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CHARACTERIZATIONS OF LIPSCHITZ SPACE VIA COMMUTATORS OF SOME BILINEAR INTEGRAL OPERATORS

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Abstract. In this article, we give some characterizations of Lipschitz space via commutators of bilinear singular integral operators and bilinear fractional integral operators, respectively.

1. Introduction and preliminaries

Let $0 < \gamma < 1$, and let b be a locally integrable function on $\mathbb{R}^n$. We say that b *belongs to the (homogeneous) Lipschitz space* $\Lambda_\gamma(\mathbb{R}^n)$ if there is a constant $C > 0$ such that

$$\bigl|b(x) - b(y)\bigr| \le C|x - y|^\gamma$$

for any $x, y \in \mathbb{R}^n$. Moreover, the norm $\|\cdot\|_{\Lambda_\gamma}$ is the infimum of C. (For more details about the Lipschitz space, we refer readers to [9], [10], and [11].) Let b be a locally integrable function on $\mathbb{R}^n$, and let T be a *Calderón–Zygmund* singular integral operator. The commutator $[b, T]$ is defined by

$$[b, T](f) = bT(f) - T(bf).$$

Johnson [8] gave a characterization of Lipschitz space by the boundedness of commutator $[b, T]$. He proved that $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in \Lambda_\gamma(\mathbb{R}^n)$, where $1 < p < \infty$, $0 < \gamma < 1$, and $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$. Paluszyński [13]

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obtained that $[b,T]$ is bounded from $L^p(\mathbb{R}^n)$ to $\dot{F}^{\gamma}_{p,\infty}(\mathbb{R}^n)$ if and only if $b \in \Lambda_\gamma(\mathbb{R}^n)$, where $0<\gamma<1$, $1<p<\infty$, and $\dot{F}^{\gamma}_{p,\infty}(\mathbb{R}^n)$ is the homogeneous *Triebel–Lizorkin* space with the equivalent norm

$$\|b\|_{\dot{F}^{\gamma}_{p,\infty}(\mathbb{R}^n)} \approx \left\| \sup_{Q\ni x} \frac{1}{|Q|^{1+\frac{1}{\gamma}}} \int_Q \big|b(y)-b_Q\big|\,dy \right\|_{L^p(\mathbb{R}^n)}.$$

Shi, Zhang, and Huang [15] established that $[b,T]$ is a bounded operator from $L^p(\mathbb{R}^n)$ to $C^{p,\beta}(\mathbb{R}^n)$ if and only if $b\in\Lambda_\gamma(\mathbb{R}^n)$, where $1<p<\infty$, $-\frac{n}{p}\le\beta<0$, $\gamma=\beta+\frac{n}{p}<1$, and $C^{p,\beta}(\mathbb{R}^n)$ is the *Morrey–Campanato* space defined as

$$C^{p,\beta}(\mathbb{R}^n)=\left\{f : \|f\|_{C^{p,\beta}(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}\Big(\frac{1}{|Q|}\int_Q \big|f(x)-f_Q\big|^p\,dx\Big)^{\frac{1}{p}}<\infty\right\}.$$

Let $0<\alpha<n$. The fractional integral operator is defined by

$$I_\alpha f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy,$$

and the commutator generated by I_α and b is defined by

$$[b,I_\alpha](f)=bI_\alpha(f)-I_\alpha(bf).$$

Paluszyński [13] proved that $[b,I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ if and only if $b\in\Lambda_\gamma(\mathbb{R}^n)$, where $1<p<\infty$, $0<\gamma<1$, and $\frac{1}{p}-\frac{1}{r}=\frac{\gamma+\alpha}{n}$. Shi, Zhang, and Huang [15] obtained that $[b,I_\alpha]$ is a bounded operator from $M^{p,\beta}(\mathbb{R}^n)$ to $M^{q,\tilde{\beta}}(\mathbb{R}^n)$ if and only if $b\in\Lambda_\gamma(\mathbb{R}^n)$, where $1<p<\infty$, $-\frac{n}{p}\le\beta<0$, $\frac{1}{q}=\frac{1}{p}-\frac{\gamma+\alpha}{n}$, $\tilde{\beta}=\alpha+\beta+\gamma$, and $M^{p,\beta}(\mathbb{R}^n)$ is the *Morrey* space defined as

$$M^{p,\beta}(\mathbb{R}^n)=\left\{f : \|f\|_{M^{p,\beta}(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}\Big(\frac{1}{|Q|}\int_Q \big|f(x)\big|^p\,dx\Big)^{\frac{1}{p}}<\infty\right\}.$$

The main aim of this article is to give some characterizations of Lipschitz space via commutators of bilinear singular integral operators and bilinear fractional integral operators. Throughout the article, the constant C will be used to denote a constant which is independent of the main parameters, but which may vary from line to line. The symbol $A\sim B$ means that $C_1B\le A\le C_2B$, where $C_1,C_2>0$.

2. Main results

2.1. **Definitions and theorems.** First, let us recall some definitions.

Definition 2.1 ([6, Section 2], [14]). Let T be a bilinear operator initially defined on the 2-fold product of Schwartz spaces and taking values in the space of tempered distributions,

$$T:\mathscr{S}(\mathbb{R}^n)\times\mathscr{S}(\mathbb{R}^n)\longrightarrow\mathscr{S}'(\mathbb{R}^n).$$

The bilinear singular integral operator is defined by

$$T(f,g)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K(x,y,z)f(y)g(z)\,dy\,dz \quad \text{for } x\notin(\operatorname{supp} f\cap\operatorname{supp} g),$$

where the kernel K is a function defined on $(\mathbb{R}^n)^3 \setminus \{(x,y,z) \in (\mathbb{R}^n)^3 : x = y = z\}$ and there is an $\alpha > 0$ such that

(i) $|K(x,y,z)| \leq \frac{A}{(|x-y|+|x-z|+|y-z|)^{2n}}$;

(ii) $|K(x,y,z) - K(x',y,z)| \leq \frac{A|x-x'|^\alpha}{(|x-y|+|x-z|+|y-z|)^{2n+\alpha}}$ if $|x-x'| \leq \frac{1}{2}\max(|x-y|, |x-z|, |y-z|)$;

(iii) $|K(x,y,z) - K(x,y',z)| \leq \frac{A|y-y'|^\alpha}{(|x-y|+|x-z|+|y-z|)^{2n+\alpha}}$ if $|y-y'| \leq \frac{1}{2}\max(|x-y|, |x-z|, |y-z|)$;

(iv) $|K(x,y,z) - K(x,y,z')| \leq \frac{A|z-z'|^\alpha}{(|x-y|+|x-z|+|y-z|)^{2n+\alpha}}$ if $|z-z'| \leq \frac{1}{2}\max(|x-y|, |x-z|, |y-z|)$.

Lastly, we say that the bilinear singular integral operator is of *convolution type* if the kernel $K(x,y,z)$ is actually of the form $K(x-y, x-z)$.

Definition 2.2 ([2, Section 1]). Let $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, and let T be a bilinear singular integral operator. The commutators $[b,T]_i$ $(i = 1,2)$ are defined by

$$[b,T]_1(f,g) = T(bf,g) - bT(f,g) \quad \text{and} \quad [b,T]_2(f,g) = T(f,bg) - bT(f,g).$$

Definition 2.3 ([2, Section 1]). For $0 < \alpha < 2n$, the bilinear fractional integral operator is defined by

$$I_\alpha(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(y)g(z)}{(|x-y|+|x-z|)^{2n-\alpha}}\,dy\,dz.$$

Definition 2.4 ([2, Proposition 3.3]). Let $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, and let I_α be a bilinear fractional integral operator. The commutators $[b, I_\alpha]_i$ $(i = 1,2)$ are defined by

$$[b,I_\alpha]_1(f,g) = I_\alpha(bf,g) - bI_\alpha(f,g) \qquad \text{and} \qquad [b,I_\alpha]_2(f,g) = I_\alpha(f,bg) - bI_\alpha(f,g).$$

Definition 2.5 ([12, Definition 3.1]). The bilinear Hardy–Littlewood maximal operator M is defined as follows:

$$M(f,g)(x) = \sup_{Q \ni x}\Big(\frac{1}{|Q|}\int_Q |f(y)|\,dy\Big)\Big(\frac{1}{|Q|}\int_Q |g(z)|\,dz\Big).$$

Definition 2.6 ([5, Theorem 3.1]). For $0 < \alpha < 2n$, the maximal sub-bilinear operator is defined by

$$M_\alpha(f,g)(x) = \sup_{Q \ni x}|Q|^{\frac{\alpha}{n}}\Big(\frac{1}{|Q|}\int_Q |f(y)|\,dy\Big)\Big(\frac{1}{|Q|}\int_Q |g(z)|\,dz\Big).$$

The main results in this article are the following theorems.

Theorem 2.7. *Let T be a bilinear Calderón–Zygmund operator of convolution type, and let $b \in \Lambda_\gamma(\mathbb{R}^n)$, $1 < p_1, p_2 < \infty$, $0 < \gamma < 1$, and $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\gamma}{n}$. Then the commutator $[b,T]_i$ $(i = 1,2)$ satisfies*

$$\big\|[b,T]_i(f,g)\big\|_{L^q} \leq C\|b\|_{\Lambda_\gamma(\mathbb{R}^n)}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.$$

Conversely, if $[b,T]_i$ $(i = 1,2)$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where T is a bilinear Calderón–Zygmund operator of convolution type with a homogeneous kernel K of degree $(-2n)$, and the Fourier series of $\frac{1}{K}$ is absolutely convergent on some ball $B \in \mathbb{R}^{2n}$, then $b \in \Lambda_\gamma(\mathbb{R}^n)$.

Theorem 2.8. *Let T be a bilinear Calderón–Zygmund operator of convolution type, and let $b \in \Lambda_\gamma(\mathbb{R}^n)$, $1 < p < \infty$, $-\frac{2n}{p} \le \beta < 0$, $\gamma = \frac{\beta}{2} + \frac{n}{p} < 1$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then the commutator $[b,T]_i$ $(i = 1, 2)$ satisfies*

$$\big\|[b,T]_i(f,g)\big\|_{C^{p,\beta}(\mathbb{R}^n)} \le C\|b\|_{\Lambda_\gamma(\mathbb{R}^n)}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.$$

Conversely, if $[b,T]_i$ $(i = 1, 2)$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $C^{p,\beta}(\mathbb{R}^n)$, where T is a bilinear Calderón–Zygmund operator of convolution type with a homogeneous kernel K of degree $(-2n)$, and the Fourier series of $\frac{1}{K}$ is absolutely convergent on some ball $B \in \mathbb{R}^{2n}$, then $b \in \Lambda_\gamma(\mathbb{R}^n)$.

Theorem 2.9. *Let $1 < p_1, p_2 < \infty$, $0 < \gamma < 1$, and $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\gamma+\alpha}{n}$. Then the following statements are equivalent:*

(a) $b \in \Lambda_\gamma(\mathbb{R}^n)$,
(b) $[b, I_\alpha]_i$ *$(i = 1, 2)$ is a bounded operator from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Theorem 2.10. *Let $1 < p_1, p_2 < \infty$, $-\frac{2n}{p} \le \beta < 0$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\gamma+\alpha}{n}$, and $\tilde{\beta} = \alpha + 2\beta + \gamma$. Then the following statements are equivalent:*

(a) $b \in \Lambda_\gamma(\mathbb{R}^n)$,
(b) $[b, I_\alpha]_i$ *$(i = 1, 2)$ is a bounded operator from $M^{p_1,\beta}(\mathbb{R}^n) \times M^{p_2,\beta}(\mathbb{R}^n)$ to $M^{q,\tilde{\beta}}(\mathbb{R}^n)$.*

2.2. **Proofs of theorems.** In order to prove our theorems, we need the following lemmas.

Lemma 2.11. *Let $1 < p < \infty$, and let $-\frac{2n}{p} \le \beta < 0$. Then there is a constant $C > 0$ such that for any $f \in C^{p,\beta}(\mathbb{R}^n)$ and $a > 0$,*

$$\|f\|_{C^{p,\beta}(\mathbb{R}^n)} \sim \sup_Q \inf_{a\in\mathbb{R}} \Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}} \int_Q \big|f(x) - a\big|^p\,dx\Big)^{\frac{1}{p}}.$$

Proof. We only need to consider the part of the proof of Lemma 2.11 which will be used in the proof of Theorem 2.8; that is, for any cube Q, we have

$$\begin{aligned}
\Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}} \int_Q \big|f(x) - f_Q\big|^p\,dx\Big)^{\frac{1}{p}} &\le \Big(\frac{1}{|Q|^{\frac{p\beta}{2n}}}\frac{1}{|Q|} \int_Q \big(\big|f(x) - a\big| + |a - f_Q|\big)^p\,dx\Big)^{\frac{1}{p}} \\
&\le C\frac{1}{|Q|^{\frac{\beta}{2n}}}\Big(\frac{1}{|Q|} \int_Q \big[\big|f(x) - a\big|^p + |a - f_Q|^p\big]\,dx\Big)^{\frac{1}{p}} \\
&\le C\frac{1}{|Q|^{\frac{\beta}{2n}}}\Big(\Big(\frac{1}{|Q|} \int_Q \big|f(x) - a\big|^p\,dx\Big)^{\frac{1}{p}} + |a - f_Q|\Big) \\
&\le C\frac{1}{|Q|^{\frac{\beta}{2n}}}\Big(\frac{C}{|Q|} \int_Q \big|f(x) - a\big|^p\,dx\Big)^{\frac{1}{p}} \\
&\le C\Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}} \int_Q \big|f(x) - a\big|^p\,dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

□

Lemma 2.12 ([7, Definition 3.3], [6]). *Let $1 < p_1, p_2 < \infty$, and let $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then there exists a constant $C > 0$ such that*

$$\big\|T(f,g)\big\|_{L^p} \leq C\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.$$

Lemma 2.13 ([7, Proposition 1.1]). *Let $0 < \alpha < 2n$, $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then there exists a constant $C > 0$ such that*

$$\big\|I_\alpha(f,g)\big\|_{L^q} \leq C\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.$$

Lemma 2.14 ([12, Theorem 3.7], [3]). *Let $1 < p_1, p_2 < \infty$, and let $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then there exists a constant $C > 0$ such that*

$$\big\|M(f,g)\big\|_{L^p} \leq C\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.$$

Lemma 2.15 ([13, Lemma 1.5]). *Let $1 \leq p < \infty$, and let $0 < \gamma < 1$. Then we have*

$$\|f\|_{\Lambda_\gamma(\mathbb{R}^n)} \approx \sup_Q \frac{1}{|Q|^{1+\frac{\gamma}{n}}} \int_Q \big|f(x) - f_Q\big|\, dx \approx \sup_Q \Big(\frac{1}{|Q|^{1+\frac{p\gamma}{n}}} \int_Q \big|f(x) - f_Q\big|^p\, dx\Big)^{\frac{1}{p}}.$$

For $p = \infty$, the formula should be interpreted appropriately as

$$\|f\|_{\Lambda_\gamma(\mathbb{R}^n)} \approx \sup_Q \frac{1}{|Q|^{\frac{\gamma}{n}}} \sup_{x\in Q} \big|f(x) - f_Q\big|.$$

Lemma 2.16 ([4, Lemma 4.1]). *Let $B^* \subset B$. Then there exists a constant $C > 0$ such that*

$$|f_{B^*} - f_B| \leq C\|f\|_{\Lambda_\gamma(\mathbb{R}^n)}|B|^{\frac{\gamma}{n}}.$$

Lemma 2.17 ([1, Theorem 1.1]). *If $\operatorname{diam}(\Omega) < \infty$ and there exists a positive constant c such that $\Omega(x_0, l) \geq \mathrm{c}l^n$ for every $x_0 \in \Omega$ and $l \in (0, \operatorname{diam}\Omega)$, then $M^{p,\beta}(\Omega) = C^{p,\beta}(\Omega)$.*

Proof of Theorem 2.7. We first give the proof of necessity and only consider the case $i = 1$. Let $B = B((y_0, z_0), \delta\sqrt{2n}) \subset \mathbb{R}^{2n}$ be the ball for which we can express $\frac{1}{k(y,z)}$ as an absolutely convergent Fourier series of the form

$$\frac{1}{k(y,z)} = \sum_j a_j e^{v_j\cdot(y,z)}.$$

Set $y_1 = \delta^{-1}y_0$ and $z_1 = \delta^{-1}z_0$. Note that for all (y, z) such that

$$\big(|y - y_1|^2 + |z - z_1|^2\big)^{\frac{1}{2}} < \sqrt{2n},$$

we have

$$\frac{1}{k(y,z)} = \frac{\delta^{-2n}}{k(\delta y, \delta z)} = \delta^{-2n} \sum_j a_j e^{i\delta v_j\cdot(y,z)}.$$

Let $Q = Q(x_0, r)$ be an arbitrary cube in $\mathbb{R}^n$, set $\tilde{y} = x_0 - ry_1$ and $\tilde{z} = x_0 - rz_1$, and take $Q' = Q(\tilde{y}, r) \subset \mathbb{R}^n$ and $Q'' = Q(\tilde{z}, r) \subset \mathbb{R}^n$. Then for any $x \in Q$ and $y \in Q'$, we have

$$\Big|\frac{x-y}{r} - y_1\Big| \leq \Big|\frac{x - x_0}{r}\Big| + \Big|\frac{y - \tilde{y}}{r}\Big| \leq \sqrt{n},$$

and similarly for $x \in Q$ and $z \in Q''$. Then we have that

$$\Big(\Big|\frac{x-y}{r} - y_1\Big|^2 + \Big|\frac{x-z}{r} - z_1\Big|^2\Big)^{\frac{1}{2}} \leq \sqrt{2n}.$$

Let $s(x) = \operatorname{sgn}(b(x) - b_{Q'})$. We have the following estimate:

$$\begin{aligned}
\int_Q \big|b(x) - b_{Q'}\big|\,dx &= \int_Q \big(b(x) - b_{Q'}\big)s(x)\,dx \\
&= \frac{1}{|Q'|}\int_Q\int_{Q'}\big(b(x) - b(y)\big)\,dy s(x)\,dx \\
&= \frac{1}{|Q''|}\frac{1}{|Q'|}\int_Q\int_{Q''}\int_{Q'}\big(b(x) - b(y)\big)\,dz\,dy s(x)\,dx \\
&\leq r^{-2n}\int_{\mathbb{R}^n} s(x)\chi_Q(x) \\
&\quad \times \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(b(x) - b(y)\big)\frac{r^{2n}k(x-y, x-z)}{k(\frac{x-y}{r}, \frac{x-z}{r})} \\
&\quad \times \chi_{Q'}(y)\chi_{Q''}(z)\,dz\,dy\,dx.
\end{aligned}$$

Let $f_j(x) = e^{-i\frac{\delta}{r}v_j^1\cdot y}\chi_{Q'}(y)$, $g_j(x) = e^{-i\frac{\delta}{r}v_j^2\cdot z}\chi_{Q''}(z)$, and $h_j(x) = e^{i\frac{\delta}{r}v_j\cdot(x,x)}s(x) \times \chi_Q(x)$. Then we have

$$\begin{aligned}
&\int_Q \big|b(x) - b_{Q'}\big|\,dx \\
&\quad \leq C\sum_j a_j \int_{\mathbb{R}^n} h_j(x)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(b(x) - b(y)\big)k(x-y, x-z)f_j(y)g_j(z)\,dz\,dy\,dx \\
&\quad = C\sum_j a_j \int_{\mathbb{R}^n}[b,T]_1(f_j, g_j)(x)h_j(x)\,dx \\
&\quad \leq C\sum_j |a_j| \int_{\mathbb{R}^n}\big|[b,T]_1(f_j, g_j)(x)\big|\big|h_j(x)\big|\,dx \\
&\quad \leq C\sum_j |a_j|\Big(\int_{\mathbb{R}^n}\big|h_j(x)\big|^{q'}\,dx\Big)^{\frac{1}{q'}}\Big(\int_{\mathbb{R}^n}\big|[b,T]_1(f_j, g_j)(x)\big|^q\,dx\Big)^{\frac{1}{q}} \\
&\quad \leq C\sum_j |a_j|\|h_j\|_{L^{q'}}\|f_j\|_{L^{p_1}}\|g_j\|_{L^{p_2}} \leq C\sum_j |a_j||Q|^{\frac{1}{q'}}|Q|^{\frac{1}{p_1}}|Q|^{\frac{1}{p_2}} \\
&\quad \leq C\sum_j |a_j||Q|^{1+\frac{\gamma}{n}}.
\end{aligned}$$

It follows that

$$\sup_Q \frac{1}{|Q|^{1+\frac{\gamma}{n}}}\int_Q \big|b(x) - b_Q\big|\,dx \leq C.$$

Next we give the proof of sufficiency. We note that

$$\begin{aligned}
&\big|[b,T]_1(f,g)(x)\big| \\
&\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big|b(x)-b(y)\big|\big|k(x-y,x-z)\big|\big|f(y)g(z)\big|\,dz\,dy \\
&\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \bigg(\frac{|b(x)-b(y)|}{|x-y|^\gamma}|x-y|^\gamma\bigg)\big|k(x-y,x-z)\big|\big|f(y)g(z)\big|\,dz\,dy \\
&\le C\|b\|_{\Lambda_\gamma(\mathbb{R}^n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^\gamma \frac{|f(y)g(z)|}{(|x-y|+|x-z|)^{2n}}\,dz\,dy \\
&\le C\|b\|_{\Lambda_\gamma(\mathbb{R}^n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(|x-y|^\gamma+|x-z|^\gamma\big) \frac{|f(y)g(z)|}{(|x-y|+|x-z|)^{2n}}\,dz\,dy \\
&\le C\|b\|_{\Lambda_\gamma(\mathbb{R}^n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(y)g(z)|}{(|x-y|+|x-z|)^{2n-\gamma}}\,dz\,dy \\
&= C\|b\|_{\Lambda_\gamma} I_\alpha\big(|fg|\big)(x).
\end{aligned}$$

By Lemma 2.13, we have

$$\big\|[b,T]_1(f,g)\big\|_{L^q} \le C\|b\|_{\Lambda_\gamma(\mathbb{R}^n)}\|I_\gamma\big(|fg|\big)\|_{L^q} \le C\|b\|_{\Lambda_\gamma(\mathbb{R}^n)}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}. \qquad \square$$

Proof of Theorem 2.8. In a way similar to the proof of Theorem 2.7, we first give the proof of necessity and only consider the case $i=1$. Then we have

$$\begin{aligned}
\int_Q \big|b(x)-b_{Q'}\big|\,dx &\le C\Sigma|a_j|\int_{\mathbb{R}^n}\big|[b,T]_1(f_j,g_j)(x)\big|\big|h_j(x)\big|\,dx \\
&\le C\Sigma|a_j|\bigg(\int_{\mathbb{R}^n}\big|h_j(x)\big|^{p'}\,dx\bigg)^{\frac{1}{p'}}\bigg(\int_{\mathbb{R}^n}\big|[b,T]_1(f_j,g_j)(x)\big|^p\,dx\bigg)^{\frac{1}{p}}.
\end{aligned}$$

The fact that $M^{p,\beta}(Q)=C^{p,\beta}(Q)$ allows us to have

$$\begin{aligned}
\int_Q \big|b(x)-b_{Q'}\big|\,dx &\le C\Sigma|a_j||Q|^{\frac{1}{p'}}|Q|^{\frac{1}{p}+\frac{\beta}{2n}}\bigg(\frac{1}{1+\frac{p\beta}{2n}}\int_{\mathbb{R}^n}\big|[b,T]_1(f_j,g_j)(x)\big|^p\,dx\bigg)^{\frac{1}{p}} \\
&\le C|Q|^{\frac{1}{p'}}|Q|^{\frac{1}{p}+\frac{\beta}{2n}}\big\|[b,T]_1(f_j,g_j)\big\|_{C^{p,\beta}} \\
&\le C|Q|^{1+\frac{\beta}{2n}}|Q|^{\frac{1}{p_1}}|Q|^{\frac{1}{p_2}} \\
&= C|Q|^{1+\frac{\gamma}{n}}.
\end{aligned}$$

Thus,

$$\sup_Q \frac{1}{|Q|^{1+\frac{\gamma}{n}}}\int_Q \big|b(x)-b_Q\big|\,dx \le C.$$

Next we give the proof of sufficiency. For a cube $Q=Q(x_0,r)\subset\mathbb{R}^n$, let $b\in\Lambda_\gamma(\mathbb{R}^n)$, $f_1=f\chi_{2Q}$, $g_1=g\chi_{2Q}$, $f_2=f-f_1$, and $g_2=g-g_1$. By Lemma 2.11, choose $a=T((b-b_Q)f_1,g_2)(x_0)+T((b-b_Q)f_2,g_1)(x_0)+T((b-b_Q)f_2,g_2)(x_0)$.

The fact that $[b,T](f,g)(x)=[b-b_Q,T](f,g)(x)$ allows us to have

$$\begin{aligned}
&\Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}}\int_Q \big|[b,T]_1(f,g)(x)-[b,T]_1(f,g)_Q\big|^p dx\Big)^{\frac{1}{p}}\\
&\quad=\Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}}\int_Q \big|[b-b_Q,T]_1(f,g)(x)-[b-b_Q,T]_1(f,g)_Q\big|^p\,dx\Big)^{\frac{1}{p}}\\
&\quad\le C\Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}}\int_Q \big|T\big((b-b_Q)f,g\big)(x)-(b-b_Q)T(f,g)-a\big|^p\,dx\Big)^{\frac{1}{p}}\\
&\quad\le \Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}}\int_Q \big|(b-b_Q)T(f,g)(x)\big|^p\,dx\Big)^{\frac{1}{p}}\\
&\qquad+\Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}}\int_Q \big|T\big((b-b_Q)f_1,g_1\big)(x)\big|^p\,dx\Big)^{\frac{1}{p}}\\
&\qquad+\Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}}\int_Q \big|T\big((b-b_Q)f_1,g_2\big)(x)-T\big((b-b_Q)f_1,g_2\big)(x_0)\big|^p\,dx\Big)^{\frac{1}{p}}\\
&\qquad+\Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}}\int_Q \big|T\big((b-b_Q)f_2,g_1\big)(x)-T\big((b-b_Q)f_2,g_1\big)(x_0)\big|^p\,dx\Big)^{\frac{1}{p}}\\
&\qquad+\Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}}\int_Q \big|T\big((b-b_Q)f_2,g_2\big)(x)-T\big((b-b_Q)f_2,g_2\big)(x_0)\big|^p\,dx\Big)^{\frac{1}{p}}\\
&\quad=: I+II+III+IV+V.
\end{aligned}$$

By Lemma 2.12, Lemma 2.14, and $\gamma=\frac{\beta}{2}+\frac{n}{p}$, we have

$$\begin{aligned}
I&\le \frac{1}{|Q|^{\frac{\gamma}{n}}}\|b-b_Q\|_{L^\infty}\big\|T(f,g)\big\|_{L^p}\le \|b\|_{\Lambda_\gamma(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)},\\
II&\le \frac{1}{|Q|^{\frac{\gamma}{n}}}\big\|T\big((b-b_Q)f_1,g_1\big)\big\|_{L^p}\le \|b\|_{\Lambda_\gamma(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}.
\end{aligned}$$

Note that for $x\in Q$, $y\in 2Q$, and $z\in(2Q)^c$, we have $|x-y|+|x-z|\sim |x_0-y|+|x_0-z|$. Then we can derive that

$$\begin{aligned}
&\big|T\big((b-b_Q)f_1,g_2\big)(x)-T\big((b-b_Q)f_1,g_2\big)(x_0)\big|\\
&\quad=\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(K(x-y,x-z)-K(x_0-y,x_0-z)\big)\big(b(y)-b_Q\big)\\
&\qquad\times f_1(y)g_2(z)\,dy\,dz\Big|\\
&\quad\le\int_{2Q}\int_{(2Q)^c}\frac{|x-x_0|}{(|x_0-y|+|x_0-z|)^{2n+1}}\big|b(y)-b_Q\big|\big|f(y)g(z)\big|\,dy\,dz\\
&\quad\le C\sum_{k=2}^{\infty}\frac{1}{2^k|2^kQ|^2}\int_{2Q}\int_{2^kQ\setminus 2^{k-1}Q}\big|b(y)-b_Q\big|\big|f(y)g(z)\big|\,dy\,dz\\
&\quad\le C\sum_{k=2}^{\infty}\frac{1}{2^k|2^kQ|^2}\int_{2^kQ}\int_{2^kQ}\big|b(y)-b_{2^kQ}\big|+|b_Q-b_{2^kQ}|\big|f(y)g(z)\big|\,dy\,dz.
\end{aligned}$$

It follows that

$$\begin{aligned} III &\le \Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}}\int_Q\Big|\sum_{k=2}^{\infty}\frac{2^{-k}}{|2^kQ|^2}\int_{2^kQ}\int_{2^kQ}\big|b(y)-b_{2^kQ}\big|\big|f(y)g(z)\big|\,dy\,dz\Big|^p\,dx\Big)^{\frac{1}{p}}\\ &\quad+\Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}}\int_Q\Big|\sum_{k=2}^{\infty}\frac{2^{-k}}{|2^kQ|^2}\int_{2^kQ}\int_{2^kQ}|b_Q-b_{2^kQ}|\big|f(y)g(z)\big|\,dy\,dz\Big|^p\,dx\Big)^{\frac{1}{p}}\\ &:= III_1+III_2;\\ III_1 &\le \frac{1}{|Q|^{\frac{\gamma}{n}}}\sum_{k=2}^{\infty}\frac{1}{2^k}\big\|M\big(|b-b_{2^kQ}||fg|\big)\big\|_{L^p(2^kQ)}\\ &\le \sum_{k=2}^{\infty}\frac{1}{2^k}\frac{1}{|Q|^{\frac{\gamma}{n}}}\big\|(b-b_{2^kQ})f\big\|_{L^{p_1}(2^kQ)}\|g\|_{L^{p_2}(2^kQ)}\\ &\le \sum_{k=2}^{\infty}\frac{2^{k\gamma}}{2^k}\frac{1}{|2^kQ|^{\frac{\gamma}{n}}}\|b-b_{2^kQ}\|_{\infty}\|f\|_{L^{p_1}(2^kQ)}\|g\|_{L^{p_2}(2^kQ)}\\ &\le \sum_{k=2}^{\infty}\frac{1}{2^{k(1-\gamma)}}\|b\|_{\Lambda_\gamma}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\\ &\le C\|b\|_{\Lambda_\gamma}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)};\\ III_2 &\le \sum_{k=2}^{\infty}\frac{1}{2^{k(1-\gamma)}}\|b\|_{\Lambda_\gamma}\big\|M\big(|fg|\big)\big\|_{L^p(\mathbb{R}^n)}\\ &\le C\|b\|_{\Lambda_\gamma}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}. \end{aligned}$$

Then we have the estimates for III:

$$III\le C\|b\|_{\Lambda_\gamma}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}.$$

Similarly, we have

$$\begin{aligned} &\big|T\big((b-b_Q)f_2,g_1\big)(x)-T\big((b-b_Q)f_2,g_1\big)(x_0)\big|\\ &\quad=\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(K(x-y,x-z)-K(x_0-y,x_0-z)\big)\big(b(y)-b_Q\big)f_2(y)g_1(z)\,dy\,dz\Big|\\ &\quad\le\int_{(2Q)^c}\int_{2Q}\frac{|x-x_0|}{(|x_0-y|+|x_0-z|)^{2n+1}}\big|b(y)-b_Q\big|\big|f(y)g(z)\big|\,dy\,dz\\ &\quad\le C\sum_{k=2}^{\infty}\frac{1}{2^k|2^kQ|^2}\int_{2^kQ}\int_{2^kQ}\big|b(y)-b_Q\big|\big|f(y)g(z)\big|\,dy\,dz. \end{aligned}$$

It follows that

$$\begin{aligned} IV &\le\Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}}\int_Q\Big|\sum_{k=2}^{\infty}\frac{1}{2^k|2^kQ|^2}\int_{2^kQ}\int_{2^kQ}\big|b(y)-b_Q\big|\big|f(y)g(z)\big|\,dy\,dz\Big|^p\,dx\Big)^{\frac{1}{p}}\\ &\le C\|b\|_{\Lambda_\gamma}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}. \end{aligned}$$

We note that

$$
\begin{aligned}
&\big|T\big((b-b_Q)f_2,g_2\big)(x)-T\big((b-b_Q)f_2,g_2\big)(x_0)\big|\\
&\quad=\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(K(x-y,x-z)-K(x_0-y,x_0-z)\big)\big(b(y)-b_Q\big)f_2(y)g_2(z)\,dy\,dz\Big|\\
&\quad\le C\int_{(2Q)^c}\int_{(2Q)^c}\frac{|x-x_0|}{(|x_0-y|+|x_0-z|)^{2n+1}}\big|b(y)-b_Q\big|\big|f(y)g(z)\big|\,dy\,dz\\
&\quad\le C\sum_{k=2}^{\infty}\frac{2^{-k}}{|2^kQ|^2}\int_{2^kQ\setminus 2^{k-1}Q}\int_{2^kQ\setminus 2^{k-1}Q}\big(\big|b(y)-b_{2^kQ}\big|+|b_Q-b_{2^kQ}|\big)|fg|\,dy\,dz\\
&\quad\le C\sum_{k=2}^{\infty}\frac{1}{2^k|2^kQ|^2}\int_{2^kQ}\int_{2^kQ}\big(\big|b(y)-b_{2^kQ}\big|+|b_Q-b_{2^kQ}|\big)\big|f(y)g(z)\big|\,dy\,dz.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
V&\le\Big(\frac{1}{|Q|^{1+\frac{p\beta}{2n}}}\int_Q\Big|\sum_{k=2}^{\infty}\frac{2^{-k}}{|2^kQ|^2}\int_{(2^kQ)^2}\big(|b-b_{2^kQ}|+|b_Q-b_{2^kQ}|\big)|fg|\,dy\,dz\Big|^p\,dx\Big)^{\frac{1}{p}}\\
&\le C\|b\|_{\Lambda_\gamma}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}.
\end{aligned}
$$

The proof of Theorem 2.8 is finished. □

Proof of Theorem 2.9. (a) ⇒ (b) Since $b\in\Lambda_\gamma(\mathbb{R}^n)$, we have the following estimate:

$$
\begin{aligned}
\big|[b,I_\alpha]_1(f,g)(x)\big|&=\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{(|x-y|+|x-z|)^{2n-\alpha}}\big|f(y)g(z)\big|\,dy\,dz\Big|\\
&\le\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|b(x)-b(y)|}{(|x-y|+|x-z|)^{2n-\alpha}}f(y)g(z)\,dy\,dz\\
&\le C\|b\|_{\Lambda_\gamma}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(y)g(z)|}{(|x-y|+|x-z|)^{2n-(\alpha+\gamma)}}\,dy\,dz\\
&\le C\|b\|_{\Lambda_\gamma}I_{\alpha+\gamma}\big(|fg|(x)\big).
\end{aligned}
$$

By Lemma 2.13, $[b,I_\alpha]_1$ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

(b) ⇒ (a) This proof can be handled in much the same way as that of Theorem 2.7, thus we omit the details here. □

Proof of Theorem 2.10. (a) ⇒ (b) The fact that $b\in\Lambda_\gamma(\mathbb{R}^n)$ allows us to have the following:

$$
\begin{aligned}
\big|[b,I_\alpha]_1(f,g)(x)\big|&\le C\|b\|_{\Lambda_\gamma}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(y)g(z)|}{(|x-y|+|x-z|)^{2n-(\alpha+\gamma)}}\,dy\,dz\\
&\le C\|b\|_{\Lambda_\gamma}I_{\alpha+\gamma}\big(|fg|(x)\big).
\end{aligned}
$$

We have

$$\begin{aligned}\big\|I_{\alpha+\gamma}\big(|fg|(x)\big)\big\|_{M^{q,\tilde\beta}} &= \sup_Q \frac{1}{|Q|^{\frac{\tilde\beta}{n}+\frac{1}{q}}}\Big(\int_Q \big|I_{\alpha+\gamma}\big(|fg|(x)\big)\big|^q\Big)^{\frac{1}{q}}\\ &\le \sup_Q \frac{1}{|Q|^{\frac{2\beta}{n}}}\bigg(\frac{1}{Q}\int_Q \big|f(x)\big|^{p_1}\bigg)^{\frac{1}{p_1}}\bigg(\frac{1}{Q}\int_Q \big|g(x)\big|^{p_2}\bigg)^{\frac{1}{p_2}}\\ &\le C\|f\|_{M^{p_1,\beta}}\|g\|_{M^{p_2,\beta}}.\end{aligned}$$

Therefore, $\|[b,I_\alpha]_1(f,g)(x)\|_{M^{q,\tilde\beta}} \le C\|b\|_{\Lambda_\gamma}\|f\|_{M^{p_1,\beta}}\|g\|_{M^{p_2,\beta}}$.

(b) $\Rightarrow$ (a) Let $s(x)=\operatorname{sgn}(b(x)-b_{Q'})$, $f(y)=\chi_{Q'}(y)$, and $g(z)=\chi_{Q''}(z)$. We have

$$\begin{aligned}\int_Q \big|b(x)-b_{Q'}\big|\,dx &= \int_Q \big(b(x)-b_{Q'}\big)s(x)\,dx\\ &= \frac{1}{|Q|''}\frac{1}{|Q|'}\int_{Q''}\int_{Q'}\int_Q \big(b(x)-b(y)\big)s(x)\,dz\,dy\,dx\\ &= \frac{1}{|Q|''}\frac{1}{|Q|'}\int_{Q''}\int_{Q'}\int_Q \frac{(b(x)-b(y))}{|x-y|+|x-z|^{2n-\alpha}}\\ &\qquad\times\big(|x-y|+|x-z|\big)^{2n-\alpha}s(x)\,dz\,dy\,dx\\ &\le C|Q|^{-\frac{\alpha}{n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(b(x)-b(y))}{|x-y|+|x-z|^{2n-\alpha}}\\ &\qquad\times s(x)\chi_Q(x)\chi_{Q'}(y)\chi_{Q''}(z)\,dz\,dy\,dx\\ &\le C|Q|^{-\frac{\alpha}{n}}\int_{\mathbb{R}^n}\chi_Q(x)\big|[I_\alpha,b]_1(f,g)(x)\big|\,dx\\ &\le C|Q|^{-\frac{\alpha}{n}+1}\Big(\frac{1}{|Q|}\int_Q \big|[I_\alpha,b]_1(f,g)(x)\big|^q\,dx\Big)^{\frac{1}{q}}\\ &\le C|Q|^{-\frac{\alpha}{n}+1+\frac{\tilde\beta}{n}}\big\|[I_\alpha,b]_1(f,g)(x)\big\|_{M^{\tilde\beta,q}}\\ &\le C|Q|^{-\frac{\alpha}{n}+1+\frac{\tilde\beta}{n}}\|f\|_{M^{p_1,\beta}}\|g\|_{M^{p_2,\beta}}\\ &\le C|Q|^{-\frac{\alpha}{n}+1+\frac{\tilde\beta}{n}}|Q|^{-\frac{\beta}{n}}|Q|^{-\frac{\beta}{n}}\\ &\le C|Q|^{1+\frac{\gamma}{n}}.\end{aligned}$$

Therefore, we have

$$\frac{1}{|Q|^{1+\frac{\gamma}{n}}}\int_Q \big|b(x)-b_{Q'}\big|\,dx \le C. \qquad \square$$

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