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SOME OPERATOR INEQUALITIES FOR UNITARILY INVARIANT NORMS

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ABSTRACT. This note aims to present some operator inequalities for unitarily invariant norms. First, a Zhan-type inequality for unitarily invariant norms is given. Moreover, some operator inequalities for the Cauchy–Schwarz type are also established.

1. Introduction

Throughout this article, let $\mathbf{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex separable Hilbert space $(\mathcal{H}, \langle , \rangle)$. For self-adjoint operators A, B, the order relation $A \leq B$ means that $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in \mathcal{H}$. In particular, if $0 \leq (<)\ A$, then A is called *positive* (*invertible positive*). Here $|||\cdot|||$ denotes a unitarily invariant norm defined on a two-sided ideal $K_{|||\cdot|||}$ that is included in C_∞ (the set of compact operators), which has the basic property $|||UAV||| = |||A|||$ for every $A \in K_{|||\cdot|||}$ and all unitary operators $U, V \in \mathbf{B}(\mathcal{H})$. If $\dim \mathcal{H} = n$, then we identify $\mathbf{B}(\mathcal{H})$ with the algebra M_n of all $n \times n$ matrices with entries in $\mathbb{C}$.

Bhatia and Davis [2] proved the following: Let A, B, $X \in M_n$ with A, $B > 0$. Then the inequality

$$2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \leq |||A^vXB^{1-v} + A^{1-v}XB^v||| \leq |||AX + XB||| \tag{1.1}$$

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holds for every unitarily invariant norm $|||\cdot|||$ and $v \in [0,1]$. The second inequality in (1.1) is one of the most essential inequalities in operator theory, which is often called the *Heinz inequality*.

Replacing A and B by A^2 and B^2 in inequality (1.1), respectively, let $r = 2v$ for $v \in [0,1]$. Then the Heinz inequality gives

$$|||A^r X B^{2-r} + A^{2-r} X B^r||| \leq |||A^2 X + X B^2|||. \tag{1.2}$$

In [10], Zhan proved the following result by introducing two parameters r and t. Let A, B, $X \in M_n$ with A, $B > 0$. Then

$$(2+t)|||A^r X B^{2-r} + A^{2-r} X B^r||| \leq 2|||A^2 X + tAXB + XB^2||| \tag{1.3}$$

holds for any unitarily invariant norm $|||\cdot|||$ and $(t,r) \in (-2,2] \times [\frac{1}{2}, \frac{3}{2}]$. Obviously, inequality (1.3) is a generalization of inequality (1.2) when $\dim \mathcal{H} = n$ and $r \in [\frac{1}{2}, \frac{3}{2}]$. The tool used for proving this result was based on the induced Schur product norm.

Another important norm inequality is the well-known norm inequalities of the Cauchy–Schwarz type obtained by Hiai and Zhan [5, Theorem 1], which says the following. Let A, B, $X \in M_n$ with A, $B > 0$. For every positive real number r and every unitarily invariant norm $|||\cdot|||$, the function

$$g(t) = \big|\big|\big| |A^t X B^{1-t}|^r \big|\big|\big| \cdot \big|\big|\big| |A^{1-t} X B^t|^r \big|\big|\big|$$

is convex on the interval $[0,1]$ and attains its minimum at $t = \frac{1}{2}$. Consequently, it is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$. Hence the following norm inequality holds (see [5, Corollary 2]):

$$\begin{aligned} \big|\big|\big| |A^{\frac{1}{2}} X B^{\frac{1}{2}}|^r \big|\big|^2 &\leq \big|\big|\big| |A^t X B^{1-t}|^r \big|\big|\big| \cdot \big|\big|\big| |A^{1-t} X B^t|^r \big|\big|\big| \\ &\leq \big|\big|\big| |AX|^r \big|\big|\big| \cdot \big|\big|\big| |XB|^r \big|\big|\big|. \end{aligned} \tag{1.4}$$

It should be mentioned that inequality (1.4) also holds for operators, where A, B, $X \in \mathbf{B}(\mathcal{H})$ with A, $B \geq 0$ and $X \in K_{|||\cdot|||}$. Indeed, the key inequality $|||\,|A^{\frac{1}{2}} X B^{\frac{1}{2}}|^r\|^2 \leq |||\,|AX|^r||| \cdot |||\,|XB|^r|||$ obtained by Bhatia and Davis [3, Theorem 1] and used to prove the convexity of $g(t) = |||\,|A^t X B^{1-t}|^r||| \cdot |||\,|A^{1-t} X B^t|^r|||$ (see [5, Theorem 1]) holds for operators, where A, B, $X \in \mathbf{B}(\mathcal{H})$ with A, $B \geq 0$ and $X \in K_{|||\cdot|||}$. Therefore, Hiai and Zhan [5, Corollary 2] actually proved that inequality (1.4) holds for operators. Let p and q be two nonnegative real numbers with $p > 0$ or $q > 0$. Putting $t = \frac{p}{p+q}$, and replacing A and B by A^{p+q} and B^{p+q} in inequality (1.4), respectively, we have

$$\big|\big|\big| |A^p X B^q|^r \big|\big|\big| \cdot \big|\big|\big| |A^q X B^p|^r \big|\big|\big| \leq \big|\big|\big| |A^{p+q} X|^r \big|\big|\big| \cdot \big|\big|\big| |X B^{p+q}|^r \big|\big|\big|, \tag{1.5}$$

where A, B, $X \in \mathbf{B}(\mathcal{H})$ with A, $B \geq 0$ and $X \in K_{|||\cdot|||}$.

Recently, unitarily invariant norms of Heinz inequality for matrices and Hilbert space operators have been obtained. These forms can be found in [7], [6], and the references therein. The related Cauchy–Schwarz inequality has been given in [1] and [4, Theorems 2, 3, and 4], respectively.

In this note, we study operator inequalities for unitarily invariant norms. Precisely, we present a generalization of inequality (1.3) for operators. Moreover, we also give some operator inequalities for the Cauchy–Schwarz type.

2. Zhan-type inequality for operators

In this section, we present a generalization of Zhan's inequality for unitarily invariant norms. To achieve our goal, we need the following lemmas; the first lemma was obtained by Kittaneh [8, Corollary 1], which is often called the *generalized version of the* CPR *inequality.*

Lemma 2.1. *Let R, S, $T \in \mathbf{B}(\mathcal{H})$ with R and S invertible and $T \in \mathrm{K}_{|||\cdot|||}$. Then, for every unitarily invariant norm $|||\cdot|||$, inequality*

$$2|||T||| \le |||R^*TS^{-1} + R^{-1}TS^*||| \tag{2.1}$$

holds, where R^* is the conjugate transpose operator of R.

The matrix version of the next lemma was obtained by Sababheh in [9, Theorem 2.8]; we point out that it is also true for operators.

Lemma 2.2. *Let $|||\cdot|||$ be any unitarily invariant norm on $\mathrm{K}_{|||\cdot|||}$, and A, B, $X \in \mathbf{B}(\mathcal{H})$ with A, $B \ge 0$ and $X \in \mathrm{K}_{|||\cdot|||}$. Then, for every unitarily invariant norm $|||\cdot|||$ and $p \ge q \ge r \ge 0$,*

$$|||A^pXB^q + A^qXB^p||| \le |||A^{p+r}XB^{q-r} + A^{q-r}XB^{p+r}|||. \tag{2.2}$$

Proof. The proof is the same as that of [9, Theorem 2.8]. For the reader's convenience, we give its proof again. When $p = 0$, the result holds obviously, and so we only need to prove it holds for $p > 0$. By inequality (1.5), we get

$$|||A^pXB^q + A^qXB^p||| \le |||A^{p+q}X + XB^{p+q}|||.$$

Hence we obtain

$$\begin{aligned} |||A^pXB^q + A^qXB^p||| &= |||A^{p-q+r}(A^{q-r}XB^{q-r})B^r + A^r(A^{q-r}XB^{q-r})B^{p-q+r}||| \\ &\le |||A^{p-q+2r}(A^{q-r}XB^{q-r}) + (A^{q-r}XB^{q-r})B^{p-q+2r}||| \\ &= |||A^{p+r}XB^{q-r} + A^{q-r}XB^{p+r}|||. \end{aligned}$$

This completes the proof. □

Later in this article we present a Zhan-type inequality for unitarily invariant norms.

Theorem 2.3. *Let A, B, $X \in \mathbf{B}(\mathcal{H})$ with A, $B > 0$ and $X \in \mathrm{K}_{|||\cdot|||}$ and $p \ge q \ge r \ge 0$. Then inequality*

$$\begin{aligned} &(t+2)|||A^{\frac{3p+q}{2}}XB^{\frac{3q+p}{2}} + A^{\frac{3q+p}{2}}XB^{\frac{3p+q}{2}}||| \\ &\le 4|||A^{\frac{3p+q+2r}{2}}XB^{\frac{3q+p-2r}{2}} + A^{\frac{3q+p-2r}{2}}XB^{\frac{3p+q+2r}{2}}||| - 2(2-t)|||A^{p+q}XB^{p+q}||| \\ &\le |||A^{2(p+q)}X + XB^{2(p+q)} + tA^{p+q}XB^{p+q}||| \end{aligned} \tag{2.3}$$

holds for any unitarily invariant norm $|||\cdot|||$ and $t \in (-2, 2]$.

Proof. Thanks to inequality (2.2),

$$|||A^pXB^q + A^qXB^p||| \le |||A^{p+r}XB^{q-r} + A^{q-r}XB^{p+r}|||,$$

and the Heinz inequality

$$|||A^pXB^q + A^qXB^p||| \le |||A^{p+q}X + XB^{p+q}|||,$$

we have

$$\begin{aligned} |||A^pXB^q + A^qXB^p||| &\le |||A^{p+r}XB^{q-r} + A^{q-r}XB^{p+r}||| \\ &\le |||A^{p+q}X + XB^{p+q}|||. \end{aligned} \tag{2.4}$$

Replacing X by $A^{-\frac{p+q}{2}}XB^{-\frac{p+q}{2}}$ in inequality (2.4), we obtain

$$\begin{aligned} |||A^{\frac{p-q}{2}}XB^{\frac{q-p}{2}} + A^{\frac{q-p}{2}}XB^{\frac{p-q}{2}}||| &\le |||A^{\frac{p-q+2r}{2}}XB^{\frac{q-p-2r}{2}} + A^{\frac{q-p-2r}{2}}XB^{\frac{p-q+2r}{2}}||| \\ &\le |||A^{\frac{p+q}{2}}XB^{-\frac{p+q}{2}} + A^{-\frac{p+q}{2}}XB^{\frac{p+q}{2}}|||. \end{aligned} \tag{2.5}$$

Thanks to

$$\begin{aligned} &A^{\frac{p+q}{2}}(A^{\frac{p+q}{2}}XB^{-\frac{p+q}{2}} + A^{-\frac{p+q}{2}}XB^{\frac{p+q}{2}})B^{-\frac{p+q}{2}} \\ &\quad + A^{-\frac{p+q}{2}}(A^{\frac{p+q}{2}}XB^{-\frac{p+q}{2}} + A^{-\frac{p+q}{2}}XB^{\frac{p+q}{2}})B^{\frac{p+q}{2}} \\ &= A^{p+q}XB^{-(p+q)} + A^{-(p+q)}XB^{p+q} + 2X \end{aligned}$$

and the generalized version of the C-P-R inequality (2.1), $2|||X||| \le |||S^{-1}XT + SXT^{-1}|||$, where S and T are two invertible self-adjoint operators and $X \in K_{|||\cdot|||}$, we deduce that

$$\begin{aligned} &2|||A^{\frac{p+q}{2}}XB^{-\frac{p+q}{2}} + A^{-\frac{p+q}{2}}XB^{\frac{p+q}{2}}||| \\ &\quad \le |||A^{p+q}XB^{-(p+q)} + A^{-(p+q)}XB^{p+q} + 2X|||. \end{aligned} \tag{2.6}$$

Relations (2.5) and (2.6) give

$$\begin{aligned} &2|||A^{\frac{p-q}{2}}XB^{\frac{q-p}{2}} + A^{\frac{q-p}{2}}XB^{\frac{p-q}{2}}||| \\ &\quad \le 2|||A^{\frac{p-q+2r}{2}}XB^{\frac{q-p-2r}{2}} + A^{\frac{q-p-2r}{2}}XB^{\frac{p-q+2r}{2}}||| \\ &\quad \le |||A^{p+q}XB^{-(p+q)} + A^{-(p+q)}XB^{p+q} + 2X|||. \end{aligned} \tag{2.7}$$

On the other hand, due to

$$\begin{aligned} &A^{p+q}XB^{-(p+q)} + A^{-(p+q)}XB^{p+q} + 2X \\ &\quad = A^{p+q}XB^{-(p+q)} + A^{-(p+q)}XB^{p+q} + tX + (2-t)X, \end{aligned}$$

we have

$$\begin{aligned} &|||A^{p+q}XB^{-(p+q)} + A^{-(p+q)}XB^{p+q} + 2X||| \\ &\quad \le |||A^{p+q}XB^{-(p+q)} + A^{-(p+q)}XB^{p+q} + tX||| + (2-t)|||X|||. \end{aligned} \tag{2.8}$$

Combining (2.7) with (2.8), we get

$$\begin{aligned} &4|||A^{\frac{p-q}{2}}XB^{\frac{q-p}{2}} + A^{\frac{q-p}{2}}XB^{\frac{p-q}{2}}||| - 2(2-t)|||X||| \\ &\quad \le 2|||A^{\frac{p-q+2r}{2}}XB^{\frac{q-p-2r}{2}} + A^{\frac{q-p-2r}{2}}XB^{\frac{p-q+2r}{2}}||| - 2(2-t)|||X||| \\ &\quad \le 2|||A^{p+q}XB^{-(p+q)} + A^{-(p+q)}XB^{p+q} + tX|||. \end{aligned} \tag{2.9}$$

Once again, using the generalized version of the C-P-R inequality, we have

$$\begin{aligned}(t+2)|||A^{\frac{p-q}{2}}XB^{\frac{q-p}{2}}+A^{\frac{q-p}{2}}XB^{\frac{p-q}{2}}|||\\ \leq 4|||A^{\frac{p-q}{2}}XB^{\frac{q-p}{2}}+A^{\frac{q-p}{2}}XB^{\frac{p-q}{2}}|||-2(2-t)|||X|||.\end{aligned} \tag{2.10}$$

It follows from inequalities (2.9) and (2.10) that

$$\begin{aligned}&(t+2)|||A^{\frac{p-q}{2}}XB^{\frac{q-p}{2}}+A^{\frac{q-p}{2}}XB^{\frac{p-q}{2}}|||\\ &\leq 2|||A^{\frac{p-q+2r}{2}}XB^{\frac{q-p-2r}{2}}+A^{\frac{q-p-2r}{2}}XB^{\frac{p-q+2r}{2}}|||-2(2-t)|||X|||\\ &\leq 2|||A^{p+q}XB^{-(p+q)}+A^{-(p+q)}XB^{p+q}+tX|||.\end{aligned} \tag{2.11}$$

Replacing X by $A^{p+q}XB^{p+q}$ in inequality (2.11), we get the desired result (2.3). This completes the proof. □

Remark 2.4. By Theorem 2.3, we also have the following result. Let A, B, $X \in \mathbf{B}(\mathcal{H})$ with A, $B > 0$ and $X \in K_{|||\cdot|||}$ and $q \geq p \geq r \geq 0$. Then inequality

$$\begin{aligned}&(t+2)|||A^{\frac{3q+p}{2}}XB^{\frac{3p+q}{2}}+A^{\frac{3p+q}{2}}XB^{\frac{3q+p}{2}}|||\\ &\leq 4|||A^{\frac{3q+p+2r}{2}}XB^{\frac{3p+q-2r}{2}}+A^{\frac{3p+q-2r}{2}}XB^{\frac{3q+p+2r}{2}}|||-2(2-t)|||A^{q+p}XB^{q+p}|||\\ &\leq 2|||A^{2(q+p)}X+XB^{2(q+p)}+tA^{q+p}XB^{q+p}|||\end{aligned}$$

holds for any unitarily invariant norm $|||\cdot|||$ and $t \in (-2, 2]$.

Based on Theorem 2.3 and Remark 2.4, we obtain the following operator inequality.

Corollary 2.5. *Let A, B, $X \in \mathbf{B}(\mathcal{H})$ with A, $B > 0$ and $X \in \mathrm{K}_{|||\cdot|||}$ and p, $q > 0$. Then inequality*

$$\begin{aligned}(t+2)|||A^{\frac{3p+q}{2}}XB^{\frac{3q+p}{2}}+A^{\frac{3q+p}{2}}XB^{\frac{3p+q}{2}}|||\\ \leq 2|||A^{2(p+q)}X+XB^{2(p+q)}+tA^{p+q}XB^{p+q}|||\end{aligned} \tag{2.12}$$

holds for any unitarily invariant norm $|||\cdot|||$ and $t \in (-2, 2]$.

Remark 2.6. Putting $p+q=1$ and $r_1=\frac{3q+p}{2}$, then $2-r_1=\frac{3p+q}{2}$ and $r_1=\frac{1}{2}+q \in [\frac{1}{2}, \frac{3}{2}]$, inequality (2.12) becomes (1.3). Thus inequality (2.12) is a generalization of inequality (1.3) for operators.

Remark 2.7. By Corollary 2.5, when $t = 1$, we get

$$|||A^{\frac{3p+q}{2}}XB^{\frac{3q+p}{2}}+A^{\frac{3q+p}{2}}XB^{\frac{3p+q}{2}}||| \leq |||A^{2(p+q)}X+XB^{2(p+q)}|||; \tag{2.13}$$

hence, when $p = q = \frac{1}{2}$, by inequality (2.13), we get

$$2|||AXB||| \leq |||A^2X+XB^2|||.$$

This is just the well-known arithmetic–geometric norm inequality due to Bhatia and Davis [2].

3. Cauchy–Schwarz-type inequality for operators

In this section, we mainly present some Cauchy–Schwarz operator inequalities for unitarily invariant norms. First, we have the following theorem.

Theorem 3.1. *Let A, B, $X \in \mathbf{B}(\mathcal{H})$ with $A, B > 0$ and $X \in \mathrm{K}_{|||\cdot|||}$ and $p \geq q \geq s \geq 0$. Then inequality*

$$\left|\left|\left| |A^pXB^q|^r \right|\right|\right| \cdot \left|\left|\left| |A^qXB^p|^r \right|\right|\right| \leq \left|\left|\left| |A^{p+s}XB^{q-s}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{q-s}XB^{p+s}|^r \right|\right|\right| \tag{3.1}$$

holds for any unitarily invariant norm $|||\cdot|||$ and $r > 0$.

Proof. By inequality (1.5), we get

$$\begin{aligned}
&\left|\left|\left| |A^pXB^q|^r \right|\right|\right| \cdot \left|\left|\left| |A^qXB^p|^r \right|\right|\right| \\
&= \left|\left|\left| \left|A^{p-q+s}(A^{q-s}XB^{q-s})B^s\right|^r \right|\right|\right| \cdot \left|\left|\left| \left|A^s(A^{q-s}XB^{q-s})B^{p-q+s}\right|^r \right|\right|\right| \\
&\leq \left|\left|\left| \left|A^{p-q+2s}(A^{q-s}XB^{q-s})\right|^r \right|\right|\right| \cdot \left|\left|\left| \left|(A^{q-s}XB^{q-s})B^{p-q+2s}\right|^r \right|\right|\right| \\
&= \left|\left|\left| |A^{p+s}XB^{q-s}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{q-s}XB^{p+s}|^r \right|\right|\right|.
\end{aligned}$$

This completes the proof. □

Remark 3.2. By inequality (3.1), we have

$$\left|\left|\left| |A^pXB^q|^r \right|\right|\right| \cdot \left|\left|\left| |A^qXB^p|^r \right|\right|\right| \leq \left|\left|\left| |A^{q+s}XB^{p-s}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{p-s}XB^{q+s}|^r \right|\right|\right| \tag{3.2}$$

for $q \geq p \geq s \geq 0$. By inequality (1.5), we easily see that inequalities (3.1) and (3.2) are the refinements of inequality (1.5).

Based on Theorem 3.1, we obtain the following result.

Theorem 3.3. *Let A, B, $X \in \mathbf{B}(\mathcal{H})$ with $A, B > 0$ and $X \in \mathrm{K}_{|||\cdot|||}$, $p \geq q \geq s \geq 0$ and $r > 0$. Then the function*

$$f(s) = \left|\left|\left| |A^{p+s}XB^{q-s}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{q-s}XB^{p+s}|^r \right|\right|\right|$$

is increasing on $[0, q]$.

Proof. Let $0 \leq s_1 < s_2 \leq q$. Then, by inequality (3.1), we have

$$\begin{aligned}
f(s_1) &= \left|\left|\left| |A^{p+s_1}XB^{q-s_1}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{q-s_1}XB^{p+s_1}|^r \right|\right|\right| \\
&\leq \left|\left|\left| |A^{p+s_1+(s_2-s_1)}XB^{q-s_1-(s_2-s_2)}|^r \right|\right|\right| \\
&\quad \times \left|\left|\left| |A^{q-s_1-(s_2-s_2)}XB^{p+s_1+(s_2-s_1)}|^r \right|\right|\right| \\
&= \left|\left|\left| |A^{p+s_2}XB^{q-s_2}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{q-s_2}XB^{p+s_2}|^r \right|\right|\right| \\
&= f(s_2).
\end{aligned}$$

This completes the proof. □

Remark 3.4. Noting that

$$f(0) = \left|\left|\left| |A^pXB^q|^r \right|\right|\right| \cdot \left|\left|\left| |A^qXB^p|^r \right|\right|\right|$$

and

$$f(q) = \left|\left|\left| |A^{p+q}X|^r \right|\right|\right| \cdot \left|\left|\left| |XB^{p+q}|^r \right|\right|\right|,$$

then inequality (1.5) can be written as $f(0) \le f(q)$. However, by Theorem 3.3, we have $f(0) \le f(r) \le f(q)$ for $0 < r < q$. This implies the intermediate inequality interpolate the Cauchy–Schwarz inequality increasingly.

The following corollary is a consequence of Theorem 3.3.

Corollary 3.5. *Let $A, B, X \in \mathbf{B}(\mathcal{H})$ with $A, B > 0$ and $X \in \mathrm{K}_{|||\cdot|||}$, $t \in [0,1]$ and $r > 0$. Then the function*

$$g(t) = \left|\left|\left| |A^t X B^{1-t}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{1-t} X B^t|^r \right|\right|\right|$$

is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$.

Proof. If $0 \le t \le \frac{1}{2}$, then

$$g(t) = \left|\left|\left| |A^{\frac{1}{2}-(\frac{1}{2}-t)} X B^{\frac{1}{2}+(\frac{1}{2}-t)}|^r \right|\right|\right| \cdot \left|\left|\left| |A^{\frac{1}{2}+(\frac{1}{2}-t)} X B^{\frac{1}{2}-(\frac{1}{2}-t)}|^r \right|\right|\right|$$

can be viewed as $|||\,|A^{p+s}XB^{q-s}|^r|||\cdot|||\,|A^{q-s}XB^{p+s}|^r\|$ with $p = q = \frac{1}{2}$ and $s = \frac{1}{2} - t$; thus $g(t)$ is decreasing on $[0, \frac{1}{2}]$ due to the increasing of $f(s)$ by Theorem 3.3. As with the proof of the increasing of $g(s)$ on $[\frac{1}{2}, 1]$, the details are omitted here.

This completes the proof. □

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