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CONVEX CONES OF GENERALIZED MULTIPLY MONOTONE FUNCTIONS AND THE DUAL CONES

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ABSTRACT. Let n and k be nonnegative integers such that $1 \le k \le n+1$. The convex cone $\mathscr{F}_+^{k:n}$ of all functions f on an arbitrary interval $I \subseteq \mathbb{R}$ whose derivatives $f^{(j)}$ of orders $j = k-1, \dots, n$ are nondecreasing is characterized. A simple description of the convex cone dual to $\mathscr{F}_+^{k:n}$ is given. In particular, these results are useful in, and were motivated by, applications in probability. In fact, the results are obtained in a more general setting with certain generalized derivatives of f of the jth order in place of $f^{(j)}$. Somewhat similar results were previously obtained, in terms of Tchebycheff–Markov systems, in the case when the left endpoint of the interval I is finite, with certain additional integrability conditions; such conditions fail to hold in the mentioned applications. Development of substantially new methods was needed to overcome the difficulties.

1. INTRODUCTION

In applications in probability (see, e.g., [11], [5], [6], [20], [21], [3], [26], [23], [25], and references therein), one is concerned with stochastic domination, defined by a formula of the form

$$X \overset{\mathscr{F}}{\succcurlyeq} Y \overset{\text{def}}{\iff} X \succcurlyeq Y (\operatorname{mod} \mathscr{F}) \overset{\text{def}}{\iff} \mathsf{E} f(X) \ge \mathsf{E} f(Y) \quad \text{for all } f \in \mathscr{F},$$

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where X and Y are random variables (r.v.'s) and $\mathscr{F}$ is a class of functions, assuming the expectations are appropriately defined.

In the case when $\mathscr{F}$ is the class of all nondecreasing functions, the relation $\overset{\mathscr{F}}{\succcurlyeq}$ is called the *first-order stochastic dominance*.

In general, the functions $f \in \mathscr{F}$ may be referred to as the *test functions*.

Unless $\mathsf{E}f(X) = \mathsf{E}f(Y) = \infty$ or $\mathsf{E}f(X) = \mathsf{E}f(Y) = -\infty$, the inequality $\mathsf{E}f(X) \ge \mathsf{E}f(Y)$ can be rewritten as $\nu(f) := \int f \,\mathrm{d}\nu \ge 0$, where ν is the signed measure (say, $\nu_{X,Y}$) equal to the difference between the probability distributions of the r.v.'s X and Y.

More generally, one may allow ν to belong to a larger set (say, N) of signed measures on an interval I, which are not necessarily the differences between two probability measures. Usually, the set N is assumed to be a convex cone. One can define the cone dual to $\mathscr{F}$ by the formula

$$\hat{\mathscr{F}} := \big\{\nu \in \mathrm{N}\colon \nu(f) \ge 0 \text{ for all } f \in \mathscr{F}\big\}.$$

Thus, at least in the case when the r.v.'s X and Y are such that $\mathsf{E}f(X)$ and $\mathsf{E}f(Y)$ are both finite for all $f \in \mathscr{F}$, one will have $X \overset{\mathscr{F}}{\succcurlyeq} Y \iff \nu_{X,Y} \in \hat{\mathscr{F}}$.

For any α and β in $\mathbb{Z} \cup \{\infty\}$, let $\overline{\alpha, \beta} := \{j \in \mathbb{Z}\colon \alpha \le j \le \beta\}$. In what follows, assume that the values of indices i, j, k, ℓ, m, n are each in the set $\overline{0, \infty}$, unless specified otherwise.

Classes of test functions of particular interest in the mentioned applications are

$$\begin{aligned}\mathscr{F}_+^{k:n} &:= \mathscr{F}_+^{k:n}(I) \\ &:= \{f \in \mathscr{D}^n\colon f^{(j)} \text{ is nondecreasing for each } j \in \overline{k-1, n}\},\end{aligned} \tag{1.1}$$

where

$$1 \le k \le n + 1; \tag{1.2}$$

I is an interval in $\mathbb{R}$ with endpoints a and b such that $a < b$ and, for each $j \in \overline{1, n}$, $\mathscr{D}^j = \mathscr{D}^j(I)$ is the set of all $(j-1)$-times differentiable functions $f \in \mathbb{R}^I$ such that the function $f^{(j-1)}$ is absolutely continuous with an almost everywhere derivative (denoted here by $f^{(j)}$) that is (i) right-continuous on the interval $I \setminus \{b\}$ and (ii) left-continuous at the point b in the case when $b \in I$. Also, let $\mathscr{D}^0$ be the set of all Borel-measurable functions in $\mathbb{R}^I$. Then, in particular, $\mathscr{F}_+^{1:0}(I)$ will be the set of nondecreasing functions $f \in \mathbb{R}^I$.

The functions of the class $\mathscr{F}_+^{n:n-1}$ are widely known as *n-convex functions* (see, e.g., [18] and references therein).

On the other hand, the nonnegative functions g whose "horizontal reflections" $g(-\cdot)$ belong to the class $\mathscr{F}_+^{1:n-1}$ are also widely known as *n-monotone functions* (see, e.g., [28], [17], [16]). In particular, in [16], multiply monotone distributions are compared using s-convex stochastic orders—on $[0, \infty)$, with four applications to insurance; other applications to insurance problems were given in [8]. Clearly, the notion of the n-monotonicity is an extension of Bernstein's complete monotonicity.

Note that the class $\mathscr{F}_+^{k:n}$ (or, rather, its "reflection" $\mathscr{F}_-^{k:n}$ defined in (5.21)) may be considered a generalization/extension of the class of all completely monotone functions on $(0,\infty)$ (in the Bernstein sense). Indeed, the latter class coincides with $\bigcap_{n=0}^\infty \mathscr{F}_-^{1:n}([0,\infty))$; cf. [23, Proposition 3.4] and its proof therein.

The case of stochastic domination mod $\mathscr{F}_+^{k:n}$ with $a>-\infty$ has been systematically studied in the literature (see, e.g., [11], [30], [12], [18], [5], [6]). In this case, one can rely on the Taylor expansion

$$f(x)=\sum_{i\in\overline{0,n}} f^{(i)}(a+)\frac{(x-a)^i}{i!}+\int_I \mathrm{d}f^{(n)}(t)\frac{(x-t)_+^n}{n!} \tag{1.3}$$

for $x\in I$; as usual, we let $u_+:=0\vee u$ and $u_+^v:=(u_+)^v$ for all $u\in\mathbb{R}$ and $v\in[0,\infty)$, along with the convention $0^0:=1$. Note that $f^{(i)}\ge 0$ and hence $f^{(i)}(a+)\ge 0$ for any $f\in\mathscr{F}_+^{k:n}$ and any $i\in\overline{k,n}$. It is also clear that the functions $I\ni x\mapsto c(x-a)^i$, $I\ni x\mapsto (x-a)^j$, and $I\ni x\mapsto (x-t)_+^n$ belong to the set $\mathscr{F}_+^{k:n}$ for all $c\in\mathbb{R}$, $i\in\overline{0,k-1}$, $j\in\overline{k,n}$, and $t\in I$. So, assuming appropriate integrability conditions, one has the following characterization of the dual cone $\hat{\mathscr{F}}_+^{k:n}$: a signed measure $\nu\in\mathrm{N}$ is in $\hat{\mathscr{F}}_+^{k:n}$ if and only if all of the following three conditions hold:

(i) $\int_I (x-a)^i\nu(\mathrm{d}x)=0$ for all $i\in\overline{0,k-1}$;
(ii) $\int_I (x-a)^j\nu(\mathrm{d}x)\ge 0$ for all $j\in\overline{k,n}$;
(iii) $\int_I (x-t)_+^n\nu(\mathrm{d}x)\ge 0$ for all $t\in I$.

Such a characterization of the dual cone is very useful, as it reduces the verification of the inequality $\nu(f)\ge 0$ for all test functions $f\in\mathscr{F}_+^{k:n}$ to the verification of this inequality just in the case when f is in a certain set of polynomials and their "positive parts" $x\mapsto(x-t)_+^n$. One may note that, in the case when $k\le n$ (cf. (1.2)), the conjunction of the above conditions (i) and (ii) is equivalent to that of conditions

(i′) $\int_I x^i\nu(\mathrm{d}x)=0$ for all $i\in\overline{0,k-1}$;
(ii′) $\int_I x^k\nu(\mathrm{d}x)\ge 0$;
(ii″) $\int_I (x-a)^j\nu(\mathrm{d}x)\ge 0$ for all $j\in\overline{k+1,n}$.

Alas, Taylor expansion (1.3) does not seem to make sense when $a=-\infty$ and $n\ge 1$, and then the entire argument no longer holds (cf., e.g., [5, Remark 3.6]).

On the other hand, it is the case when $I=\mathbb{R}$ and hence $a=-\infty$ that is of foremost interest in applications to probability (see [20], [3], [23], [25]), as the distribution of the r.v. X in those applications may be normal (e.g., in [20]) or a convolution of a normal distribution and a Poisson one (e.g., in [23]), whose support set will then be the entire real line, or with a support set bounded from above rather than from below (e.g., in [20], [3], [23], [25]). In general as well it is desirable to allow the support sets of both X and Y not to be a priori bounded either from above or below.

Because of the lack of the Taylor expansion (1.3), it is much more difficult to obtain a characterization of the dual cone $\hat{\mathscr{F}}_+^{k:n}$ in the case when $a=-\infty$ and $k\ne n+1$. The first step here is to observe that for any $f\in\mathscr{F}_+^{k:n}$ one has

$f^{(j)}(a+) = 0$ for all $j \in \overline{k+1, n}$, which results in the following Taylor expansion of the function $f^{(k)}$ "at the point $-\infty+ := (-\infty)+$":

$$\begin{aligned} f^{(k)}(x_k) &= \sum_{i\in\overline{k,n}} f^{(i)}(-\infty+)\frac{(x_k+\infty)^{i-k}}{(i-k)!} + \int_I \mathrm{d}f^{(n)}(t)\frac{(x_k-t)_+^{n-k}}{(n-k)!} \\ &= f^{(k)}(-\infty+) + \int_I \mathrm{d}f^{(n)}(t)\frac{(x_k-t)_+^{n-k}}{(n-k)!} \end{aligned} \tag{1.4}$$

for $x_k \in I$; here and elsewhere, we are assuming the conventions $0 \cdot c := 0$ for any $c \in [-\infty, \infty]$ and $\infty^0 := 1$. Next, we fix an arbitrary $z \in Y$ and, for any $y \in I \cap (-\infty, z)$, truncate the above Taylor expansion of the kth derivative $f^{(k)}$ by replacing the integral $\int_I$ in (1.4) with $\int_{I\cap[y,\infty)}$; let us denote the resulting function by $(f^{(k)})_y$. Finally, the so-truncated kth derivative is lifted back up, in the sense that a function g_y is constructed so that the conditions $(g_y)^{(k)} = (f^{(k)})_y$ and $(g_y)^{(i)}(z) = f^{(i)}(z)$ for all $i \in \overline{0, k-1}$ hold. In fact, g_y is completely determined by these conditions and is given by formulas (4.27), (4.33), and (4.34). Moreover, g_y approximates f in the sense of (4.35). So, g_y may be considered an approximate Taylor expansion of f at $a = -\infty$. As Remark 4.7 shows, in general functions $f \in \mathscr{F}_+^{k:n}$ admit only of such an approximate Taylor expansion of f at $a = -\infty$; that is, one cannot do without the truncation described above.

However, this approximate Taylor expansion of f is enough to obtain a desired characterization of the dual cone $\hat{\mathscr{F}}_+^{k:n}$ in the case when $a = -\infty$ and $k \neq n+1$, which is as follows: a signed measure $\nu \in \mathrm{N}$ is in $\hat{\mathscr{F}}_+^{k:n}$ if and only if

($\mathrm{i}_{-\infty}$) $\int_I x^i \nu(\mathrm{d}x) = 0$ for all $i \in \overline{0, k-1}$;
($\mathrm{ii}_{-\infty}$) $\int_I x^k \nu(\mathrm{d}x) \geq 0$;
($\mathrm{iii}_{-\infty}$) $\int_I (x-t)_+^n \nu(\mathrm{d}x) \geq 0$ for all $t \in I$.

One can see that conditions ($\mathrm{i}_{-\infty}$), ($\mathrm{ii}_{-\infty}$), and ($\mathrm{iii}_{-\infty}$) are, respectively, the same as conditions (i′), (ii′), and (iii) on page 866; however, condition (ii″) from page 866 "disappears" when $a = -\infty$.

The case when $k = n+1$ is overall simpler (than the just discussed case $k \neq n+1$) but has a certain peculiarity to it, which will be addressed later in this paper.

In fact, we consider a more general version of the class $\mathscr{F}_+^{k:n}$, defined in (1.1), by replacing the operators $f \mapsto f^{(j)}$ of multiple differentiation with more general differential operators, of the form

$$E^j := E^j_{w_0,\dots,w_j} := (R_{w_j}D)\cdots(R_{w_1}D)R_{w_0}, \tag{1.5}$$

where D is the usual differentiation operator, $w_0, \dots, w_n$ are positive smooth enough functions, and $R_w f := f/w$ for any function $f \in \mathbb{R}^I$. Thus, the operator E^j is the alternating composition of the operators of the division by positive functions and the differentiation operator. The functions $w_0, \dots, w_n$ may be referred to as the *gauge* functions. In the *unit-gauge* case, with $w_0 = \cdots = w_n = 1$, the operator $E^j_{w_0,\dots,w_j}$ reduces back to D^j, the operator of the j-fold differentiation.

As shown in [24, Proposition 2.1], a special case of nonunit gauge functions—with $w_0 = 1$ and $w_1 = \cdots = w_n = \psi'$ for a general continuous function ψ'—arises from the unit gauges by a (generally nonlinear) change of scale.

One reason to consider general gauge functions $w_0, \dots, w_n$ is to encompass, in particular, the corresponding results in [11], [30], and [12] on dual cones, defined in terms of extended complete Tchebycheff systems. Details on this are given in Section 5.3. The theory and applications of Tchebycheff systems have a long and rich history (see, e.g., [12], [15]). Perhaps even more importantly, the general, not necessarily unit-gauge, case is the most interesting and challenging aspect of the new theory.

Dealing with general, not necessarily unit, gauge functions $w_0, \dots, w_n$ requires overcoming more difficulties. One of them is that such an explicit representation of the approximation g_y of f as the one mentioned above and given by formulas (4.27), (4.33), and (4.34) for the unit-gauge case is then no longer available. Here, to be used in place of usual polynomials, generalized polynomials are introduced, depending on the sequence $\mathbf{w} := (w_0, \dots, w_n)$ of gauge functions; rather naturally, a function p is called a $\mathbf{w}$-*polynomial of degree* j if the function $E^j_{w_0,\dots,w_j} p$ is a nonzero constant.

Another notable distinction from the unit-gauge case is that, in general, in place of the set $\overline{k+1,n}$ in condition (ii$''$) on page 866 for the unit-gauge case, one may get any given subset of $\overline{k+1,n}$, depending on the choice of the gauge functions $w_0, \dots, w_n$, as follows from Proposition 3.4. No phenomenon of this kind appears to have been observed before.

In distinction with the mentioned classical theory on dual cones presented in [11], [30], and [12], the more general setting considered in this paper requires an extra chain of $\mathbf{w}$-polynomials, defined in Section 3.4, in addition to the chain of $(\mathbf{w}; t)$-monomials considered in Section 3.3 and already present in the classical theory.

An even more general treatment of the subject is given in the preprint version (see [24]) of this paper, where no smoothness conditions on the gauge functions w_j are imposed, except for being Borel-measurable and locally bounded.

Closely related moment problems for generalized polynomials on a semi-infinite interval in $\mathbb{R}$ and on $\mathbb{R}$ itself were considered by Kreĭn and Nudel′man [15, Chapter V]. Essentially, the method used there is compactification of the (semi-)infinite interval—which, however, requires additional restrictions on the limit behavior of certain generalized polynomials or their ratios near the infinite endpoint(s). No such additional restrictions are assumed in the present paper.

The paper is organized as follows. In Sections 2–4, we develop necessary, mostly quite novel, tools to provide a convenient description of the cone $\mathscr{F}^{k:n}_+$ of generalized monotone functions. These developments culminate in Theorem 4.6, according to which every function $f \in \mathscr{F}^{k:n}_+$ is approximated in a monotone manner by a function g_y, which is the sum of two summands: (i) a member of a certain set of generalized polynomials and (ii) a mixture of the "positive parts" (defined by (3.12)) of generalized polynomials, belonging to another set. This new approximative representation of the functions $f \in \mathscr{F}_{+}^{k:n}$ enables us to provide a description, in Section 5, of the cones dual to the cones $\mathscr{F}_{+}^{k:n}$, for any subinterval I of $\mathbb{R}$ and any gauge functions. This description of the dual cones is quite convenient in the desired applications and looks quite similar to the known descriptions of this kind, such as the mentioned ones in [11], [30], and [12], which latter were obtained assuming $a > -\infty$ and/or certain integrability conditions. However, without such additional conditions, quite substantial difficulties needed to be overcome. The close relations of our results with the Tchebycheff systems are discussed in Section 5.3. Applications are presented in Section 6.

2. Compositions of operators of gauged differentiation

Take any interval $I \subseteq \mathbb{R}$ of nonzero length; a particular possibility is that $I = \mathbb{R}$. Let

$$a := \inf I \qquad \text{and} \qquad b := \sup I,$$

so that

$$-\infty \le a < b \le \infty. \tag{2.1}$$

Let the ligature $\mathscr{RC} = \mathscr{RC}(I)$ (for “right-continuous”) denote the set of all functions in $\mathbb{R}^I$ that are (i) right-continuous on the interval $I \setminus \{b\}$ and (ii) left-continuous at the point b in the case when $b \in I$.

Next, introduce

$$\begin{aligned} \mathscr{D} := \mathscr{D}(I) := {} & \text{the set of all functions } f \in \mathbb{R}^I \text{ such that} \\ & \text{there is a function } Df \in \mathscr{RC} \text{ satisfying the condition} \\ & f(x) = f(z) + \int_z^x (Df)(u)\,\mathrm{d}u \quad \text{for all } x \text{ and } z \text{ in } I. \end{aligned} \tag{2.2}$$

Here and elsewhere, $\int_z^x := -\int_x^z$ if $x < z$. Any function $f \in \mathscr{D}$ is continuous and even absolutely continuous. Moreover, for any function $f \in \mathscr{D}$ and any point $x \in I$, one has the following: (i) if $x \neq b$, then the right derivative of f at x (exists and) necessarily equals $(Df)(x)$; (ii) the left derivative of f at b (exists and) necessarily equals $(Df)(b)$. Therefore, in view of the condition $Df \in \mathscr{RC}$, the “generalized derivative” function Df is uniquely determined for each $f \in \mathscr{D}$. Thus, one has the linear operator

$$D\colon \mathscr{D} \to \mathscr{RC}.$$

More generally, for each $j \in \overline{1,n}$, let $\mathscr{D}^j = \mathscr{D}^j(I)$ be the set of all $(j-1)$-times differentiable functions $f \in \mathbb{R}^I$ such that the $(j-1)$th derivative of f is in $\mathscr{D}$. In particular, $\mathscr{D}^1 = \mathscr{D}$. Let also $\mathscr{D}^0$ be the set of all Borel-measurable functions in $\mathbb{R}^I$.

For each $j \in \overline{0,n}$, let w_j be a (strictly) positive function of class C^{n-j} on the interval I; that is, $(D^{n-j}w_j)(x)$ is defined for all $x \in I$ and is continuous in $x \in I$. Let

$$\mathbf{w} := (w_0, \dots, w_n). \tag{2.3}$$

For each positive function w on I, define the linear operator R_w by the formula

$$R_w f := \frac{f}{w} \tag{2.4}$$

for all $f\colon I \to \mathbb{R}$.

For each $j \in \overline{0,n}$, let

$$D_j := D_{\mathbf{w},j} := DR_{w_j} \qquad \text{and} \qquad E_j := E_{\mathbf{w},j} := R_{w_j}D. \tag{2.5}$$

Further, let

$$D^0 := D^0_{\mathbf{w}} := \mathrm{id} \qquad \text{and} \qquad E^0 := E^0_{\mathbf{w}} := R_{w_0}, \tag{2.6}$$

where id denotes the identity operator: $\mathrm{id}f = f$ for any function f. Now, for all $j \in \overline{1,n}$ define the linear operators $D^j_{\mathbf{w}}$ and E^j recursively by the formulas

$$\begin{aligned} &D^j_{\mathbf{w}} := D_{j-1}D^{j-1}_{\mathbf{w}} = D_{j-1}\cdots D_0 = DR_{w_{j-1}}\cdots DR_{w_0} \qquad \text{and} \\ &E^j := E^j_{\mathbf{w}} := E_jE^{j-1} = E_j\cdots E_1E^0 = R_{w_j}D\cdots R_{w_1}DR_{w_0} = R_{w_j}D^j_{\mathbf{w}}. \end{aligned} \tag{2.7}$$

It is assumed here that the operators $D^j_{\mathbf{w}}$ and E^j have the same domain, which is the class $\mathscr{D}^j = \mathscr{D}^j(I)$, defined previously in this section.

We introduce now the notation

$$f^{(j)} := f^{(j)}_{\mathbf{w}} := E^j f = \frac{D^j_{\mathbf{w}} f}{w_j} \tag{2.8}$$

for $j \in \overline{0,n}$ and $f \in \mathscr{D}^j$. Note that

$$\begin{aligned} &f \in \mathscr{D}^n \\ &\quad\Longrightarrow \begin{cases} f^{(0)},\dots,f^{(n-1)} \text{ are absolutely continuous, } f^{(n)} \in \mathscr{RC}, \\ f = f^{(0)}w_0, \\ f^{(j)}(x) = f^{(j)}(z) + \int_z^x f^{(j+1)}(u)w_{j+1}(u)\,\mathrm{d}u \\ \quad \text{for all } j \in \overline{0,n-1}, x \in I, z \in I. \end{cases} \end{aligned} \tag{2.9}$$

Since w_j was assumed to be of class C^{n-j}, it follows that for $j \in \overline{0,n}$ and $f \in \mathscr{D}^j$ the function $f^{(j)}$ is in $\mathscr{RC}$; moreover, for $f \in \mathscr{D}^n$ and $j \in \overline{0,n-1}$ the function $f^{(j)}$ is continuous.

The functions w_j may be referred to as the *gauge* functions.

Concerning these functions, the simplest and most common case is when $w_j = 1$ for all j, which may be referred to as the *unit-gauge* case. In that case, for each $j \in \overline{0,n-1}$ each of the "gauged" higher-order derivatives $f^{(j)}$ and $D^j_{\mathbf{w}}f$ coincides with the usual jth derivative of f.

Remark 2.1. Instead of the class $\mathscr{RC}$ (defined right after (2.1)), one could use any other appropriate class of functions—as long as the condition that Df belongs to that other class together with the condition that $f(x) = f(z) + \int_z^x (Df)(u)\,\mathrm{d}u$ for all x and z in I (cf. (2.2)) uniquely determine the function Df. However, our particular choice of the class $\mathscr{RC}$ should be sufficient for most applications.

Moreover, the conditions on the gauge functions $w_0,\dots,w_n$ (imposed in the paragraph containing (2.3)) can be significantly relaxed. For example, in [24] it is only assumed that the gauges are locally bounded (strictly) positive Borel-measurable functions. Of course, then the definitions of the compositional differential operators $D^j_{\mathbf{w}}$ and $E^j_{\mathbf{w}}$ have to be correspondingly adjusted.

We conclude this section with a proposition that allows us to compare the values of two functions on an interval given a comparison between their gauged higher-order derivatives and the same "initial" conditions at a point of the interval. In the proof of that proposition and elsewhere, the following definition will be useful.

For $i \in \overline{0,n}$, let S^i denote the ith power of the left-shift operator (say, S), so that

$$\mathsf{S} = \mathsf{S}^1 \qquad \text{and} \qquad \mathsf{S}^i\mathbf{w} = \mathbf{v}, \quad \text{where } \mathbf{v} = (v_0, \dots, v_{n-i}) = (w_i, \dots, w_n). \tag{2.10}$$

Proposition 2.2. *Take any $z \in I$ and any $k \in \overline{0,n}$. Suppose that functions f and g in $\mathscr{D}^k$ are such that $f^{(j)}(z) = g^{(j)}(z)$ for all $j \in \overline{0,k-1}$. Then the inequality $f^{(k)} \ge g^{(k)}$ on $I \cap [z,\infty)$ implies that $f \ge g$ on $I \cap [z,\infty)$. Similarly, the inequality $f^{(k)} \ge g^{(k)}$ on $I \cap (-\infty,z]$ implies that $(-1)^k(f-g) \ge 0$ on $I \cap (-\infty,z]$.*

Proof. In view of the recursive definition of E^j in (2.7), this proof can be naturally done by induction in k. If $k = 0$, then in view of (2.8) and (2.6), there is nothing to prove. Suppose now that $k \in \overline{1,n}$. Assume that $f^{(k)} \ge g^{(k)}$ on $I \cap (-\infty,z]$. Without loss of generality, $g = 0$ (otherwise, replace f by $f - g$). In view of (2.8), (2.7), and (2.10), one has $f^{(j)} = h^{(j-1)}_{\mathsf{S}\mathbf{w}}$ for all $j \in \overline{1,k}$, where

$$h := Df^{(0)} = DR_{w_0}f.$$

The conditions

(i) $k \in \overline{1,n}$,
(ii) $f^{(j)}(z) = g^{(j)}(z)$ for all $j \in \overline{0,k-1}$,
(iii) $f^{(k)} \ge g^{(k)}$ on $I \cap (-\infty,z]$,
(iv) $g = 0$, and
(v) $f^{(j)} = h^{(j-1)}_{\mathsf{S}\mathbf{w}}$ for all $j \in \overline{1,k}$ imply

$$f^{(0)}(z) = 0, \tag{2.11}$$

$h^{(i)}_{\mathsf{S}\mathbf{w}}(z) = 0$ for all $i \in \overline{0,k-2}$, and $h^{(k-1)}_{\mathsf{S}\mathbf{w}} \ge 0$ on $I \cap (-\infty,z]$. So, by induction, $(-1)^{k-1}h \ge 0$ on $I \cap (-\infty,z]$. Therefore, in view of (2.11),

$$(-1)^k f^{(0)}(x) = -\int_z^x (-1)^{k-1} Df^{(0)}(u)\,\mathrm{d}u = -\int_z^x (-1)^{k-1} h(u)\,\mathrm{d}u \ge 0$$

for all $x \in I \cap (-\infty,z]$. Thus, the part of Proposition 2.2 concerning the interval $I \cap (-\infty,z]$ is proved. The part concerning the interval $I \cap [z,\infty)$ is proved quite similarly. □

3. w-POLYNOMIALS

Take any $k \in \overline{-1,n}$.

3.1. **w-Polynomials: Basic definitions.** If $k \ge 0$, let us say that a function p is a $\mathbf{w}$*-polynomial of degree* $\le k$ (on I) if $p \in \mathscr{D}^k$ and $p^{(k)}$ is a constant. Let us further say that the only $\mathbf{w}$-polynomial of degree ≤ -1 is the zero function on I. Denote the set of all $\mathbf{w}$-polynomials of degree $\le k$ by $\mathscr{P}^{\le k}$ or, in detailed notation, by $\mathscr{P}^{\le k}_{\mathbf{w}}$. In particular, $\mathscr{P}^{\le -1} = \{0\}$. Let

$$\mathscr{P}^{\le k}_{+} := \mathscr{P}^{\le k}_{\mathbf{w};+} := \{p \in \mathscr{P}^{\le k} \colon p^{(k)} \ge 0\}. \tag{3.1}$$

In particular, $\mathscr{P}^{\le -1}_{+} = \mathscr{P}^{\le -1} = \{0\}$, $\mathscr{P}^{\le 0} = \{cw_0 \colon c \in \mathbb{R}\}$, and $\mathscr{P}^{\le 0}_{+} = \{cw_0 \colon c \in [0, \infty)\}$. We then define the set of all $\mathbf{w}$-polynomials of degree k as

$$\mathscr{P}^{k} := \mathscr{P}^{k}_{\mathbf{w}} := \mathscr{P}^{\le k} \setminus \mathscr{P}^{\le k-1} \quad \text{for } k \in \overline{0,n}, \text{with } \mathscr{P}^{-1} := \mathscr{P}^{\le -1} = \{0\}.$$

So, for any $k \in \overline{0,n}$,

$$\mathscr{P}^{k} = \{p \in \mathscr{D}^{k} \colon p \in \mathscr{D}^{k} \text{ and } p^{(k)} \text{ is a nonzero constant}\}.$$

In the unit-gauge case, the sets $\mathscr{P}^{k}$ and $\mathscr{P}^{\le k}$ coincide with the sets of usual polynomial functions on I of degree k and of degree $\le k$, respectively.

3.2. **w-Polynomials: An interpolation/tangency property.** The following interpolation/tangency property of the $\mathbf{w}$-polynomials is an extension of the corresponding property of the usual polynomials.

Proposition 3.1. *For each $z \in I$ and each $(c_0, \dots, c_k) \in \mathbb{R}^{k+1}$, there is a unique $\mathbf{w}$-polynomial $p \in \mathscr{P}^{\le k}$ such that $p^{(j)}(z) = c_j$ for all $j \in \overline{0,k}$; moreover, this $\mathbf{w}$-polynomial p is continuous.*

Remark 3.2. In particular, Proposition 3.1 implies that any $\mathbf{w}$-polynomial is continuous—because, obviously, for any $p \in \mathscr{P}^{\le k}$ and any $z \in I$ there is some finite sequence $(c_0, \dots, c_k) \in \mathbb{R}^{k+1}$ such that $p^{(j)}(z) = c_j$ for all $j \in \overline{0,k}$.

Proof of Proposition 3.1. The proof is naturally done by induction in k. If $k = -1$, there is almost nothing to prove, because then the set $\overline{0,k}$ is empty and the set $\mathscr{P}^{\le k}_{\mathbf{w}}$ is a singleton one, consisting of just one $\mathbf{w}$-polynomial, 0, which is obviously continuous. Suppose now that $k \in \overline{0,n}$. Then (cf. the proof of Proposition 2.2), the condition that $p \in \mathscr{P}^{\le k}_{\mathbf{w}}$ & $p^{(j)}_{\mathbf{w}}(z) = c_j$ for all $j \in \overline{0,k}$ can be rewritten as $p(z) = c_0 w_0(z)$ & $q \in \mathscr{P}^{\le k-1}_{\mathsf{S}\mathbf{w}}$ & $q^{(i)}_{\mathsf{S}\mathbf{w}}(z) = c_{i+1}$ for all $i \in \overline{0,k-1}$, where $q := Dp^{(0)} = DR_{w_0}p$. By induction, the condition $q^{(i)}_{\mathsf{S}\mathbf{w}}(z) = c_{i+1}$ for all $i \in \overline{0,k-1}$ determines a unique $\mathsf{S}\mathbf{w}$-polynomial $q \in \mathscr{P}^{\le k-1}_{\mathsf{S}\mathbf{w}}$, and this q is continuous. It remains to note that the conditions $DR_{w_0}p = q$ and $p(z) = c_0 w_0(z)$ imply that $p(x) = w_0(x)(c_0 + \int_z^x q(u)\,\mathrm{d}u)$ for all $x \in I$ and thus determine a unique $p \in \mathscr{P}^{\le k}_{\mathbf{w}}$; moreover, this p is continuous (since $c_0 + \int_z^x q(u)\,\mathrm{d}u$ is continuous in x). □

3.3. **A chain of w-polynomials vanishing at a point.** Take any

$$t \in \{a\} \cup I \setminus \{b\} = [a, b). \tag{3.2}$$

For j and m in $\overline{0,n}$ such that $j \le m$, define the functions $p_{t;j,m} \colon I \to (-\infty, \infty]$ recursively by the conditions

$$\begin{aligned} &p_{t;m,m}(x_m) = w_m(x_m) \quad \text{for all } x_m \in I; \\ &p_{t;j,m}(x_j) = w_j(x_j) \int_{t+}^{x_j} \mathrm{d}x_{j+1} p_{t;j+1,m}(x_{j+1}) \quad \text{for all } x_j \in I \text{ if } j < m. \end{aligned} \tag{3.3}$$

In the case when, for a given triple (t,j,m), one has $p_{t;j,m}(x_j)\in\mathbb{R}$ for all $x_j\in I$, let us identify $p_{t;j,m}$ with the function whose graph is the same as that of $p_{t;j,m}$ but the codomain is $\mathbb{R}$.

Consider first the case when $t\in I$. Then, by induction and the continuity of the functions $w_0,\dots,w_n$, the functions $p_{t;j,m}$ are real-valued and continuous, so that

$$p_{t;j,m}\in\mathscr{RC}. \tag{3.4}$$

Furthermore, by (2.5) and (2.2),

$$p_{t;j+1,m}=D_jp_{t;j,m}\qquad\text{and}\qquad p_{t;j,m}(t)=0\quad\text{if } j<m. \tag{3.5}$$

Hence, by (2.8) and (2.7), $p_{t;j,m}$ is an $\mathsf{S}^j\mathbf{w}$-polynomial of degree $m-j$, satisfying the conditions

$$(p_{t;j,m})^{(i)}_{\mathsf{S}^j\mathbf{w}}(t)=0\quad\text{for all } i\in\overline{0,m-j-1}\qquad\text{and}\qquad(p_{t;j,m})^{(m-j)}_{\mathsf{S}^j\mathbf{w}}=1; \tag{3.6}$$

by Proposition 3.1, such a polynomial is unique. It follows from (3.6) that

$$(p_{t;j,m})^{(i)}_{\mathsf{S}^j\mathbf{w}}(t)=\mathrm{I}\{i=m-j\}\quad\text{for all } i\in\overline{0,n},$$

where $\mathrm{I}\{\cdot\}$ denotes the indicator function. So, again by Proposition 3.1, for each $k\in\overline{j,n}$, the $\mathsf{S}^j\mathbf{w}$-polynomials $p_{t;j,j},\dots,p_{t;j,k}$ form a basis of the linear space $\mathscr{P}^{\le k-j}_{\mathsf{S}^j\mathbf{w}}$. More specifically, each $\mathsf{S}^j\mathbf{w}$-polynomial p of degree $k-j$ can be uniquely represented by a linear combination of the basis $\mathsf{S}^j\mathbf{w}$-polynomials $p_{t;j,j},\dots,p_{t;j,k}$, as follows:

$$p=\sum_{i=0}^{k-j}p^{(i)}_{\mathsf{S}^j\mathbf{w}}(t)p_{t;j,j+i}. \tag{3.7}$$

Consider now the remaining case $t\notin I$, so that, by the condition (3.2), $t=a$ and $a\notin I$. Then, since $w_i>0$ for all $i\in\overline{0,n}$, the function $p_{a;j,m}$ is strictly positive on I but may take the value ∞ at some point of the interval I; in such a case, it is easy to see that $p_{a;j,m}=\infty$ everywhere on I. In fact, for each pair $(j,m)\in\overline{0,n}^2$ such that $j\le m$, one has the following dichotomy: either (i) $p_{a;j,m}=\infty$ everywhere on I or (ii) $p_{a;j,m}$ is in $\mathscr{P}^{\le k-j}_{\mathsf{S}^j\mathbf{w}}$ and hence finite and continuous, so that (3.4) holds in the latter case. Introduce the following "finiteness" sets for the functions $p_{a;j,m}$:

$$\begin{aligned}F:=F_{\mathbf{w}}&:=\big\{(j,m)\in\overline{0,n}^2\colon j\le m\ \&\ p_{a;j,m}<\infty\big\},\\ F_{\bullet m}:=F_{\mathbf{w};\bullet m}&:=\big\{j\colon (j,m)\in F\big\}=\{j\in\overline{0,m}\colon p_{a;j,m}<\infty\},\\ F_{j,n}:=F_{\mathbf{w};j,n}&:=\big\{m\colon (j,m)\in F\big\}=\{m\in\overline{j,n}\colon p_{a;j,m}<\infty\}.\end{aligned} \tag{3.8}$$

In view of (3.3),

$$F_{\bullet m}=\overline{j_m,m}\quad\text{for some } j_m\in\overline{0,m}. \tag{3.9}$$

In particular, $m\in F_{\bullet m}$ and hence $F_{\bullet m}\ne\emptyset$. Similarly, $j\in F_{j,n}$ and hence $F_{j,n}\ne\emptyset$.

In the unit-gauge case, for $j < m$ and $x_j \in I$,

$$p_{t;j,m}(x_j) = \frac{(x_j - t)^{m-j}}{(m-j)!} \tag{3.10}$$

if $t \neq -\infty$, and $p_{-\infty;j,m} = \infty$; also, $p_{t;m,m} = 1$.

Remark 3.3. In view of (3.10), the polynomials $p_{t;0,i}$ (of degree i with $i \in \overline{0,n}$) defined in accordance with (3.3) may be referred to as the *canonical* $(\mathbf{w};t)$*-monomials.*

One may refer to the set $\overline{j,n} \setminus F_{j,n}$ as the set of "missing" degrees m of canonical $(\mathbf{w};a)$-monomials $p_{a;j,m}$, for $m \in \overline{j,n}$. The following proposition states that, if $a \notin I$, then the set of "missing" degrees can be any subset of the set $\overline{j+1,n}$, depending on the choice of the gauge functions $w_0, \dots, w_n$.

Proposition 3.4. *Suppose that $a \notin I$. Then, for any $j \in \overline{0,n}$ and any given set $M \subseteq \overline{j+1,n}$, one can construct a sequence $\mathbf{w} = (w_0, \dots, w_n)$ of real-valued continuous functions on I such that the missing-degrees set $\overline{j,n} \setminus F_{\mathbf{w};j,n}$ coincides with M.*

Proof. In the case when $a = -\infty$, for each $j \in \overline{0,n}$ take some $\lambda_j \in \mathbb{R}$, and let $w_j(x_j) := \exp(\lambda_j x_j)$ for all $x_j \in I$. Then it is not hard to verify by induction in $m - j$ that, for all j and m in $\overline{0,n}$ such that $j < m$ and for all $x_j \in I$,

$$\begin{aligned} &p_{a;j,m}(x_j) \\ &\quad = \begin{cases} \exp\{(\sum_{i \in \overline{j,m}} \lambda_i) x_j\} / \prod_{i \in \overline{j,m-1}} \sum_{s \in \overline{i+1,m}} \lambda_s & \text{if } \lambda_m + \Lambda_{j,m} > 0, \\ \infty & \text{otherwise,} \end{cases} \end{aligned} \tag{3.11}$$

where

$$\Lambda_{j,m} := \min_{i \in \overline{j,m-1}} \sum_{s \in \overline{i+1,m-1}} \lambda_s,$$

with the usual convention that the sum of an empty family is 0.

The case when $a > -\infty$ (and hence $a \in \mathbb{R} \setminus I$) can be considered quite similarly. In this case, one may let $w_j(x_j) := (x_j - a)^{\lambda_j - 1}$ for all $x_j \in I$. Then (3.11) holds for all $x_j \in I$ if the exponent $(\sum_{i \in \overline{j,m}} \lambda_i) x_j$ therein is replaced by $(-1 + \sum_{i \in \overline{j,m}} \lambda_i) \ln(x_j - a)$.

So, in either case, whether $a = -\infty$ or $a > -\infty$, for the corresponding constructed sequence $\mathbf{w}$ and any $m \in \overline{j+1,n}$ one has $m \in F_{\mathbf{w};j,n} \setminus \{j\}$ if and only if $\lambda_m + \Lambda_{j,m} > 0$. Since for any given j and m the real number $\Lambda_{j,m}$ depends only on $(\lambda_s)_{s \in \overline{j+1,m-1}}$, one can choose λ_m recursively in $m \in \overline{j+1,n}$ so that the finiteness condition $\lambda_m + \Lambda_{j,m} > 0$ in (3.11) be satisfied if and only if m is not in the prescribed subset M of the set $\overline{j+1,n}$. $\square$

The definitions of F, $F_{\bullet m}$, $F_{j,n}$, and j_m by formulas (3.8) and (3.9) continue to make sense even when $a \in I$, and

$$a \in I \implies F = \left\{(j,m) \in \overline{0,n}^2 \colon j \le m\right\}, \quad F_{\bullet m} = \overline{0,m}, F_{j,n} = \overline{j,n}, j_m = 0.$$

Note also that, for each $j \in F_{\bullet m} = \overline{j_m, m}$, the function $p_{a;j,m}$ is an (everywhere positive) $\mathsf{S}^j\mathbf{w}$-polynomial of degree $\le m-j$, whether $a \in I$ or not.

In the unit-gauge case, for all j and m in $\overline{0,n}$ one has (i) $F_{\bullet m} = \overline{0,m}$ and $F_{j,n} = \overline{j,n}$ if $a > -\infty$, and (ii) $F_{\bullet m} = \{m\}$ and $F_{j,n} = \{j\}$ if $a = -\infty$.

Again for j and m in $\overline{0,n}$ such that $j \le m$, define the "positive parts" of the functions $p_{t;j,m}$ by the formula

$$p^+_{t;j,m}(x_j) := p_{t;j,m}(x_j)\mathrm{I}\{x_j \ge t\} \tag{3.12}$$

for all $x_j \in I$. Here and subsequently, the convention

$$\infty \cdot 0 = 0 \cdot \infty = 0 \tag{3.13}$$

is used.

By (3.12), (3.3), and the positivity of the w_j's,

$$p^+_{t;j,m} \ge 0 \tag{3.14}$$

for all j and m in $\overline{0,n}$ such that $j \le m$. Also, recall the definition of the class $\mathscr{RC}$ right after (2.1) and note that, by (3.4),

$$p^+_{t;j,m} \in \mathscr{RC} \quad \text{for } (j,m) \in F. \tag{3.15}$$

Moreover, one has the following.

Lemma 3.5. *Suppose that $j < m$ and $(j,m) \in F$. Then*

$$p^+_{t;j+1,m} = D_j p^+_{t;j,m}. \tag{3.16}$$

Proof. Take any x_j and z in I. In view of (2.5), (2.2), (3.12), and (3.15), it is enough to show that

$$\begin{aligned} \frac{p_{t;j,m}(x_j)}{w_j(x_j)}\mathrm{I}\{x_j \ge t\} = {} & \frac{p_{t;j,m}(z)}{w_j(z)}\mathrm{I}\{z \ge t\} \\ & + \int_z^{x_j} \mathrm{d}x_{j+1} p_{t;j+1,m}(x_{j+1})\mathrm{I}\{x_{j+1} \ge t\}. \end{aligned} \tag{3.17}$$

In the case when $x_j \ge t$ and $z \ge t$, (3.17) follows by the first equality in (3.5).

In the case when $x_j \ge t$ and $z < t$, the integral in (3.17) equals $\int_t^{x_j} \mathrm{d}x_{j+1} \times p_{t;j+1,m}(x_{j+1})$, and so, (3.17) follows by the two equalities in (3.5).

The case of $x_j < t$ and $z \ge t$ is quite similar to that of $x_j \ge t$ and $z < t$, as the roles of x_j and z are interchangeable.

In the case when $x_j < t$ and $z < t$, (3.17) is obvious, as each of the three indicators in (3.17) equals 0. ☐

In the unit-gauge case, for $j < m$ and $x_j \in I$,

$$p^+_{t;j,m}(x_j) = \frac{(x_j - t)_+^{m-j}}{(m-j)!} \tag{3.18}$$

if $t \ne -\infty$, and $p^+_{-\infty;j,m} = \infty$; also, $p^+_{t;m,m}(x_m) = \mathrm{I}\{x_m \ge t\}$ for $x_m \in I$.

3.4. **Another chain of w-polynomials vanishing at a point.** Fix an arbitrary

$$z \in (a,b) \tag{3.19}$$

and recall (3.8).

Take any $(k,j) \in F$ and $i \in \overline{0,k}$, and define the functions $p_{a,z;i:k:j} \colon I \to \mathbb{R}$ by the conditions

$$p_{a,z;k:k:j}(x_k) = p_{a;k,j}(x_k) \quad \text{for all } x_k \in I; \tag{3.20}$$

$$p_{a,z;i:k:j}(x_i) = w_i(x_i) \int_z^{x_i} \mathrm{d}x_{i+1} p_{a,z;i+1:k:j}(x_{i+1}) \quad \text{for all } x_i \in I \text{ if } i < k; \tag{3.21}$$

then $p_{a,z;i:k:j} \in \mathscr{P}^{j-i}_{\mathsf{S}^i\mathbf{w}}$.

Indeed, by (3.8), (3.20), and (3.21), the functions $p_{a,z;i:k:j}$ are nonnegative and finite. Also, by (2.5) and (2.2),

$$D_i p_{a,z;i:k:j} = p_{a,z;i+1:k:j} \quad \text{if } i < k. \tag{3.22}$$

Hence, $D^{k-i}_{\mathsf{S}^i\mathbf{w}} p_{a,z;i:k:j} = p_{a,z;k:k:j} = p_{a;k,j}$. Moreover, (cf. (3.6)) $(p_{a;k,j})^{(j-k)}_{\mathsf{S}^k\mathbf{w}} = 1$. Hence, $(p_{a;k,j})^{(j-k+1)}_{\mathsf{S}^k\mathbf{w}} = 0$,

$$(p_{a,z;i:k:j})^{(j-i)}_{\mathsf{S}^i\mathbf{w}} = 1, \tag{3.23}$$

and $(p_{a,z;i:k:j})^{(j-i+1)}_{\mathsf{S}^i\mathbf{w}} = 0$, which indeed yields

$$p_{a,z;i:k:j} \in \mathscr{P}^{j-i}_{\mathsf{S}^i\mathbf{w}}. \tag{3.24}$$

In the unit-gauge case, for all $i \in \overline{0,k}$ and $x_i \in I$,

$$p_{a,z;i:k:j}(x_i) = \frac{1}{(j-i)!}\Big[(x_i-a)^{j-i} - \sum_{\gamma=0}^{k-i-1} \binom{j-i}{\gamma}(z-a)^{j-i-\gamma}(x_i-z)^\gamma\Big] \tag{3.25}$$

if $a > -\infty$ and $j \in \overline{k,\infty}$, and

$$p_{a,z;i:k:k}(x_i) = \frac{(x_i-z)^{k-i}}{(k-i)!} \tag{3.26}$$

whether $a = -\infty$ or $a > -\infty$. Recall here that the generalized polynomials $p_{a,z;i:k:j}$ were defined for $(k,j) \in F$ and $i \in \overline{0,k}$; recall also that, in the unit-gauge case, $F = \{(j,m) \in \overline{0,n}^2 \colon j \le m\}$ if $a > -\infty$ and $F = \{(m,m) \colon m \in \overline{0,n}\}$ if $a = -\infty$. Besides, in the case when $a > -\infty$,

$$p_{a,z;i:k:j}(x_i) \underset{z\downarrow a}{\longrightarrow} \frac{(x_i-a)^{j-i}}{(j-i)!}$$

for all $i \in \overline{0,k}$ and $x_i \in I$.

4. Convex cones of generalized multiply monotone functions

4.1. **Convex cones $\mathscr{H}_+^{i:n}$ of generalized multiply monotone functions.** Let M_+ denote the set of all nonnegative measures μ defined on the Borel σ-algebra over I such that $\mu(I\cap\{b\})=0$.

For $j\in\overline{0,n}$, $\mu\in\mathrm{M}_+$, and $x\in I$, let

$$h_{j;\mu}(x):=h_{j:n;\mu}(x):=\int_I\mu(\mathrm{d}t)p^+_{t;j,n}(x), \tag{4.1}$$

so that $h_{j;\mu}(x)\in[0,\infty]$. Note that, if $\mu\in\mathrm{M}_+$ is such that $h_{j;\mu}(x)<\infty$ for all $x\in I$, then one has a function $h_{j;\mu}\colon I\to\mathbb{R}$.

For each $i\in\overline{0,n}$, let $\mathscr{H}_+^{i:n}$ denote the set of all functions $h\colon I\to\mathbb{R}$ such that (i) $h(x)=h_{i;\mu}(x)$ for some $\mu\in\mathrm{M}_+$ and all $x\in I$ and (ii) $h_{j;\mu}(x)<\infty$ for all $j\in\overline{i,n}$ and all $x\in I$.

Lemma 4.1. *Take any $i\in\overline{0,n}$ and any $\mu\in\mathrm{M}_+$ such that $h_{i;\mu}\in\mathscr{H}_+^{i:n}$. Then $h_{i;\mu}\in\mathscr{RC}$. Also, the function $h_{i;\mu}/w_i$ is nondecreasing. Moreover, if $i\in\overline{0,n-1}$, then $h_{i;\mu}\in\mathscr{D}$ (recall (2.2)) and*

$$D_ih_{i;\mu}=h_{i+1;\mu}. \tag{4.2}$$

Proof. Take any $t\in I\setminus\{b\}$. By (3.3), the function $p_{t;i,n}/w_i$ is nonnegative and nondecreasing on the interval $I\cap[t,\infty)$. So, by (3.12), the function $p^+_{t;i,n}/w_i$ is nonnegative and nondecreasing on the interval I. Moreover, by (3.4), (3.12), and the condition $t\in I\setminus\{b\}$, one has $p^+_{t;i,n}\in\mathscr{RC}$. Next,

$$\frac{h_{i;\mu}(x_i)}{w_i(x_i)}=\int_I\mu(\mathrm{d}t)\frac{p^+_{t;i,n}(x_i)}{w_i(x_i)}=\int_{I\setminus\{b\}}\mu(\mathrm{d}t)\frac{p^+_{t;i,n}(x_i)}{w_i(x_i)}$$

for all $x_i\in I$, because $\mu\in\mathrm{M}_+$ and hence $\mu(I\cap\{b\})=0$. So, by dominated convergence, the condition $p^+_{t;i,n}\in\mathscr{RC}$ for $t\in I\setminus\{b\}$ implies $h_{i;\mu}\in\mathscr{RC}$. It also follows that $h_{i;\mu}/w_i$ is nondecreasing.

Now suppose that $i\in\overline{0,n-1}$. Then $h_{i+1;\mu}\in\mathscr{H}_+^{i+1:n}$, and so, $h_{i+1;\mu}\in\mathscr{RC}$. Also, by (4.1) and (3.17), for any x_i and z in I,

$$\begin{aligned}\int_z^{x_i}\mathrm{d}x_{i+1}h_{i+1;\mu}(x_{i+1})&=\int_z^{x_i}\mathrm{d}x_{i+1}\int_I\mu(\mathrm{d}t)p^+_{t;i+1,n}(x_{i+1})\\&=\int_I\mu(\mathrm{d}t)\int_z^{x_i}\mathrm{d}x_{i+1}p^+_{t;i+1,n}(x_{i+1})\\&=\int_I\mu(\mathrm{d}t)\Big(\frac{p^+_{t;i,n}(x_i)}{w_i(x_i)}-\frac{p^+_{t;i,n}(z)}{w_i(z)}\Big)\\&=\frac{h_{i;\mu}(x_i)}{w_i(x_i)}-\frac{h_{i;\mu}(z)}{w_i(z)}.\end{aligned}$$

In view of (2.5) and (2.2), this verifies (4.2) and thus completes the proof of Lemma 4.1. $\square$

4.2. **Convex cones $\mathscr{F}_+^{k:n}$ of generalized multiply monotone functions.** Recall condition (1.2). Recall also (2.8) and introduce the class of functions

$$\begin{aligned} \mathscr{F}_+^{k:n} &:= \mathscr{F}_+^{k:n}(I) \\ &:= \{f \in \mathscr{D}^n \colon f^{(j)} \text{ is nondecreasing for each } j \in \overline{k-1,n}\}. \end{aligned} \tag{4.3}$$

Clearly, $\mathscr{F}_+^{k:n}$ is a convex cone. For instance, in the unit-gauge case $\mathscr{F}_+^{1:0}$ is the cone of all nondecreasing functions in $\mathbb{R}^I$, $\mathscr{F}_+^{1:1}$ is the cone of all continuous nondecreasing convex functions in $\mathbb{R}^I$, and $\mathscr{F}_+^{2:1}$ is the cone of all continuous convex functions in $\mathbb{R}^I$. Also in the unit-gauge setting, special cases of the cones $\mathscr{F}_+^{k:n}$ (or similar to them) and cones in a sense dual to those cones were considered, more or less explicitly, in a number of papers, including the following: [9], [27], and [19] for $(k,n)=(4,3)$; [2] and [3] (dealing with the cone $\mathscr{H}_+^{0:2}$; cf. Proposition 4.2 in the present paper, below); [20] (dealing with the cone $\mathscr{H}_+^{0:5}$; cf. Theorem 6.1 in the present paper); [14] for $(k,n)=(2,2)$; [21] for $(k,n)=(1,3)$; [23] for $n\in\{2,3\}$ and $k\in\overline{1,n}$; [25] for $(k,n)\in\{(1,2),(1,3)\}$; [22] (dealing with the cone $\mathscr{H}_+^{0:3}$).

Proposition 4.2. $\mathscr{H}_+^{0:n} \subseteq \mathscr{F}_+^{k:n}$.

Proof. Take any $h \in \mathscr{H}_+^{0:n}$, so that $h = h_{0;\mu}$ for some $\mu \in \mathrm{M}_+$. By (4.1) and (3.14), $h_{i;\mu} \ge 0$ for each $i \in \overline{0,n}$. So, by (2.8) and (4.2), $h_{0;\mu}^{(i)} = (D_{\mathbf{w}}^i h_{0;\mu})/w_i = h_{i;\mu}/w_i \ge 0$ for each $i \in \overline{0,n}$. Hence, $h_{0;\mu}^{(i)}$ is nondecreasing for each $i \in \overline{0,n-1}$. Also, in view of (4.1), (3.12), and (3.3), for each $x \in I$,

$$h_{0;\mu}^{(n)}(x) = \frac{h_{n;\mu}(x)}{w_n(x)} = \frac{1}{w_n(x)} \int_I \mu(\mathrm{d}t) p_{t;n,n}^+(x) = \mu\big(I \cap (-\infty, x]\big),$$

which is nondecreasing in x. Thus, by (4.3), $h = h_{0;\mu} \in \mathscr{F}_+^{k:n}$. $\square$

Important bounding properties for the functions in the class $\mathscr{F}_+^{k:n}$ are given by the following.

Proposition 4.3. *Take any $f \in \mathscr{F}_+^{k:n}$ and any $z \in I$.*

(I) *There exists a $\mathbf{w}$-polynomial $p \in \mathscr{P}^{\le k-1}$ such that*
  (i) *if k is even, then $f \ge p$ on the interval I;*
  (ii) *if k is odd, then*
    (*) *$f \ge p$ on the interval $I \cap [z, \infty)$;*
    (**) *$f \le p$ on the interval $I \cap (-\infty, z]$.*

  Moreover, one may assume that this $\mathbf{w}$-polynomial p depends on f and z only via the values of $f^{(0)}(z), \dots, f^{(k-1)}(z)$.

(II) *If $k \le n$, then there exists a $\mathbf{w}$-polynomial $q \in \mathscr{P}_+^{\le k}$ such that*
  (i) *if k is odd, then $f \ge q$ on the interval I;*
  (ii) *if k is even, then*
    (*) *$f \ge q$ on the interval $I \cap [z, \infty)$;*
    (**) *$f \le q$ on the interval $I \cap (-\infty, z]$.*

  Moreover, one may assume that this $\mathbf{w}$-polynomial q depends on f and z only via the values of $f^{(0)}(z), \dots, f^{(k)}(z)$.

Proof. By Proposition 3.1, there exists a unique $\mathbf{w}$-polynomial $p \in \mathscr{P}^{\le k-1}$ such that $p^{(i)}(z) = f^{(i)}(z)$ for all $i \in \overline{0, k-1}$; moreover, then the condition $p \in \mathscr{P}^{\le k-1}$ implies that $p^{(k-1)}(x) = p^{(k-1)}(z) = f^{(k-1)}(z)$ for all $x \in I$. On the other hand, the condition $f \in \mathscr{F}_+^{k:n}$ implies that the function $f^{(k-1)}$ is nondecreasing. Therefore, $f^{(k-1)} \ge p^{(k-1)}$ on the interval $I \cap [z, \infty)$, and $f^{(k-1)} \le p^{(k-1)}$ on the interval $I \cap (-\infty, z]$. To complete the proof of Proposition 4.3(I), it remains to recall Proposition 2.2.

Part (II) of Proposition 4.3 is proved similarly, by letting q be the unique $\mathbf{w}$-polynomial in $\mathscr{P}^{\le k}$ such that $p^{(i)}(z) = f^{(i)}(z)$ for all $i \in \overline{0, k}$. Here the additional condition $k \le n$ (together with the condition $f \in \mathscr{F}_+^{k:n}$) implies, in view of (2.8), that the function $f^{(k)}$ is nonnegative, and so, $q^{(k)}(x) = q^{(k)}(z) = f^{(k)}(z) \ge 0$ for all $x \in I$. Therefore and because $q \in \mathscr{P}^{\le k}$, it follows that $q \in \mathscr{P}_+^{\le k}$. Moreover, since $f^{(k)}$ is nondecreasing, it follows that $f^{(k)} \ge q^{(k)}$ on the interval $I \cap [z, \infty)$, and $f^{(k)} \le q^{(k)}$ on the interval $I \cap (-\infty, z]$. □

4.3. **Generalized Taylor expansion at the left endpoint a of the interval I of the generalized derivatives $f^{(j)}$ for $j \in \overline{k, n}$ of a function f in $\mathscr{F}_+^{k:n}$.** Take any $f \in \mathscr{F}_+^{k:n}$. It follows by (2.8) that

$$\text{for each } j \in \overline{k, n}, \quad \begin{cases} f^{(j)} \text{ is nonnegative and nondecreasing, and so,} \\ \text{there exists a limit } f^{(j)}(a+) \in [0, \infty). \end{cases} \tag{4.4}$$

Now one can state the following generalized Taylor expansion.

Lemma 4.4. *For all $j \in \overline{k, n}$*

$$\begin{gathered} f^{(j)} w_j = p_j + h_j, \quad \textit{where} \\ p_j := \sum_{i \in \overline{j, n}} f^{(i)}(a+) p_{a;j,i} \textit{ and } h_j := \int_I \mathrm{d} f^{(n)}(t) p^+_{t;j,n}. \end{gathered} \tag{4.5}$$

The integral in (4.5) is understood in the "pointwise" sense, so that $h_j(x_j) = \int_I \mathrm{d} f^{(n)}(t) p^+_{t;j,n}(x_j)$ for all $x_j \in I$; the latter integral exists (in $[0, \infty]$), since $p^+_{t;j,n} \ge 0$ and the function $f^{(n)}$ is nondecreasing.

Proof of Lemma 4.4. This is done by downward induction in j, starting with $j = n$. Indeed, by the definitions of p_j in (4.5) and of $p_{t;j,m}$ in (3.3),

$$p_n = f^{(n)}(a+) w_n.$$

By the definitions of h_j in (4.5) and of $p^+_{t;j,m}$ in (3.12), for all $x_n \in I$,

$$h_n(x_n) = \int_I \mathrm{d} f^{(n)}(t) w_n(x_n) \mathrm{I}\{x_n \ge t\} = \big(f^{(n)}(x_n) - f^{(n)}(a+)\big) w_n(x_n);$$

here we also used the fact that $f^{(n)} \in \mathscr{RC}$, which was noted in the paragraph following (2.8). So, the equality $f^{(j)} w_j = p_j + h_j$ holds for $j = n$. Suppose now this equality holds for some $j \in \overline{k+1, n}$. It remains to show then that this equality holds with $j-1$ instead of j. By (2.8) and the induction assumption, for all $x_{j-1}\in I$,

$$\begin{aligned} f^{(j-1)}(x_{j-1})-f^{(j-1)}(a+)&=\int_{a+}^{x_{j-1}}\mathrm{d}x_j f^{(j)}(x_j)w_j(x_j)\\ &=J_1(x_{j-1})+J_2(x_{j-1}), \end{aligned}\tag{4.6}$$

where

$$\begin{aligned} J_1(x_{j-1})&:=\int_{a+}^{x_{j-1}}\mathrm{d}x_j p_j(x_j)\\ &=\sum_{i\in\overline{j,n}}f^{(i)}(a+)\int_{a+}^{x_{j-1}}\mathrm{d}x_j p_{a;j,i}(x_j)\\ &=\sum_{i\in\overline{j,n}}f^{(i)}(a+)\frac{p_{a;j-1,i}(x_{j-1})}{w_{j-1}(x_{j-1})}=\frac{p_{j-1}(x_{j-1})}{w_{j-1}(x_{j-1})}-f^{(j-1)}(a+) \end{aligned}\tag{4.7}$$

by (4.5) and (3.3), whereas

$$\begin{aligned} J_2(x_{j-1})&:=\int_{a+}^{x_{j-1}}\mathrm{d}x_j h_j(x_j)\\ &=\int_{a+}^{x_{j-1}}\mathrm{d}x_j\int_I \mathrm{d}f^{(n)}(t)p^{+}_{t;j,n}(x_j)\\ &=\int_I \mathrm{d}f^{(n)}(t)\int_{a+}^{x_{j-1}}\mathrm{d}x_j p_{t;j,n}(x_j)\mathrm{I}\{x_j\ge t\}\\ &=\int_I \mathrm{d}f^{(n)}(t)\int_{t+}^{x_{j-1}}\mathrm{d}x_j p_{t;j,n}(x_j)\mathrm{I}\{x_{j-1}\ge t\}\\ &=\int_I \mathrm{d}f^{(n)}(t)\frac{p_{t;j-1,n}(x_{j-1})}{w_{j-1}(x_{j-1})}\mathrm{I}\{x_{j-1}\ge t\}\\ &=\int_I \mathrm{d}f^{(n)}(t)\frac{p^{+}_{t;j-1,n}(x_{j-1})}{w_{j-1}(x_{j-1})}=\frac{h_{j-1}(x_{j-1})}{w_{j-1}(x_{j-1})} \end{aligned}\tag{4.8}$$

by (4.5), (3.14), the Fubini theorem, (3.12), (3.3), again (3.12), and again (4.5). Now (4.6), (4.7), and (4.8) indeed yield $f^{(j-1)}w^{(j-1)}=p^{(j-1)}+h^{(j-1)}$. $\square$

Take any $j\in\overline{k,n}$, as in Lemma 4.4. Since $p_{a;j,i}\ge 0$ for $i\in\overline{j,n}$, it follows from Lemma 4.4 and (4.4) that the values of the function h_j are all in $[0,\infty)$, that is, are finite and nonnegative. It also follows, in view of (3.8), that necessarily

$$f^{(i)}(a+)=0 \quad \text{for all } i\in\overline{j,n}\setminus F_{j,n}. \tag{4.9}$$

Moreover, in view of (4.5) and (3.13),

$$p_j=\sum_{i\in F_{j,n}}f^{(i)}(a+)p_{a;j,i}. \tag{4.10}$$

In the unit-gauge case,

$$F_{i,n} = \begin{cases} \overline{i,n} & \text{if } a > -\infty \text{ and } i \le n, \\ \{i\} & \text{if } a = -\infty \text{ and } i \le n, \\ \emptyset & \text{if } i > n, \end{cases} \tag{4.11}$$

and (4.5) becomes the almost usual Taylor expansion (of the function $f^{(j)}$ "at the point $a+$") given by the formula

$$f^{(j)}(x_j) = \sum_{i\in\overline{j,n}} f^{(i)}(a+)\frac{(x_j-a)^{i-j}}{(i-j)!} + \int_I \mathrm{d}f^{(n)}(t)\frac{(x_j-t)_+^{n-j}}{(n-j)!} \tag{4.12}$$

for $j \in \overline{k,n}$ and $x_j \in I$. Here in the case when $a = -\infty$ one necessarily has $f^{(i)}(a+) = 0$ for all $i \in \overline{j+1,n}$ (cf. (4.9)), and then the sum in (4.12) reduces simply to $f^{(j)}(-\infty)$. For simplicity, we let

$$g(-\infty) := g\big((-\infty)+\big) \tag{4.13}$$

for any function g.

Note that the set $\overline{k,n}$ is empty if $k = n+1$, and then (4.4), Lemma 4.4, and (4.12) become vacuous. However, the definition of p_j in (4.5) and the expression of p_j in (4.10) make sense even for $j = n+1$, if one uses the standard convention that the sum of any empty family is 0, so that

$$p_{n+1} = 0. \tag{4.14}$$

4.4. **Truncation of the generalized Taylor expansion at the point a of the generalized derivative $f^{(k)}$ of a function f in $\mathscr{F}_+^{k:n}$.** Recall (3.19). Take then any

$$y \in (a, z], \tag{4.15}$$

recall (3.8), and introduce the function

$$\tilde{g}_y := \tilde{g}_{z,y} := \sum_{j\in F_{k,n}} f^{(j)}(a+)p_{a,z;0:k:j} + h_{0,y}, \tag{4.16}$$

where $p_{a,z;0:k:j}$ is understood according to (3.20)–(3.21) and

$$h_{i,y} := \int_{I\cap[y,\infty)} \mathrm{d}f^{(n)}(t)p_{t;i,n}^+ \tag{4.17}$$

for $i \in \overline{0,n}$. The latter integral (understood in the "pointwise" sense, similarly to the integral expressing h_j in (4.5)) exists (in $[0,\infty]$), again because $p_{t;j,n}^+ \ge 0$ and the function $f^{(n)}$ is nondecreasing. In fact, $h_{i,y}(x_i) = \int_{I\cap[y,\infty)} \mathrm{d}f^{(n)}(t)p_{t;i,n}^+(x_i) = \int_{[y,y\vee x_i]} \mathrm{d}f^{(n)}(t)p_{t;i,n}(x_i) < \infty$ for all $x_i \in I$—because, by (3.3) and the continuity of the functions $w_0,\dots,w_n$, the expression $p_{t;i,n}(x_i)$ is locally bounded in $t \in I$ for each $x_i \in I$ (actually, $p_{t;i,n}(x_i)$ is locally bounded in $(t,x_i) \in I^2$). So, in view of (4.1),

$$h_{i,y} = h_{i;\mu_{n,y}} \in \mathscr{H}_+^{i:n}, \tag{4.18}$$

where the measure $\mu_{n,y} \in \mathrm{M}_+$ is defined by the condition that $\mu_{n,y}(I\cap(-\infty,x]) = f^{(n)}(x \vee y) - f^{(n)}(y)$ for all $x \in I$; note here that $\mu_{n,y}(I \cap \{b\}) = \mu_{n,y}(\{b\}) = f^{(n)}(b) - f^{(n)}(b-) = 0$ if $b \in I$, since $f^{(n)} \in \mathscr{RC}$, and trivially $\mu_{n,y}(I \cap \{b\}) = \mu_{n,y}(\emptyset) = 0$ if $b \notin I$. So, in either case, $\mu_{n,y}(I \cap \{b\}) = 0$.

Moreover, Lemma 4.1 immediately yields.

Lemma 4.5. *Take any $i \in \overline{0,n}$. Then $h_{i,y} \in \mathscr{RC}$. Also, the function $h_{i;y}/w_i$ is nondecreasing. Moreover, if $i \in \overline{0,n-1}$, then $h_{i,y} \in \mathscr{D}$ and*

$$D_i h_{i,y} = h_{i+1,y}. \tag{4.19}$$

Now combine (4.16), (2.8), (2.7), (3.22), (3.20), (4.10), and (4.19) to conclude that

$$(\tilde{g}_y)^{(k)} w_k = p_k + h_{k,y} \tag{4.20}$$

(here one may want to recall that in the case when $k = n+1$ one has $\tilde{g}_y = h_{0,y}$ and, by (4.14), $p_k = 0$). Similarly (but using (3.21) instead of (3.20)), one can also observe that

$$(\tilde{g}_y)^{(i)}(z) = h_{i,y}(z) \quad \text{for all } i \in \overline{0,k-1}, \tag{4.21}$$

since $p_{a,z;i:k:j}(z) = 0$ if $i < k$.

Recalling again that $f^{(n)} \in \mathscr{RC}$, one has $\int_{I\cap\{a\}} \mathrm{d}f^{(n)}(t) p^+_{t;i,n} = 0$. So, on comparing (4.20) with (4.5), one concludes that

$$(\tilde{g}_y)^{(k)} \underset{y\downarrow a}{\nearrow} f^{(k)} \tag{4.22}$$

(pointwise, on I).

4.5. **Lifting the truncated generalized Taylor expansion of $f^{(k)}$ to an approximation $g_y \in \mathscr{P}_+^{k:n} + \mathscr{H}_+^{0:n}$ of a function $f \in \mathscr{F}_+^{k:n}$.** In accordance with Proposition 3.1, let $q_{k;z,y}$ be the unique $\mathbf{w}$-polynomial in $\mathscr{P}^{\le k-1}$ such that

$$(q_{k;z,y})^{(i)}(z) = (f - \tilde{g}_y)^{(i)}(z) \quad \text{for all } i \in \overline{0,k-1}.$$

Let now

$$g_y = g_{z,y} := q_{k;z,y} + \tilde{g}_y. \tag{4.23}$$

Then

$$(g_y)^{(i)}(z) = f^{(i)}(z) \quad \text{for all } i \in \overline{0,k-1} \tag{4.24}$$

and

$$(g_y)^{(k)} = (\tilde{g}_y)^{(k)}, \tag{4.25}$$

so that, by (4.22),

$$(g_y)^{(k)} \underset{y\downarrow a}{\nearrow} f^{(k)}. \tag{4.26}$$

In view of (4.16), one can rewrite (4.23) as

$$g_y = P_{z,y} + R_{z,y}, \tag{4.27}$$

where

$$P_{z,y} := q_{k;z,y} + \sum_{j\in F_{k,n}} f^{(j)}(a+)p_{a,z;0:k:j} \qquad \text{and} \qquad R_{z,y} := h_{0,y}. \tag{4.28}$$

Take any $j \in F_{k,n} \subseteq \overline{k,n}$. By (2.7), (3.22), and (3.20), $D^k_{\mathbf{w}} p_{a,z;0:k:j} = p_{a,z;k:k:j} = p_{a;k,j}$. Therefore and by (2.8) and (3.5), for each $s \in \overline{k,j}$ one has $p^{(s)}_{a,z;0:k:j} = p_{a;s,j}/w_s$, which is nonnegative and nondecreasing, by (3.3). So, $p^{(s)}_{a,z;0:k:j}$ is nondecreasing for each $s \in \overline{k-1,j}$. Also, by (3.23), $p^{(s)}_{a,z;0:k:j} = 0$ for each $s \in \overline{j+1,n}$. We conclude that $p^{(s)}_{a,z;0:k:j}$ is nondecreasing for each $s \in \overline{k-1,n}$. Also, by (3.24), $p_{a,z;0:k:j} \in \mathscr{P}^j \subseteq \mathscr{P}^{\le n}$. So,

$$p_{a,z;0:k:j} \in \mathscr{P}^{k:n}_+ := \mathscr{P}^{\le n} \cap \mathscr{F}^{k:n}_+ \quad \text{for } j \in F_{k,n}. \tag{4.29}$$

Hence, by the condition $q_{k;z,y} \in \mathscr{P}^{\le k-1}$,

$$P_{z,y} \in \mathscr{P}^{k:n}_+. \tag{4.30}$$

Thus, (4.27) may be considered as a Taylor-type expansion of the function g_y (which latter is in turn an approximation to f, as seen from (4.35) below); at that, $R_{z,y}$ may be considered the remainder term, which vanishes when the function $f^{(n)}$ is constant on the interval $I\cap[y,\infty)$. In view of (4.28), (4.18), and Proposition 4.2,

$$R_{z,y} \in \mathscr{H}^{0:n}_+ \subseteq \mathscr{F}^{k:n}_+. \tag{4.31}$$

It follows from (4.27), (4.30), and (4.31) that

$$g_y \in \mathscr{P}^{k:n}_+ + \mathscr{H}^{0:n}_+ \subseteq \mathscr{F}^{k:n}_+. \tag{4.32}$$

In view of (4.11), in the unit-gauge case with $a = -\infty$ and $k \le n$, the summands $P_{z,y}$ and $R_{z,y}$ in (4.27) are as follows: for all $x_0 \in I$,

$$\begin{aligned} P_{z,y}(x_0) &= \sum_{i=0}^{k-1} c_{i;z,y}\frac{(x_0-z)^i}{i!} + f^{(k)}(-\infty)\frac{(x_0-z)^k}{k!}, \quad \text{with} \\ c_{i;z,y} &:= f^{(i)}(z) - (\tilde g_y)^{(i)}(z) \\ &= f^{(i)}(z) - h_{i,y}(z) \\ &= f^{(i)}(z) - \int_{I\cap[y,\infty)} \mathrm{d}f^{(n)}(x_n)\frac{(z-x_n)_+^{n-i}}{(n-i)!}, \end{aligned} \tag{4.33}$$

and

$$R_{z,y}(x_0) = \int_{I\cap[y,\infty)} \mathrm{d}f^{(n)}(x_n)\frac{(x_0-x_n)_+^n}{n!}. \tag{4.34}$$

Here, the second expression for $c_{i;z,y}$ is obtained by (4.21); if $a > -\infty$ or $k = n+1$, then the expression for $P_{z,y}$ is simpler than the one in (4.33).

By (4.30), Remark 3.2, Lemma 4.5, and the continuity of the w_i's, the functions $P_{z,y}$ and $h_{0,y}$ are locally bounded, for each y. Similarly, by (4.25) and (4.20), $(g_y)^{(k)}w_k$ is locally bounded, for each y. In particular, $D^k_{\mathbf{w}} g_z$ is locally bounded. Now by (2.9), Proposition 2.2, and monotone convergence, one immediately obtains the following approximative representation of any function $f \in F_+^{k:n}$ by mixtures of $\mathbf{w}$-polynomials and "positive parts" thereof.

Theorem 4.6. *For any $f \in F_+^{k:n}$ and g_y as in (4.27)–(4.28),*

$$g_y \underset{y\downarrow a}{\nearrow} f \quad \text{on } I \cap [z,\infty) \qquad \text{and} \qquad (-1)^k(f-g_y) \underset{y\downarrow a}{\searrow} 0 \quad \text{on } I\cap(-\infty,z]. \tag{4.35}$$

Remark 4.7. In view of (4.35) and (4.32), the cone $\mathscr{F}_+^{k:n}$ of functions f on I can be viewed as the closure, in a certain topology, of the cone $\mathscr{P}_+^{k:n} + \mathscr{H}_+^{0:n}$. One may, therefore, ask whether these two cones are the same, that is, whether $\mathscr{F}_+^{k:n} = \mathscr{P}_+^{k:n} + \mathscr{H}_+^{0:n}$. However, in general, this is not the case for any n and k as in (1.2). For example, in the unit-gauge setting, let $I = \mathbb{R}$ and

$$f(x) := g(x)\mathrm{I}\{x \le 0\} + p(x)\mathrm{I}\{x>0\} \tag{4.36}$$

for all $x \in \mathbb{R}$, where $g(x) := (-1)^k(1-x)^{k-1/2}$ and $p(x) := \sum_{i=0}^{n+1} g^{(i)}(0)x^i/i!$. Then, $f^{(j)}(x) = g^{(j)}(x)\mathrm{I}\{x\le 0\} + \sum_{i=j}^{n+1} g^{(i)}(0)x^{i-j}\mathrm{I}\{x>0\}/(i-j)! > 0$ for all $j \in \overline{k, n+1}$ and $x \in \mathbb{R}$, and hence, $f \in \mathscr{F}_+^{k:n}$. On the other hand, $f \notin \mathscr{P}_+^{k:n} + \mathscr{H}_+^{0:n}$. Indeed, take any $q \in \mathscr{P}_+^{k:n}$ and $h \in \mathscr{H}_+^{0:n}$. Then, by [21, p. 619, Lemma 2 and p. 606, (1)], $h(-\infty) = 0$; so, for $x \to -\infty$, either $q(x)+h(x) \sim c|x|^k$ for some $c \in (0,\infty)$ or $q(x)+h(x) = O(|x|^{k-1})$, depending on whether the degree of the polynomial q is n or $< n$, whereas $|f(x)| \sim |x|^{k-1/2}$.

Quite similarly, one can show that $\mathscr{F}_+^{k:n} \ne \mathscr{P}_+^{k:n} + \mathscr{H}_+^{0:n}$ for any interval I with the left endpoint $a = -\infty$, again in the unit-gauge case and again for any n and k as in (1.2). Moreover, in view of [24, Remark 2.3], it is easy to see that $\mathscr{F}_+^{k:n} \ne \mathscr{P}_+^{k:n} + \mathscr{H}_+^{0:n}$ for any given nonzero-length interval I and any n and k as in (1.2), with an appropriate choice of gauge functions $w_0, \dots, w_n$.

The idea of construction (4.36) comes from [21] (cf. Propositions 1 and 2 therein). As mentioned before, in [21] the special case with $n = 3$, $k = 1$, and unit-gauge w_j was considered.

5. Convex cone dual to $\mathscr{F}_+^{k:n}$

Let us recall that condition (1.2) continues to hold in this section. Here, we shall define and completely characterize the convex cone dual to any set $\mathscr{G}$ such that

$$\mathscr{P}_+^{k:n} \cup \mathscr{P}_{0:n}^+ \subseteq \mathscr{G} \subseteq \mathscr{F}_+^{k:n}, \tag{5.1}$$

where $\mathscr{P}_+^{k:n}$ is as defined in (4.29) and

$$\mathscr{P}_{0:n}^+ := \{p_{t;0,n}^+ \colon t \in I\}, \tag{5.2}$$

with $p_{t;0,n}^+$ defined according to (3.12).

Note that

$$\mathscr{P}^{\le k-1} \subseteq \mathscr{P}_+^{\le k}, \tag{5.3}$$

$$\mathscr{P}^{\le k-1} \subseteq \mathscr{G}, \tag{5.4}$$

$$k \le n \implies \mathscr{P}_+^{\le k} \subseteq \mathscr{G}. \tag{5.5}$$

Indeed, by the definition, if $p \in \mathscr{P}^{\le k-1}$, then $p^{(k-1)}$ is a constant and $p^{(k)} = 0$, whence $p \in \mathscr{P}_+^{\le k}$ by (3.1), and $p \in \mathscr{P}^{\le n} \cap \mathscr{F}_+^{k:n} = \mathscr{P}_+^{k:n} \subseteq \mathscr{G}$ by (1.2), (4.3), (4.29), and (5.1). This yields (5.3) and (5.4).

If now $k \le n$ and $p \in \mathscr{P}_+^{\le k}$, then obviously $p \in \mathscr{P}^{\le n}$, and also $p^{(k)} \ge 0$ and $p^{(k+1)} = 0$, whence, in view of (2.8), $p^{(k-1)}$ is nondecreasing and $p^{(j)}$ is constant for each $j \in \overline{k,n}$, so that, by (4.3), $p \in \mathscr{F}_+^{k:n}$. Thus, recalling again the definition of $\mathscr{P}_+^{k:n}$ in (4.29) and the first set inclusion in (5.1), one obtains (5.5).

Also, one has the following.

Proposition 5.1. $\mathscr{P}_{0:n}^+ \subseteq \mathscr{F}_+^{1:n} \subseteq \mathscr{F}_+^{k:n}$.

Proof. Take any $t \in I$. In view of (3.16), (2.8), and (3.14), $(p_{t;0,n}^+)^{(j)} = p_{t;j,n}^+/w_j \ge 0$ for all $j \in \overline{0,n}$. In particular, $(p_{t;0,n}^+)^{(n)}(x_0) = \mathrm{I}\{x_0 \ge t\}$ is obviously nondecreasing in $x_0 \in I$. It also follows that for each $j \in \overline{0,n-1}$, one has $(p_{t;0,n}^+)^{(j+1)} w_{j+1} = p_{t;j+1,n}^+ \ge 0$ and hence, by (2.8), $(p_{t;0,n}^+)^{(j)}$ is nondecreasing. So, $p_{t;0,n}^+ \in \mathscr{F}_+^{1:n} \subseteq \mathscr{F}_+^{k:n}$. Now Proposition 5.1 follows by (5.2). ☐

By (4.29) and Proposition 5.1, there always is a set $\mathscr{G}$ satisfying conditions (5.1), which will be the only conditions generally imposed on $\mathscr{G}$ in this paper. In particular, the set $\mathscr{G}$ will not have to be convex or a cone. However, the cone dual to $\mathscr{G}$, to be denoted by $\hat{\mathscr{G}}$ and defined later in this section, will be a convex cone indeed. Moreover, it will turn out that in most cases the dual cone $\hat{\mathscr{G}}$ will not depend on the choice of $\mathscr{G}$ as long as conditions (5.1) are satisfied—the only exception in this regard being the case when all of the following conditions hold:

$$k = n + 1,\ k \text{ is odd, and } a \notin I. \tag{5.6}$$

So, unless this exceptional case takes place, the dual cone $\hat{\mathscr{G}}$ will coincide with $\hat{\mathscr{F}}_+^{k:n}$.

5.1. **Admissible set of measures.** In accordance with the general definition of the dual cone (see, e.g., [10, Chapter III, Section 5] or [29, p. 7]), it appears natural to define the cone $\hat{\mathscr{G}}$ dual to the set $\mathscr{G}$ of functions on I as consisting of signed measures on the Borel σ-algebra—say, $\mathfrak{B}$—over I. However, we shall take a more general approach by letting $\hat{\mathscr{G}}$ be a set of ordered pairs (ν_1, ν_2) of nonnegative (not necessarily finite) measures on $\mathfrak{B}$ such that $\nu_1(f) \ge \nu_2(f)$ for all $f \in \mathscr{G}$. Here and subsequently, we use the common definition $\nu(f) := \int_I f \,\mathrm{d}\nu$ for a Borel-measurable function $f\colon I \to \mathbb{R}$ and a nonnegative measure ν on $\mathfrak{B}$, if the integral exists in the extended sense, that is, if at least one of the values $\nu(f_+)$ or $\nu(f_-)$ is finite, and in such a case we let $\nu(f) := \nu(f_+) - \nu(f_-)$; as usual, $f_+ := f \vee 0$ and $f_- := (-f)_+$. For brevity (unless otherwise indicated), when we say that $\nu(f)$ satisfies a certain condition, it will actually mean that $\nu(f)$ *exists and* satisfies that condition. For example, if we say $\nu(f) > -\infty$, it actually means that $\nu(f)$ exists and does not equal $-\infty$ (which is equivalent to the statement that $\nu(f_-) < \infty$).

Of course, if at least one of the nonnegative measures ν_1, ν_2 is finite, then one can introduce the signed measure $\nu := \nu_1 - \nu_2$; if, moreover, at least one of the integrals $\nu_1(f), \nu_2(f)$ is finite, then one can also let $\nu(f) := \nu_1(f) - \nu_2(f)$ and write the usual duality condition $\nu(f) \ge 0$ instead of $\nu_1(f) \ge \nu_2(f)$. However, such additional restrictions on the finiteness of one of the measures ν_1, ν_2 or one of the integrals $\nu_1(f), \nu_2(f)$ are unnecessary for our results on the dual cone or in the relevant applications.

Yet, to ensure that the dual cone $\hat{\mathscr{G}}$ be convex, one cannot allow two pairs (ν_1, ν_2) and (ρ_1, ρ_2) of measures to both belong to $\hat{\mathscr{G}}$ if $\{\nu_j(f), \rho_j(f)\} = \{\infty, -\infty\}$ for some $f \in \mathscr{G}$ and some $j \in \{1, 2\}$ because in that case the integral $(\nu_j + \rho_j)(f)$ would not exist and thus the pair $(\nu_1, \nu_2) + (\rho_1, \rho_2) = (\nu_1 + \rho_1, \nu_2 + \rho_2)$ could not possibly belong to $\hat{\mathscr{G}}$. For this reason, only pairs (ν_1, ν_2) of nonnegative measures such that $\nu_1(f) \wedge \nu_2(f) > -\infty$ for all $f \in \mathscr{G}$ will be allowed to belong to the dual cone $\hat{\mathscr{G}}$; such pairs of measures may be referred to as *admissible*.

To formalize this approach to admissibility (which works well in the applications), let us first introduce the notation N_+ for the set of all nonnegative (not necessarily finite) measures on $\mathfrak{B}$. Next, introduce the set

$$\mathrm{N}_+(\mathscr{G}) := \big\{\nu \in \mathrm{N}_+ \colon \nu(f) > -\infty \text{ for all } f \in \mathscr{G}\big\}, \tag{5.7}$$

which may be referred to as the *admissible set* (of nonnegative measures corresponding to the set $\mathscr{G}$ of functions). One has the following characterization of this admissible set.

Proposition 5.2. *Take any $\nu \in \mathrm{N}_+$.*

(i) *If $k \le n$, then*

$$\nu \in \mathrm{N}_+(\mathscr{G}) \iff \nu(p) > -\infty \quad \text{for all } p \in \mathscr{P}_+^{\le k}. \tag{5.8}$$

(ii) *If k is even or $a \in I$, then*

$$\begin{aligned} \nu \in \mathrm{N}_+(\mathscr{G}) &\iff \nu(p) > -\infty \quad \text{for all } p \in \mathscr{P}^{\le k-1} \\ &\iff \nu(p) \in \mathbb{R} \quad \text{for all } p \in \mathscr{P}^{\le k-1}. \end{aligned} \tag{5.9}$$

(iii) *If the exceptional case (5.6) takes place, then*

$$\begin{aligned} \nu \in \mathrm{N}_+(\mathscr{F}_+^{k:n}) &\iff \nu(p) \in \mathbb{R} \quad \text{for all } p \in \mathscr{P}^{\le k-1} \\ &\qquad \& \ \nu\big((a, \tilde{a})\big) = 0 \text{ for some } \tilde{a} \in I \\ &\implies \nu \in \mathrm{N}_+(\mathscr{G}). \end{aligned} \tag{5.10}$$

Thus (given the condition (5.1)), the admissible set $\mathrm{N}_+(\mathscr{G})$ does not actually depend on the choice of $\mathscr{G}$, except for the case (5.6). In that exceptional case, (5.10) shows that the admissible set $\mathrm{N}_+(\mathscr{F}_+^{k:n})$ is inconveniently too small, consisting only of measures ν with support $\operatorname{supp}\nu$ bounded away from the left endpoint a of the interval I; in particular, in the important case when $a = -\infty$, the set $\operatorname{supp}\nu$ will have to be bounded from below, which would rule out applications without such a restriction. Allowing $\mathscr{G}$ to differ from $\mathscr{F}_+^{k:n}$ was motivated by this inconvenience. Indeed, if the class $\mathscr{G}$ is smaller $\mathscr{F}_+^{k:n}$, then, in view of (5.7), the admissible set $\mathrm{N}_+(\mathscr{G})$ may turn out to be a large enough extension of the too small class $\mathrm{N}_+(\mathscr{F}_+^{k:n})$ of measures. In particular, a sensible choice of $\mathscr{G}$ in the exceptional case (5.6) appears to be given by the formula

$$\mathscr{G} = \{f \in \mathscr{F}_+^{n+1:n}\colon f \ge p \text{ for some } p \in \mathscr{P}^{\le n}\}, \tag{5.11}$$

so that the equivalences in (5.9) obviously continue to hold even in the exceptional case (5.6).

Proof of Proposition 5.2. The implication $\implies$ in part (i) of this proposition follows immediately by (5.7) and (5.5), whereas the reverse implication $\impliedby$ there follows by parts (II)(i) and (I)(i) of Proposition 4.3 and (5.3).

If k is even, then the first equivalence in part (ii) of Proposition 5.2 follows by (5.7), part (I)(i) of Proposition 4.3, and (5.4). If k is odd and $a \in I$, then the just-mentioned equivalence follows by (5.7), part (I)(ii)(*) of Proposition 4.3 (with $z = a$), and (5.4). As for the second equivalence in part (ii) of Proposition 5.2, it follows because $-p \in \mathscr{P}^{\le k-1}$ for any $p \in \mathscr{P}^{\le k-1}$.

To complete the proof of Proposition 5.2, it remains to prove its part (iii). To do this, suppose first that $\nu((a,\tilde{a})) = 0$ for some $\tilde{a} \in I$. Then one can replace ν by its restriction to the Borel σ-algebra over the reduced interval $I \cap [\tilde{a}, \infty)$ in place of I and, accordingly, replace the functions in $\mathscr{G}$ and the functions $w_0, \dots, w_n$ by their respective restrictions to the interval $I \cap [\tilde{a}, \infty)$, which obviously contains its left endpoint $\tilde{a}$. Now the implication $\implies$ in the last line in (5.10) and, in particular, the implication $\impliedby$ in the first line there follow immediately by the already verified part (ii) of Proposition 5.2.

The implication $\nu \in \mathrm{N}_+(\mathscr{F}_+^{k:n}) \implies \nu(p) \in \mathbb{R}$ for all $p \in \mathscr{P}^{\le k-1}$ in (5.10) follows by (5.4), (5.1), and the second equivalence in (5.9) (which latter holds whether or not the exceptional case (5.6) takes place).

Thus, it remains to verify the implication $\nu \in \mathrm{N}_+(\mathscr{F}_+^{k:n}) \implies \nu((a,\tilde{a})) = 0$ for some $\tilde{a} \in I$ in (5.10). Toward this end, assume, on the contrary, that $\nu((a,\tilde{a})) > 0$ for all $\tilde{a} \in I$. Take any sequence $(t_i)_{i\in\mathbb{N}}$ in I such that $t_i \to a$ (as $i \to \infty$). Then, by the assumption, $\nu((a,t_i)) > 0$ for all $i \in \mathbb{N}$. Now, introduce the functions $p^-_{t;j,m}\colon I \to \mathbb{R}$ by the formula

$$p^-_{t;j,m}(x_j) := p_{t;j,m}(x_j)\mathrm{I}\{x_j < t\} \tag{5.12}$$

for all t and $x_j \in I$ (cf. (3.12)). The conditions in (5.6) that $k = n+1$ and k is odd imply that n is even. So, by (5.12) and (3.3), $p^-_{t;0,n} \ge 0$ (on I) and $p^-_{t;0,n} > 0$ on $I \cap (-\infty, t) = (a,t)$ for any $t \in I$. Now, take any $i \in \mathbb{N}$. Then, recalling that $\nu((a,t_i)) > 0$, take any $\gamma_i \in \mathbb{R}$ such that

$$0 < \gamma_i \le \nu(p^-_{t_i;0,n}); \tag{5.13}$$

in particular, if $\nu(p^-_{t_i;0,n}) < \infty$, then one may take $\gamma_i = \nu(p^-_{t_i;0,n})$. Now, introduce the function

$$f := -\sum_{i=1}^\infty \frac{1}{\gamma_i} p^-_{t_i;0,n} = -\int_I \mu(\mathrm{d}t) p^-_{t;0,n}, \tag{5.14}$$

where μ is the nonnegative measure defined by the formula

$$\mu(g) := \sum_{i=1}^{\infty} \frac{1}{\gamma_i} g(t_i)$$

for all nonnegative functions g on I. Since $p^-_{t;0,n} \geq 0$ for all $t \in I$, it follows that $f \leq 0$. Moreover, in view of the condition $a \notin I$ in (5.6), for any $x \in I$, there is some $i_x \in \mathbb{N}$ such that for all $i \in \overline{i_x, \infty}$, one has $t_i < x$ and hence $p^-_{t_i;0,n}(x) = 0$, so that $f(x) = -\sum_{i=1}^{i_x-1} \frac{1}{\gamma_i} p^-_{t_i;0,n}(x) > -\infty$. Therefore, $f(x) \in (-\infty, 0]$ for all $x \in I$. Next (cf. (5.14), Lemma 4.1, (3.3)),

$$\begin{aligned} f^{(n)}(x) &= -\frac{1}{w_n(x)} \int_I \mu(\mathrm{d}t) p^-_{t;n,n}(x) \\ &= -\frac{1}{w_n(x)} \int_I \mu(\mathrm{d}t) p_{t;n,n}(x) \mathrm{I}\{x < t\} \\ &= -\int_I \mu(\mathrm{d}t) \mathrm{I}\{x < t\} = -\mu\big(I \cap (x, \infty)\big) \end{aligned}$$

is nondecreasing in $x \in I$, so that $f \in \mathscr{F}_+^{n+1:n} = \mathscr{F}_+^{k:n}$. On the other hand, by (5.14) and (5.13),

$$\nu(f) = -\sum_{i=1}^{\infty} \frac{1}{\gamma_i} \nu(p^-_{t_i;0,n}) \leq -\sum_{i=1}^{\infty} 1 = -\infty.$$

So, the assumption that the condition "$\nu((a, \tilde{a})) = 0$ for some $\tilde{a} \in I$" is violated has led to the conclusion that $\nu \notin \mathrm{N}_+(\mathscr{F}_+^{k:n})$. This completes the proof of Proposition 5.2(iii) as well. □

Suppose, for example, that the exceptional case (5.6) with $n = 0$ takes place in the unit-gauge setting. Then, $\mathscr{F}_+^{k:n} = \mathscr{F}_+^{n+1:n} = \mathscr{F}_+^{1:0}$ is the set of all nondecreasing functions $f\colon I \to \mathbb{R}$. In this situation, with $a \notin I$, it is rather clear that, for any nonnegative measure ν on $\mathfrak{B}$ with $\inf \operatorname{supp} \nu = a$, one can choose a function $f \in \mathscr{F}_+^{1:0} = \mathscr{F}_+^{k:n}$ growing so fast (from $-\infty$ up) on I that $\nu(f) = -\infty$. This simple observation was the main idea behind the above proof of part (iii) of Proposition 5.2. Again for the exceptional case (5.6) with $n = 0$ in the unit-gauge setting, the set $\mathscr{G}$ as in (5.11) is the set of all nondecreasing functions in $\mathscr{RC}$ that are *also bounded from below*, which latter appears to be a rather natural additional condition to impose on the functions in $\mathscr{F}_+^{1:0}$.

The function f defined by formula (5.14) in the proof of part (iii) of Proposition 5.2 may be considered a generalized spline of order n. For instance, in the unit-gauge setting with $n = 2$ and $k = n + 1 = 3$, that function f will be continuously differentiable, with the graph consisting of countably many parabolic arcs.

5.2. **Dual cone.** Define the dual cone $\hat{\mathscr{G}}$ by the formula

$$\begin{aligned} \hat{\mathscr{G}} &:= \big\{(\nu_1, \nu_2) \in \mathrm{N}_+(\mathscr{G}) \times \mathrm{N}_+(\mathscr{G})\colon \nu_1(f) \geq \nu_2(f) \text{ for all } f \in \mathscr{G}\big\} \\ &= \big\{(\nu_1, \nu_2) \in \mathrm{N}_+ \times \mathrm{N}_+\colon \nu_1(f) \geq \nu_2(f) > -\infty \text{ for all } f \in \mathscr{G}\big\}. \end{aligned} \tag{5.15}$$

Theorem 5.3. *Take any $(\nu_1,\nu_2)\in \mathrm{N}_+(\mathscr{G})\times \mathrm{N}_+(\mathscr{G})$. Then $(\nu_1,\nu_2)\in\hat{\mathscr{G}}$ if and only if all of the following conditions hold:*

(i) *$\nu_1(p)=\nu_2(p)\in\mathbb{R}$ for all $p\in\mathscr{P}^{\le k-1}$;*

(ii) *$\nu_1(p)\ge\nu_2(p)$ for all $p\in\mathscr{P}_+^{k:n}$ [$=\mathscr{P}^{\le n}\cap\mathscr{F}_+^{k:n}$, by (4.29)];*

(iii) *$\nu_1(p_{t;0,n}^+)\ge\nu_2(p_{t;0,n}^+)$ for all $t\in I$.*

Thus, the verification of the condition on $(\nu_1,\nu_2)\in \mathrm{N}_+(\mathscr{G})\times \mathrm{N}_+(\mathscr{G})$ that $\nu_1(f)\ge\nu_2(f)$ for all $f\in\mathscr{F}_+^{k:n}$ reduces to the verification of this inequality just for certain **w**-polynomials and their "positive parts."

Remark 5.4. For each $(\nu_1,\nu_2)\in \mathrm{N}_+(\mathscr{G})\times \mathrm{N}_+(\mathscr{G})$, condition (i) of Theorem 5.3 follows from condition (ii) there. Indeed, suppose that condition (ii) holds, and then take any $(\nu_1,\nu_2)\in \mathrm{N}_+(\mathscr{G})\times \mathrm{N}_+(\mathscr{G})$ and any $p\in\mathscr{P}^{\le k-1}$. Then $\{p,-p\}\subseteq\mathscr{P}^{\le k-1}\subseteq\mathscr{G}$ by (5.4), whence $\nu_1(p)\ge\nu_2(p)>-\infty$ and $-\nu_1(p)=\nu_1(-p)\ge\nu_2(-p)=-\nu_2(p)$, with $\nu_2(-p)>-\infty$, which yields $\nu_1(p)=\nu_2(p)\in\mathbb{R}$. Thus, again, for each $(\nu_1,\nu_2)\in \mathrm{N}_+(\mathscr{G})\times \mathrm{N}_+(\mathscr{G})$, conditions (ii) and (iii) of Theorem 5.3 already suffice for $(\nu_1,\nu_2)\in\hat{\mathscr{G}}$. Moreover, in the case when $k=n+1$, one has $\mathscr{P}^{\le k-1}=\mathscr{P}^{\le n}=\mathscr{P}_+^{k:n}$, because $p^{(n)}$ is constant and hence nondecreasing for any $p\in\mathscr{P}^{\le n}$; therefore, in this case conditions (i) and (ii) of Theorem 5.3 are just equivalent to each other.

Proof of Theorem 5.3. That condition (ii) in Theorem 5.3 is necessary for $(\nu_1,\nu_2)\in\hat{\mathscr{G}}$ follows immediately from (5.15) and (5.1). Next, condition (i) follows from condition (ii) by Remark 5.4. The necessity of condition (iii) in Theorem 5.3 follows immediately from (5.1) and (5.2). Thus, the "only if" part of Theorem 5.3 is verified.

Let us now consider the "if" part of the theorem. Suppose that conditions (i)–(iii) of Theorem 5.3 hold. Take any $z\in(a,b)$ and then $y\in(a,z]$, as in (3.19) and (4.15). Take also any $f\in\mathscr{G}$. Then, by (5.1), $f\in\mathscr{F}^{k:n}$, and so, the function $f^{(n)}$ is nondecreasing and hence the corresponding Lebesgue–Stieltjes measure $\mathrm{d}f^{(n)}$ is nonnegative. So, by the definition of $R_{z,y}$ in (4.28), (4.17), condition (iii) of Theorem 5.3, the Fubini theorem, and (3.14),

$$\nu_1(R_{z,y})\ge\nu_2(R_{z,y})\ge 0. \tag{5.16}$$

By (4.30), (5.1), conditions (ii) of Theorem 5.3 and $(\nu_1,\nu_2)\in \mathrm{N}_+(\mathscr{G})\times \mathrm{N}_+(\mathscr{G})$, and (5.7),

$$\nu_1(P_{z,y})\ge\nu_2(P_{z,y})>-\infty. \tag{5.17}$$

It follows by (4.27), (5.16), and (5.17) that

$$\nu_1(g_y)\ge\nu_2(g_y). \tag{5.18}$$

Also, by (4.32), $g_z\in\mathscr{F}_+^{k:n}$. So, by part I of Proposition 4.3, there exists a **w**-polynomial $p_z\in\mathscr{P}^{\le k-1}$ such that (i) if k is even, then $g_z\ge p_z$ and (ii) if k is odd, then $g_z\ge p_z$ on $I\cap[z,\infty)$ and $g_z\le p_z$ on $I\cap(-\infty,z]$. In view of (4.35), one concludes that

(i) if k is even, then $g_y\ge p_z$ and $g_y\underset{y\downarrow a}{\nearrow} f$ on the interval I;

(ii) if k is odd, then

(*) $g_y \geq p_z$ and $g_y \underset{y\downarrow a}{\nearrow} f$ on the interval $I \cap [z,\infty)$;

(**) $g_y \leq p_z$ and $g_y \underset{y\downarrow a}{\searrow} f$ on the interval $I \cap (-\infty,z]$.

By Theorem 5.3(i), $\nu_1(p_z) = \nu_2(p_z) \in \mathbb{R}$. Thus, by the Lebesgue monotone convergence theorem and conclusions (i) and (ii) above,

$$\begin{aligned}
&\int_{I\cap[z,\infty)} g_y \,\mathrm{d}\nu_j \underset{y\downarrow a}{\longrightarrow} \int_{I\cap[z,\infty)} f\,\mathrm{d}\nu_j \qquad \text{and} \\
&\int_{I\cap(-\infty,z)} g_y \,\mathrm{d}\nu_j \underset{y\downarrow a}{\longrightarrow} \int_{I\cap(-\infty,z)} f\,\mathrm{d}\nu_j
\end{aligned} \tag{5.19}$$

for $j \in \{1,2\}$. In view of the condition $(\nu_1,\nu_2) \in \mathrm{N}_+(\mathscr{G})\times\mathrm{N}_+(\mathscr{G})$ and the definition (5.7), $\nu_1(f)\wedge\nu_2(f) > -\infty$ or, equivalently, $\nu_1(f_-)\vee\nu_2(f_-) < \infty$, whence

$$\int_{I\cap[z,\infty)} f\,\mathrm{d}\nu_1 \wedge \int_{I\cap[z,\infty)} f\,\mathrm{d}\nu_2 \wedge \int_{I\cap(-\infty,z)} f\,\mathrm{d}\nu_1 \wedge \int_{I\cap(-\infty,z)} f\,\mathrm{d}\nu_2 > -\infty. \tag{5.20}$$

It follows from (5.19) and (5.20) that $\nu_j(g_y) = \int_{I\cap[z,\infty)} g_y\,\mathrm{d}\nu_j + \int_{I\cap(-\infty,z)} g_y\,\mathrm{d}\nu_j \underset{y\downarrow a}{\longrightarrow} \int_{I\cap[z,\infty)} f\,\mathrm{d}\nu_j + \int_{I\cap(-\infty,z)} f\,\mathrm{d}\nu_j = \nu_j(f)$, again for $j\in\{1,2\}$; condition (5.20) is used here to show that the integrals $\int_{I\cap[z,\infty)} f\,\mathrm{d}\nu_j$ and $\int_{I\cap(-\infty,z)} f\,\mathrm{d}\nu_j$ can be added. So, by (5.18), $\nu_1(f)\geq\nu_2(f)$, for any $f\in\mathscr{G}$. In view of (5.15), the proof of the "if" part of Theorem 5.3 is now completed as well. □

Theorem 5.3 can be restated in the following "basis" form.

Theorem 5.5. *Take any $(\nu_1,\nu_2)\in\mathrm{N}_+(\mathscr{G})\times\mathrm{N}_+(\mathscr{G})$, and also take any s and z in I. Then, $(\nu_1,\nu_2)\in\hat{\mathscr{G}}$ if and only if all of the following conditions hold:*

(i′) $\nu_1(p_{s;0,i}) = \nu_2(p_{s;0,i}) \in \mathbb{R}$ *for all* $i\in\overline{0,k-1}$*;*
(ii′) $\nu_1(p_{a,z;0:k:j}) \geq \nu_2(p_{a,z;0:k:j})$ *for all* $j\in F_{k,n}$*;*
(iii) $\nu_1(p^+_{t;0,n}) \geq \nu_2(p^+_{t;0,n})$ *for all* $t\in I$*.*

(Recall here the definitions (3.3) and (3.20), (3.21), (3.8) concerning the **w**-polynomials $p_{s;0,i}$ for $i\in\overline{0,k-1}$ and $p_{a,z;0:k:j}$ for $j\in F_{k,n}$.)

Proof of Theorem 5.5. First here, by (3.7), the **w**-polynomials $p_{s;0,0},\dots,p_{s;0,k-1}$ constitute a basis of the linear space $\mathscr{P}^{\leq k-1}$, and so, condition (i′) of Theorem 5.5 is equivalent to condition (i) of Theorem 5.3.

One can similarly see that the conjunction of conditions (i′) and (ii′) of Theorem 5.5 is equivalent to condition (ii) of Theorem 5.3 (which latter is in turn equivalent to the conjunction of conditions (i) and (ii) of Theorem 5.3).

Alternatively, one can note that, by (4.29) and condition (ii) of Theorem 5.3, for each $(\nu_1,\nu_2)\in\mathrm{N}_+(\mathscr{G})\times\mathrm{N}_+(\mathscr{G})$ condition (ii′) of Theorem 5.5 is necessary for $\nu\in\hat{\mathscr{G}}$. On the other hand, condition (ii) of Theorem 5.3 was used in the proof of the "if" part of Theorem 5.3 only to obtain the conclusion (5.17). However, the same conclusion can be obviously obtained by using the definition of $P_{z,y}$ in (4.28), formula (4.4), and conditions (i′) and (ii′) of Theorem 5.5—instead of (4.30) and condition (ii) of Theorem 5.3. Thus, Theorem 5.5 is proved. □

In the unit-gauge case, Theorem 5.5 immediately results in the following corollaries, in view of (3.10), (3.18), (3.25), (3.26), and (4.11).

Corollary 5.6. *Suppose that $w_0 = \dots = w_n = 1$. Suppose also that $a = -\infty$. Take any $(\nu_1, \nu_2) \in \mathrm{N}_+(\mathscr{G}) \times \mathrm{N}_+(\mathscr{G})$, and also take any real s and z. Then $(\nu_1, \nu_2) \in \hat{\mathscr{G}}$ if and only if all of the following conditions hold:*

(i) $\int_I (x-s)^i \nu_1(\mathrm{d}x) = \int_I (x-s)^i \nu_2(\mathrm{d}x) \in \mathbb{R}$ *for all* $i \in \overline{0, k-1}$*;*
(ii) $\int_I (x-z)^k \nu_1(\mathrm{d}x) \ge \int_I (x-z)^k \nu_2(\mathrm{d}x)$ *if* $k \le n$*;*
(iii) $\int_I (x-t)_+^n \nu_1(\mathrm{d}x) \ge \int_I (x-t)_+^n \nu_2(\mathrm{d}x)$ *for all* $t \in I$*.*

Corollary 5.7. *Suppose that $w_0 = \dots = w_n = 1$. Suppose also that $a > -\infty$. Take any $(\nu_1, \nu_2) \in \mathrm{N}_+(\mathscr{G}) \times \mathrm{N}_+(\mathscr{G})$, and also take any real s. Then $(\nu_1, \nu_2) \in \hat{\mathscr{G}}$ if and only if all of the following conditions hold:*

(i) $\int_I (x-s)^i \nu_1(\mathrm{d}x) = \int_I (x-s)^i \nu_2(\mathrm{d}x) \in \mathbb{R}$ *for all* $i \in \overline{0, k-1}$*;*
(ii) $\int_I (x-a)^j \nu_1(\mathrm{d}x) \ge \int_I (x-a)^j \nu_2(\mathrm{d}x)$ *for all* $j \in \overline{k, n}$*;*
(iii) $\int_I (x-t)_+^n \nu_1(\mathrm{d}x) \ge \int_I (x-t)_+^n \nu_2(\mathrm{d}x)$ *for all* $t \in I$*.*

One may note here that the part of condition (ii) in Corollary 5.7 for $j = n$ follows from condition (iii) there. Therefore, one may replace the specification $j \in \overline{k, n}$ in Corollary 5.7(ii) by $j \in \overline{k, n-1}$.

Corollary 5.6 immediately results in the following statement, which will be useful in probabilistic applications such as ones considered in [21], [20], and [25].

Corollary 5.8. *Suppose that $w_0 = \dots = w_n = 1$. Suppose also that $a = -\infty$. Let X and Y be any r.v.'s with values in the interval I and with distributions belonging to the admissible set $\mathrm{N}_+(\mathscr{G})$ (characterized in Proposition 5.2). Take any real s and z. Then,*

$$\mathsf{E} f(X) \ge \mathsf{E} f(Y) \quad \textit{for all } f \in \mathscr{G}$$

if and only if all of the following conditions hold:

(i) $\mathsf{E}(X-s)^i = \mathsf{E}(Y-s)^i \in \mathbb{R}$ *for all* $i \in \overline{1, k-1}$*;*
(ii) $\mathsf{E}(X-z)^k \ge \mathsf{E}(Y-z)^k$ *if* $k \le n$*;*
(iii) $\mathsf{E}(X-t)_+^n \ge \mathsf{E}(Y-t)_+^n$ *for all* $t \in I$*.*

Clearly, similar probabilistic formulations can be immediately obtained based on Theorems 5.3 and 5.5 and Corollary 5.7.

By using the reflection transformation $\mathbb{R} \ni x \mapsto -x$, one immediately obtains the corresponding results for the "reflected" class of functions

$$\mathscr{F}_-^{k:n}(I) := \{ f^- \colon f \in \mathscr{F}_+^{k:n}(-I) \}, \tag{5.21}$$

where $-I := \{-x \colon x \in I\}$ and $f^-(x) := f(-x)$ for all $x \in I$. For instance, in view of part (i) of Proposition 5.2, one has the following "reflected" counterpart of Corollary 5.8, where for simplicity we shall consider only the case when $k \le n$.

Corollary 5.9. *Suppose that $k \le n$ and $b = \infty$. Let X and Y be any r.v.'s with values in the interval I and such that $\mathsf{E}p(X) \wedge \mathsf{E}p(Y) > -\infty$ for all (usual) polynomials p of degree $\le k$ with $(-1)^k p^{(k)} \ge 0$. Take any real s and z. Then,*

$$\mathsf{E} f(X) \ge \mathsf{E} f(Y) \quad \textit{for all } f \in \mathscr{F}_-^{k:n}(I)$$

if and only if all of the following conditions hold:

(i) $\mathsf{E}(X-s)^i = \mathsf{E}(Y-s)^i \in \mathbb{R}$ *for all* $i \in \overline{1,k-1}$*;*
(ii) $\mathsf{E}(z-X)^k \ge \mathsf{E}(z-Y)^k$*;*
(iii) $\mathsf{E}(t-X)_+^n \ge \mathsf{E}(t-Y)_+^n$ *for all* $t \in I$*.*

5.3. **Relations with Tchebycheff systems.** Let $(u_0,\dots,u_n)$ be a sequence of real-valued functions of class C^n on a finite closed interval $[a,b] \subset \mathbb{R}$. According to [12, p. 376, Theorem 1.1], $(u_0,\dots,u_n)$ is an extended complete Tchebycheff system (or, briefly, an *ECT*-system) on $[a,b]$ if and only if the Wronskian

$$W_k(x) := W_k(u_0,\dots,u_k)(x) := \det\big[u_i^{(j)}(x)\big]_{i,j=0}^k$$

is (strictly) positive for all $k \in \overline{0,n}$ and $x \in I$; here, ${}^{(j)}$ denotes the usual jth derivative. This characterization of *ECT*-systems, as well as their definition given at the very beginning of [12, Chapter XI], can be extended almost verbatim to sequences $(u_0,\dots,u_n)$ of functions of class C^n defined on any interval $I \subseteq \mathbb{R}$.

Let $\mathbf{W} = \mathbf{W}_n$ denote the set of all sequences $\mathbf{w} = (w_0,\dots,w_n)$ of smooth positive functions on I described in the paragraph containing formula (2.3).

The following proposition states that the *ECT*-systems can be described as the sequences of $\mathbf{w}$-polynomials graded by degree, with positive coefficients of their highest-degree $\mathbf{w}$-monomials.

Proposition 5.10. *Let* $(u_0,\dots,u_n)$ *be a sequence of real-valued functions of class* C^n *on* I*. Then,* $(u_0,\dots,u_n)$ *is an* ECT*-system if and only if for some* $\mathbf{w} \in \mathbf{W}$ *and all* $\alpha \in \overline{0,n}$ *the function* u_α *is a* $\mathbf{w}$*-polynomial of degree* α *with* $(u_\alpha)_{\mathbf{w}}^{(\alpha)} > 0$*.*

Proof. Take any t and s in I such that $t < s$. By [12, p. 379, Theorem 1.2] and [12, p. 379, Remark 1.2], $(u_0,\dots,u_n)$ is an *ECT*-system on the interval $[t,s]$ if and only if, for some $\mathbf{w} \in \mathbf{W}$ and some nonsingular lower-triangular real matrix $L = [\ell_{\alpha,i}]_{\alpha,i=0}^n$, one has $\tilde u_\alpha = p_{t;0,\alpha}$ on $[t,s]$ for all $\alpha \in \overline{0,n}$, where $[\tilde u_0,\dots,\tilde u_n]^T := L^{-1}[u_0,\dots,u_n]^T$, and ^{T} stands for the matrix transposition; cf. here [12, p. 378, (1.5)] and (3.3). That is, in view of Remark 3.3, $(u_0,\dots,u_n)$ is an *ECT*-system on the interval $[t,s]$ if and only if, for some $\mathbf{w} \in \mathbf{W}$ and all $\alpha \in \overline{0,n}$, the function u_α is a linear combination $\sum_{i=0}^\alpha \ell_{\alpha,i} p_{t;0,i}$ on the interval $[t,s]$ of the canonical $(\mathbf{w};t)$-monomials $p_{t;0,0},\dots,p_{t;0,\alpha}$, with $\ell_{\alpha,\alpha} > 0$.

On the other hand, by (3.7) (with $j=0$ and $k=\alpha$), a function p is a $\mathbf{w}$-polynomial of degree α on $[t,s]$ with $p_{\mathbf{w}}^{(\alpha)} > 0$ if and only if p is a linear combination $\sum_{i=0}^\alpha \ell_{\alpha,i} p_{t;0,i}$ on the interval $[t,s]$ of the canonical $(\mathbf{w};t)$-monomials $p_{t;0,0},\dots,p_{t;0,\alpha}$, with $\ell_{\alpha,\alpha} = p_{\mathbf{w}}^{(\alpha)}(t) > 0$.

Thus, $(u_0,\dots,u_n)$ is an *ECT*-system on $[t,s]$ if and only if for some $\mathbf{w} \in \mathbf{W}$ and all $\alpha \in \overline{0,n}$ the function u_α is a $\mathbf{w}$-polynomial of degree α on $[t,s]$ with $(u_\alpha)^{(\alpha)} > 0$.

At that, by [12, p. 380, Remark 1.3], for any point $x \in [t,s]$, the values of all the gauge functions $w_0,\dots,w_n$ at x do not depend on the choice of t and s, as long as the condition $x \in [t,s]$ holds; rather, these values are completely determined by the values of the functions $u_0,\dots,u_n$ in a neighborhood of x.

This completes the proof of Proposition 5.10. ∎

As before, let $(u_0, \dots, u_n)$ be a sequence of real-valued functions of class C^n on a finite closed interval $[a, b] \subset \mathbb{R}$. According to [12, p. 375, Definition 1.1], the cone $C(u_0, \dots, u_n)$ of functions f from the *open* interval (a, b) to $\mathbb{R}$ is defined by the condition

$$f \in C(u_0, \dots, u_n) \overset{\text{def}}{\Longleftrightarrow} \det\bigl[u_i(t_j)\bigr]_{i,j=0}^{n+1} \geq 0$$

for all $t_0, \dots, t_n$ such that $a < t_0 < \dots < t_{n+1} < b$, where $u_{n+1} := f$.

Beginning with [12, Chapter XI, Section 2], it is assumed there that the u_i's are what was referred to in our Remark 3.3 as the *canonical* $(\mathbf{w}; a)$-*monomials*; that is, $u_i = p_{a;0,i}$ for all $i \in \overline{0, n}$. At that, it is tacitly assumed in [12] that all these u_i's are finite on $[a, b]$; compare (3.8). Under all these conditions, for $I = (a, b)$, according to [12, p. 386, Theorem 2.1], the cone $C(u_0, \dots, u_n)$ in [12] essentially coincides with our cone $\mathscr{F}_+^{n+1:n}$, and hence, $\bigcap_{j=k-1}^n C(u_0, \dots, u_j)$ essentially coincides with $\mathscr{F}_+^{k:n}$; here one may recall definition (4.3) of $\mathscr{F}_+^{k:n}$. More precisely,

$$\bigcap_{j=k-1}^{n} C(u_0, \dots, u_j) \lhd \mathscr{F}_+^{k:n}, \tag{5.22}$$

where the symbol $\lhd$ stands for the following: "is contained in, and would coincide with if the class $\mathscr{RC}$ in the definition (2.2) of $\mathscr{D}$ and Df were replaced by the narrower class of all functions in $\mathbb{R}^I$ that are right-continuous on the interval $I \setminus \{b\}$ and left-continuous on the interval $I \setminus \{a\}$" (cf. Remark 2.1).

A result similar to a special case of Theorem 5.5 was stated as two separate theorems in [12, p. 407]: Theorem 5.1 for $k = n+1$ and Theorem 5.2 for $k \in \overline{1, n}$, in the notation of the present paper; the symbol k in the present paper corresponds to $k + 1$ in [12, p. 407, Theorem 5.2]. The latter two theorems in [12] are based on the papers [11] and [30], respectively. In the mentioned special case when I is a finite open interval—so that $I = (a, b)$, $-\infty < a < b < \infty$—Theorems 5.1 and 5.2 in [12, p. 407] characterize convex cones that are in a certain sense dual to the cones $C(u_0, \dots, u_n)$ and $\bigcap_{j=k-1}^n C(u_0, \dots, u_j)$, respectively, where, for each $i \in \overline{0, n}$, the function u_i coincides with the $(\mathbf{w}; a)$-monomial $p_{a;0,i}$ and—somewhat tacitly but crucially—is assumed to be finite; compare again the definition of the u_i's in [12, p. 381, (2.1)] and the definition of the $p_{t;j,m}$'s in (3.3) (in the present paper).

There are a number of differences between our Theorem 5.5 and [12, p. 407, Theorems 5.1, 5.2]. One is that the dual cone in [12] is defined in a traditional manner, as a set of signed measures ν rather than a set of ordered pairs (ν_1, ν_2) of nonnegative measures (cf. the beginning of the discussion in Section 5.1); at that, the total variation of ν in [12, p. 407, Theorems 5.1, 5.2] was assumed to be finite. Also, in these theorems in [12] the interval I is assumed to be finite and open, whereas we allow I to be any interval in $\mathbb{R}$. Moreover, our treatment appears to be more direct, as we define the classes $\mathscr{F}_+^{k:n}$ of multiply monotone functions directly in terms of the gauge functions w_i, rather than in terms of specific $(\mathbf{w}; a)$-monomials $u_i = p_{a;0,i}$. Our method of proof is also more direct, without an explicit characterization or use of the extreme rays of the cones $\mathscr{F}_+^{k:n}$.

However, the most significant difference between [12, p. 407, Theorems 5.1, 5.2] and our Theorem 5.5 is that the former ones impose the mentioned additional, ostensibly innocuous condition of the finiteness of the $(\mathbf{w}; a)$-monomials $u_i = p_{a;0,i}$ for all $i \in \overline{0, n}$. This additional condition rules out, among others, the unit-gauge case with $a = -\infty$, the most important case in applications such as ones considered in [21], [20], and [25], which, in fact, motivated the present paper.

In [7], stochastic orderings for comparing discrete r.v.'s valued in an arbitrary ordered finite grid of nonnegative points were studied, with an emphasis on the effect on such orderings caused by an addition of a point. It appears that the theory presented in this paper can be extended to cones of generalized multiply monotone functions defined on a (not necessarily finite) one-dimensional grid rather than on an interval $I \subseteq \mathbb{R}$. One would then have to use finite differences in place of derivatives.

6. Applications

Here, we shall present applications of our main results to generalized moment comparison inequalities for sums of independent r.v.'s and (super)martingales. Other illustrations and applications, including ones concerning solutions of compositional systems of linear differential inequalities and refinements of the Chebyshev integral association inequality for generalized multiply monotone functions (extending results of Andersson [1], Karlin and Ziegler [13], and Borell [4]), can be found in [24].

The following theorem concerns normal domination of (super)martingales with conditionally bounded differences, which may be further applied to concentration of measure for separately Lipschitz functions, as shown in [20, Section 4]. Let $(S_0, S_1, \dots)$ be a supermartingale relative to a filter $(H_{\le 0}, H_{\le 1}, \dots)$ of σ-algebras, with $S_0 \le 0$ almost surely and differences $X_i := S_i - S_{i-1}$ for $i \in \overline{1, \infty}$. Let E_j and Var_j denote the conditional expectation and variance, respectively, given $H_{\le j}$.

Theorem 6.1. *Suppose that for every $i \in \overline{1, \infty}$ there exist $H_{\le (i-1)}$-measurable r.v.'s C_{i-1} and D_{i-1} and a positive real number s_i such that*

$$C_{i-1} \le X_i \le D_{i-1} \qquad \text{and} \tag{6.1}$$

$$D_{i-1} - C_{i-1} \le 2s_i \tag{6.2}$$

almost surely. Then, for all $f \in \mathscr{F}_+^{1:5}$ (in the unit-gauge case) and all $n \in \overline{1, \infty}$,

$$\mathsf{E} f(S_n) \le \mathsf{E} f(sZ), \tag{6.3}$$

where

$$s := \sqrt{s_1^2 + \cdots + s_n^2}$$

and $Z \sim N(0, 1)$.

If, moreover, $(S_0, S_1, \dots)$ is a martingale relative to $(H_{\le 0}, H_{\le 1}, \dots)$ with $S_0 = 0$ almost surely, then inequality (6.3) holds for all $f \in \mathscr{F}_+^{2:5}$.

Proof. By [20, Theorem 2.1], $\mathsf{E}(S_n - t)_+^5 \le \mathsf{E}(sZ - t)_+^5$ for all $t \in \mathbb{R}$. Also, the conditions that $S_0 \le 0$ almost surely and $(S_0, S_1, \dots)$ be a supermartingale yield

$\mathsf{E}S_n \le 0 = \mathsf{E}sZ$. So, by Corollary 5.8 with $I = \mathbb{R}$, inequality (6.3) holds for all $f \in \mathscr{F}_+^{1:5}$.

Assuming now that $(S_0, S_1, \dots)$ is a martingale with $S_0 = 0$, one has $\mathsf{E}S_n = 0 = \mathsf{E}sZ$. Also, it then follows that $\mathsf{E}S_n^2 = \sum_1^n \mathsf{E}X_i^2 = \sum_1^n \mathsf{E}\,\mathsf{Var}_{i-1}X_i \le \sum_1^n \mathsf{E}|C_{i-1} \times D_{i-1}| \le \frac{1}{4}\sum_1^n \mathsf{E}(|C_{i-1}|+|D_{i-1}|)^2 = \frac{1}{4}\sum_1^n \mathsf{E}(D_{i-1}-C_{i-1})^2 \le \sum_1^n s_i^2 = s^2 = \mathsf{E}(sZ)^2$; the first inequality in the above chain of equalities and inequalities follows by [20, (2.12)] and the third equality in this chain follows because, by (6.1), $C_{i-1} \le \mathsf{E}_{i-1}X_i = 0 \le D_{i-1}$ almost surely. So, under the additional conditions stated in the last sentence of Theorem 6.1, (6.3) holds indeed for all $f \in \mathscr{F}_+^{2:5}$. □

Recalling the definition of the set $\mathscr{H}_+^{i:n}$ in the beginning of Section 4.1 and Proposition 4.2, one sees that Theorem 6.1 provides an extension of inequality (6.3), from all $f \in \mathscr{H}_+^{0:5}$ to all $f \in \mathscr{F}_+^{1:5}$ (or, under the additional, martingale conditions on $(S_0, S_1, \dots)$, even to all $f \in \mathscr{F}_+^{2:5}$). Quite similarly one can extend other results in [20], including: (i) Theorem 2.6 as far as it concerns (2.3); (ii) inequality (4.3); and (iii) Theorem 4.4 and Corollary 4.8 as far as they concern (4.3) (all the references in this sentence are to [20]).

Among other applications is the main result in [25], whose proof is based, in part, on Corollary 5.9 in the present paper. In particular, that result in [25] implies the following.

Theorem 6.2. *Let $X_1, \dots, X_n$ be any* nonnegative *independent r.v.'s such that for some nonnegative real numbers $m, m_1, \dots, m_n, s, s_1, \dots, s_n$ one has $0 < s \le m^2/n$,*

$$\mathsf{E}X_i \ge m_i \qquad \text{and} \qquad \mathsf{E}X_i^2 \le s_i$$

for all $i \in \overline{1,n}$ and

$$m_1 + \dots + m_n \ge m \qquad \text{and} \qquad s_1 + \dots + s_n \le s.$$

Let $Y_1, \dots, Y_n$ denote any independent identically distributed r.v.'s such that

$$\mathsf{P}\Big(Y_1 = \frac{s}{m}\Big) = 1 - \mathsf{P}(Y_1 = 0) = \frac{m^2}{ns}.$$

Then,

$$\mathsf{E}f(S_n) \le \mathsf{E}f(Y_1 + \dots + Y_n) \le \mathsf{E}f\Big(\frac{s}{m}\Pi_{m^2/s}\Big) \le \mathsf{E}f(m + Z\sqrt{s})$$

for all $f \in \mathscr{F}_-^{1:3}$, where Π_λ is any r.v. having the Poisson distribution with parameter $\lambda \in (0, \infty)$ and Z is any standard normal r.v.

Variants of this result for the classes $\mathscr{F}_-^{2:2}$, $\mathscr{F}_-^{3:2}$, $\mathscr{F}_-^{2:3}$, $\mathscr{F}_-^{3:3}$, and $\mathscr{F}_-^{4:3}$ of functions in place of $\mathscr{F}_-^{1:3}$ are given in [25, Remark 2.5].

Similarly, it would be quite convenient to use Corollary 5.6 of the present paper in place of [21, Lemmas 1–4] and in place of [23, Lemmas 4.5, 4.6]; those lemmas in [21] and [23] were proved in a rather ad hoc manner.

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