-1}^2 \\ \vdots & \vdots & \cdots & \vdots \\ z^{n-1} & G_1^{n-1} & \cdots & G_{n-1}^{n-1} \end{vmatrix}.$$

We have

$$\begin{aligned} \|R_k\| &= \Big\| \Delta_{11}(z) \sum_{i=k}^{\infty} a_i z^i - \Delta_{12}(z) \sum_{i=k}^{\infty} a_i G_1^i + \cdots + (-1)^{1+n} \Delta_{1n}(z) \sum_{i=k}^{\infty} a_i G_{n-1}^i \Big\| \\ &\leq \big(\|\Delta_{11}\| + \|\Delta_{12}\| + \cdots + \|\Delta_{1n}\| \big) \varepsilon = L\varepsilon. \end{aligned}$$

Hence $g_1(z) = \sum_{k=0}^{\infty} a_k Z_k$.

It is easy to see that $Z_0 = \Delta(z)$, $Z_1 = Z_2 = \cdots = Z_{n-1} = 0$. When $k \geq n$, set

$$P_k(u) = \begin{vmatrix} u^k & G_1^k(z) & \cdots & G_{n-1}^k(z) \\ u & G_1(z) & \cdots & G_{n-1}(z) \\ u^2 & G_1^2(z) & \cdots & G_{n-1}^2(z) \\ \vdots & \vdots & \cdots & \vdots \\ u^{n-1} & G_1^{n-1}(z) & \cdots & G_{n-1}^{n-1}(z) \end{vmatrix}.$$

It is obvious that $P_k(z) = Z_k$. Note that $P_k(G_1(z)) = 0$ since in the determinant expression the first two columns are the same. This implies that $P_k(u)$ has a factor $u - G_1(z)$. Therefore, $P_k(z) = Z_k$ has a factor $z - G_1(z)$. Similarly, Z_k has factors $z - G_2(z), \ldots, z - G_{n-1}(z)$. If setting

$$P_k(u) = \begin{vmatrix} z^k & u^k & G_2^k(z) & \cdots & G_{n-1}^k(z) \\ z & u & G_2(z) & \cdots & G_{n-1}(z) \\ z^2 & u^2 & G_2^2(z) & \cdots & G_{n-1}^2(z) \\ \vdots & \vdots & \cdots & \vdots \\ z^{n-1} & u^{n-1} & G_2^{n-1}(z) & \cdots & G_{n-1}^{n-1}(z) \end{vmatrix},$$

then $P_k(G_1(z)) = Z_k$. Since $P_k(G_2(z)) = 0$, $P_k(u)$ has a factor $G_2(z) - u$, and so $P_k(G_1(z)) = Z_k$ has a factor $G_2(z) - G_1(z)$. Similarly, Z_k has factors $G_3(z) - G_1(z), \ldots, G_{n-1}(z) - G_1(z)$. By the same arguments, Z_k has factors $G_3 - G_2, G_4 - G_2, \ldots, G_{n-1} - G_{n-2}$. Hence Z_k has a factor $\Delta(z)$, and $g_1(z) = \Delta(z) f_1(z)$. Lemma 2.7 indicates that $f_1 \in R(\mathbb{D})$, $g_1 \in \operatorname{ran} M_\Delta$, and $f_1 = M_\Delta^{-1} g_1$. Thus

$$\frac{g_1(z)}{\Delta(z)} = (M_\Delta^{-1} T_1 g)(z).$$

By the same argument,

$$\frac{g_k(z)}{\Delta(z)} = (M_\Delta^{-1} T_k g)(z) \qquad \text{and} \qquad A = \sum_{k=1}^{n} M_{h_k} M_\Delta^{-1} T_k.$$

Conversely, for arbitrary $h_1, h_2, \ldots, h_n$ in $R(\mathbb{D})$, define $A \in \mathcal{L}(R(\mathbb{D}))$ by

$$A = \sum_{k=1}^{n} M_{h_k} M_\Delta^{-1} T_k.$$

For $g \in R(\mathbb{D})$,

$$\begin{aligned}
(AM_B g)(z) &= \big[A(Bg)\big](z) \\
&= \sum_{k=1}^{n} h_k(z) M_\Delta^{-1} \Big(\sum_{j=1}^{n} (-1)^{k+j} \Delta_{kj(z)} B\big(G_{j-1}(z)\big) g\big(G_{j-1}(z)\big)\Big) \\
&= \sum_{k=1}^{n} h_k(z) M_\Delta^{-1} \Big(\sum_{j=1}^{n} (-1)^{k+j} \Delta_{kj}(z) B(z) g\big(G_{j-1}(z)\big)\Big) \\
&= B(z) \sum_{k=1}^{n} h_k(z) M_\Delta^{-1} \Big(\sum_{j=1}^{n} (-1)^{k+j} \Delta_{kj}(z) g\big(G_{j-1}(z)\big)\Big) \\
&= \Big[M_B \Big(\sum_{k=1}^{n} M_{h_k} M_\Delta^{-1} T_k\Big) g\Big](z) \\
&= (M_B A g)(z).
\end{aligned}$$

The third equality is because of $B(G_{j-1}(z)) = B(z)$ for $j = 1, 2, \ldots, n$. In fact, by the argument before Lemma 2.4, for each $z_0 \in \mathbb{D}$,

$$B(z) - B(z_0) = (z - z_0)\big(z - G_1(z_0)\big) \cdots \big(z - G_{n-1}(z_0)\big) B_{z_0}(z),$$

where $B_{z_0}(z) \neq 0$ for $z \in \mathbb{D}$. If setting $z = G_{j-1}(z_0)$ for $j = 1, 2, \ldots, n$, then $B(G_{j-1}(z_0)) - B(z_0) = 0$ for each $z_0 \in \mathbb{D}$; that is, $B(z) = B(G_{j-1}(z))$. Hence $AM_B = M_B A$ and $A \in \mathcal{A}'(M_B)$. The proof of Theorem 2.8 is complete. □

Remark 2.9. In Theorem 2.8, the zeros of $B(z)$ are not necessarily nonzero. If all the zeros of $B(z)$ are zero, then we get $B(z) = z^n$, and so Theorem 2.8 generalizes the result in [8]. If some are zero and others are nonzero, then Theorem 2.8 is true for the general finite Blaschke product $B(z) = \alpha z^k \prod_{i=1}^{m} \frac{z - a_i}{1 - \overline{a_i} z}$.

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