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EXTENDED SPECTRUM AND EXTENDED EIGENSPACES OF QUASINORMAL OPERATORS

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Abstract. We say that a complex number λ is an extended eigenvalue of a bounded linear operator T on a Hilbert space $\mathcal{H}$ if there exists a nonzero bounded linear operator X acting on $\mathcal{H}$, called the *extended eigenvector associated to* λ, and satisfying the equation $TX = \lambda XT$. In this article, we describe the sets of extended eigenvalues and extended eigenvectors for the quasinormal operators.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable complex Hilbert space, and denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. If T is an operator in $\mathcal{B}(\mathcal{H})$, then a complex number λ is an extended eigenvalue of T if there is a nonzero operator X such that $TX = \lambda XT$. We denote by the symbol $\sigma_{\text{ext}}(T)$ the set of extended eigenvalues of T. The subspace generated by extended eigenvectors corresponding to λ will be denoted by $E_{\text{ext}}(T, \lambda)$.

The concepts of extended eigenvalue and extended eigenvector are closely related with the generalization of the famous Lomonosov theorem on the existence of nontrivial hyperinvariant subspace for the compact operators on a Banach space (which were done by Brown in [6] and by Kim, Moore, and Pearcy in [13]),

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and is stated as follows: *If an operator T on a Banach space has a nonzero compact eigenvector, then T has a nontrivial hyperinvariant subspace.* (The special case where T commutes with a nonzero compact operator is the original theorem of Lomonosov [15].)

Extended eigenvalues and their corresponding extended eigenvectors were studied by several authors (see, e.g., [1], [3], [7], [11], and [14]). On the finite-dimensional spaces, the extended eigenvalues and eigenvectors of operators were studied by Biswas and Petrovic [4]. In [3], Biswas, Lambert, and Petrovic introduced these notions and showed that $\sigma_{\text{ext}}(V) =]0, +\infty[$, where V is the well-known integral Volterra operator on the space $L^2[0, 1]$. In [11], Karaev gave a complete description of the set of extended eigenvectors of V. The set of all extended eigenvectors of the Volterra integration operator was described in terms of the Duhamel operator and the composition operator in [12]. Also, Gürdal obtained important results in this subject, which gave extended eigenvalues and extended eigenvectors of integration operators on the Wiener algebra in [10]. Recall that Shkarin [16] has shown that there exist compact quasinilpotent operators for which the extended spectrum is reduced to $\{1\}$. Let us note that this allows us to classify this type of operator. Recently, the present authors in [7] gave an accurate and practical description of the set of extended eigenvectors of normal operators.

In the present article, we completely solve the problem of extended spectrum and extended eigenvectors for a more general class of operators, that is, the quasinormal operators. In Section 2, we introduce the sets of intertwining values of a couple of operators and λ-intertwining operators associated with a couple of operators and an intertwining value. We give a complete description of the set of intertwining values associated with a quasinormal operator and an operator of the form $A \otimes S$, where A is an injective positive operator and S is the usual forward shift on the Hardy space H^2. This is the main result of the paper, and it is used several times throughout. In particular, we apply this result to describe the extended spectrum of a pure quasinormal operator. Theorem 3.1 gives a description of extended eigenvectors for any injective subnormal operator. In particular, we describe in Section 3 the set of extended eigenvectors related to the canonical decomposition of a subnormal operator into a sum of a normal and a pure subnormal operator. Section 4 is devoted to the complete description of the extended eigenvalues and the extended eigenspaces of a general quasinormal operator. The last section deals with the relationship between extended eigenvectors of some pure quasinormal operators and their minimal normal extensions.

2. Intertwining values and associated intertwining operators of quasinormal operators

In this section, we characterize the set of extended eigenvalues of a quasinormal operator. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is quasinormal if it commutes with its modulus $|T| := (T^*T)^{1/2}$; that is, $T|T| = |T|T$. Furthermore, T is pure if it has no reducing subspaces $\mathcal{M} \neq \{0\}$ such that $T|_{\mathcal{M}}$ is normal. Since the case of normal operators has been accomplished in [7], we will focus in this section on the case of pure quasinormal operators. First we will show some auxiliary results.

Proposition 2.1. *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$. Then $\sigma_{\text{ext}}(T_1)\sigma_{\text{ext}}(T_2) \subset \sigma_{\text{ext}}(T_1 \otimes T_2)$, where $T_1 \otimes T_2$ is the tensor product of T_1 and T_2.*

Proof. Let $\lambda_i \in \sigma_{\text{ext}}(T_i)$ and $X_i \in E_{\text{ext}}(T_i, \lambda_i)\backslash\{0\}$, $i = 1, 2$. If we consider $X := X_1 \otimes X_2$, then X is a nonzero operator in $E_{\text{ext}}(T_1 \otimes T_2, \lambda_1\lambda_2)$, which implies that

$$\lambda_1\lambda_2 \in \sigma_{\text{ext}}(T_1 \otimes T_2). \qquad \square$$

Now, if we denote by S the unilateral shift (which is supposed to act on the Hardy space H^2), then Brown proved the following theorem (see [5, Theorem 1]).

Theorem 2.2. *An operator $T \in \mathcal{B}(\mathcal{H})$ is a pure quasinormal operator if and only if there is an injective positive operator A on a Hilbert space $\mathfrak{L}$ such that T is unitarily equivalent to $A \otimes S$ acting on $\mathfrak{L} \otimes H^2$.*

Remark 2.3. The two following remarks will be frequently used in the rest of this article.

(1) Let $T = V_T|T| \in \mathcal{B}(\mathcal{H})$ be the polar decomposition of a pure quasinormal operator T. The subspace $\mathfrak{L}_T = \mathcal{H} \ominus V_T\mathcal{H}$ is invariant by $|T|$, and we can choose the positive operator A in the above theorem by setting $A := A_T = |T|_{|}\mathfrak{L}_T$. In this case we will denote by $U_T \in \mathcal{B}(\mathcal{H}, \mathfrak{L}_T \otimes H^2)$ the unitary operator such that $A_T \otimes S = U_T T U_T^*$. Proposition 2.1 and Theorem 2.2 already show that $\sigma_{\text{ext}}(A_T) \cdot \mathbb{D}^c \subseteq \sigma_{\text{ext}}(T)$. We will frequently identify the space $\mathfrak{L}_T \otimes H^2$ with the space $\bigoplus_{k=0}^{\infty} \mathfrak{L}_T$.

(2) Let $\mathcal{H}$ be a Hilbert space, and let $T \in \mathcal{B}(\mathcal{H})$. Suppose that there exist $S, U \in \mathcal{B}(\mathcal{H})$ with U an invertible operator such that $T = U^{-1}SU$. Then one can easily verify that $\sigma_{\text{ext}}(T) = \sigma_{\text{ext}}(S)$. Moreover, for all $\lambda \in \sigma_{\text{ext}}(T)$, we have that $E_{\text{ext}}(T, \lambda) = UE_{\text{ext}}(S, \lambda)U^{-1}$.

For our purpose, we introduce the following useful sets of operators. Letting $(A, B) \in \mathcal{B}(\mathcal{H}_1) \times \mathcal{B}(\mathcal{H}_2)$ and $r \in \mathbb{R}_+^*$, we define $\mathcal{A}_r(A, B)$ by setting

$$\mathcal{A}_r(A, B) = \big\{L \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) : \exists c \geq 0 \text{ such that } \forall x \in \mathcal{H}_2, \\ \forall n \in \mathbb{N}, \|r^{-n}A^nLx\| \leq c\|B^nx\|\big\}.$$

When $\mathcal{H}_1 = \mathcal{H}_2 := \mathcal{H}$ and $A = B$, the set $\mathcal{A}_r(A, A)$ is denoted as $\mathcal{A}_r(A)$, or if no confusion is possible, then we write simply $\mathcal{A}_r$. Moreover, if the positive operator A is invertible, then the set $\mathcal{A}_{|\lambda|}$ is defined by

$$\mathcal{A}_{|\lambda|} = \big\{L \in \mathcal{B}(\mathcal{H}) : \sup_{n\in\mathbb{N}} \|\lambda^{-n}A^nLA^{-n}\| < +\infty\big\}.$$

In addition, for $|\lambda| = 1$, we get the Deddens algebra given in [8]. We also define the intertwining values associated here with the couple of operators (A, B) by setting $\Lambda_{\text{int}}(A, B) = \{\lambda \in \mathbb{C} : \exists X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) \setminus \{0\} \text{ such that } AX = \lambda XB\}$. Also, if $\lambda \in \Lambda_{\text{int}}(A, B)$, then we denote by $E_{\text{int}}(A, B, \lambda)$ the space of λ-intertwining operators $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, that is, operators such that $AX = \lambda XB$. When $A = B$, $\Lambda_{\text{int}}(A, A)$ is exactly the extended spectrum of the operator A, and if $\lambda \in \sigma_{\text{ext}}(A)$, then the space of λ-intertwining operators is exactly $E_{\text{ext}}(T, \lambda)$. The next result will be used several times from now on.

Proposition 2.4. *Let R be an injective quasinormal operator acting on a Hilbert space $\mathcal{H}$, and let A be an injective positive operator on a Hilbert space $\mathfrak{L}$. Then $\lambda \in \Lambda_{\text{int}}(R, A \otimes S)$ if and only if $\mathcal{A}_{|\lambda|}(R, A) \neq \{0\}$.*

Proof. Assume that $\lambda \in \Lambda_{\text{int}}(R, A \otimes S)$, and let

$$X = [X_0, \dots, X_n, \dots] \in \mathcal{B}\Big(\bigoplus_{k=0}^{+\infty} \mathfrak{L}, \mathcal{H}\Big)$$

be a nonzero operator satisfying $RX = \lambda X(A \otimes S)$. Since R is injective, we have $\lambda \neq 0$. An easy calculation shows that $RX_k = \lambda X_{k+1} A$, and hence $R^k X_0 = \lambda^k X_k A^k$ for any $k \in \mathbb{N}$. On the one hand, since the range of A^k is dense, it implies that X_0 is necessarily nonnull. On the other hand, we see that

$$|\lambda|^{-n} \|R^n X_0 x\| \leq \|X\| \|A^n x\| \quad (\forall n \in \mathbb{N}, \forall x \in \mathfrak{L}).$$

Consequently, X_0 is a nonzero element of $\mathcal{A}_{|\lambda|}(R, A)$.

Reciprocally, if $L \in \mathcal{A}_{|\lambda|}(R, A) \setminus \{0\}$, then we define, for all $n \in \mathbb{N}$, the operator

$$\begin{aligned} \breve{X}_n : \quad & \operatorname{Im} A^n \to \mathcal{H}, \\ & A^n x \mapsto \lambda^{-n} R^n L x. \end{aligned}$$

Since $L \in \mathcal{A}_{|\lambda|}(R, A)$, there is $c \geq 0$ such that, for all $y \in \operatorname{Im} A^n$, $\|\breve{X}_n y\| \leq c\|y\|$. Also, $\operatorname{Im} A^n$ is dense in $\mathfrak{L}$, and thus $\breve{X}_n$ has a (unique bounded) extension on $\mathfrak{L}$, which will be denoted by X_n. It remains to verify that $RX_n = \lambda X_{n+1} A$ for all $n \in \mathbb{N}$, and so we let $x \in \mathfrak{L}$ and $y = A^n x$. Then

$$\begin{aligned} RX_n y &= RX_n A^n x = R\breve{X}_n A^n x = \lambda^{-n} R^{n+1} L x \\ &= \lambda \lambda^{-(n+1)} R^{n+1} L x = \lambda \breve{X}_{n+1} A^{n+1} x = \lambda X_{n+1} A y. \end{aligned}$$

By density, we get $RX_n = \lambda X_{n+1} A$, and hence $X \in E_{\text{int}}(R, A \otimes S, \lambda) \setminus \{0\}$, as we wanted. $\square$

Let T be a self-adjoint operator acting on a Hilbert space H. Denote by $m_T = \inf\{\langle Tx, x\rangle : \|x\| = 1\}$. We observe that $m_T = 1/\|T^{-1}\|$ when T is an invertible positive operator. Also denote as usual by $\sigma(T)$ and $\sigma_p(T)$ the spectrum and the point spectrum of T, respectively. We write $\mathbb{D}(z_0, r[$ for the open unit disc of radius r centered at z_0 (resp., $\mathbb{D}(z_0, r]$ for the closed disc with the same center and the same radius) and $\mathbb{D}(z_0, r[^c$ (resp., $\mathbb{D}(z_0, r]^c$) for the complementary set. The following theorem is our main result and will be used several times again below.

Theorem 2.5. *Let $R \in \mathcal{B}(\mathcal{H})$ be an injective quasinormal operator, and let $A \in \mathcal{B}(\mathfrak{L})$ be an injective positive operator. Then we have the following:*

(1) *if $(m_{|R|}, \|A\|) \in \sigma_p(|R|) \times \sigma_p(A)$, then*

$$\Lambda_{\text{int}}(R, A \otimes S) = \mathbb{D}\Big(0, \frac{m_{|R|}}{\|A\|}\Big[^c;$$

(2) *if* $(m_{|R|}, \|A\|) \notin \sigma_p(|R|) \times \sigma_p(A)$, *then*

$$\Lambda_{\text{int}}(R, A \otimes S) = \mathbb{D}\Big(0, \frac{m_{|R|}}{\|A\|}\Big]^c.$$

Proof. The first step consists of proving the inclusion

$$\mathbb{D}\Big(0, \frac{m_{|R|}}{\|A\|}\Big]^c \subseteq \Lambda_{\text{int}}(R, A \otimes S).$$

Let ε be in $]0, \|A\|[$. Denote by E^A (resp., $E^{|R|}$) the spectral measure of A (resp., $|R|$). Then we can choose a nonzero vector a (resp., a nonzero vector b) in $E^{|R|}([m_{|R|}, m_{|R|} + \varepsilon])(\mathcal{H})$ (resp., in $E^A([\|A\| - \varepsilon, \|A\|])(\mathfrak{L})$) because $m_{|R|} = \inf\{\lambda : \lambda \in \sigma(|R|)\}$ (resp., $\|A\| = \sup\{\lambda : \lambda \in \sigma(A)\}$). Observe that b can be written under the form $b = A^n b_n$, where

$$b_n = \Big(\int_{\|A\|-\varepsilon}^{\|A\|} t^{-n}\, dE^A(t)\Big) b.$$

Set $L = a \otimes b$. Since R is quasinormal, we have

$$\begin{aligned}\|R^n L x\| &= \||R|^n a\| |\langle x, b\rangle| = \||R|^n a\| \|b_n\| \Big| \Big\langle A^n x, \frac{b_n}{\|b_n\|} \Big\rangle \Big| \\ &\le \||R|^n a\| \|b_n\| \|A^n x\| \le \|a\| \|b\| \Big(\frac{m_{|R|} + \varepsilon}{\|A\| - \varepsilon}\Big)^n \|A^n x\|.\end{aligned}$$

Hence the nonnull operator L belongs to $\in \mathcal{A}_{\frac{m_{|R|}+\varepsilon}{\|A\|-\varepsilon}}(R, A)$. Applying Proposition 2.4, we see that $\mathbb{D}(0, \frac{m_{|R|}+\varepsilon}{\|A\|-\varepsilon}[^c \subseteq \Lambda_{\text{int}}(R, A \otimes S)$. Since ε could be arbitrarily chosen in $]0, \|A\|[$, we obtained the desired inclusion.

The second step is to prove that $\Lambda_{\text{int}}(R, A \otimes S) \subseteq \mathbb{D}(0, \frac{m_{|R|}}{\|A\|}[^c$. Letting $\lambda \in \Lambda_{\text{int}}(R, A \otimes S)$, we know from Proposition 2.4 that there exists a nonnull operator $L \in \mathcal{A}_{|\lambda|}(R, A)$. Recall that, since the operator R is quasinormal, we have $|R^n| = |R|^n$. Therefore, there exists an absolute positive constant C such that $(m_{|R|})^{2n} L^* L \le L^* |R|^{2n} L = L^* R^{*n} R^n L \le C^2 |\lambda|^{2n} A^{2n}$. Then we necessarily have

$$\frac{m_{|R|}}{\|A\|} \Big(\frac{\|L\|}{C}\Big)^{\frac{1}{n}} \le |\lambda|,$$

and letting $n \to \infty$, we obtained the desired conclusion. We are now in position to prove the announced assertions.

(1) By hypothesis, there exists a couple of unit eigenvectors $(u, v) \in \mathcal{H} \times \mathfrak{L}$ such that $|R| u = m_{|R|} u$ and $Av = \|A\| v$. We can see that the operator $L = u \otimes v$ is in $\mathcal{A}_{m_{|R|}/\|A\|}(R, A)$. From Proposition 2.4, we deduce that the circle $\mathcal{C}(0, m_{|R|}/\|A\|)$ centered in 0 and of radius $m_{|R|}/\|A\|$ is contained in $\Lambda_{\text{int}}(R, A \otimes S)$. Using the first two steps of the proof, we can conclude.

(2) Suppose that $\lambda \in \Lambda_{\text{int}}(R, A \otimes S)$ with $|\lambda| = \frac{m_{|R|}}{\|A\|}$. Then there exists $X \neq 0$ such that $RX = \lambda X (A \otimes S)$. Since R is injective, then $\lambda \neq 0$, and hence $|R|$ is invertible ($m_{|R|} > 0$). As in the proof of the last proposition, we write $X = [X_0, \ldots, X_n, \ldots]$, and we get $R^n X_0 = \lambda^n X_n A^n$. Let $R = V|R|$ be the polar

decomposition of the operator R. Since R is injective, we see that V is an isometry. Choosing $x \in \mathfrak{L}$ and $y = R^{*n}b \in \mathrm{Im}(R^{*n})$, we derive that

$$|\langle X_0 x, y\rangle| = \Big(\frac{m_{|R|}}{\|A\|}\Big)^n |\langle X_n A^n x, b\rangle| \leq \|X\| \Big\| \Big(\frac{A}{\|A\|}\Big)^n x \Big\| m_{|R|}^n \|b\|.$$

Since R is quasinormal, the isometry V commutes with $|R|$. Then observe that we can choose b in the closure of the range of R^n which is contained in the range of V^n, and hence we can write $b = V^n c$. Therefore, we get $\||R|^{-n}y\| = \|V^{*n}b\| = \|V^{*n}V^n c\| = \|c\| = \|b\|$. Then, using the density of the range of R^{*n}, for all $(x, y) \in \mathfrak{L} \times \mathcal{H}$, we obtain

$$|\langle X_0 x, y\rangle| \leq \|X\| \Big\| \Big(\frac{A}{\|A\|}\Big)^n x \Big\| \|m_{|R|}^n |R|^{-n} y\|.$$

But

$$\|m_{|R|}^n |R|^{-n} y\|^2 = \int_{m_{|R|}}^{\|R\|} m_{|R|}^{2n} \frac{1}{t^{2n}}\, dE_{y,y}^{|R|}(t) \underset{n\to+\infty}{\longrightarrow} E_{y,y}^{|R|}\big(\{m_{|R|}\}\big).$$

Similarly, we see that

$$\Big\| \Big(\frac{A}{\|A\|}\Big)^n x \Big\|^2 \underset{n\to+\infty}{\longrightarrow} E_{x,x}^{A}\big(\{\|A\|\}\big).$$

According to the assumptions of (2), we must have at least one of the two spectral projections $E^{|R|}(\{m_{|R|}\})$ or $E^{A}(\{\|A\|\})$ null. Thus, the three previous facts show that $X_0 = 0$, which implies that $X = 0$, and so we get a contradiction. Consequently, using Proposition 2.4, it follows that the circle $\mathcal{C}(0, m_{|R|}/\|A\|)$ does not intersect $\Lambda_{\text{int}}(R, A \otimes S)$. From the first two steps of the proof, we derive that $\Lambda_{\text{int}}(R, A \otimes S) = \mathbb{D}(0, \frac{m_{|T|}}{\|T\|}]^c$. This finishes the proof of Theorem 2.5. □

Corollary 2.6. *Let T be a pure quasinormal operator acting on a Hilbert space $\mathcal{H}$. The extended spectrum of T is $\mathbb{D}(0, \frac{m_{|T|}}{\|T\|}[^c$ when $m_{|T|}$ and $\|T\|$ are in $\sigma_p(|T|)$, and is $\mathbb{D}(0, \frac{m_{|T|}}{\|T\|}]^c$ when $m_{|T|}$ and $\|T\|$ are not both in $\sigma_p(|T|)$.*

Proof. Applying Theorem 2.2 and taking into account Remark 2.3, we see that T is unitarily equivalent to the operator $A_T \otimes S$ acting on the Hilbert space $\mathfrak{L}_T \otimes H^2$, where $\mathfrak{L}_T = \mathcal{H} \ominus V_T\mathcal{H}$. We set for simplicity $A := A_T$. We clearly have $m_{|T|} = m_A$, $\|T\| = \|A\|$, and $\sigma_p(|T|) = \sigma_p(A)$. Therefore, from now on, we may assume that T is under the form $A \otimes S$ and $\mathcal{H} = \mathfrak{L}_T \otimes H^2$. Then, it suffices to apply Theorem 2.5 with $R = A \otimes S$. □

3. Case of subnormal operators

It is known that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace and a normal operator $N \in \mathcal{B}(\mathcal{K})$ such that $S = N|_{\mathcal{H}}$. This normal extension is minimal (MNE) if $\mathcal{K}$ has no proper subspace that reduces N and contains $\mathcal{H}$. In addition, we know that every quasinormal operator is subnormal, and so we show the following theorem in the more general case, that is, the subnormal one. In particular, it remains true for quasinormal operators.

Theorem 3.1. *Let $N \in \mathcal{B}(\mathcal{E})$ and $T \in \mathcal{B}(\mathcal{F})$ be normal and pure subnormal operators, respectively, such that the operator $R = N \oplus T \in \mathcal{B}(\mathcal{E} \oplus \mathcal{F})$ is injective. Let $\lambda \in \sigma_{\text{ext}}(Z)$, and let*

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \in E_{\text{ext}}(Z, \lambda).$$

Then $X_3 = 0$, $X_1 \in E_{\text{ext}}(N, \lambda)$, $X_4 \in E_{\text{ext}}(T, \lambda)$, and $X_2 \in E_{\text{int}}(N, T, \lambda)$.

Proof. The hypotheses imply that

$$\begin{cases} NX_1 = \lambda X_1 N, \\ NX_2 = \lambda X_2 T, \\ TX_3 = \lambda X_3 N, \\ TX_4 = \lambda X_4 T. \end{cases} \tag{3.1}$$

Clearly, it suffices to show that $X_3 = 0$, and so let

$$M = \begin{bmatrix} T & Y \\ 0 & T_1 \end{bmatrix} \in \mathcal{B}(\mathcal{F} \oplus \mathcal{G})$$

be the MNE of T, and consider the following operators defined on $\mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G}$ by

$$\tilde{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & T & Y \\ 0 & 0 & T_1 \end{bmatrix}, \qquad \tilde{N} = \begin{bmatrix} N & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \tilde{X} = \begin{bmatrix} 0 & 0 & 0 \\ X_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then formulas (3.1) imply that $\tilde{M}\tilde{X} = \lambda \tilde{X}\tilde{N}$, but both $\tilde{M}$ and $\tilde{N}$ are normal operators. By using the Fuglede–Putnam theorem, it follows that $\tilde{M}^*\tilde{X} = \bar{\lambda}\tilde{X}\tilde{N}^*$. From this, we have easily that $T^*X_3 = \bar{\lambda}X_3N^*$. Hence, for all $m, n \in \mathbb{N}$, we get the following system:

$$\begin{cases} T^n X_3 = \lambda^n X_3 N^n, \\ T^{*m} X_3 = \bar{\lambda}^m X_3 N^{*m}, \end{cases}$$

which implies, since N is normal, that

$$T^{*m}T^n X_3 = T^n T^{*m} X_3.$$

Consequently,

$$(T^{*m}T^n - T^nT^{*m})X_3 = 0 \quad (\forall m, n \in \mathbb{N}),$$

which means that

$$\operatorname{Im}(X_3) \subset \bigcap_{m,n \in \mathbb{N}} \ker(T^{*m}T^n - T^nT^{*m}) := \mathcal{M}.$$

Now, let $x \in \mathcal{M}$. Then

$$T^{*m}T^n(Tx) = T^{*m}T^{n+1}x = T^{n+1}T^{*m}x = T^nT^{*m}(Tx).$$

Thus $\mathcal{M} \in \mathcal{L}at(T)$. Furthermore, if $x \in \mathcal{M}$, then

$$T^{*m}T^n(T^*x) = T^{*m}(T^nT^*x) = T^{*m+1}T^nx = T^nT^{*m}(T^*x).$$

Hence $\mathcal{M} \in \mathcal{L}at(T^*)$. From this, $\mathcal{M}$ is a reducing subspace for T. Therefore, there are two operators $M_1 \in \mathcal{B}(\mathcal{M})$, $M_1 \in \mathcal{B}(\mathcal{M}^\perp)$ such that $T = M_1 \oplus M_2$. Moreover, in $\mathcal{M} \oplus \mathcal{M}^\perp$, the operators TT^* and T^*T have the following representations:

$$TT^* = M_1M_1^* \oplus M_2M_2^*, \qquad T^*T = M_1^*M_1 \oplus M_2^*M_2.$$

Finally, let $x \in \mathcal{M}$. Then $TT^*x = T^*Tx$, which implies that $M_1M_1^*x = M_1^*M_1x$, and so M_1 is normal. Then we get $\mathcal{M} = 0$ since T is pure. Consequently, $X_3 = 0$, and the proof is complete. □

4. Extended eigenvalues and extended eigenspaces of quasinormal operators

The following theorem describes the spaces of extended eigenvectors of a pure quasinormal operator. We will use the notation introduced in Remark 2.3.

Theorem 4.1. *Let T be a pure quasinormal operator acting on a Hilbert space $\mathcal{H}$. Let $\lambda \in \sigma_{\mathrm{ext}}(T)$. Then*

$$\begin{aligned} E_{\mathrm{ext}}(T,\lambda) = \textit{weak}^*\textit{-span}\big\{&U_T^*(I \otimes S^m)\operatorname{diag}(L, X_{1,1}, \dots X_{n,n}, \dots)U_T : \\ &m \in \mathbb{N}, L \in \mathcal{A}_{|\lambda|}(A_T)\big\}, \end{aligned}$$

where, for every $n \in \mathbb{N}$, extension $X_{n,n}$ is the (unique bounded) extension on $\mathfrak{L}_T$ of the operator

$$\begin{aligned} \breve{X}_{n,n} : \quad &\operatorname{Im} A_T^n \to \mathfrak{L}_T \\ &A_T^n x \mapsto \lambda^{-n} A_T^n L x. \end{aligned}$$

Proof. As usual, we set $A := A_T$, $\mathfrak{L}_T = \mathcal{H} \ominus V_T\mathcal{H}$, and $\mathcal{A}_{|\lambda|} := \mathcal{A}_{|\lambda|}(A_T)$. Let $X_0 \in \mathcal{B}(\mathcal{H})$ be a nonzero solution of $TX_0 = \lambda X_0 T$. Then we have seen that $X_0 = U_T^* X U_T$, where $X \in \mathcal{B}(\mathfrak{L}_T \otimes H^2)$ is a solution of the equation $(A \otimes S)X = \lambda X(A \otimes S)$. Let $(A_{i,j})_{i,j\geq 0}$ be the matrix of $A \otimes S$ in $\mathfrak{L} \otimes H^2$; that is,

$$A_{i,j} = \begin{cases} A & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Consider for all $\alpha \in \overline{\mathbb{D}}$ the operator J_α whose matrix in $\mathfrak{L} \otimes H^2$ is defined by

$$(J_\alpha)_{i,j} = \begin{cases} \alpha^i I & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then one can verify that $J_\alpha(A \otimes S) = \alpha(A \otimes S)J_\alpha$. In particular, we have that $J_0(A \otimes S) = 0$. Now let $\lambda \in \sigma_{\mathrm{ext}}(T)$, and let $X = (X_{i,j})$ be a nonzero operator acting on $\mathfrak{L} \otimes H^2$, with $\lambda \in \mathbb{C}$ (necessarily nonzero) verifying $(A \otimes S)X = \lambda X(A \otimes S)$. A left composition by J_0 gives

$$0 = J_0(A \otimes S)X = \lambda J_0 X(A \otimes S) = J_0 X(A \otimes S).$$

But

$$\big(J_0X(A \otimes S)\big)_{i,j} = \begin{cases} X_{0,j+1}A & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

which implies that $X_{0,j+1} = 0$ for all j since A has a dense range. In addition, we know that $(A \otimes S)X = \lambda X(A \otimes S)$ implies that $(A \otimes S)^n X = \lambda^n X(A \otimes S)^n$ for all $n \in \mathbb{N}$. The same process gives $X_{n,m} = 0$ for all $0 < n < m$. Consequently, X has a lower triangular matrix. Thus, if we denote by $X(m)$ the operator whose matrix is

$$\big(X(m)\big)_{i,j} = \begin{cases} X_{i,j} & \text{if } j = m + i, \\ 0 & \text{otherwise,} \end{cases}$$

j for all $m \in \mathbb{Z}$, then we can prove that

$$X = \underset{n\to+\infty}{weak^{*}lim}\Big(\sum_{k=0}^{n}\Big(1 - \frac{k}{n+1}\Big)X(-k)\Big).$$

Moreover, we observe that there exists an operator Y acting on $\mathfrak{L} \otimes H^2$ such that $X(-n) = (I \otimes S^n)(Y(0))$ for all $n \in \mathbb{N}$. Furthermore, one can verify that $(A \otimes S)X(-n) = \lambda X(-n)(A \otimes S)$ if and only if $(A \otimes S)Y(0) = \lambda Y(0)(A \otimes S)$. Therefore, we are reduced to examining the case where $X = X(0)$.

We have

$$\big((A \otimes S)X(0)\big)_{i,j} = \begin{cases} AX_{i-1,i-1} & \text{if } i = j + 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\big(\lambda X(0)(A \otimes S)\big)_{i,j} = \begin{cases} \lambda X_{i,i}A & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for all $n \in \mathbb{N}$, we have $AX_{n,n} = \lambda X_{n+1,n+1}A$. Thus we get for all n $\lambda^{-n}A^nX_{0,0} = X_{n,n}A^n$. On the one hand, since the range of A is dense, it implies that $X_{0,0}$ is necessarily nonnull. On the other hand, we see that

$$|\lambda|^{-n}\|A^nX_{0,0}x\| \le \|X\|\|A^nx\| \quad (\forall n \in \mathbb{N}, \forall x \in \mathfrak{L}).$$

Consequently, $X_{0,0}$ is a nonzero element of $\mathcal{A}_{|\lambda|} := \mathcal{A}_{|\lambda|}(A)$.

Reciprocally, if $L \in \mathcal{A}_{|\lambda|} \setminus \{0\}$, then we define, for all $n \in \mathbb{N}$, the operator

$$\begin{aligned} \breve{X}_{n,n} : \quad & \operatorname{Im} A^n \to \mathfrak{L}, \\ & A^nx \mapsto \lambda^{-n}A^nLx. \end{aligned}$$

Since $L \in \mathcal{A}_{|\lambda|}$, there is $c \ge 0$ such that, for all $y \in \operatorname{Im} A^n$, $\|\breve{X}_{n,n}y\| \le c\|y\|$. Also, $\operatorname{Im} A^n$ is dense in $\mathfrak{L}$, and thus $\breve{X}_{n,n}$ has a (unique bounded) extension on $\mathfrak{L}$, which will be denoted by $X_{n,n}$. It remains to verify that $AX_{n,n} = \lambda X_{n+1,n+1}A$ for all $n \in \mathbb{N}$. Let $x \in \mathfrak{L}$, and $y = A^nx$. Then

$$\begin{aligned} AX_{n,n}y &= AX_{n,n}A^nx = A\breve{X}_{n,n}A^nx = \lambda^{-n}A^{n+1}Lx \\ &= \lambda\lambda^{-n-1}A^{n+1}Lx = \lambda\breve{X}_{n+1,n+1}A^{n+1}x = \lambda X_{n+1,n+1}Ay \end{aligned}$$

by density, and we get $AX_{n,n} = \lambda X_{n+1,n+1}A$, which in turn implies that $X \in E_{\text{ext}}(T, \lambda) \setminus \{0\}$, as we wanted. □

Remark 4.2. Let A be an injective positive operator on a Hilbert space $\mathfrak{L}$, and let $|\lambda| \le 1$. Then we can easily verify that $\mathcal{A}_{|\lambda|} := \mathcal{A}_{|\lambda|}(A)$ is an algebra, but that is not true in general when $|\lambda| > 1$ (see example (4.5)). Recall that $\mathcal{A}_1$ is the

Deddens algebra given in [8]. Finally, if we denote by $D_{A,L,\lambda}$ the operator defined on $\mathfrak{L} \otimes H^2$ by

$$(D_{A,L,\lambda})_{i,j\in\mathbb{N}} = \begin{cases} \lambda^{-i} A^i L A^{-i} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

then we get the following corollary.

Corollary 4.3. *Let A be an invertible positive operator on a Hilbert space $\mathfrak{L}$, and let $T = A \otimes S$. If $\lambda \in \sigma_{\text{ext}}(T)$, then*

$$E_{\text{ext}}(T,\lambda) = weak^*\text{-}span\big\{(I \otimes S^n) D_{A,L,\lambda} : n \in \mathbb{N}, L \in \mathcal{A}_{|\lambda|}(A)\big\}.$$

Example 4.4. Let $T = A \otimes S$ be such that

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \quad (\alpha > \beta > 0),$$

and let

$$L = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (a,b,c,d \in \mathbb{C}).$$

If $\lambda \in \mathbb{C}^*$, then we get

$$\lambda^{-n} A^n L A^{-n} = \begin{bmatrix} \lambda^{-n} a & (\frac{\alpha}{\lambda\beta})^n b \\ (\frac{\beta}{\lambda\alpha})^n c & \lambda^{-n} d \end{bmatrix}.$$

Then we distinguish the following cases:

(1) if $|\lambda| \geq \frac{\alpha}{\beta} > 1$, then $\mathcal{A}_{|\lambda|} = \mathcal{B}(\mathfrak{L}) = \mathcal{M}_2(\mathbb{C})$,
(2) if $1 \leq |\lambda| < \frac{\alpha}{\beta}$, then

$$\mathcal{A}_{|\lambda|} = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}, a,c,d \in \mathbb{C} \right\},$$

(3) if $\frac{\beta}{\alpha} \leq |\lambda| < 1$, then

$$\mathcal{A}_{|\lambda|} = \left\{ \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}, c \in \mathbb{C} \right\},$$

(4) if $|\lambda| < \frac{\beta}{\alpha}$, then $\mathcal{A}_{|\lambda|} = \{0\}$.

Example 4.5. Let $T = A \otimes S$ be such that

$$A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \quad (\alpha > \beta > \gamma > 0),$$

and let

$$L_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then one can verify that

$$\|\lambda^{-n}A^nL_1A^{-n}\| = |\lambda|^{-n}\Big(\frac{\alpha}{\beta}\Big)^n,$$
$$\|\lambda^{-n}A^nL_2A^{-n}\| = |\lambda|^{-n}\Big(\frac{\beta}{\gamma}\Big)^n,$$

and

$$\|\lambda^{-n}A^nL_1L_2A^{-n}\| = |\lambda|^{-n}\Big(\frac{\alpha}{\gamma}\Big)^n.$$

Now, let α, β, γ, and λ be such that

$$\frac{\alpha}{\beta} = \frac{\beta}{\gamma} = |\lambda|.$$

Then, clearly, $L_1, L_2 \in \mathcal{A}_{|\lambda|}$, but $L_1L_2 \notin \mathcal{A}_{|\lambda|}$.

In the next result, we describe completely the extended eigenspaces of a general quasinormal operator.

Theorem 4.6. *Let R be an injective quasinormal operator on a Hilbert space $\tilde{\mathcal{H}}$, and let $R = N \oplus T \in \mathcal{B}(\mathcal{E} \oplus \mathcal{H})$ be its canonical decomposition into a direct sum of a normal operator $N \in \mathcal{B}(\mathcal{E})$ and a pure quasinormal operator $T \in \mathcal{B}(\mathcal{H})$. Let $\lambda \in \sigma_{\mathrm{ext}}(R)$. Then $E_{\mathrm{ext}}(R, \lambda)$ is the following subspace of $\mathcal{B}(\mathcal{E} \oplus \mathcal{H})$:*

$$\left\{ \begin{bmatrix} U & VU_T \\ 0 & W \end{bmatrix} : U \in E_{\mathrm{ext}}(N, \lambda), V \in E_{\mathrm{int}}(N, A_T \otimes S, \lambda), W \in E_{\mathrm{ext}}(T, \lambda) \right\},$$

where $E_{\mathrm{int}}(N, A_T \otimes S, \lambda)$ is the set of operators $V \in \mathcal{B}(\bigoplus_{k=0}^{+\infty} \mathfrak{L}_T, \mathcal{E})$ whose matrix form are given by $V = [V_0, \ldots, V_n, \ldots]$, where $V_0 \in \mathcal{A}_{|\lambda|}(N, A_T)$, and for every $n \in \mathbb{N}^*$, V_n is the (unique bounded) extension on $\mathfrak{L}$ of the operator

$$\begin{aligned} \breve{V}_n : \quad & \operatorname{Im} A^n \to \mathcal{E}, \\ & A^n x \mapsto \lambda^{-n} N^n V_0 x. \end{aligned}$$

Proof. Let X be an extended eigenvector of R associated with the extended eigenvalue λ. According to Theorem 3.1, it suffices to describe the upper off-diagonal coefficient X_2 in the matrix representation of X with respect to the orthogonal direct sum $\tilde{\mathcal{H}} = \mathcal{E} \oplus \mathcal{H}$. Clearly, we have $X_2 = VU_T$, where $V = [V_0, \ldots, V_n, \ldots] \in E_{\mathrm{int}}(N, A_T \otimes S, \lambda)$. For convenience, we write $A = A_T$. We see that $N^nV_0 = \lambda^nV_nA^n$ for every $n \in \mathbb{N}$. Thus we have $\|N^nV_0x\| \le \|V\||\lambda|^n\|A^nx\|$, and hence $V_0 \in \mathcal{A}_{|\lambda|}(N, A)$.

Conversely, by using assumptions, we get that $NV = \lambda VA \otimes S$ and we get any matrix of the form

$$X = \begin{bmatrix} U & VU_T \\ 0 & W \end{bmatrix},$$

where $U \in E_{\mathrm{ext}}(N, \lambda)$ and $W \in E_{\mathrm{ext}}(T, \lambda)\}$ is an extended eigenvector of R. This ends the proof. $\square$

We can now describe the extended spectrum of a general quasinormal operator.

Corollary 4.7. *Let R, N, T, and $\tilde{\mathcal{H}}$ be as in Theorem 4.6. Then*

$$\sigma_{\text{ext}}(R) = \sigma_{\text{ext}}(N \oplus T) = \sigma_{\text{ext}}(N) \cup \mathbb{D}\Big(0, \frac{m_{|N|} \wedge m_{|T|}}{\|T\|}\Big[^c$$

if one of the following assumptions holds:

(1) $m_{|N|} < m_{|T|}$*, and* $(m_{|N|}, \|T\|) \in \sigma_p(|N|) \times \sigma_p(|T|)$*;*
(2) $m_{|N|} = m_{|T|}$*, and* $(m_{|N|}, \|T\|) \in \sigma_p(|N|) \times \sigma_p(T)$ *or* $(m_{|T|}, \|T\|) \in \sigma_p(|T|)^2$*;*
(3) $m_{|N|} > m_{|T|}$*, and* $(m_{|T|}, \|T\|) \in \sigma_p(|T|)^2$*.*

Otherwise we have

$$\sigma_{\text{ext}}(R) = \sigma_{\text{ext}}(N) \cup \mathbb{D}\Big(0, \frac{m_{|N|} \wedge m_{|T|}}{\|T\|}\Big]^c.$$

Proof. Using Theorem 4.6, we see that

$$\sigma_{\text{ext}}(R) = \sigma_{\text{ext}}(N) \cup \sigma_{\text{ext}}(T) \cup \Lambda_{\text{int}}(N, A_T \otimes S).$$

Taking into account Corollary 2.6, we see that the proof rests on an application of Theorem 2.5. □

5. Lifting of eigenvectors of pure quasinormal operators

In [17, Theorems 1 and 3], the author gives a necessary and sufficient condition that an operator commuting with a quasinormal operator has an extension commuting with the normal extension of the quasinormal operator. The following theorem can be proved in a similar way as in Theorem 10 of [9]. Applying Berberian's trick concerning 2×2 operator matrices (see [2]), it can also be seen as a direct consequence of Corollary 12 of [9]. This theorem will be useful in this section.

Theorem 5.1. *Let $T_i \in \mathcal{B}(\mathcal{H}_i)$ be an injective quasinormal operator with the polar decomposition $T_i = V_i|T_i|$, and let $N_i \in \mathcal{B}(\mathcal{K}_i)$ be the MNE of T_i with the polar decomposition $N_i = U_i|N_i|$, $i = 1, 2$. If $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, then the following are equivalent:*

(1) *X has a (unique) extension $\hat{X} \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$ such that $N_1\hat{X} = \hat{X}N_2$,*
(2) *$V_1X = XV_2$, and $|T_1|X = X|T_2|$.*

Now, let A be a positive operator, and denote by U the bilateral shift. Then $A \otimes U$ is the MNE of $A \otimes S$, and the preceding theorem implies the following corollary.

Corollary 5.2. *Let A be an injective positive operator on a Hilbert space $\mathfrak{L}$, and let X be a bounded operator on $\mathfrak{L} \otimes H^2$. Then the following assertions are equivalent:*

(1) *X has a (unique) extension $\hat{X} \in \mathcal{B}(\mathfrak{L} \otimes L^2)$ such that $(A \otimes U)\hat{X} = \lambda\hat{X}(A \otimes U)$,*
(2) *$(A \otimes I)X = |\lambda|X(A \otimes I)$, and $(I \otimes S)X = \lambda/|\lambda|X(I \otimes S)$.*

Now, we use similar arguments to those used in the proof of Theorem 2.5 and Theorem 4.1 to establish the following result.

Theorem 5.3. *Let A be an invertible positive operator on a Hilbert space $\mathfrak{L}$, and let $N = A \otimes U$. Denote by $a := \|A\|\|A^{-1}\|$. If $(\|A\|, \|A^{-1}\|^{-1}) \in \sigma_p(A)^2$, then*

$$\sigma_{\text{ext}}(N) = \Big\{ z \in \mathbb{C} : \frac{1}{a} \leq |z| \leq a \Big\}.$$

Otherwise we have

$$\sigma_{\text{ext}}(N) = \Big\{ z \in \mathbb{C} : \frac{1}{a} < |z| < a \Big\}.$$

Moreover, if $\lambda \in \sigma_{\text{ext}}(N)$, then

$$E_{\text{ext}}(N, \lambda) = \textit{weak*-span}\big\{ (I \otimes U^m)\hat{D}_{A,L,\lambda} : m \in \mathbb{Z}, L \in \hat{\mathcal{A}}_{|\lambda|} \big\},$$

where $\hat{D}_{A,L,\lambda}$ is the operator defined by

$$(\hat{D}_{A,L,\lambda})_{i,j\in\mathbb{Z}} = \begin{cases} \lambda^{-i}A^iLA^{-i} & \textit{if } i = j, \\ 0 & \textit{otherwise,} \end{cases}$$

and

$$\hat{\mathcal{A}}_{|\lambda|} = \big\{ L \in \mathcal{B}(\mathfrak{L}) : \sup_{n\in\mathbb{Z}} \|\lambda^{-n}A^nLA^{-n}\| < +\infty \big\}.$$

Indeed, if $(\|A\|, \|A^{-1}\|^{-1}) \in \sigma_p(A)^2$, then it suffices to consider $L_1 = u \otimes v$ and $L_2 = v \otimes u$, where u and v are eigenvectors of A associated with $\|A\|$ and $\|A^{-1}\|^{-1}$, respectively.

Otherwise, we use the inequality

$$|\langle Lx, y\rangle| \leq C \Big\| \Big(\frac{A}{\|A\|}\Big)^m x \Big\| \Big\| \Big(\frac{A^{-1}}{\|A^{-1}\|}\Big)^m y \Big\|$$

(which is available for any $L \in \hat{\mathcal{A}}_a$ and any $m \in \mathbb{N}$) in order to show that L is necessarily null (see proof of Theorem 2.5). We proceed similarly for proving that $\hat{\mathcal{A}}_{a^{-1}} = \{0\}$.

Theorem 5.4. *Let A be an invertible positive operator on a Hilbert space $\mathfrak{L}$, $T = A \otimes S$, $N = A \otimes U$, $\lambda \in \sigma_{\text{ext}}(N)$, and $X \in \mathcal{B}(\mathfrak{L} \otimes H^2)$. Then X has a (unique) extension $\hat{X} \in \mathcal{B}(\mathfrak{L} \otimes L^2)$ such that*

$$N\hat{X} = \lambda\hat{X}N$$

if and only if

$$X \in \textit{weak*-span}\big\{ (I \otimes S^n)D_{A,L,\lambda} : n \in \mathbb{N}, L \in E_{\text{ext}}(A, |\lambda|) \big\}.$$

Proof. Let $\hat{X} = (\hat{X}_{i,j})_{i,j\in\mathbb{Z}}$ be an operator acting on $\mathfrak{L} \otimes L^2$ such that

$$N\hat{X} = \lambda\hat{X}N. \tag{5.1}$$

Consider for all $\alpha \in \mathbb{T}$ the operator $\hat{J}_\alpha$ whose matrix in $\mathfrak{L} \otimes L^2$ is defined by

$$(\hat{J}_\alpha)_{i,j} = \begin{cases} \alpha^i I & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then one can verify that $\hat{J}_\alpha N = \alpha N \hat{J}_\alpha$. Hence let $\alpha, \beta \in \mathbb{T}$. If we apply to both sides of equation (5.1) the operator $\hat{J}_\alpha$ from the left and the operator $\hat{J}_\beta$ from the right, then we get

$$N\hat{J}_\alpha \hat{X} \hat{J}_\beta = \frac{\lambda}{\alpha\beta} \hat{J}_\alpha \hat{X} \hat{J}_\beta N.$$

Now, let $m \in \mathbb{Z}$, $\theta \in [0, 2\pi]$, and put $\alpha = \beta^{-1} = e^{i\theta}$. Then the last equation implies that

$$N \int_0^{2\pi} e^{-im\theta} \hat{J}_{e^{i\theta}} \hat{X} \hat{J}_{e^{-i\theta}}\, dm(\theta) = \lambda \int_0^{2\pi} e^{-im\theta} \hat{J}_{e^{i\theta}} \hat{X} \hat{J}_{e^{-i\theta}}\, dm(\theta) N,$$

where the integrals are well defined in the Bochner sense. Hence

$$N\hat{X}(m) = \lambda \hat{X}(m) N,$$

where $\hat{X}(m)$ is the operator acting on $\mathfrak{L} \otimes L^2$ whose matrix is given by

$$\big(\hat{X}(m)\big)_{i,j} = \begin{cases} \hat{X}_{i,j} & \text{if } i = m + j, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, one can easily verify that $\hat{J}_{e^{i\theta}} \hat{X} \hat{J}_{e^{-i\theta}}$ is an extension of the operator $J_{e^{i\theta}} X J_{e^{-i\theta}}$ so that $\hat{X}(m)$ is an extension of $X(m) = (I \otimes S^m) D_{A,L,\lambda}$. Also, by using Theorem 5.4, there exists $L \in \hat{\mathcal{A}}_{|\lambda|}$ such that

$$\hat{X}(m) = (I \otimes U^m) \hat{D}_{A,L,\lambda}.$$

Now, suppose that $m < 0$. In this case, $L = 0$. Indeed, if $L \neq 0$, then $\mathfrak{L} \otimes H^2 \notin \mathcal{L}at(\hat{X}(m))$, which means that there is no bounded operator on $\mathfrak{L} \otimes H^2$ for which $\hat{X}(m)$ is an extension. Now assume that $m \geq 0$. Then $\hat{X}(m)$ is an extension of the operator $X(m)$, and by using the Corollary 5.2, we have an equivalence with the two following equations:

$$(A \otimes I)X = |\lambda| X (A \otimes I) \quad \text{and} \quad (I \otimes S)X = \lambda/|\lambda| X (I \otimes S).$$

One can easily verify that the last equalities are equivalent to

$$AL = |\lambda| LA,$$

which means that $L \in E_{\text{ext}}(A, |\lambda|)$. The converse is easy and will be left to the reader. □

Remark 5.5. Let A be an invertible positive operator on a Hilbert space $\mathfrak{L}$ such that $(\|A\|, \|A^{-1}\|^{-1}) \in \sigma_p(A)^2$, $T = A \otimes S$, and $N = A \otimes U$ the MNE of T. As a direct result of the last theorem, we can summarize the relationship between extended eigenvectors of T and N in the four following cases.

(1) If $|\lambda| \in [1/a, a]$ and let

$$X = (I \otimes S^n) D_{A,L,\lambda} \quad (n \in \mathbb{N}).$$

Suppose that $L \in E_{\text{ext}}(A, |\lambda|)$. Then X has a (unique) extension $\hat{X} \in E_{\text{ext}}(N, \lambda)$.

(2) With the same hypotheses, if we suppose that $L \notin E_{\text{ext}}(A, |\lambda|)$, then X does not have any extension in $E_{\text{ext}}(N, \lambda)$.

(3) Let $|\lambda| \in [1/a, a]$ and $\hat{X} \in E_{\text{ext}}(N, \lambda)\backslash\{0\}$ be such that

$$\hat{X} = (I \otimes U^m)\hat{D}_{A,L,\lambda} \quad (m < 0).$$

Then there is no bounded operator on $\mathfrak{L} \otimes H^2$ for which $\hat{X}$ is an extension.

(4) If $|\lambda| > a$, and

$$X = (I \otimes S^n)D_{A,L,\lambda} \quad (n \in \mathbb{N}, L \in \mathcal{A}_{|\lambda|}\backslash\{0\}),$$

then $X \in E_{\text{ext}}(T, \lambda)$, but it has no extension in $E_{\text{ext}}(N, \lambda)$.

When $(\|A\|, \|A^{-1}\|^{-1}) \notin \sigma_p(A)^2$, the reader can adapt this remark by using Theorem 5.3 and conclude.

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