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DUALITY FOR IDEALS OF LIPSCHITZ MAPS

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Abstract. We develop a systematic approach to the study of ideals of Lipschitz maps from a metric space to a Banach space, inspired by classical theory on using Lipschitz tensor products to relate ideals of operator/tensor norms for Banach spaces. We study spaces of Lipschitz maps from a metric space to a dual Banach space that can be represented canonically as the dual of a Lipschitz tensor product endowed with a Lipschitz cross-norm, and we show that several known examples of ideals of Lipschitz maps (Lipschitz maps, Lipschitz p-summing maps, maps admitting Lipschitz factorization through subsets of L_p-space) admit such a representation. Generally, we characterize when the space of a Lipschitz map from a metric space to a dual Banach space is in canonical duality with a Lipschitz cross-norm. Finally, we introduce a concept of operators which are approximable with respect to one of these ideals of Lipschitz maps, and we identify them in terms of tensor-product notions.

Introduction

The study of ideals of linear operators between Banach spaces (i.e., families of operators that are closed under composition) has been an important tool in the study of Banach spaces. A stellar example is that of p-summing operators, as attested by the astonishing number of results and applications that can be found, for example, in [7]. In recent years, a number of ideals of Lipschitz maps (which, in particular, are generally nonlinear) inspired by well-known and very useful ideals of linear operators between Banach spaces have appeared in the literature. One

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example is the notion of Lipschitz p-summing operators between metric spaces, a nonlinear generalization of p-summing operators, which was introduced by Farmer and Johnson in [8]. Other examples of such ideals of Lipschitz maps are operators that admit a Lipschitz factorization through an L_p-space (see [11]), Lipschitz p-nuclear and Lipschitz p-integral operators (see [4]), or operators admitting a Lipschitz factorization through a subset of a Hilbert space (see [3]). If we restrict our attention to one of these ideals of maps from a fixed metric space to a fixed normed space, then the resulting space of maps is itself a normed space. Therefore, being able to identify the dual of such a space of maps would be interesting and useful. That is precisely one of the questions raised by Farmer and Johnson [8, Problem 3] for the specific case of Lipschitz p-summing maps.

Our purpose in this article is to develop a systematic approach to the duality theory for ideals of Lipschitz maps from a metric space to a Banach space, a generalization of the aforementioned question of Farmer and Johnson. This approach is inspired, on one hand, by the deep and useful connections between theories of operator ideals and theories of tensor norms for Banach spaces (see Defant and Floret [6]) and on the other hand by the second author's solution in [2] to the problem of duality for Lipschitz p-summing operators. The key idea in [2] is that of spaces of Banach-space-valued molecules, which play the role of a sort of tensor product between a metric space and a Banach space; those spaces of molecules are endowed with certain Lipschitz versions of the tensor norms described by Chevet [5] and Saphar [13], which are in canonical duality with spaces of Lipschitz p-summing maps. In [1], the present authors formalized the notion of a Lipschitz tensor product between a metric space and a normed space and studied its basic properties. In this article, we develop the duality theory that relates Lipschitz tensor products and ideals of Lipschitz maps by answering the following two questions. (1) Given an ideal of Lipschitz maps, when can it be canonically identified with the dual of a Lipschitz tensor product? (2) Given a Lipschitz tensor product, when can its dual space be canonically identified with an ideal of Lipschitz maps?

Let us now describe the contents of this paper. Section 1 gathers some preliminary results on the Lipschitz tensor product $X \boxtimes E$ between a metric space X and a normed space E. In Section 2, we introduce and study the space $\mathrm{Lip}_\alpha(X, E^*)$ of E^*-valued α-Lipschitz operators defined on X, that is, operators from X to E^* that induce a continuous functional on a given Lipschitz tensor product with a Lipschitz cross-norm α (denoted by $X \boxtimes_\alpha E$). The α-Lipschitz operators are in fact Lipschitz maps, which justify the terminology. Moreover, we show that several known examples of ideals of Lipschitz maps—namely, Lipschitz maps, Lipschitz p-summing maps, and maps admitting a Lipschitz factorization through a subset of an L_p-space—are associated to Lipschitz cross-norms in this way.

Section 3 addresses the duality theory for α-Lipschitz operators and contains the main result of this paper: the space of E^*-valued α-Lipschitz operators defined on X is canonically isometrically isomorphic to the dual of the Lipschitz tensor product $X \boxtimes_\alpha E$. This canonical identification is the basis of our study of the duality for ideals of Lipschitz maps. The section is completed by studying the several topologies on the space of E^*-valued α-Lipschitz operators defined on X. Thus the main questions we are pursuing in this paper can be rephrased in the following. *When is the space of α-Lipschitz operators an ideal? Given an ideal of Lipschitz maps, when can it be represented as a space of α-Lipschitz maps?*

In Section 4, we take a small detour from the main theme of the paper to work out several results dealing with approximations. We show that under minimal assumptions on the cross-norm α, the simplest (that is, the so-called *Lipschitz finite-rank*) Lipschitz operators from X to E^* are all α-Lipschitz. We also study the Lipschitz operators from X into E^* that are limits in the α-Lipschitz norm of sequences of Lipschitz finite-rank operators, which are called *α-Lipschitz approximable operators*.

In Section 5, we formalize the notion of ideals of Lipschitz maps, which we have called *Banach ideals of Lipschitz operators*. We introduce these ideals, and give some sufficient conditions and other necessary ones on $\mathrm{Lip}_\alpha(X, E^*)$ and the space of α-Lipschitz approximable operators to be such a Banach ideal of Lipschitz operators.

In Section 6, we look at spaces of maps from X to E^* which are not necessarily ideals but nevertheless are in duality with Lipschitz cross-norms. We introduce the concept of a Lipschitz operator Banach space, and give simple conditions on α that characterize when $\mathrm{Lip}_\alpha(X, E^*)$ is one such space. As was already mentioned, it is proved in Section 2 that, if α is a Lipschitz cross-norm on $X \boxtimes E$, then $\mathrm{Lip}_\alpha(X, E^*)$ can be identified with the dual of the space $X \boxtimes_\alpha E$. We now prove a converse result, characterizing those Lipschitz operator Banach spaces that are canonically isometrically isomorphic to the dual of $X \boxtimes_\alpha E$ for some Lipschitz cross-norm α on $X \boxtimes E$ (in terms of the compactness of their unit balls with respect to one of the topologies introduced in Section 3).

1. Notation and preliminary results

Given two metric spaces (X, d_X) and (Y, d_Y), let us recall that a map $f\colon X \to Y$ is said to be *Lipschitz* if there exists a real constant $C \geq 0$ such that $d_Y(f(x), f(y)) \leq C d_X(x, y)$ for all $x, y \in X$. The least constant C for which the preceding inequality holds will be denoted by $\mathrm{Lip}(f)$; that is,

$$\mathrm{Lip}(f) = \sup\Big\{\frac{d_Y(f(x), f(y))}{d_X(x, y)} : x, y \in X, x \neq y\Big\}.$$

A pointed metric space X is a metric space with a basepoint in X, that is, a designated special point, which we will always denote by 0. As usual, $\mathbb{K}$ denotes the field of real or complex numbers. We will consider a normed space E over $\mathbb{K}$ as a pointed metric space with the distance defined by its norm and the zero vector as the basepoint. As is customary, B_E and S_E stand for the closed unit ball of E and the unit sphere of E, respectively.

Given two pointed metric spaces X and Y, we denote by $\mathrm{Lip}_0(X, Y)$ the set of all basepoint-preserving Lipschitz maps from X to Y. If E is a Banach space, then $\mathrm{Lip}_0(X,E)$ is a Banach space under the Lipschitz norm given by

$$\mathrm{Lip}(f) = \sup\Big\{ \frac{\|f(x)-f(y)\|}{d(x,y)} : x,y \in X, x \neq y \Big\}.$$

The elements of $\mathrm{Lip}_0(X,E)$ are known as *Lipschitz operators*. The space $\mathrm{Lip}_0(X,\mathbb{K})$ is called the *Lipschitz dual of* X, and it will be denoted by $X^{\#}$.

For two vector spaces E and F, $L(E,F)$ stands for the vector space of all linear operators from E into F. In the case that E and F are Banach spaces, $\mathcal{L}(E,F)$ represents the Banach space of all bounded linear operators from E to F endowed with the canonical norm of operators. In particular, the algebraic dual $L(E,\mathbb{K})$ and the topological dual $\mathcal{L}(E,\mathbb{K})$ are denoted by E' and E^*, respectively. For each $e \in E$ and $e^* \in E'$, we frequently will write $\langle e^*, e\rangle$ instead of $e^*(e)$.

Throughout this article, unless otherwise stated, X will denote a pointed metric space with basepoint 0 and E a Banach space.

We now recall some concepts and facts whose proofs can be found in [1]. Let X be a pointed metric space, and let E be a Banach space. The *Lipschitz tensor product* $X \boxtimes E$ is the linear span of all linear functionals $\delta_{(x,y)} \boxtimes e$ on $\mathrm{Lip}_0(X,E^*)$ of the form

$$(\delta_{(x,y)} \boxtimes e)(f) = \big\langle f(x) - f(y), e \big\rangle$$

for $(x,y) \in X^2$ and $e \in E$. A norm α on $X \boxtimes E$ is a *Lipschitz cross-norm* if

$$\alpha(\delta_{(x,y)} \boxtimes e) = d(x,y)\|e\|$$

for all $(x,y) \in X^2$ and $e \in E$. We denote by $X \boxtimes_\alpha E$ the linear space $X \boxtimes E$ with norm α, and we denote by $X \widehat{\boxtimes}_\alpha E$ the completion of $X \boxtimes_\alpha E$. A Lipschitz cross-norm α on $X \boxtimes E$ is called *dualizable* if, given $g \in X^{\#}$ and $\phi \in E^*$, we have

$$\Big|\sum_{i=1}^n \big(g(x_i) - g(y_i)\big)\langle \phi, e_i\rangle\Big| \leq \mathrm{Lip}(g)\|\phi\|\alpha\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big)$$

for all $\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, and it is called *uniform* if, given $h \in \mathrm{Lip}_0(X,X)$ and $T \in \mathcal{L}(E,E)$, we have

$$\alpha\Big(\sum_{i=1}^n \delta_{(h(x_i),h(y_i))} \boxtimes T(e_i)\Big) \leq \mathrm{Lip}(h)\|T\|\alpha\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big)$$

for all $\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$.

For each $\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, the *Lipschitz injective norm* on $X \boxtimes E$ is defined by

$$\varepsilon\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big) = \sup\Big\{\Big|\sum_{i=1}^n \big(g(x_i) - g(y_i)\big)\langle \phi, e_i\rangle\Big| : g \in B_{X^{\#}}, \phi \in B_{E^*}\Big\}.$$

For each $u \in X \boxtimes E$, the *Lipschitz projective norm* π and the *Lipschitz p-nuclear* norm d_p for $1 < p < \infty$ are defined on $X \boxtimes E$ as

$$\pi(u) = \inf\Big\{\sum_{i=1}^n d(x_i,y_i)\|e_i\| : u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big\},$$

$$\begin{aligned}
d_1(u) = \; & \inf\Big\{\big(\sup_{g\in B_{X^\#}}\big(\lambda_i \max_{1\le i\le n}\big|g(x_i)-g(y_i)\big|\big)\big)\Big(\sum_{i=1}^n \|e_i\|\Big)\colon \\
& u=\sum_{i=1}^n \lambda_i\delta_{(x_i,y_i)}\boxtimes e_i, \{\lambda_i\}_{i=1}^n\subset\mathbb{R}^+\Big\}, \\
d_p(u) = \; & \inf\Big\{\Big(\sup_{g\in B_{X^\#}}\Big(\sum_{i=1}^n \lambda_i^{p'}\big|g(x_i)-g(y_i)\big|^{p'}\Big)^{\frac{1}{p'}}\Big)\Big(\sum_{i=1}^n \|e_i\|^p\Big)^{\frac{1}{p}}\colon \\
& u=\sum_{i=1}^n \lambda_i\delta_{(x_i,y_i)}\boxtimes e_i, \{\lambda_i\}_{i=1}^n\subset\mathbb{R}^+\Big\}, \\
d_\infty(u) = \; & \inf\Big\{\Big(\sup_{g\in B_{X^\#}}\Big(\sum_{i=1}^n \lambda_i\big|g(x_i)-g(y_i)\big|\Big)\Big)\big(\max_{1\le i\le n}\|e_i\|\big)\colon \\
& u=\sum_{i=1}^n \lambda_i\delta_{(x_i,y_i)}\boxtimes e_i, \{\lambda_i\}_{i=1}^n\subset\mathbb{R}^+\Big\}.
\end{aligned}$$

It is known that ε, π, and d_p for $p\in[1,\infty]$ are uniform and dualizable Lipschitz cross-norms on $X\boxtimes E$ and $d_1=\pi$. Moreover, ε is the least dualizable Lipschitz cross-norm on $X\boxtimes E$, and π is the greatest Lipschitz cross-norm on $X\boxtimes E$, and

$$\pi(u)=\sup\big\{\big|u(f)\big|\colon f\in \mathrm{Lip}_0(X,E^*),\ \mathrm{Lip}(f)\le 1\big\}$$

for all $u\in X\boxtimes E$. In fact, a norm α on $X\boxtimes E$ is a dualizable Lipschitz cross-norm if and only if $\varepsilon\le\alpha\le\pi$.

If $g\in X^\#$ and $\phi\in E^*$, then we consider the linear functional $g\boxtimes\phi$ on $X\boxtimes E$ defined by

$$(g\boxtimes\phi)\Big(\sum_{i=1}^n \delta_{(x_i,y_i)}\boxtimes e_i\Big)=\sum_{i=1}^n\big(g(x_i)-g(y_i)\big)\langle\phi,e_i\rangle.$$

The *associated Lipschitz tensor product* of $X\boxtimes E$, denoted by $X^\#\circledast E^*$, is the linear span of all linear functionals $g\boxtimes\phi$ on $X\boxtimes E$ for $g\in X^\#$ and $\phi\in E^*$. A norm β on $X^\#\circledast E^*$ is called a *Lipschitz cross-norm* if

$$\beta(g\boxtimes\phi)=\mathrm{Lip}(g)\|\phi\|$$

for all $g\in X^\#$ and $\phi\in E^*$. Denote by $X^\#\circledast_\beta E^*$ the linear space $X^\#\circledast E^*$ with norm β, and denote by $X^\#\widehat{\circledast}_\beta E^*$ the completion of $X^\#\circledast_\beta E^*$.

Given a dualizable Lipschitz cross-norm α on $X\boxtimes E$, the map $\alpha'\colon X^\#\circledast E^*\to\mathbb{R}$ defined by

$$\alpha'\Big(\sum_{j=1}^m g_j\boxtimes\phi_j\Big)=\sup\Big\{\Big|\Big(\sum_{j=1}^m g_j\boxtimes\phi_j\Big)\Big(\sum_{i=1}^n \delta_{(x_i,y_i)}\boxtimes e_i\Big)\Big|\colon \alpha\Big(\sum_{i=1}^n \delta_{(x_i,y_i)}\boxtimes e_i\Big)\le 1\Big\}$$

for $\sum_{j=1}^m g_j\boxtimes\phi_j\in X^\#\circledast E^*$ is a Lipschitz cross-norm on $X^\#\circledast E^*$ called the *associated Lipschitz norm of* α, and clearly $X^\#\circledast_{\alpha'}E^*$ is a normed linear subspace of $(X\widehat{\boxtimes}_\alpha E)^*$.

For $h \in \mathrm{Lip}_0(X,Y)$ and $T \in \mathcal{L}(E,F)$, we also consider the linear operator $h \boxtimes T$ from $X \boxtimes E$ to $Y \boxtimes F$ given by

$$(h \boxtimes T)\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big) = \sum_{i=1}^{n} \delta_{(h(x_i),h(y_i))} \boxtimes T(e_i).$$

2. Cross-norm-Lipschitz operators

In this section, we introduce the concept that will give rise to the canonical association between Lipschitz cross-norms and ideals of Lipschitz maps. It is the concept of a cross-norm-Lipschitz operator from X to E^*, which is an operator that induces a bounded functional on $X \boxtimes E$ endowed with a Lipschitz cross-norm. To be precise, we have the following definition.

Definition 2.1. Let α be a Lipschitz cross-norm on $X \boxtimes E$. A basepoint-preserving map $f\colon X \to E^*$ is said to be an α-Lipschitz operator if there exists a real constant $C \geq 0$ such that

$$\Big|\sum_{i=1}^{n} \big\langle f(x_i) - f(y_i), e_i \big\rangle\Big| \leq C\alpha\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big)$$

for all $\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$. The infimum of such constants C is denoted by $\mathrm{Lip}_\alpha(f)$ and called the *α-Lipschitz norm of* f. The set of all α-Lipschitz operators from X into E^* is denoted by $\mathrm{Lip}_\alpha(X,E^*)$.

The following lemma justifies the terminology used in Definition 2.1 since every α-Lipschitz operator turns out to be a Lipschitz operator.

Lemma 2.2. *Let α be a Lipschitz cross-norm on $X \boxtimes E$. Then every α-Lipschitz operator $f\colon X \to E^*$ is Lipschitz, and* $\mathrm{Lip}(f) \leq \mathrm{Lip}_\alpha(f)$.

Proof. Let $f \in \mathrm{Lip}_\alpha(X,E^*)$. For $x, y \in X$ and $e \in E$, we have

$$\big|\big\langle f(x) - f(y), e \big\rangle\big| \leq \mathrm{Lip}_\alpha(f)\alpha(\delta_{(x,y)} \boxtimes e) = \mathrm{Lip}_\alpha(f)d(x,y)\|e\|;$$

hence, $\|f(x) - f(y)\| \leq \mathrm{Lip}_\alpha(f)d(x,y)$, so $f \in \mathrm{Lip}_0(X,E^*)$, with $\mathrm{Lip}(f) \leq \mathrm{Lip}_\alpha(f)$. $\square$

Remark 2.3. Note that, if $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$ and $f \in \mathrm{Lip}_0(X,E^*)$, then

$$u(f) = \sum_{i=1}^{n} \big\langle f(x_i) - f(y_i), e_i \big\rangle,$$

and therefore f is in $\mathrm{Lip}_\alpha(X,E^*)$ if and only if $|u(f)| \leq C\alpha(u)$ for all $u \in X \boxtimes E$. Moreover,

$$\begin{aligned}\mathrm{Lip}_\alpha(f) &= \min\big\{C \geq 0\colon \big|u(f)\big| \leq C\alpha(u), \forall u \in X \boxtimes E\big\}\\ &= \sup\big\{\big|u(f)\big|\colon u \in X \boxtimes E, \alpha(u) \leq 1\big\}\\ &= \sup\big\{\big|u(f)\big|\colon u \in X \boxtimes E, \alpha(u) = 1\big\}.\end{aligned}$$

Lemma 2.4. *Let α be a Lipschitz cross-norm on $X \boxtimes E$. Then $\mathrm{Lip}_\alpha(X,E^*)$ is a normed space with the α-Lipschitz norm.*

Proof. Let $f, g \in \mathrm{Lip}_\alpha(X, E^*)$, and let $\lambda \in \mathbb{K}$. Clearly, $\mathrm{Lip}_\alpha(f) \geq 0$. Assume that $f \neq 0$. Then, for some $x \in X$ and $e \in E$, $\langle f(x), e\rangle \neq 0$ (i.e., $\langle \delta_{(x,0)} \boxtimes e, f\rangle \neq 0$). This implies that $\delta_{(x,0)} \boxtimes e \neq 0$, and thus $\alpha(\delta_{(x,0)} \boxtimes e) > 0$. Then, $\mathrm{Lip}_\alpha(f) \geq |\langle f(x), e\rangle| / \alpha(\delta_{(x,0)} \boxtimes e) > 0$, as required. Next we use Remark 2.3. For any $u \in X \boxtimes E$, we obtain

$$\left|u(\lambda f)\right| = \left|\lambda u(f)\right| = |\lambda|\left|u(f)\right| \leq |\lambda| \mathrm{Lip}_\alpha(f)\alpha(u),$$

and therefore $\lambda f \in \mathrm{Lip}_\alpha(X, E^*)$ and $\mathrm{Lip}_\alpha(\lambda f) \leq |\lambda| \mathrm{Lip}_\alpha(f)$. From this inequality, it follows that if $\lambda = 0$, then $\mathrm{Lip}_\alpha(\lambda f) = 0 = |\lambda| \mathrm{Lip}_\alpha(f)$, and if $\lambda \neq 0$, then we have $\mathrm{Lip}_\alpha(f) = \mathrm{Lip}_\alpha(\lambda^{-1}(\lambda f)) \leq |\lambda|^{-1} \mathrm{Lip}_\alpha(\lambda f)$, and hence $|\lambda| \mathrm{Lip}_\alpha(f) \leq \mathrm{Lip}_\alpha(\lambda f)$. This proves that $\mathrm{Lip}_\alpha(\lambda f) = |\lambda| \mathrm{Lip}_\alpha(f)$. Finally, for all $u \in X \boxtimes E$,

$$\left|u(f+g)\right| = \left|u(f) + u(g)\right| \leq \left|u(f)\right| + \left|u(g)\right| \leq \left(\mathrm{Lip}_\alpha(f) + \mathrm{Lip}_\alpha(g)\right)\alpha(u),$$

and so $f + g \in \mathrm{Lip}_\alpha(X, E^*)$ and $\mathrm{Lip}_\alpha(f+g) \leq \mathrm{Lip}_\alpha(f) + \mathrm{Lip}_\alpha(g)$. This completes the proof of the lemma. □

We now identify the space of all Lipschitz operators from X into E^* with the space of all π-Lipschitz operators.

Lemma 2.5. *The sets* $\mathrm{Lip}_0(X, E^*)$ *and* $\mathrm{Lip}_\pi(X, E^*)$ *are equal. Moreover,* $\mathrm{Lip}(f) = \mathrm{Lip}_\pi(f)$ *for all* $f \in \mathrm{Lip}_0(X, E^*)$.

Proof. Let $f \in \mathrm{Lip}_0(X, E^*)$. Since $|u(f)| \leq \mathrm{Lip}(f)\pi(u)$ for all $u \in X \boxtimes E$, we infer that $f \in \mathrm{Lip}_\pi(X, E^*)$ and $\mathrm{Lip}_\pi(f) \leq \mathrm{Lip}(f)$. The lemma now follows by Lemma 2.2. □

In [8], Farmer and Johnson introduced the notion of Lipschitz p-summing operators between metric spaces for $1 \leq p < \infty$ (see [2] for the case $p = \infty$). Let us recall that, if X and Y are pointed metric spaces, a map $f \in \mathrm{Lip}_0(X, Y)$ is said to be *Lipschitz p-summing* ($1 \leq p \leq \infty$) if there exists a constant $C \geq 0$ such that, regardless of the natural number n and regardless of the choice of points $x_1, \ldots, x_n, y_1, \ldots, y_n$ in X and positive reals $\lambda_1, \ldots, \lambda_n$, we have the inequality

$$\left(\sum_{i=1}^n \lambda_i d\big(f(x_i), f(y_i)\big)^p\right)^{\frac{1}{p}} \leq C \sup_{g \in B_{X^\#}} \left(\sum_{i=1}^n \lambda_i \left|g(x_i) - g(y_i)\right|^p\right)^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty,$$

$$\max_{1 \leq i \leq n} \lambda_i d\big(f(x_i), f(y_i)\big) \leq C \sup_{g \in B_{X^\#}} \left(\max_{1 \leq i \leq n} \lambda_i \left|g(x_i) - g(y_i)\right|\right) \quad \text{if } p = \infty.$$

The infimum of such constants is denoted by $\pi_p^L(f)$ and called the *Lipschitz p-summing norm of f*. If E is a Banach space, then the set $\Pi_p^L(X, E^*)$ of all Lipschitz p-summing operators from X into E^* with the norm π_p^L is a Banach space (see [8], [2]). If p' is the conjugate index of $p \in [1, \infty]$, we next identify the Lipschitz p-summing operators from X to E^* with the $d_{p'}$-Lipschitz operators.

Theorem 2.6. *Let* $1 \leq p \leq \infty$. *Then* $\mathrm{Lip}_{d_p}(X, E^*) = \Pi_{p'}^L(X, E^*)$, *and* $\mathrm{Lip}_{d_p}(f) = \pi_{p'}^L(f)$ *for every* $f \in \mathrm{Lip}_{d_p}(X, E^*)$.

Proof. Let $f \in \Pi^L_{p'}(X, E^*)$, and let $u \in X \boxtimes E$. If $\sum_{i=1}^n \lambda_i \delta_{(x_i,y_i)} \boxtimes e_i$ is a representation of u, then

$$\begin{aligned}
\big|u(f)\big| &= \Big|\sum_{i=1}^n \big\langle \lambda_i\big(f(x_i) - f(y_i)\big), e_i\big\rangle\Big| \\
&\leq \sum_{i=1}^n \lambda_i \big\|f(x_i) - f(y_i)\big\| \|e_i\| \\
&\leq \Big(\sum_{i=1}^n \lambda_i^{p'} \big\|f(x_i) - f(y_i)\big\|^{p'}\Big)^{\frac{1}{p'}} \Big(\sum_{i=1}^n \|e_i\|^p\Big)^{\frac{1}{p}} \\
&\leq \pi^L_{p'}(f) \Big(\sum_{i=1}^n \|e_i\|^p\Big)^{\frac{1}{p}} \sup_{g \in B_{X^\#}} \Big(\sum_{i=1}^n \lambda_i^{p'} \big|g(x_i) - g(y_i)\big|^{p'}\Big)^{\frac{1}{p'}}
\end{aligned}$$

in the case $1 < p < \infty$. When $p = 1$, we have

$$\begin{aligned}
\big|u(f)\big| &\leq \sum_{i=1}^n \lambda_i \big\|f(x_i) - f(y_i)\big\| \|e_i\| \\
&\leq \big(\max_{1 \leq i \leq n} \lambda_i \big\|f(x_i) - f(y_i)\big\|\big) \sum_{i=1}^n \|e_i\| \\
&\leq \pi^L_\infty(f) \sup_{g \in B_{X^\#}} \big(\max_{1 \leq i \leq n} \lambda_i \big|g(x_i) - g(y_i)\big|\big) \sum_{i=1}^n \|e_i\|,
\end{aligned}$$

and, for $p = \infty$, we have

$$\begin{aligned}
\big|u(f)\big| &\leq \sum_{i=1}^n \lambda_i \big\|f(x_i) - f(y_i)\big\| \|e_i\| \\
&\leq \big(\max_{1 \leq i \leq n} \|e_i\|\big) \sum_{i=1}^n \lambda_i \big\|f(x_i) - f(y_i)\big\| \\
&\leq \pi^L_1(f) \big(\max_{1 \leq i \leq n} \|e_i\|\big) \sup_{g \in B_{X^\#}} \Big(\sum_{i=1}^n \lambda_i \big|g(x_i) - g(y_i)\big|\Big).
\end{aligned}$$

Taking the infimum over all such representations of u, we have $|u(f)| \leq \pi^L_{p'}(f) d_p(u)$. Since u was arbitrary in $X \boxtimes E$, it follows that $f \in \mathrm{Lip}_{d_p}(X, E^*)$ and $\mathrm{Lip}_{d_p}(f) \leq \pi^L_{p'}(f)$.

Conversely, let $f \in \mathrm{Lip}_{d_p}(X, E^*)$, and let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in X$. Let $\varepsilon > 0$. Then, for each $i \in \{1, \ldots, n\}$, there exists $e_i \in E$ with $\|e_i\| \leq 1 + \varepsilon$ such that $\langle f(x_i) - f(y_i), e_i\rangle = \|f(x_i) - f(y_i)\|$. It is elementary that the map $T \colon \mathbb{K}^n \to \mathbb{K}$, defined by

$$T(t_1, \ldots, t_n) = \sum_{i=1}^n t_i \big\|f(x_i) - f(y_i)\big\|, \quad \forall (t_1, \ldots, t_n) \in \mathbb{K}^n,$$

is linear and continuous on $(\mathbb{K}^n, \|\cdot\|_p)$ with

$$\|T\| = \begin{cases} (\sum_{i=1}^n \|f(x_i)-f(y_i)\|^{p'})^{1/p'} & \text{if } 1<p\leq\infty, \\ \max_{1\leq i\leq n}\|f(x_i)-f(y_i)\| & \text{if } p=1. \end{cases}$$

For any $(t_1,\dots,t_n)\in\mathbb{K}^n$ with $\|(t_1,\dots,t_n)\|_p\leq 1$, we have

$$\begin{aligned}\left|T(t_1,\dots,t_n)\right| &= \left|\sum_{i=1}^n \left\langle f(x_i)-f(y_i), t_ie_i\right\rangle\right| \\ &= \left|\sum_{i=1}^n \left\langle \delta_{(x_i,y_i)}\boxtimes(t_ie_i), f\right\rangle\right| \leq \mathrm{Lip}_{d_p}(f)d_p\Big(\sum_{i=1}^n \delta_{(x_i,y_i)}\boxtimes(t_ie_i)\Big).\end{aligned}$$

If $1<p<\infty$, then it follows that

$$\begin{aligned}\left|T(t_1,\dots,t_n)\right| &\leq \mathrm{Lip}_{d_p}(f)\Big(\sum_{i=1}^n \|t_ie_i\|^p\Big)^{\frac{1}{p}} \sup_{g\in B_{X^\#}}\Big(\sum_{i=1}^n\left|g(x_i)-g(y_i)\right|^{p'}\Big)^{\frac{1}{p'}} \\ &\leq \mathrm{Lip}_{d_p}(f)(1+\varepsilon)\sup_{g\in B_{X^\#}}\Big(\sum_{i=1}^n\left|g(x_i)-g(y_i)\right|^{p'}\Big)^{\frac{1}{p'}}.\end{aligned}$$

Consequently, we have

$$\Big(\sum_{i=1}^n\left\|f(x_i)-f(y_i)\right\|^{p'}\Big)^{\frac{1}{p'}} \leq \mathrm{Lip}_{d_p}(f)(1+\varepsilon)\sup_{g\in B_{X^\#}}\Big(\sum_{i=1}^n\left|g(x_i)-g(y_i)\right|^{p'}\Big)^{\frac{1}{p'}},$$

and since ε was arbitrary, we deduce that

$$\Big(\sum_{i=1}^n\left\|f(x_i)-f(y_i)\right\|^{p'}\Big)^{\frac{1}{p'}} \leq \mathrm{Lip}_{d_p}(f)\sup_{g\in B_{X^\#}}\Big(\sum_{i=1}^n\left|g(x_i)-g(y_i)\right|^{p'}\Big)^{\frac{1}{p'}},$$

and so $f\in\Pi^L_{p'}(X,E^*)$ with $\pi^L_{p'}(f)\leq\mathrm{Lip}_{d_p}(f)$. Reasoning similarly, we arrive at the same conclusion for the cases $p=1$ and $p=\infty$. Indeed, if $p=1$, we have

$$\begin{aligned}\left|T(t_1,\dots,t_n)\right| &\leq \mathrm{Lip}_{d_1}(f)\Big(\sum_{i=1}^n\|t_ie_i\|\Big)\sup_{g\in B_{X^\#}}\big(\max_{1\leq i\leq n}\left|g(x_i)-g(y_i)\right|\big) \\ &\leq \mathrm{Lip}_{d_1}(f)(1+\varepsilon)\sup_{g\in B_{X^\#}}\big(\max_{1\leq i\leq n}\left|g(x_i)-g(y_i)\right|\big),\end{aligned}$$

which gives

$$\max_{1\leq i\leq n}\left\|f(x_i)-f(y_i)\right\| \leq \mathrm{Lip}_{d_1}(f)\sup_{g\in B_{X^\#}}\big(\max_{1\leq i\leq n}\left|g(x_i)-g(y_i)\right|\big),$$

and so $f\in\Pi^L_\infty(X,E^*)$ with $\pi^L_\infty(f)\leq\mathrm{Lip}_{d_1}(f)$. For $p=\infty$, we have

$$\begin{aligned}\left|T(t_1,\dots,t_n)\right| &\leq \mathrm{Lip}_{d_\infty}(f)\big(\max_{1\leq i\leq n}\|t_ie_i\|\big)\sup_{g\in B_{X^\#}}\Big(\sum_{i=1}^n\left|g(x_i)-g(y_i)\right|\Big) \\ &\leq \mathrm{Lip}_{d_\infty}(f)(1+\varepsilon)\sup_{g\in B_{X^\#}}\Big(\sum_{i=1}^n\left|g(x_i)-g(y_i)\right|\Big);\end{aligned}$$

hence

$$\sum_{i=1}^{n} \big\| f(x_i) - f(y_i) \big\| \le \mathrm{Lip}_{d_\infty}(f) \sup_{g \in B_{X^\#}} \Big(\sum_{i=1}^{n} \big| g(x_i) - g(y_i) \big| \Big),$$

and so $f \in \Pi_1^L(X, E^*)$ with $\pi_1^L(f) \le \mathrm{Lip}_{d_\infty}(f)$. $\square$

A similar description can be obtained for the class of maps admitting a Lipschitz factorization through a subset of an L_p-space. This has been proved, though stated in a slightly different language, in [3] for $p = 2$. Let us recall the basic definitions. For any pointed metric spaces X and Y, $f \in \mathrm{Lip}_0(X, Y)$, and $1 \le p \le \infty$, consider the infimum of $\mathrm{Lip}(R) \cdot \mathrm{Lip}(S)$ taken over all factorizations of the form

$$\begin{tikzcd} & Z \arrow[dr, "S"] & \\ X \arrow[ur, "R"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

where μ is a measure and Z is a subset of $L_p(\mu)$. We will denote this infimum by $\gamma_p^{\mathrm{Lip}}(f)$, inspired by the notation of a similar situation in Banach space theory and by $\Gamma_p^{\mathrm{Lip}}(X, Y)$, the set of all maps in $\mathrm{Lip}_0(X, Y)$ admitting such a factorization. For a pointed metric space X and a Banach space E, it is not hard to show that $(\Gamma_p^{\mathrm{Lip}}(X, E), \gamma_p^{\mathrm{Lip}})$ is a Banach space. For $x_j, x_j', y_i, y_i' \in X, \lambda_i, \mu_j \in \mathbb{R}$, and $1 \le p \le \infty$, we write $(\lambda_i, y_i, y_i')_{i=1}^n \prec_p (\mu_j, x_j, x_j')_{j=1}^m$ if, for every $f \in X^\#$,

$$\sum_{i=1}^{n} \big| \lambda_i \big[f(y_i) - f(y_i') \big] \big|^p \le \sum_{j=1}^{m} \big| \mu_j \big[f(x_j) - f(x_j') \big] \big|^p.$$

Equivalently, this means that there exists a linear map $A = (a_{ij}) \colon \ell_p^m \to \ell_p^n$ of norm at most one such that, for each $1 \le i \le n$,

$$\lambda_i \delta_{(y_i, y_i')} = \sum_{j=1}^{m} a_{ij} \mu_j \delta_{(x_j, x_j')}$$

(see [3, Lemma 3.2]).

Definition 2.7. Let X be a pointed metric space, let E be a Banach space, and let $1 \le p \le \infty$. For $u \in X \boxtimes E$, define

$$w_p(u) = \inf \Big\{ \Big(\sum_{i=1}^{n} \|e_i\|^p \Big)^{1/p} \Big(\sum_{j=1}^{m} \mu_j^{p'} d(x_j, x_j')^{p'} \Big)^{1/p'} : u = \sum_{i=1}^{n} \lambda_i \delta_{(y_i, y_i')} \boxtimes e_i \Big\},$$

where the infimum is taken over the representations of u in the form $\sum_{i=1}^n \lambda_i \delta_{(y_i, y_i')} \boxtimes e_i$ with $x_j, x_j', y_i, y_i' \in X$, $\lambda_i, \mu_j \in \mathbb{R}$, $e_i \in E$, and $(\lambda_i, y_i, y_i')_{i=1}^n \prec_{p'} (\mu_j, x_j, x_j')_{j=1}^m$.

Similar arguments to those in [3] for $p = 2$ show that the norm w_p is a uniform and dualizable Lipschitz cross-norm on $X \boxtimes E$. Furthermore, the Banach space of all w_p-Lipschitz operators from X to E^* can be identified with the Banach space $\Gamma_{p'}^{\mathrm{Lip}}(X, E^*)$ according to the following restatement of [3, Theorem 4.5] (while in that article the proof is written out in the special case $p = 2$, the details carry over to the general case).

Theorem 2.8. *Let $1 \leq p \leq \infty$. Then* $\mathrm{Lip}_{w_p}(X, E^*) = \Gamma_{p'}^{\mathrm{Lip}}(X, E^*)$, *and* $\mathrm{Lip}_{w_p}(f) = \gamma_{p'}^{\mathrm{Lip}}(f)$ *for every* $f \in \mathrm{Lip}_{w_p}(X, E^*)$.

3. Duality for spaces of cross-norm-Lipschitz operators

As was expected from the definition, we now verify that, if α is a Lipschitz cross-norm on $X \boxtimes E$, then there is a canonical identification between the normed space $\mathrm{Lip}_\alpha(X, E^*)$ and the dual space of $X \widehat{\boxtimes}_\alpha E$. In particular, $\mathrm{Lip}_\alpha(X, E^*)$ will be a Banach space.

Theorem 3.1. *Let α be a Lipschitz cross-norm on $X \boxtimes E$. Then* $\mathrm{Lip}_\alpha(X, E^*)$ *is isometrically isomorphic to* $(X \widehat{\boxtimes}_\alpha E)^*$ *via the map* $\Lambda_0\colon \mathrm{Lip}_\alpha(X, E^*) \to (X \widehat{\boxtimes}_\alpha E)^*$ *defined by*

$$\Lambda_0(f)(u) = \sum_{i=1}^{n} \big\langle f(x_i) - f(y_i), e_i \big\rangle$$

for $f \in \mathrm{Lip}_\alpha(X, E^*)$ *and* $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$. *Its inverse is the map* Λ_0^{-1} *from* $(X \widehat{\boxtimes}_\alpha E)^*$ *to* $\mathrm{Lip}_\alpha(X, E^*)$ *given by*

$$\big\langle \Lambda_0^{-1}(\varphi)(x), e \big\rangle = \varphi(\delta_{(x,0)} \boxtimes e)$$

for $\varphi \in (X \widehat{\boxtimes}_\alpha E)^*$, $x \in X$, *and* $e \in E$.

Proof. Let $f \in \mathrm{Lip}_\alpha(X, E^*)$, and let $\Lambda(f)$ be the linear functional on $X \boxtimes E$ defined by

$$\Lambda(f)(u) = \sum_{i=1}^{n} \big\langle f(x_i) - f(y_i), e_i \big\rangle$$

for $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$. Note that $\Lambda(f) \in (X \boxtimes_\alpha E)^*$ and $\|\Lambda(f)\| \leq \mathrm{Lip}_\alpha(f)$ since

$$\big|\Lambda(f)(u)\big| = \Big|\sum_{i=1}^{n} \big\langle f(x_i) - f(y_i), e_i \big\rangle\Big| \leq \mathrm{Lip}_\alpha(f)\alpha(u)$$

for all $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$. By the denseness of $X \boxtimes_\alpha E$ in $X \widehat{\boxtimes}_\alpha E$, there is a unique continuous extension $\Lambda_0(f)$ of $\Lambda(f)$ to $X \widehat{\boxtimes}_\alpha E$. Let $\Lambda_0\colon \mathrm{Lip}_\alpha(X, E^*) \to (X \widehat{\boxtimes}_\alpha E)^*$ be the map so defined. Since $\Lambda\colon \mathrm{Lip}_0(X, E^*) \to (X \boxtimes E)'$ is a linear monomorphism by [1, Corollary 1.8], it follows easily that so is Λ_0.

In order to see that Λ_0 is a surjective isometry, let φ be in $(X \widehat{\boxtimes}_\alpha E)^*$. Define the mapping $f\colon X \to E^*$ by

$$\big\langle f(x), e \big\rangle = \varphi(\delta_{(x,0)} \boxtimes e) \quad (x \in X, e \in E).$$

It is easy to check that $f(x)$ is a well-defined bounded linear functional on E and that f is well defined. Note that $\langle f(x) - f(y), e \rangle = \varphi(\delta_{(x,y)} \boxtimes e)$ for all $x, y \in X$ and $e \in E$. For any $\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, we have

$$\Big|\sum_{i=1}^{n} \big\langle f(x_i) - f(y_i), e_i \big\rangle\Big| = \Big|\varphi\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big)\Big| \leq \|\varphi\| \alpha\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big),$$

and therefore $f \in \mathrm{Lip}_\alpha(X, E^*)$ and $\mathrm{Lip}_\alpha(f) \leq \|\varphi\|$. For any $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, we obtain

$$\begin{aligned}\Lambda_0(f)(u) &= \sum_{i=1}^n \big\langle f(x_i) - f(y_i), e_i \big\rangle = \sum_{i=1}^n \varphi(\delta_{(x_i,y_i)} \boxtimes e_i) \\ &= \varphi\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big) = \varphi(u).\end{aligned}$$

Hence $\Lambda_0(f) = \varphi$ on a dense subspace of $X \widehat{\boxtimes}_\alpha E$, and, consequently, $\Lambda_0(f) = \varphi$. Moreover, $\mathrm{Lip}_\alpha(f) \leq \|\varphi\| = \|\Lambda_0(f)\|$, as required. Finally, it follows that $\langle \Lambda_0^{-1}(\varphi)(x), e \rangle = \langle f(x), e \rangle = \varphi(\delta_{(x,0)} \boxtimes e)$ for $\varphi \in (X \widehat{\boxtimes}_\alpha E)^*$, $x \in X$, and $e \in E$. $\square$

Theorem 3.1 and Lemma 2.5 give the next result of [10, Theorems 4.1, 5.8].

Corollary 3.2. *The space* $\mathrm{Lip}_0(X, E^*)$ *is isometrically isomorphic to* $(X \widehat{\boxtimes}_\pi E)^*$.

From Theorems 3.1 and 2.6, we derive the following description for the space of Lipschitz p-summing operators from X into E^* (compare it with [2, Theorem 4.3]).

Corollary 3.3. *For* $1 \leq p \leq \infty$*, the space* $\Pi_p^L(X, E^*)$ *is isometrically isomorphic to* $(X \widehat{\boxtimes}_{d_{p'}} E)^*$.

Similarly, Theorems 3.1 and 2.8 give the next identification stated in [3, Corollary 4.6] for $p = 2$.

Corollary 3.4. *For* $1 \leq p \leq \infty$*, the space* $\Gamma_p^{\mathrm{Lip}}(X, E^*)$ *is isometrically isomorphic to* $(X \widehat{\boxtimes}_{w_{p'}} E)^*$.

Since $\mathrm{Lip}_\alpha(X, E^*)$ is a dual space by Theorem 3.1, we may consider it equipped with its weak* topology.

Definition 3.5. Let α be a Lipschitz cross-norm on $X \boxtimes E$. The weak* topology (in short, w*) on $\mathrm{Lip}_\alpha(X, E^*)$ is the weak* topology on $(X \widehat{\boxtimes}_\alpha E)^*$, that is, the topology induced by the linear space of linear functionals $\kappa_{X \widehat{\boxtimes}_\alpha E}(X \widehat{\boxtimes}_\alpha E)$ on $(X \widehat{\boxtimes}_\alpha E)^*$, where $\kappa_{X \widehat{\boxtimes}_\alpha E}$ is the canonical injection from $X \widehat{\boxtimes}_\alpha E$ into $(X \widehat{\boxtimes}_\alpha E)^{**}$.

We also introduce on $\mathrm{Lip}_\alpha(X, E^*)$ another topology that we will use later.

Definition 3.6. Let α be a Lipschitz cross-norm on $X \boxtimes E$. The weak* Lipschitz operator topology (in short, w*Lo) on $\mathrm{Lip}_\alpha(X, E^*)$ is the topology induced by the linear space $X \boxtimes E$ of linear functionals on $\mathrm{Lip}_\alpha(X, E^*)$.

The following facts on w*Lo can be deduced from the theory on topologies induced by families of functions (see, for example, [12, Section 2.4]).

Remark 3.7. Let α be a Lipschitz cross-norm on $X \boxtimes E$.

(i) Here w*Lo is a locally convex topology on $\mathrm{Lip}_\alpha(X, E^*)$, and the dual space of $\mathrm{Lip}_\alpha(X, E^*)$ with respect to this topology is $X \boxtimes E$. Since the family of functions $X \boxtimes E$ is separating, we have that w*Lo is completely regular.

(ii) If $\{f_\gamma\}$ is a net in $\mathrm{Lip}_\alpha(X, E^*)$ and $f \in \mathrm{Lip}_\alpha(X, E^*)$, then $\{f_\gamma\}$ converges to f in the w*Lo topology if and only if $\{u(f_\gamma)\}$ converges to $u(f)$ for each $u \in X \boxtimes E$.

(iii) If $B(X, E^*)$ is a linear subspace of $\mathrm{Lip}_\alpha(X, E^*)$ and $\mathrm{Lip}_\alpha(X, E^*)$ is equipped with the w*Lo topology, then the relative w*Lo topology of $\mathrm{Lip}_\alpha(X, E^*)$ on $B(X, E^*)$ agrees with the topology induced by the linear space of linear functionals on $B(X, E^*)$ given by $\{u|_{B(X,E^*)} : u \in X \boxtimes E\}$.

Corollary 3.8. *Let α be a Lipschitz cross-norm on $X \boxtimes E$.*

(i) *A net $\{f_\gamma\}$ in $\mathrm{Lip}_\alpha(X, E^*)$ converges to $f \in \mathrm{Lip}_\alpha(X, E^*)$ in the weak* topology if and only if $\{u(f_\gamma)\}$ converges to $u(f)$ for every $u \in X \widehat{\boxtimes}_\alpha E$.*

(ii) *On $\mathrm{Lip}_\alpha(X, E^*)$, the weak* Lipschitz operator topology is weaker than the weak* topology. Moreover, on bounded subsets of $\mathrm{Lip}_\alpha(X, E^*)$, both topologies agree.*

Proof. (i) Let $\Lambda_0 \colon \mathrm{Lip}_\alpha(X, E^*) \to (X \widehat{\boxtimes}_\alpha E)^*$ be the isometric isomorphism defined in Theorem 3.1. We have

$$
\begin{aligned}
&\{f_\gamma\} \to f \quad \text{in } \big(\mathrm{Lip}_\alpha(X, E^*), w^*\big) \\
&\quad \Leftrightarrow \big\{\Lambda_0(f_\gamma)\big\} \to \Lambda_0(f) \quad \text{in } \big((X \widehat{\boxtimes}_\alpha E)^*, w^*\big) \\
&\quad \Leftrightarrow \big\{\big\langle \kappa_{X \widehat{\boxtimes}_\alpha E}(u), \Lambda_0(f_\gamma)\big\rangle\big\} \to \big\langle \kappa_{X \widehat{\boxtimes}_\alpha E}(u), \Lambda_0(f)\big\rangle, \quad \forall u \in X \widehat{\boxtimes}_\alpha E \\
&\quad \Leftrightarrow \big\{\Lambda_0(f_\gamma)(u)\big\} \to \Lambda_0(f)(u), \quad \forall u \in X \widehat{\boxtimes}_\alpha E \\
&\quad \Leftrightarrow \big\{u(f_\gamma)\big\} \to u(f), \quad \forall u \in X \widehat{\boxtimes}_\alpha E.
\end{aligned}
$$

(ii) Let $\{f_\gamma\}$ be a net in $\mathrm{Lip}_\alpha(X, E^*)$ which converges to $f \in \mathrm{Lip}_\alpha(X, E^*)$ in the w* topology. By (i), $\{u(f_\gamma)\}$ converges to $u(f)$ for each $u \in X \widehat{\boxtimes}_\alpha E$. In particular, $\{u(f_\gamma)\}$ converges to $u(f)$ for each $u \in X \boxtimes E$ since $X \boxtimes E \subset X \widehat{\boxtimes}_\alpha E$. This means that $\{f_\gamma\}$ converges to f in the w*Lo topology. Hence the identity on $\mathrm{Lip}_\alpha(X, E^*)$ is a continuous bijection from the w* topology to the w*Lo topology, and thus the latter topology is weaker than the former, as required. On a bounded subset of $\mathrm{Lip}_\alpha(X, E^*)$, the w* topology is compact and the w*Lo topology is Hausdorff, and so both topologies must coincide. □

4. Cross-norm-Lipschitz approximable operators

The concepts of Lipschitz finite-rank operators and Lipschitz approximable operators from X into E were introduced in [9]. We now study the relation between Lipschitz finite-rank operators and cross-norm-Lipschitz operators of X into E^*.

Let us recall that a Lipschitz operator $f \in \mathrm{Lip}_0(X, E^*)$ is considered Lipschitz finite-rank if the linear span $\mathrm{lin}(f(X))$ of $f(X)$ in E^* is finite-dimensional, in which case the rank of f, denoted by $\mathrm{rank}(f)$, is defined as the dimension of $\mathrm{lin}(f(X))$. We denote by $\mathrm{Lip}_{0F}(X, E^*)$ the linear space of all Lipschitz finite-rank operators from X into E^*. For every $g \in X^\#$ and $\phi \in E^*$, the map $g \cdot \phi \colon X \to E^*$, defined by $(g\cdot\phi)(x)=g(x)\phi$ for all $x\in X$, is in $\mathrm{Lip}_{0F}(X,E^*)$ with $\mathrm{Lip}(g\cdot\phi)=\mathrm{Lip}(g)\|\phi\|$ by [1, Lemma 1.5]. Furthermore, every operator $f\in\mathrm{Lip}_{0F}(X,E^*)$ can be expressed in the form $f=\sum_{j=1}^m g_j\cdot\phi_j$, where $m=\mathrm{rank}(f)$, $g_1,\dots,g_m\in X^\#$ and $\phi_1,\dots,\phi_m\in E^*$.

Theorem 4.1. *Let α be a dualizable Lipschitz cross-norm on $X\boxtimes E$. For every $g\in X^\#$ and $\phi\in E^*$, the map $g\cdot\phi$ belongs to $\mathrm{Lip}_\alpha(X,E^*)$ and $\mathrm{Lip}_\alpha(g\cdot\phi)=\mathrm{Lip}(g)\|\phi\|$. As a consequence, $\mathrm{Lip}_{0F}(X,E^*)$ is contained in $\mathrm{Lip}_\alpha(X,E^*)$.*

Proof. Let $g\in X^\#$, and let $\phi\in E^*$. Since the Lipschitz injective norm ε is the least dualizable Lipschitz cross-norm on $X\boxtimes E$ by [1, Theorem 5.2], we have

$$\begin{aligned}\Big|\sum_{i=1}^n\big\langle(g\cdot\phi)(x_i)-(g\cdot\phi)(y_i),e_i\big\rangle\Big|&=\Big|\sum_{i=1}^n\big(g(x_i)-g(y_i)\big)\langle\phi,e_i\rangle\Big|\\&\le\mathrm{Lip}(g)\|\phi\|\varepsilon\Big(\sum_{i=1}^n\delta_{(x_i,y_i)}\boxtimes e_i\Big)\\&\le\mathrm{Lip}(g)\|\phi\|\alpha\Big(\sum_{i=1}^n\delta_{(x_i,y_i)}\boxtimes e_i\Big)\end{aligned}$$

for all $\sum_{i=1}^n\delta_{(x_i,y_i)}\boxtimes e_i\in X\boxtimes E$, and so $g\cdot\phi\in\mathrm{Lip}_\alpha(X,E^*)$ and $\mathrm{Lip}_\alpha(g\cdot\phi)\le\mathrm{Lip}(g)\|\phi\|$. The converse inequality follows from Lemma 2.2. Since the Lipschitz operators $g\cdot\phi$ generate linearly the space $\mathrm{Lip}_{0F}(X,E^*)$ and $\mathrm{Lip}_\alpha(X,E^*)$ is a linear space, we conclude that $\mathrm{Lip}_{0F}(X,E^*)$ is contained in $\mathrm{Lip}_\alpha(X,E^*)$. □

Let us recall that a Lipschitz operator from X into E^* is said to be *Lipschitz approximable* if it is the limit in the Lipschitz norm Lip of a sequence of Lipschitz finite-rank operators from X to E^* (see [9]). Since the Banach spaces $(\mathrm{Lip}_0(X,E^*),\mathrm{Lip})$ and $(\mathrm{Lip}_\pi(X,E^*),\mathrm{Lip}_\pi)$ coincide by Lemma 2.5, it is natural to introduce the following class of Lipschitz operators.

Definition 4.2. Let α be a dualizable Lipschitz cross-norm on $X\boxtimes E$. A Lipschitz operator $f\in\mathrm{Lip}_\alpha(X,E^*)$ is said to be *α-Lipschitz approximable* if it is the limit in the α-Lipschitz norm Lip_α of a sequence of Lipschitz finite-rank operators from X to E^*.

Therefore, the space of all α-Lipschitz approximable operators from X into E^*, provided that α is a dualizable Lipschitz cross-norm on $X\boxtimes E$, is the closure of the space $\mathrm{Lip}_{0F}(X,E^*)$ in $(\mathrm{Lip}_\alpha(X,E^*),\mathrm{Lip}_\alpha)$.

Theorem 4.3. *Let α be a dualizable Lipschitz cross-norm on $X\boxtimes E$, and let α' be the associated Lipschitz norm of α.*

(i) $(\mathrm{Lip}_{0F}(X,E^*),\mathrm{Lip}_\alpha)$ *is isometrically isomorphic to $X^\#\circledast_{\alpha'}E^*$ via the map $K\colon X^\#\circledast_{\alpha'}E^*\to\mathrm{Lip}_{0F}(X,E^*)$ given by*

$$K\Big(\sum_{j=1}^m g_j\boxtimes\phi_j\Big)=\sum_{j=1}^m g_j\cdot\phi_j.$$

(ii) *The space of all α-Lipschitz approximable operators from X into E^* is isometrically isomorphic to $X^\#\widehat{\circledast}_{\alpha'}E^*$.*

Proof. By [1, Theorem 2.5], the map $K\colon X^{\#} \circledast E^* \to \mathrm{Lip}_{0F}(X, E^*)$ given by

$$K\Big(\sum_{j=1}^{m} g_j \boxtimes \phi_j\Big) = \sum_{j=1}^{m} g_j \cdot \phi_j$$

is a linear bijection. For any $\sum_{j=1}^{m} g_j \boxtimes \phi_j \in X^{\#} \circledast E^*$, we have

$$\begin{aligned}
&\alpha'\Big(\sum_{j=1}^{m} g_j \boxtimes \phi_j\Big) \\
&\quad = \sup\Big\{\Big|\Big(\sum_{j=1}^{m} g_j \boxtimes \phi_j\Big)\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big)\Big| : \alpha\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big) \leq 1\Big\} \\
&\quad = \sup\Big\{\Big|\sum_{i=1}^{n}\Big\langle \Big(\sum_{j=1}^{m} g_j \cdot \phi_j\Big)(x_i) - \Big(\sum_{j=1}^{m} g_j \cdot \phi_j\Big)(y_i), e_i\Big\rangle\Big| : \\
&\qquad \alpha\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big) \leq 1\Big\} \\
&\quad = \mathrm{Lip}_{\alpha}\Big(\sum_{j=1}^{m} g_j \cdot \phi_j\Big)
\end{aligned}$$

by using [1, Lemmas 2.2 and 1.4] and Remark 2.3. Hence K is an isometry from $X^{\#} \circledast_{\alpha'} E^*$ onto $(\mathrm{Lip}_{0F}(X, E^*), \mathrm{Lip}_{\alpha})$, and this proves (i). Then (ii) follows from (i) by applying a known result of functional analysis. □

From Corollary 3.2 and Theorem 4.3, we deduce the following consequence.

Corollary 4.4. *Let α be a dualizable Lipschitz cross-norm on $X \boxtimes E$. Then $X^{\#} \widehat{\circledast}_{\alpha'} E^*$ is isometrically isomorphic to $(X \widehat{\boxtimes}_{\alpha} E)^*$ if and only if $\mathrm{Lip}_{\alpha}(X, E^*)$ is isometrically isomorphic to the space of α-Lipschitz approximable operators from X to E^*.*

5. Lipschitz operator Banach ideals

We now formalize the notion of an ideal of Lipschitz operators with a definition inspired by the analogous one for linear operators between Banach spaces.

Definition 5.1. A Banach ideal of Lipschitz operators (or simply a Lipschitz operator Banach ideal) from X to E^* is a linear subspace $A(X, E^*)$ of $\mathrm{Lip}_0(X, E^*)$ equipped with a norm $\|\cdot\|_A$ with the following properties.

- (i) The Lipschitz rank 1 operator $g \cdot \phi$ from X to E^* belongs to $A(X, E^*)$ for every $g \in X^{\#}$ and $\phi \in E^*$, and $\|g \cdot \phi\|_A \leq \mathrm{Lip}(g)\|\phi\|$.
- (ii) The linear subspace $(A(X, E^*), \|\cdot\|_A)$ is a Banach space.
- (iii) The ideal property: If $f \in A(X, E^*)$, $h \in \mathrm{Lip}_0(X, X)$, and $S \in \mathcal{L}(E^*, E^*)$, then the composition Sfh belongs to $A(X, E^*)$ and $\|Sfh\|_A \leq \|S\| \cdot \|f\|_A \cdot \mathrm{Lip}(h)$.

Our aim is to study the case when $\mathrm{Lip}_{\alpha}(X, E^*)$ is a Lipschitz operator Banach ideal. We will need the following lemma.

Lemma 5.2. *Let α be a Lipschitz cross-norm on $X \boxtimes E$, and let $\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$. Then there exists a $f \in \mathrm{Lip}_\alpha(X, E^*)$ such that $\mathrm{Lip}_\alpha(f) = 1$ and $\sum_{i=1}^n \langle f(x_i) - f(y_i), e_i\rangle = \alpha(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i)$.*

Proof. By the Hahn–Banach theorem, there exists a functional $\varphi \in (X \widehat{\boxtimes}_\alpha E)^*$ with $\|\varphi\| = 1$ such that $\varphi(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i) = \alpha(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i)$. By Theorem 3.1, there exists a function $\Lambda_0^{-1}(\varphi) \in \mathrm{Lip}_\alpha(X, E^*)$ such that $\mathrm{Lip}_\alpha(\Lambda_0^{-1}(\varphi)) = \|\varphi\|$ and $(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i)(\Lambda_0^{-1}(\varphi)) = \varphi(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i)$. Take $f = \Lambda_0^{-1}(\varphi)$, and the lemma follows. $\square$

Theorem 5.3. *Let α be a Lipschitz cross-norm on $X \boxtimes E$. Then we have the following.*

(i) *If $\mathrm{Lip}_\alpha(X, E^*)$ is a Lipschitz operator Banach ideal, then α is uniform.*
(ii) *If α is uniform and E is a reflexive Banach space, then $\mathrm{Lip}_\alpha(X, E^*)$ is a Lipschitz operator Banach ideal.*

Proof. (i) Let us assume that $\mathrm{Lip}_\alpha(X, E^*)$ is a Lipschitz operator Banach ideal. Consider $\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, and let $h \in \mathrm{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$. By Lemma 5.2, there exists $f \in \mathrm{Lip}_\alpha(X, E^*)$ with $\mathrm{Lip}_\alpha(f) = 1$ such that

$$\sum_{i=1}^n \Big\langle f\big(h(x_i)\big) - f\big(h(y_i)\big), T(e_i)\Big\rangle = \alpha\Big(\sum_{i=1}^n \delta_{(h(x_i),h(y_i))} \boxtimes T(e_i)\Big);$$

that is,

$$\sum_{i=1}^n \Big\langle T^* fh(x_i) - T^* fh(y_i), e_i\Big\rangle = \alpha\Big(\sum_{i=1}^n \delta_{(h(x_i),h(y_i))} \boxtimes T(e_i)\Big),$$

where T^* denotes the adjoint operator of T. Since $\mathrm{Lip}_\alpha(X, E^*)$ has the ideal property, then $T^* fh$ belongs to $\mathrm{Lip}_\alpha(X, E^*)$ and $\mathrm{Lip}_\alpha(T^* fh) \le \|T^*\| \,\mathrm{Lip}_\alpha(f)\mathrm{Lip}(h)$. Then we have

$$\begin{aligned}\alpha\Big(\sum_{i=1}^n \delta_{(h(x_i),h(y_i))} \boxtimes T(e_i)\Big) &\le \mathrm{Lip}_\alpha(T^* fh)\alpha\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big)\\ &\le \|T\|\mathrm{Lip}(h)\alpha\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big),\end{aligned}$$

and so α is uniform.

(ii) Notice that $\mathrm{Lip}_\alpha(X, E^*)$ is a linear subspace of $\mathrm{Lip}_0(X, E^*)$ and $(\mathrm{Lip}_\alpha(X, E^*), \mathrm{Lip}_\alpha)$ is a normed space which satisfies the conditions (i) and (ii) of Definition 5.1 by Lemmas 2.2 and 2.4 and Theorems 3.1 and 4.1. Assume that α is uniform and that E is reflexive. We only need to prove that $\mathrm{Lip}_\alpha(X, E^*)$ has the ideal property. Let $f \in \mathrm{Lip}_\alpha(X, E^*)$, let $h \in \mathrm{Lip}_0(X, X)$, and let $S \in \mathcal{L}(E^*, E^*)$. Since E is reflexive, there exists $T \in \mathcal{L}(E, E)$ such that $T^* = S$ and $\|T\| = \|S\|$. For every $\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, we have

$$\Big|\sum_{i=1}^n \Big\langle Sfh(x_i) - Sfh(y_i), e_i\Big\rangle\Big| = \Big|\sum_{i=1}^n \Big\langle fh(x_i) - fh(y_i), T(e_i)\Big\rangle\Big|$$

$$\leq \mathrm{Lip}_\alpha(f)\alpha\Big(\sum_{i=1}^{n}\delta_{(h(x_i),h(y_i))}\boxtimes T(e_i)\Big)$$
$$\leq \mathrm{Lip}_\alpha(f)\mathrm{Lip}(h)\|T\|\alpha\Big(\sum_{i=1}^{n}\delta_{(x_i,y_i)}\boxtimes e_i\Big).$$

It follows that Sfh is in $\mathrm{Lip}_\alpha(X,E^*)$ and $\mathrm{Lip}_\alpha(Sfh)\leq \|S\|\,\mathrm{Lip}_\alpha(f)\mathrm{Lip}(h)$. This completes the proof. □

Theorem 5.3 shows that for reflexive spaces there is an equivalence between the uniformity of α and Lip_α being an ideal.

We now study when α-Lipschitz approximable operators from X into E^* form a Lipschitz operator Banach ideal.

Theorem 5.4. *Let α be a uniform and dualizable Lipschitz cross-norm on the space $X\boxtimes E$. Assume that E is a reflexive Banach space. Then $(\overline{\mathrm{Lip}_{0F}}(X,E^*)$, $\mathrm{Lip}_\alpha)$ is a Lipschitz operator Banach ideal.*

Proof. We first show that $(\mathrm{Lip}_{0F}(X,E^*),\mathrm{Lip}_\alpha)$ has the ideal property. Consider the maps $f\in \mathrm{Lip}_{0F}(X,E^*)$, $h\in \mathrm{Lip}_0(X,X)$, and $S\in \mathcal{L}(E^*,E^*)$. Since

$$\mathrm{lin}\big(Sfh(X)\big)=S\big(\mathrm{lin}\big(fh(X)\big)\big)\subset S\big(\mathrm{lin}\big(f(X)\big)\big),$$

we infer that $Sfh\in \mathrm{Lip}_{0F}(X,E^*)$. The inequality $\mathrm{Lip}_\alpha(Sfh)\leq\|S\|\,\mathrm{Lip}_\alpha(f)\mathrm{Lip}(h)$ follows similarly as in the proof of the assertion (ii) of Theorem 5.3.

By Theorems 4.1 and 3.1, $(\overline{\mathrm{Lip}_{0F}}(X,E^*),\mathrm{Lip}_\alpha)$ satisfies the conditions (i) and (ii) of Definition 5.1. In order to prove that it has the ideal property, let $f\in \overline{\mathrm{Lip}_{0F}}(X,E^*)$, $h\in \mathrm{Lip}_0(X,X)$, and $S\in\mathcal{L}(E^*,E^*)$. Then we can take a sequence $\{f_n\}$ in $\mathrm{Lip}_{0F}(X,E^*)$ such that $\mathrm{Lip}_\alpha(f_n-f)\to 0$. Then $\mathrm{Lip}_\alpha(Sf_nh-Sfh)\to 0$ since

$$\mathrm{Lip}_\alpha(Sf_nh-Sfh)=\mathrm{Lip}_\alpha\big(S(f_n-f)h\big)\leq\|S\|\,\mathrm{Lip}_\alpha(f_n-f)\mathrm{Lip}(h)$$

for all $n\in\mathbb{N}$. Hence $Sfh\in\overline{\mathrm{Lip}_{0F}}(X,E^*)$. Since $\mathrm{Lip}_\alpha(Sf_nh)\leq\|S\|\,\mathrm{Lip}_\alpha(f_n)\mathrm{Lip}(h)$ for all $n\in\mathbb{N}$, we deduce that $\mathrm{Lip}_\alpha(Sfh)\leq\|S\|\,\mathrm{Lip}_\alpha(f)\mathrm{Lip}(h)$, and the theorem follows. □

6. Lipschitz operator Banach spaces

In Theorem 3.1 we have characterized $\mathrm{Lip}_\alpha(X,E^*)$ as the dual space $(X\widehat{\boxtimes}_\alpha E)^*$. Our aim in this section is to tackle the general duality problem as to when a space of maps from X to E^* is isometrically isomorphic to $(X\widehat{\boxtimes}_\alpha E)^*$ for some Lipschitz cross-norm α regardless of whether or not one has an ideal property. For that purpose, we first introduce Banach spaces of Lipschitz operators.

Definition 6.1. A Banach space of Lipschitz operators (or simply a Lipschitz operator Banach space) from X to E^* is a linear subspace $B(X,E^*)$ of $\mathrm{Lip}_0(X,E^*)$ equipped with a norm $\|\cdot\|_B$ having the properties

(i) $(B(X,E^*),\|\cdot\|_B)$ is a Banach space,
(ii) $\|f\|_B\geq \mathrm{Lip}(f)$ for every $f\in B(X,E^*)$,

(iii) for every $g \in X^{\#}$ and $\phi \in E^*$, the map $g \cdot \phi$ belongs to $B(X, E^*)$ and $\|g \cdot \phi\|_B = \mathrm{Lip}(g)\|\phi\|$.

We first characterize all Lipschitz cross-norms α on $X \boxtimes E$ for which $\mathrm{Lip}_\alpha(X, E^*)$ is a Lipschitz operator Banach space.

Theorem 6.2. *Let α be a Lipschitz cross-norm on $X \boxtimes E$. Then $\mathrm{Lip}_\alpha(X, E^*)$ is a Lipschitz operator Banach space if and only if α is dualizable.*

Proof. In view of Lemmas 2.4 and 2.2 and Theorem 3.1, $\mathrm{Lip}_\alpha(X, E^*)$ is a linear subspace of $\mathrm{Lip}_0(X, E^*)$, and $(\mathrm{Lip}_\alpha(X, E^*), \mathrm{Lip}_\alpha)$ is a normed space satisfying assumptions (i) and (ii) of Definition 6.1. Hence we only need to prove that $\mathrm{Lip}_\alpha(X, E^*)$ satisfies condition (iii) if and only if α is dualizable.

If α is dualizable, then $\mathrm{Lip}_\alpha(X, E^*)$ has the property (iii) by Theorem 4.1. Conversely, assume that every map $g \cdot \phi \colon X \to E^*$ with $g \in X^{\#}$ and $\phi \in E^*$ is in $\mathrm{Lip}_\alpha(X, E^*)$ and $\mathrm{Lip}_\alpha(g \cdot \phi) = \mathrm{Lip}(g)\|\phi\|$. Take $g \in X^{\#}$ and $\phi \in E^*$, and since

$$
\begin{aligned}
\Big|\sum_{i=1}^n \big(g(x_i) - g(y_i)\big)\langle \phi, e_i\rangle\Big| &= \Big|\sum_{i=1}^n \big\langle (g\cdot\phi)(x_i) - (g\cdot\phi)(y_i), e_i\big\rangle\Big| \\
&\le \mathrm{Lip}_\alpha(g\cdot\phi)\alpha\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big) \\
&= \mathrm{Lip}(g)\|\phi\|\alpha\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big)
\end{aligned}
$$

for all $\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, we have that α is dualizable. $\square$

Since π, ε, d_p, and w_p for $p \in [1, \infty]$ are dualizable Lipschitz cross-norms on $X \boxtimes E$, Theorem 6.2 gives the following.

Corollary 6.3. *The spaces $\mathrm{Lip}_\alpha(X, E^*)$ for $\alpha = \pi, \varepsilon, d_p, w_p$ with $1 \le p \le \infty$ are Lipschitz operator Banach spaces.*

Conversely, we will now address the problem of when a Lipschitz operator Banach space can be canonically isometrically identified with the dual of a Lipschitz tensor product endowed with a Lipschitz cross-norm. We begin with the following lemma.

Lemma 6.4. *Let $B(X, E^*)$ be a Lipschitz operator Banach space. For each element $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, define*

$$
\alpha(u) = \sup\Big\{\Big|\sum_{i=1}^n \big\langle f(x_i) - f(y_i), e_i\big\rangle\Big| \colon f \in B(X, E^*), \|f\|_B = 1\Big\}
$$

and

$$
\big\langle i(u), f\big\rangle = \sum_{i=1}^n \big\langle f(x_i) - f(y_i), e_i\big\rangle \quad \big(f \in B(X, E^*)\big).
$$

Then α is a dualizable Lipschitz cross-norm on $X \boxtimes E$, and i is a linear isometry from $X \boxtimes_\alpha E$ into $B(X, E^*)^*$.

Proof. Let $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, and let $f \in B(X,E^*)$. Note that $\langle i(u), f\rangle = u(f)$. Clearly, $i(u)$ is well defined on $B(X,E^*)$, it is linear, and $\|\langle i(u), f\rangle\| \leq \mathrm{Lip}(f)\pi(u) \leq \|f\|_B \pi(u)$ for all $f \in B(X,E^*)$. Then $i(u)$ is in $B(X,E^*)^*$, and

$$\big\|i(u)\big\| := \sup\Big\{\big|\big\langle i(u), f\big\rangle\big| \colon f \in B(X,E^*), \|f\|_B = 1\Big\} \leq \pi(u).$$

It is immediate that $i\colon X \boxtimes E \to B(X,E^*)^*$ is well defined and linear. Moreover, it is injective. Indeed, $i(u) = 0$ means that $\langle i(u), f\rangle = 0$ for all $f \in B(X,E^*)$. Since $B(X,E^*)$ contains the maps $g \cdot \phi$, it follows that $\langle u, g\cdot\phi\rangle = \langle i(u), g\cdot\phi\rangle = 0$ for all $g \in X^{\#}$ and $\phi \in E^*$, and then $u = 0$ by [1, Proposition 1.6].

Define the map α on $X \boxtimes E$ as in the statement. Notice that $\alpha(u) = \|i(u)\|$. Then α is a norm on $X \boxtimes E$, and so i is a linear isometry from $X \boxtimes_\alpha E$ into $B(X,E^*)^*$.

We claim that α is a Lipschitz cross-norm. Indeed, for any $\delta_{(x,y)} \boxtimes e \in X \boxtimes E$, we have

$$\alpha(\delta_{(x,y)} \boxtimes e) = \big\|i(\delta_{(x,y)} \boxtimes e)\big\| \leq \pi(\delta_{(x,y)} \boxtimes e) = d(x,y)\|e\|.$$

For the reverse, we may take $\phi \in S_{E^*}$ and $g \in S_{X^{\#}}$ satisfying that $\langle \phi, e\rangle = \|e\|$ and $g(x) - g(y) = d(x,y)$. For example, $g(z) = d(z,y) - d(0,y)$ for all $z \in X$. Then $g\cdot\phi \in B(X,E^*)$ with $\|g\cdot\phi\| = 1$, and we infer that

$$\alpha(\delta_{(x,y)} \boxtimes e) \geq \big|\big\langle (g\cdot\phi)(x) - (g\cdot\phi)(y), e\big\rangle\big| = \big|\big(g(x) - g(y)\big)\langle \phi, e\rangle\big| = d(x,y)\|e\|,$$

and this proves our claim.

Finally, we prove that α is dualizable. Let $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$. For any $g \in S_{X^{\#}}$ and $\phi \in S_{E^*}$, we have

$$\begin{aligned}
&\Big|\sum_{i=1}^{n}\big(g(x_i) - g(y_i)\big)\langle \phi, e_i\rangle\Big| \\
&\qquad = \Big|\sum_{i=1}^{n}\big\langle (g\cdot\phi)(x_i) - (g\cdot\phi)(y_i), e_i\big\rangle\Big| \\
&\qquad \leq \sup\Big\{\Big|\sum_{i=1}^{n}\big\langle f(x_i) - f(y_i), e_i\big\rangle\Big| \colon f \in B(X,E^*), \|f\|_B = 1\Big\},
\end{aligned}$$

and therefore $\varepsilon(u) \leq \alpha(u)$. Then α is dualizable by [1, Theorem 6.3 and Proposition 6.4]. □

We are ready to obtain the main result of this section.

Theorem 6.5. *Let $B(X,E^*)$ be a Lipschitz operator Banach space. Then the following are equivalent:*

(i) *There exists a dualizable Lipschitz cross-norm α on $X \boxtimes E$ such that $B(X,E^*) = \mathrm{Lip}_\alpha(X,E^*)$ and $\|f\|_B = \mathrm{Lip}_\alpha(f)$ for every $f \in B(X,E^*)$.*

(ii) *If f is in $\mathrm{Lip}_0(X,E^*)$ and $\{f_\gamma\}$ is a bounded net in $B(X,E^*)$ which converges to f in the weak* Lipschitz operator topology of $\mathrm{Lip}_0(X,E^*)$, then $f \in B(X,E^*)$ and $\|f\|_B \leq \sup\{\|f_\gamma\|_B \colon \gamma \in \Gamma\}$.*

Proof. Suppose that (i) holds. Let $f \in \mathrm{Lip}_0(X, E^*)$, and let $\{f_\gamma\}$ be a bounded net in $B(X, E^*)$ converging to f in the w*Lo topology of $\mathrm{Lip}_0(X, E^*)$. Denote

$$M = \sup\big\{\|f_\gamma\|_B \colon \gamma \in \Gamma\big\}.$$

If $\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$ and $\varepsilon > 0$, then we have

$$\Big|\sum_{i=1}^n \big\langle f(x_i)-f(y_i), e_i\big\rangle - \sum_{i=1}^n \big\langle f_{\gamma_0}(x_i)-f_{\gamma_0}(y_i), e_i\big\rangle\Big| = \Big|\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big)(f-f_{\gamma_0})\Big| < \varepsilon$$

for some $\gamma_0 \in \Gamma$, and therefore

$$\begin{aligned}\Big|\sum_{i=1}^n \big\langle f(x_i) - f(y_i), e_i\big\rangle\Big| &< \Big|\sum_{i=1}^n \big\langle f_{\gamma_0}(x_i) - f_{\gamma_0}(y_i), e_i\big\rangle\Big| + \varepsilon \\ &\leq \mathrm{Lip}_\alpha(f_{\gamma_0})\alpha\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big) + \varepsilon \\ &= \|f_{\gamma_0}\|_B\alpha\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big) + \varepsilon \\ &\leq M\alpha\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big) + \varepsilon.\end{aligned}$$

Since ε was arbitrary, we deduce that $f \in \mathrm{Lip}_\alpha(X, E^*)$ and $\mathrm{Lip}_\alpha(f) \leq M$. Hence $f \in B(X, E^*)$ and $\|f\|_B \leq M$.

Conversely, assume that (ii) is true. Take the dualizable Lipschitz cross-norm α on $X \boxtimes E$ and the linear isometry i from $X \boxtimes_\alpha E$ into $B(X, E^*)^*$ defined in Lemma 6.4. Next we check that $B(X, E^*) = \mathrm{Lip}_\alpha(X, E^*)$ and $\|f\|_B = \mathrm{Lip}_\alpha(f)$ for all $f \in B(X, E^*)$. To this end, we first take a function f in $B(X, E^*)$. The definition of α gives

$$\Big|\sum_{i=1}^n \big\langle f(x_i) - f(y_i), e_i\big\rangle\Big| \leq \|f\|_B\alpha\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\Big)$$

for all $\sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, and then $f \in \mathrm{Lip}_\alpha(X, E^*)$ with $\mathrm{Lip}_\alpha(f) \leq \|f\|_B$. Conversely, pick a function f in $\mathrm{Lip}_\alpha(X, E^*)$, and let $S(f) \colon i(X \boxtimes E) \to \mathbb{K}$ be the functional given by

$$\big\langle S(f), i(u)\big\rangle = \sum_{i=1}^n \big\langle f(x_i) - f(y_i), e_i\big\rangle$$

for $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$. The fact that i is injective guarantees that $S(f)$ is well defined. The linearity of $S(f)$ follows easily. Since $|\langle S(f), i(u)\rangle| = |u(f)| \leq \mathrm{Lip}_\alpha(f)\alpha(u) = \mathrm{Lip}_\alpha(f)\|i(u)\|$ for all $u \in X \boxtimes E$, it follows that $S(f)$ is continuous and $\|S(f)\| \leq \mathrm{Lip}_\alpha(f)$. Since $i(X \boxtimes E)$ is a linear subspace of $B(X, E^*)^*$, the Hahn–Banach theorem provides a functional $\widetilde{S}(f) \in B(X, E^*)^{**}$ which extends to $S(f)$ and has the same norm. Let κ_B be the canonical injection from $B(X, E^*)$ into $B(X, E^*)^{**}$. By Goldstein's theorem, there exists a net $\{f_\gamma\}$ in $B(X, E^*)$ for which $\sup\{\|f_\gamma\|_B \colon \gamma \in \Gamma\} \leq \|\widetilde{S}(f)\|$ and $\{\kappa_B(f_\gamma)\}$ converges to

$\widetilde{S}(f)$ in the weak* topology of $B(X,E^*)^{**}$. Since $i(X \boxtimes E) \subset B(X,E^*)^*$, it follows that, for each $u \in X \boxtimes E$, the net $\{\langle \kappa_B(f_\gamma), i(u)\rangle\}$ converges to $\langle \widetilde{S}(f), i(u)\rangle$; that is, $\{u(f_\gamma)\}$ converges to $u(f)$. This means that $\{f_\gamma\}$ converges to f in the weak* Lipschitz operator topology of $\mathrm{Lip}_0(X,E^*)$ by Remark 3.7. Then, by hypothesis, $f \in B(X,E^*)$, and $\|f\|_B \le \sup\{\|f_\gamma\|_B \colon \gamma \in \Gamma\} \le \mathrm{Lip}_\alpha(f)$. This finishes the proof. □

Theorem 6.5 can be reformulated as follows.

Corollary 6.6. *Let $B(X,E^*)$ be a Lipschitz operator Banach space. The following are equivalent.*

(i) *There exists a dualizable Lipschitz cross-norm α on $X \boxtimes E$ such that $B(X,E^*) = \mathrm{Lip}_\alpha(X,E^*)$ and $\|f\|_B = \mathrm{Lip}_\alpha(f)$ for all $f \in B(X,E^*)$.*
(ii) *There exists a dualizable Lipschitz cross-norm α on $X \boxtimes E$ such that $B(X,E^*)$ is isometrically isomorphic to $(X \widehat{\boxtimes}_\alpha E)^*$.*
(iii) *The closed unit ball of $B(X,E^*)$ is compact in the weak* Lipschitz operator topology of $\mathrm{Lip}_0(X,E^*)$.*

Proof. Here, the fact that (i) implies (ii) is deduced immediately, taking into account Theorem 3.1. Assume now that (ii) holds. Then $B(X,E^*)$ is isometrically isomorphic to $\mathrm{Lip}_\alpha(X,E^*)$ by Theorem 3.1. Then Alaoglu's theorem says that the closed unit ball of $B(X,E^*)$ is compact in the w* topology of $\mathrm{Lip}_\alpha(X,E^*)$, and hence by Corollary 3.8 in the w*Lo topology of $\mathrm{Lip}_\alpha(X,E^*)$. Since $\mathrm{Lip}_\alpha(X,E^*)$ is a linear subspace of $\mathrm{Lip}_0(X,E^*)$, this last topology agrees with the relative w*Lo topology of $\mathrm{Lip}_0(X,E^*)$ on $\mathrm{Lip}_\alpha(X,E^*)$ by Remark 3.7. Then (iii) follows easily.

Finally, suppose that (iii) is true. Let $f \in \mathrm{Lip}_0(X,E^*)$, and let $\{f_\gamma\}$ be a bounded net in $B(X,E^*)$ which converges to f in the w*Lo topology of $\mathrm{Lip}_0(X,E^*)$. Let $M = \sup\{\|f_\gamma\|_B \colon \gamma \in \Gamma\}$. By (iii), the closed unit ball of $B(X,E^*)$ is closed in the w*Lo topology of $\mathrm{Lip}_0(X,E^*)$. Therefore, the limit of the net $\{f_\gamma/M\}$ in the w*Lo topology of $\mathrm{Lip}_0(X,E^*)$, that is, f/M, is in the closed unit ball of $B(X,E^*)$. Hence $f \in B(X,E^*)$ and $\|f\|_B \le M$. Then the assertion (ii) of Theorem 6.5 is satisfied, and we obtain (i). □

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