Banach J. Math. Anal. 10 (2016), no. 3, 547–565
http://dx.doi.org/10.1215/17358787-3607288
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

LINEAR MAPS BETWEEN C*-ALGEBRAS PRESERVING EXTREME POINTS AND STRONGLY LINEAR PRESERVERS

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To the memory of Professor Wilhelm Kaup

Communicated by N.-C. Wong

Abstract. We study new classes of linear preservers between C*-algebras and between JB*-triples. Let E and F be JB*-triples with $\partial_e(E_1) \neq \emptyset$. We prove that every linear map $T : E \to F$ strongly preserving Brown–Pedersen quasi-invertible elements is a triple homomorphism. Among the consequences, we establish that, given two unital C*-algebras A and B, for each linear map T strongly preserving Brown–Pedersen quasi-invertible elements, there exists a Jordan *-homomorphism $S : A \to B$ satisfying $T(x) = T(1)S(x)$ for every $x \in A$. We also study the connections between linear maps strongly preserving Brown–Pedersen quasi-invertibility and other clases of linear preservers between C*-algebras like Bergmann-zero pairs preservers, Brown–Pedersen quasi-invertibility preservers, and extreme points preservers.

1. Introduction

Let X be a Banach space. In many favorable cases, the set $\partial_e(X_1)$, of all extreme points of the closed unit ball, X_1, of X, reveals many of the geometric properties of the whole Banach space X. There are spaces X with $\partial_e(X_1) = \emptyset$; however, the Krein–Milman theorem guarantees that $\partial_e(X_1)$ is nonempty when X is a dual space.

Received Jul. 22, 2015; Accepted Nov. 11, 2015.
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2010 *Mathematics Subject Classification.* Primary 47B49; Secondary 15A09, 46L05, 47B48.
Keywords. C*-algebra, JB*-triple, triple homomorphism, linear preservers, extreme points preserver, strongly Brown–Pedersen quasi-invertibility preserver.

Let A be a C*-algebra. It is known that $\partial_e(A_1) \neq \emptyset$ if and only if A is unital (see [29, Theorem I.10.2(i)]). A theorem due to R. V. Kadison establishes that the extreme points of the closed unit ball of a C*-algebra A are precisely the maximal partial isometries of A, that is, partial isometries $e \in A$ satisfying $(1 - ee^*)A(1-e^*e) = 0$ (cf. [29, Theorem I.10.2]). When A is commutative, the unitary elements in A are precisely the extreme points of the closed unit ball of A. The same statement remains true when A is a finite von Neumann algebra (cf. [24, Lemma 2]). For a general unital C*-algebra A, every unitary element in A is an extreme point of the closed unit ball of A; however, the reciprocal statement is, in general, false (e.g., a nonsurjective isometry in $B(H)$ is not a unitary element in this C*-algebra).

Let A and B be unital C*-algebras. One of the consequences derived from the Russo–Dye theorem assures that a linear mapping $T : A \to B$ mapping unitary elements in A to unitary elements in B admits a factorization of the form $T(a) = uS(a)$ $(a \in A)$, where u is a unitary in B and S is a unital Jordan *-homomorphism (cf. [27, Corollary 2]). We recall that a linear map $T : A \to B$ between Banach algebras is a *Jordan homomorphism* if $T(a^2) = T(a)^2$ for all $a \in A$ (equivalently, $T(ab + ba) = T(a)T(b) + T(b)T(a)$ for all $a, b \in A$). If A and B are unital, T is called *unital* if $T(1) = 1$. If A and B are C*-algebras and $T(a^*) = T(a)^*$, for every $a \in A$, then T is called *symmetric*. Symmetric Jordan homomorphisms are named *Jordan *-homomorphisms*.

Consequently, the problem of studying the linear maps $T : A \to B$ such that $T(\partial_e(A_1)) \subseteq \partial_e(B_1)$ is a more general challenge, which is directly motivated by the aforementioned consequence of the Russo–Dye theorem. We only know partial answers to this problem. Concretely, V. Mascioni and L. Molnár studied the linear maps on a von Neumann factor M which preserve the extreme points of the unit ball of M. They prove that if M is infinite, then every linear mapping T on M preserving extreme points admits a factorization of the form $T(a) = uS(a)$ $(a \in M)$, where u is a (fixed) unitary in M, and S is either a unital *-homomorphism or a unital *-anti-homomorphism (see [24, Theorem 1]). Theorem 2 in [24] states that, for a finite von Neumann algebra M, a linear map $T : M \to M$ preserves extreme points of the unit ball of M if and only if there exist a unitary operator $u \in M$ and a unital Jordan *-homomorphism $S : M \to M$ such that $T(a) = uS(a)$ $(a \in A)$. In [22], L. E. Labuschagne and V. Mascioni study linear maps between C*-algebras whose adjoints preserve extreme points of the dual ball.

The above results of Mascioni and L. Molnár are the most conclusive answers we know about linear maps between unital C*-algebras preserving extreme points. In this note, we shall revisit the problem in full generality. We present several counterexamples to illustrate that the conclusions proved by Mascioni and Molnár for von Neumann factors need not be true for linear mappings preserving extreme points between unital C*-algebras (cf. Remarks 5.9 and 5.10). It seems natural to ask whether a different class of linear preservers satisfies the same conclusions found by Mascioni and Molnár.

Every unital Jordan homomorphism between Banach algebras *strongly preserves invertibility*; that is, $T(a^{-1}) = T(a)^{-1}$ for every invertible element $a \in A$.

Moreover, Hua's theorem (see [15]) states that every unital additive map between skew fields that strongly preserves invertibility is either an isomorphism or an anti-isomorphism.

Let A be a Banach algebra. Recall that an element $a \in A$ is called *regular* or *von Neumann regular* if there is b in A satisfying $aba = a$ and $b = bab$. Given a and b in a C*-algebra A, we shall say that b is a *Moore–Penrose inverse* of a if $a = aba$, $b = bab$, and ab and ba are self-adjoint. It is known that every regular element a in A admits a unique Moore–Penrose inverse that will be denoted by $a^\dagger$ (see [14]). Let $A^\dagger$ denote the set of regular elements in the C*-algebra A.

We say that a linear map T between C*-algebras A and B *strongly preserves Moore–Penrose invertibility* if $T(a^\dagger) = T(a)^\dagger$ for all $a \in A^\dagger$. It is known that every Jordan *-homomorphism strongly preserves Moore–Penrose invertibility. In [25], M. Mbekhta proved that a surjective unital bounded linear map from a real rank-zero C*-algebra to a prime C*-algebra strongly preserves Moore–Penrose invertibility if and only if it is either a *-homomorphism or an *-anti-homomorphism. Recently, in [6] the first three authors of this note show that a linear map T strongly preserving Moore–Penrose invertibility between C*-algebras A and B is a Jordan *-homomorphism multiplied by a regular element of B commuting with the image of T whenever the domain C*-algebra A is unital and linearly spanned by its projections, or when A is unital and has real rank zero and T is bounded. It is also proved that every bijective linear map strongly preserving Moore–Penrose invertibility from a unital C*-algebra with essential socle is a Jordan *-isomorphism multiplied by an involutory element. The problem for linear maps strongly preserving Moore–Penrose invertibility between general C*-algebras remains open.

The set, A_q^{-1}, of quasi-invertible elements in a unital C*-algebra A was introduced by L. Brown and G. K. Pedersen as the set $A^{-1}\partial_e(A_1)A^{-1}$, where A^{-1} and $\partial_e(A_1)$ denote the set of invertible elements in A and the set of extreme points of the closed unit ball of A, respectively (see [2]). It is known that $a \in A_q^{-1}$ if and only if there exists $b \in A$ such that $B(a, b) = 0$, where $B(a, b)$ denotes the Bergmann operator on A associated with the pair (a, b) (cf. [2, Theorem 1.1] and [18, Theorem 11], and see Section 2 for details and definitions).

The notion of a quasi-invertible element was extended by F. B. Jamjoom, A. A. Siddiqui, and H. M. Tahlawi to the wider setting of JB*-triples. An element x in a JB*-triple E is called *Brown–Pedersen quasi-invertible* if there exists $y \in E$ such that $B(x, y) = 0$ (cf. [18]). The element y is called a Brown–Pedersen quasi-inverse of x. It is known that $B(x, y) = 0$ implies $B(y, x) = 0$. Moreover, the Brown–Pedersen quasi-inverse of an element is not unique. Indeed, if $B(x, y) = 0$, then it can be checked that $B(x, Q(y)(x)) = 0$, and so, for any Brown–Pedersen quasi-inverse y of x, $Q(y)(x)$ also is a Brown–Pedersen quasi-inverse of x. It is established in [18, Theorems 6 and 11] that an element x in E is Brown–Pedersen quasi-invertible if and only if it is (von Neumann) regular and its range tripotent is an extreme point of the closed unit ball of E; equivalently, there exists a complete tripotent $v \in E$ such that x is positive and invertible in $E_2(v)$. Every regular element x in E admits a unique generalized inverse, which is denoted by $x^\wedge$ (see

Section 2 for more details). In particular, the set, E_q^{-1}, of all Brown–Pedersen quasi-invertible elements in E contains all extreme points of the closed unit ball of E.

We consider in this article a new class of linear preserver. A linear map T between JB*-triples *strongly preserves Brown–Pedersen quasi-invertibility* if T preserves Brown–Pedersen quasi-invertibility and $T(x^\wedge) = T(x)^\wedge$ for every $x \in E_q^{-1}$. In the main result of this note, we prove the following: Let A and B be unital C*-algebras, considered as JB*-triples. Let $T : A \to B$ be a linear map strongly preserving Brown–Pedersen quasi-invertible elements. Then there exists a Jordan *-homomorphism $S : A \to B$ satisfying $T(x) = T(1)S(x)$ for every $x \in A$ (see Theorem 5.13).

In Section 5, we also explore the connections between linear maps strongly preserving Brown–Pedersen quasi-invertibility and other classes of linear preservers between C*-algebras such as Bergmann-zero pairs preservers, Brown–Pedersen quasi-invertibility preservers, and extreme points preservers.

The reader should have realized at this point that novelties here rely on results and tools of Jordan theory and JB*-triples (see Section 2 for definitions). The research on linear preservers on C*-algebras benefits from new results on linear preservers on JB*-triples. In Theorem 3.2, we prove that every linear map strongly preserving regularity between JB*-triples E and F with $\partial_e(E_1) \neq \emptyset$ is a triple homomorphism (i.e., it preserves triple products). We complement this result by showing that the same conclusion remains true for every bounded linear operator strongly preserving regularity from a weakly compact JB*-triple into another JB*-triple (see Theorem 4.1). The assumption of continuity cannot be dropped in the result for weakly compact JB*-triples (cf. Remark 4.2). The most significant result (Theorem 5.12) assures that every linear map strongly preserving Brown–Pedersen quasi-invertible elements between JB*-triples E and F, with $\partial_e(E_1) \neq \emptyset$, is a triple homomorphism.

2. Preliminaries

As we commented in the Introduction, in this article, we employ techniques and results in JB*-triple theory to study new classes of linear preservers between C*-algebras in connection with linear maps preserving extreme points. For this purpose, we shall regard every C*-algebra as an element in the wider class of JB*-triples. Following [20], a JB*-triple is a complex Banach space E together with a continuous triple product $\{\cdot,\cdot,\cdot\} : E \times E \times E \to E$, which is conjugate linear in the middle variable and symmetric and bilinear in the outer variables satisfying the following:

(a) $L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) - L(x,L(b,a)y)$, where $L(a,b)$ is the operator on E given by $L(a,b)x = \{a,b,x\}$;
(b) $L(a,a)$ is a Hermitian operator with nonnegative spectrum;
(c) $\|L(a,a)\| = \|a\|^2$.

For each x in a JB*-triple E, $Q(x)$ will stand for the conjugate linear operator on E defined by the assignment $y \mapsto Q(x)y = \{x,y,x\}$.

The *Bergmann operator*, $B(x,y)$, associated with a pair of elements $x,y \in E$ is the mapping defined by

$$B(x,y) = I_E - 2L(x,y) + Q(x)Q(y).$$

Every C*-algebra is a JB*-triple via the triple product given by

$$2\{x,y,z\} = xy^*z + zy^*x, \tag{2.1}$$

and every JB*-algebra is a JB*-triple under the triple product

$$\{x,y,z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*. \tag{2.2}$$

It is worth mentioning that, by the *Kaup–Banach–Stone* theorem, a linear surjection between JB*-triples is an isometry if and only if it is a triple isomorphism (cf. [20, Proposition 5.5]). We recall that a linear map $T : E \to F$ between JB*-triples is a *triple homomorphism* if

$$T\big(\{x,y,z\}\big) = \big\{T(x), T(y), T(z)\big\} \quad \text{for every } x,y,z \in E.$$

It follows, among many other consequences, that when a JB*-algebra J is a JB*-triple for a suitable triple product and the original norm, then the latter coincides with the one defined in (2.2).

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual; see [1]). It is known that the triple product of a JBW*-triple is separately weak* continuous (see [1]). The second dual of a JB*-triple E is a JBW*-triple with a product extending the product of E (see [9]).

An element e in a JB*-triple E is said to be a *tripotent* if $\{e,e,e\} = e$. Each tripotent e in E gives rise to the following decomposition of E:

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0,1,2$, $E_i(e)$ is the $\frac{i}{2}$ eigenspace of $L(e,e)$ (cf. [23, Theorem 25]). The natural projections of E onto $E_i(e)$ will be denoted by $P_i(e)$. This decomposition is termed the *Peirce decomposition* of E with respect to the tripotent e. The Peirce decomposition satisfies certain rules known as *Peirce arithmetic*:

$$\big\{E_i(e), E_j(e), E_k(e)\big\} \subseteq E_{i-j+k}(e)$$

if $i - j + k \in \{0,1,2\}$ and is zero otherwise. In addition,

$$\big\{E_2(e), E_0(e), E\big\} = \big\{E_0(e), E_2(e), E\big\} = 0.$$

We observe that, for a tripotent $e \in E$, $B(e,e) = P_0(e)$.

The Peirce space $E_2(e)$ is a JB*-algebra with product $x \circ_e y := \{x,e,y\}$ and involution $x^{\sharp_e} := \{e,x,e\}$.

A tripotent e in E is called *complete* if the equality $E_0(e) = 0$ holds. When $E_2(e) = \mathbb{C}e \neq \{0\}$, we say that e is *minimal*.

For each element x in a JB*-triple E, we shall denote $x^{[1]} := x$, $x^{[3]} := \{x,x,x\}$, and $x^{[2n+1]} := \{x,x,x^{[2n-1]}\}$ ($n \in \mathbb{N}$). The symbol E_x will stand for the JB*-subtriple generated by the element x. It is known that E_x is JB*-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω contained in $(0, \|x\|]$ such that $\Omega \cup \{0\}$ is compact, where $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at

0 with the triple product $(f,g,h) \mapsto f\overline{g}h$. It is also known that there exists a triple isomorphism Ψ from E_x onto $C_0(\Omega)$ satisfying $\Psi(x)(t) = t$ $(t \in \Omega)$ (cf. [20, Corollary 1.15] and [13]). The set $\overline{\Omega} = \mathrm{Sp}(x)$ is called the *triple spectrum* of x. We should note that $C_0(\mathrm{Sp}(x)) = C(\mathrm{Sp}(x))$ whenever $0 \notin \mathrm{Sp}(x)$.

Therefore, for each $x \in E$, there exists a unique element $y \in E_x$ satisfying $\{y,y,y\} = x$. The element y, denoted by $x^{[\frac{1}{3}]}$, is termed the *cubic root* of x. We can inductively define $x^{[\frac{1}{3^n}]} = (x^{[\frac{1}{3^{n-1}}]})^{[\frac{1}{3}]}$, $n \in \mathbb{N}$. The sequence $(x^{[\frac{1}{3^n}]})$ converges in the weak* topology of E^{**} to a tripotent denoted by $r(x)$ and called the *range tripotent* of x. The tripotent $r(x)$ is the smallest tripotent $e \in E^{**}$ satisfying the property that x is positive in the JBW*-algebra $E_2^{**}(e)$ (cf. [10, Lemma 3.3]).

Regular elements in Jordan triple systems and JB*-triples have been deeply studied in [11], [21], and [5]. An element a in a JB*-triple E is called *von Neumann regular* if there exists (a unique) $b \in E$ such that $Q(a)(b) = a$, $Q(b)(a) = b$, and $Q(a)Q(b) = Q(b)Q(a)$, or, equivalently, $Q(a)(b) = a$ and $Q(a)(b^{[3]}) = b$. The element b is called the *generalized inverse* of a. We observe that every tripotent e in E is von Neumann regular and its generalized inverse coincides with itself.

Throughout this note, we shall denote by $E^\wedge$ the set of regular elements in a JB*-triple E, and, for an element $a \in E^\wedge$, $a^\wedge$ will stand for its generalized inverse.

To simplify notation, for a C*-algebra A, let E_A denote the JB*-triple with underlying Banach space A and the triple product defined by (2.1). Let a be an element in E_A. Then the mapping $Q(a)$ is given by $Q(a)(x) = \{a,x,a\} = ax^*a$. Thus, a is Moore–Penrose invertible in A with Moore–Penrose inverse $a^\dagger$ if and only if $a \in E_A^\wedge$ and $a^\wedge = (a^\dagger)^* = (a^*)^\dagger$.

Every triple homomorphism $T : E \to F$ between JB*-triples *strongly preserves regularity*; that is, $T(x^\wedge) = T(x)^\wedge$ for every $x \in E^\wedge$. In [7], the authors characterized the triple homomorphisms between C*-algebras as the linear maps strongly preserving regularity. As a consequence, it is proved that a self-adjoint linear map from a unital C*-algebra A into a C*-algebra B is a triple homomorphism if and only if it strongly preserves Moore–Penrose invertibility (see [7, Theorem 3.5]).

3. Linear maps strongly preserving regularity on JB*-triples

It is known that a nonzero element a in a JB*-triple E is von Neumann regular if and only if $Q(a)(E)$ is closed if and only if the range tripotent $r(a)$ of a lies in E and a is a positive and invertible element in the JB*-algebra $E_2(r(a))$ (cf. [11], [21], or [5]). Moreover, when a is von Neumann regular,

$$L(a, a^\wedge) = L(a^\wedge, a) = L\big(r(a), r(a)\big)$$

and

$$Q(a)Q(a^\wedge) = Q(a^\wedge)Q(a) = P_2\big(r(a)\big)$$

(cf. [5, Theorem 3.4] and its proof). Recall that an element a in a unital Jordan algebra $J = (J, \circ)$ is *invertible* if there exists a (unique) element $b \in J$ such that $a \circ b = \mathbf{1}$ and $a^2 \circ b = a$; equivalently, U_a is invertible with inverse U_b, where U_a is defined by $U_a(x) = 2a\circ(a\circ x) - a^2 \circ x$ (see [16, p. 52, Theorem 13]). If a is invertible, its inverse is denoted by a^{-1}. Moreover, if a and b are invertible elements in the

Jordan algebra J such that $a - b^{-1}$ is also invertible, then $a^{-1} + (b^{-1} - a)^{-1}$ is invertible, and the so-called Hua's identity,

$$\left(a^{-1} - (a - b^{-1})^{-1}\right)^{-1} = \left(a^{-1} + (b^{-1} - a)^{-1}\right)^{-1} = a - U_a(b), \tag{3.1}$$

holds (see [16, p. 54, Exercise 3]).

A linear map $T : E \to F$ between JB*-triples *strongly preserves regularity* if $T(x^\wedge) = T(x)^\wedge$ for every $x \in E^\wedge$.

The next result is inspired by [6, Lemma 3.1].

Proposition 3.1. *Let E and F be JB*-triples, and let $T : E \to F$ be a linear map such that $T(x^\wedge) = T(x)^\wedge$ for every $x \in E^\wedge$. Then*

$$T(x^{[3]}) = T(x)^{[3]}$$

for every $x \in E^\wedge$.

Proof. Let $x \in E^\wedge \setminus \{0\}$. Let $e = r(x)$ be the range tripotent of x in E^{**}. As we have just mentioned, $e \in E$, and x is positive and invertible in the JB*-algebra $E_2(e)$ with inverse $x^\wedge$ and $0 \notin \mathrm{Sp}(x)$. We identify E_x (the JB*-subtriple of E generated by x) with $C(\mathrm{Sp}(x))$ in such a way that x corresponds to the function $t \mapsto t$; hence, for every $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \|x^\wedge\|^{-2}$, the element $\lambda x^\wedge - x$ is invertible in E_x, and hence invertible in $E_2(e)$ with inverse $(\lambda x^\wedge - x)^\wedge$. In this case, $x^\wedge + (\lambda x^\wedge - x)^\wedge$ is invertible in E_x (and in $E_2(e)$).

Further, the inverses of $x - \lambda x^\wedge$ and $x^\wedge - (x - \lambda x^\wedge)^\wedge$ in E_x (or in $E_2(e)$) are their generalized inverses in E (let us recall that the triple product induced on $E_2(e)$ by the Jordan *-algebra structure coincides with its original triple product, and $Q(x) = U_x \circ \sharp$ for every $x \in E$). By Hua's identity applied to $a = x$ and $b = \lambda^{-1} x$, we obtain

$$x - \lambda^{-1} x^{[3]} = \left(x^\wedge - (x - \lambda x^\wedge)^\wedge\right)^\wedge$$

(see (3.1)). Let $x \in E^\wedge$. We may assume that $T(x) \neq 0$. Since T strongly preserves regularity, $T(x)^\wedge = T(x^\wedge)$. Thus, for $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \mathrm{Min}\{\|x^\wedge\|^{-2}, \|T(x)^\wedge\|^{-2}\}$, we have

$$T(x) - \lambda^{-1} T(x)^{[3]} = \left(T(x)^\wedge - \left(T(x) - \lambda T(x)^\wedge\right)^\wedge\right)^\wedge.$$

Since T is linear and strongly preserves regularity, it follows that

$$\begin{aligned} T(x) - \lambda^{-1} T(x)^{[3]} &= \left(T(x)^\wedge - \left(T(x) - \lambda T(x)^\wedge\right)^\wedge\right)^\wedge \\ &= \left(T(x^\wedge) - T(x - \lambda x^\wedge)^\wedge\right)^\wedge \\ &= T\left(\left(x^\wedge - (x - \lambda x^\wedge)^\wedge\right)^\wedge\right) = T(x) - \lambda^{-1} T(x^{[3]}), \end{aligned}$$

and thus $T(x^{[3]}) = T(x)^{[3]}$. □

Recall that two elements a, b in a JB*-triple E are *orthogonal* (written as $a \perp b$) if $L(a, b) = 0$ (see [4, Lemma 1] for several equivalent reformulations).

Notice that a JB*-triple might contain no nontrivial tripotents (consider, e.g., the C*-algebra $C_0(0, 1]$ of all complex-valued continuous functions on $[0, 1]$ vanishing at 0). However, since the complete tripotents of a JB*-triple E coincide with the extreme points of its closed unit ball , every JBW*-triple contains a large set of complete tripotents (see [8, Theorem 3.2.3]).

Let us recall that, by Lemma 2.1 in [17], an element a in a JB*-triple E is Brown–Pedersen quasi-invertible if and only if a is regular and $\{a\}^\perp = \{0\}$, where $\{a\}^\perp = \{b \in E\colon a \perp b\}$.

Theorem 3.2. *Let E and F be JB*-triples with $\partial_e(E_1) \neq \emptyset$. Let $T : E \to F$ be a linear map strongly preserving regularity. Then T is a triple homomorphism.*

Proof. Pick a complete tripotent $e \in E$. For every $x \in E$, let $\lambda \in \mathbb{C}$ with $|\lambda| > \|P_2(e)(x)\|$. It is clear that $P_2(e)(x-\lambda e) = P_2(e)(x)-\lambda e$ is invertible in the unital JB*-algebra $E_2(e)$. It follows from [17, Lemma 2.2] that $x-\lambda e$ is Brown–Pedersen quasi-invertible. We know by Proposition 3.1 that

$$T\big((x-\lambda e)^{[3]}\big) = T(x-\lambda e)^{[3]}.$$

Since the above identity holds for every $\lambda \in \mathbb{C}$, with $|\lambda| > \|P_2(e)(x)\|$, we deduce that

$$T(x^{[3]}) = T(x)^{[3]}$$

for every $x \in E$. The polarization formula

$$8\{x,y,z\} = \sum_{k=0}^{3}\sum_{j=1}^{2} i^k(-1)^j\big(x+i^k y+(-1)^j z\big)^{[3]} \tag{3.2}$$

and the linearity of T assure that T is a triple homomorphism. □

The particularization of the previous result to the setting of C*-algebras seems to be a new result.

Corollary 3.3. *Let $T : A \to B$ be a linear map strongly preserving regularity between C*-algebras. Suppose that $\partial_e(A_1) \neq \emptyset$. Then T is a triple homomorphism.*

4. Maps strongly preserving regularity on weakly compact JB*-triples

The notions of compact and weakly compact elements in JB*-triples are due to L. J. Bunce and C.-H. Chu [3]. Recall that an element a in a JB*-triple E is said to be *compact* or *weakly compact* if the mapping $Q(a)$ is compact or weakly compact, respectively. These notions extend, in a natural way, the corresponding definitions in the settings of C*- and JB*-algebras. A JB*-triple E is weakly compact (resp., compact) if every element in E is weakly compact (resp., compact).

In a JB*-triple, the set of weakly compact elements is, in general, strictly bigger than the set of compact elements (cf. [3, Theorem 3.6]). A nonzero tripotent e in E is called *minimal* whenever $E_2(e) = \mathbb{C}e$. The socle, $\mathrm{soc}(E)$, of a JB*-triple E is the linear span of all minimal tripotents in E. Following [3], the symbol $K_0(E)$ denotes the norm closure of $\mathrm{soc}(E)$. By [3, Lemma 3.3 and Proposition 4.7], the triple ideal $K_0(E)$ coincides with the set of all weakly compact elements in E; hence a JB*-triple E is weakly compact whenever $E = K_0(E)$. Every finite sum of mutually orthogonal minimal tripotents in a JB*-triple E lies in the socle of E. It is also known that an element a in a JB*-triple E is weakly compact if and only if $L(a,a)$ is a weakly compact operator (see [3]). Therefore, for each tripotent e in the socle of E, $P_1(e) = 2L(e,e) - P_2(e) = 2L(e,e) - Q(e)^2$ is a weakly compact operator on E (cf. [12, Section 2]).

It is well known that every element in the socle of a JB*-triple is regular. Moreover, for every JB*-triple E,

$$E^\wedge + \operatorname{soc}(E) \subseteq E^\wedge.$$

Indeed, given $a \in E^\wedge$ and $x \in \operatorname{soc}(E)$,

$$(a+x) - Q(a+x)(a^\wedge) = x - 2\{a, a^\wedge, x\} - \{x, a^\wedge, x\} \in \operatorname{soc}(E) \subseteq E^\wedge.$$

By McCoy's lemma, $a + x \in E^\wedge$ (see [26]). Let E, F be JB*-triples. Let us assume that E has nonzero socle, and let $T : E \to F$ be a linear map strongly preserving regularity. The polarization formula (3.2) and Proposition 3.1 show that $T(\{x,y,z\}) = \{T(x), T(y), T(z)\}$ whenever one of the elements x, y, z is regular and the others lie in the socle.

Theorem 4.1. *Let E, F be JB*-triples with E weakly compact. Let $T : E \to F$ be a bounded linear map strongly preserving regularity. Then T is a triple homomorphism.*

Proof. We know from Proposition 3.1 that T preserves cubes of regular elements. Since every element in the socle of a JB*-triple is regular, it follows that $T(x^{[3]}) = T(x)^{[3]}$ for every $x \in \operatorname{soc}(E)$. Since $E = K_0(E) = \overline{\operatorname{soc}(E)}$, the continuity of T, together with the norm continuity of the triple product and the polarization identity, proves that T is a triple homomorphism. □

In the next example, we show that the continuity assumption cannot be dropped from the hypothesis in the previous theorem (even in the setting of C*-algebras).

Remark 4.2. Let c_0 denote the C*-algebra of all scalar null sequences. It is clear that c_0 is a weakly compact JB*-triple with $\operatorname{soc}(c_0) = c_{00}$; that is, the subspace of eventually zero sequences in c_0. Let $\{e_n\}$ denote the standard coordinate (Schauder) basis of c_0. We extend this basis via Zorn's lemma to an algebraic (Hamel) basis of c_0, say, $B = \{e_n\} \cup \{z_n\}$.

We define $T : c_0 \to c_0$ as the linear (unbounded) mapping given by

$$T(e_n) = e_n, \qquad T(z_n) = n z_n.$$

Clearly, T is not a triple homomorphism, but it strongly preserves regularity. Let us note that $c_0^\wedge = c_{00}$ and $T(c_{00}) = c_{00}$.

5. Linear maps strongly preserving Brown–Pedersen quasi-invertibility

In [12], the authors proved that Bergmann operators can be used to characterize the relation of being orthogonal in JB*-triples. More concretely, it is proved in [12, Proposition 7] that, for any element x in a JB*-triple E with $\|x\| < \sqrt{2}$, the orthogonal annihilator of x in E coincides with the set of all fixed points of the Bergmann operator $B(x,x)$. It is also obtained in the aforementioned paper that a norm one element e in a JB*-triple E is a tripotent if and only if $B(e,e)(E) = \{e\}^{\perp}$ (cf. [12, Proposition 9]).

Having in mind all the characterizations of tripotents and Brown–Pedersen quasi-invertible elements commented above, and recalling that extreme points of the closed unit ball of a JB*-triple E are precisely the complete tripotents in E, it can be deduced that the equivalence

$$e \in \partial_e(E_1) \Leftrightarrow B(e,e) = 0 \tag{5.1}$$

holds for every $e \in E$.

Let $T : E \to F$ be a linear map between JB*-triples. We introduce the following definitions.

Definition 5.1. *T preserves Brown–Pedersen quasi-invertibility* if $T(E_q^{-1}) \subseteq F_q^{-1}$; that is, T maps Brown–Pedersen quasi-invertible elements in E to Brown–Pedersen quasi-invertible elements in F.

Definition 5.2. *T preserves Bergmann-zero pairs* if

$$B(x,y) = 0 \Rightarrow B\big(T(x), T(y)\big) = 0.$$

Definition 5.3. *T strongly preserves Brown–Pedersen quasi-invertibility* if T preserves Brown–Pedersen quasi-invertibility and $T(x^\wedge) = T(x)^\wedge$ for every $x \in E_q^{-1}$.

Definition 5.4. *T preserves extreme points* if $T(\partial_e(E_1)) \subseteq \partial_e(F_1)$.

It is worth noting that all definitions above make sense for linear operators between C*-algebras. In this article, we employ Jordan techniques to study these kinds of mappings, and so we set the above definitions in the most general setting.

Suppose $T : E \to F$ is a linear mapping strongly preserving Brown–Pedersen quasi-invertibility between two JB*-triples. Suppose $u \in \partial_e(E_1)$. Then u is Brown–Pedersen quasi-invertible with $u^\wedge = u$. It follows from our assumptions that $T(u)$ is Brown–Pedersen quasi-invertible and $T(u)^\wedge = T(u^\wedge) = T(u)$. In such a case, $\{T(u), T(u), T(u)\} = Q(T(u))(T(u)) = T(u)$ is a tripotent and Brown–Pedersen quasi-invertible, which implies that $T(u) \in \partial_e(E_1)$ (cf. [17, Lemma 2.1]). We have therefore shown that every linear mapping between JB*-triples strongly preserving Brown–Pedersen quasi-invertibility also preserves extreme points.

The characterization of the extreme points of the closed unit ball of a JB*-triple given in (5.1) implies that every linear mapping between JB*-triples preserving Bergmann-zero pairs also preserves extreme points.

Clearly, a linear mapping $T : E \to F$ preserving Bergmann-zero pairs maps Brown–Pedersen quasi-invertible elements in E to Brown–Pedersen quasi-invertible elements in F.

Therefore, for every linear mapping T between JB*-triples, the following implications hold:

$$\begin{tikzcd}[column sep=large, row sep=large]
\begin{array}{c} T \text{ preserves} \\ \text{Bergmann-zero pairs} \end{array} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \begin{array}{c} T \text{ preserves} \\ \text{BP quasi-invertible elements} \end{array} \arrow[d, Rightarrow, "/" marking, "\text{(Remark 5.10)}"', shift right] \\
T \text{ preserves extreme points} \arrow[r, Rightarrow, "/" marking, "\text{(Remark 5.10)}", shift right] & \begin{array}{c} T \text{ strongly preserves} \\ \text{BP quasi-invertible elements} \end{array} \arrow[u, Rightarrow, shift right] \arrow[l, Rightarrow, shift right]
\end{tikzcd}$$

The other implications are, for the moment, unknown. We have already commented that V. Mascioni and L. Molnár characterized the linear maps on a von Neumann factor M preserving the extreme points of the unit ball of M in [24]. According to our terminology, they prove that, for a von Neumann factor M, a linear map $T : M \to M$ such that $B(T(a), T(a)) = 0$ whenever $B(a, a) = 0$ is a unital Jordan *-homomorphism multiplied by a unitary element (see [24, Theorems 1–2]).

Suppose $T : E \to E$ is a linear mapping between JB*-triples which preserves Bergmann-zero pairs. Given a Brown–Pedersen quasi-invertible element x, with generalized inverse $x^\wedge$, we have

$$B(x, x^\wedge) = B(x^\wedge, x) = B\big(r(x), r(x)\big) = 0,$$

and hence $B(T(x), T(x^\wedge)) = 0$. This shows that

$$Q\big(T(x)\big)\big(T(x^\wedge)\big) = T(x) \qquad \text{and} \qquad Q\big(T(x^\wedge)\big)\big(T(x)\big) = T(x^\wedge).$$

However, $T(x^\wedge)$ may not coincide, in general, with $T(x)^\wedge$. We shall present in Remark 5.10 an example of a linear operator preserving Bergmann-zero pairs which is not a strongly Brown–Pedersen quasi-invertibility preserver.

We mainly focus our study on maps between C*-algebras. Let A be a unital C*-algebra A. It is easy to see that, for an element a in A,

$$B(a, a)(x) = (1 - aa^*)x(1 - a^*a) \quad \text{for all } x \in A.$$

Moreover, it is also a well-known fact that the extreme points of the closed unit ball of A are precisely those elements v in A for which $(1 - vv^*)A(1 - v^*v) = \{0\}$ (see [29, Theorem I.10.2]). Hence a linear operator $T : A \to A$ preserves extreme points if and only if $B(a, a) = 0$ implies $B(T(a), T(a)) = 0$.

Let $T : A \to B$ be a linear map between unital C*-algebras which preserves extreme points. Since for every unitary element $u \in A$, $B(u, u) = 0$, it follows that $B(T(u), T(u)) = 0$, which, in particular, shows that $T(u)$ is a partial isometry; hence, T is automatically bounded and $\|T\| = 1$ (cf. [27, Section 3]). Therefore, for every self-adjoint element $a \in A$, we have

$$\big\{T(e^{ita}), T(e^{ita}), T(e^{ita})\big\} = T(e^{ita}) \quad (t \in \mathbb{R}).$$

Differentiating both sides of the above identity with respect to t, we deduce that

$$2\big\{iT(ae^{ita}), T(e^{ita}), T(e^{ita})\big\} + \big\{T(e^{ita}), iT(ae^{ita}), T(e^{ita})\big\} = iT(ae^{ita}),$$

and hence

$$2\{T(ae^{ita}),T(e^{ita}),T(e^{ita})\}-\{T(e^{ita}),T(ae^{ita}),T(e^{ita})\}=T(ae^{ita}) \tag{5.2}$$

for every $t\in\mathbb{R}$. For $t=0$, we get

$$2\{T(a),T(1),T(1)\}-\{T(1),T(a),T(1)\}=T(a);$$

equivalently,

$$T(a)=T(a)T(1)^*T(1)+T(1)T(1)^*T(a)-T(1)T(a)^*T(1) \tag{5.3}$$

for every $a=a^*$ in A.

Differentiating (5.2) with respect to t, we obtain

$$\begin{aligned}T(a^2e^{ita})&=2\{T(a^2e^{ita}),T(e^{ita}),T(e^{ita})\}-4\{T(ae^{ita}),T(ae^{ita}),T(e^{ita})\}\\&\quad+2\{T(ae^{ita}),T(e^{ita}),T(ae^{ita})\}+\{T(e^{ita}),T(a^2e^{ita}),T(e^{ita})\}\end{aligned}$$

for every $t\in\mathbb{R}$. In the case $t=0$ we get

$$\begin{aligned}T(a^2)&=2\{T(a^2),T(1),T(1)\}-4\{T(a),T(a),T(1)\}\\&\quad+2\{T(a),T(1),T(a)\}+\{T(1),T(a^2),T(1)\},\end{aligned}$$

or, equivalently,

$$\begin{aligned}T(a^2)&=T(a^2)T(1)^*T(1)+T(1)T(1)^*T(a^2)-2T(a)T(a)^*T(1)\\&\quad-2T(1)T(a)^*T(a)+2T(a)T(1)^*T(a)+T(1)T(a^2)^*T(1)\end{aligned} \tag{5.4}$$

for every $a=a^*$ in A.

Multiplying identity (5.3) by $T(1)^*$ from both sides, and taking into account that $T(1)$ is a (maximal) partial isometry, we deduce that

$$T(1)^*T(a)T(1)^*=T(1)^*T(1)T(a)^*T(1)T(1)^* \tag{5.5}$$

for every self-adjoint element $a\in A$.

Proposition 5.5. *Let A and B be unital C^*-algebras. Let $T:A\to B$ be a linear map preserving extreme points. Suppose that $T(1)$ is a unitary in B. Then there exists a unital Jordan *-homomorphism $S:A\to B$ satisfying $T(a)=T(1)S(a)$ for every $a\in A$.*

Proof. By hypothesis, $v=T(1)$ is a unitary in B. We deduce from (5.3) that

$$T(a)=vT(a)^*v$$

for every self-adjoint element $a\in A$, and hence, by linearity,

$$T(a)=vT(a^*)^*v,\qquad\text{or, equivalently,}\qquad v^*T(a)=T(a^*)^*v \tag{5.6}$$

for every $a\in A$. Therefore, the mapping $S:A\to B$, given by $S(x):=v^*T(x)$, is symmetric ($S(x^*)=S(x)^*$), and $S(1)=v^*T(1)=v^*v=1$.

Now, since $v^*v=1=vv^*$, we deduce from (5.4) and (5.6) that

$$T(a^2)=vT(a)^*T(a)$$

for every $a=a^*$ in A. Multiplying on the left by v^*, we obtain

$$S(a^2)=v^*vT(a)^*T(a)=T(a)^*T(a)=S(a)^*S(a)=S(a)^2$$

for every $a = a^*$ in A, and hence S is a Jordan *-homomorphism. It is also clear that $T(a) = vv^*T(a) = vS(a)$ for every a in A. □

Remark 5.6. Let A and B be unital C*-algebras, and let $T : A \to B$ be a linear map preserving extreme points. Let $v = T(1)$.

(a) If $v^*v = 1$, then $S(a) := v^*T(a)$ is a positive unital mapping, and therefore satisfies Kadison's generalized Schwarz inequality: $S(a^2) \geq S(a)^2$ for all $a \in A_{sa}$.

(b) If $vv^* = 1$, then $S(a) := vT(a)$ is a positive unital mapping, and therefore satisfies Kadison's generalized Schwarz inequality: $S(a^2) \geq S(a)^2$ for all $a \in A_{sa}$.

Proof. In each case, the map S is a unital contraction, and therefore positive by [27, Corollary 1]. The rest follows from [19]. □

We recall that, according to [29, Theorem 10.2], for a C*-algebra, A, the intersection $\partial_e(A_1) \cap A_{sa}$ is precisely the set of all self-adjoint unitary elements of A.

Corollary 5.7. *Let A and B be unital C^*-algebras. Let $T : A \to B$ be a symmetric linear map. If T preserves extreme points, then $T(1)$ is a self-adjoint unitary element in B, and there exists a unital Jordan *-homomorphism $S : A \to B$ satisfying $T(a) = T(1)S(a)$ for every $a \in A$.*

Proof. Suppose that T preserves extreme points. Since T is symmetric, the element $T(1)$ must be a self-adjoint extreme point of the closed unit ball of B, and hence a self-adjoint unitary element. Proposition 5.5 assures that $S(a) := T(1)T(a)$ $(a \in A)$ is a unital Jordan *-homomorphism and $T(a) = T(1)S(a)$ for every $a \in A$. □

The next result gives sufficient conditions for the reciprocal statement of Proposition 5.5 and Corollary 5.7.

Proposition 5.8. *Let $T : A \to B$ be a linear map between unital C^*-algebras. Suppose that T can be written in the form $T = vS$, where v is a unitary element in B and $S : A \to B$ is a unital Jordan *-homomorphism such that B equals the C^*-algebra generated by $S(A)$. Then T preserves extreme points.*

Proof. Suppose that $T = vS$, where v is a unitary element in B and $S : A \to B$ is a unital Jordan *-homomorphism. Since $S^{**} : A^{**} \to B^{**}$ is a unital Jordan *-homomorphism between von Neumann algebras (cf. [28, Lemma 3.1]), Theorem 3.3 in [28] implies the existence of two orthogonal central projections E and F in B^{**} such that $S_1 = S^{**} : A^{**} \to B^{**}E$ is a *-homomorphism, $S_2 = S^{**} : A^{**} \to B^{**}F$ is an *-anti-homomorphism, $E+F = 1$, and $S^{**} = S_1+S_2$. The equality $1 = S(1) = S_1(1) + S_2(1)$ implies that $S_1(1) = E$ and $S_2(1) = F$.

Take $e \in \partial_e(A_1)$. We claim that $S(e) \in \partial_e(B_1)$. Indeed, the equalities

$$\begin{aligned}
&\big(1 - S(e)S(e)^*\big)S(A)\big(1 - S(e)^*S(e)\big)\\
&\quad = \big(1 - S_1(e)S_1(e)^* - S_2(e)S_2(e)^*\big)S(A)\big(1 - S_1(e)^*S_1(e) - S_2(e)^*S_2(e)\big)\\
&\quad = \big(E - S_1(e)S_1(e)^*\big)S_1(A)\big(E - S_1(e)^*S_1(e)\big)
\end{aligned}$$

$$
\begin{aligned}
&+\big(F-S_2(e)S_2(e)^*\big)S_2(A)\big(F-S_2(e)^*S_2(e)\big)\\
&=S_1\big((1-ee^*)A(1-e^*e)\big)+S_2\big((1-e^*e)A(1-ee^*)\big)\\
&=\{0\},
\end{aligned}
$$

together with the fact that B equals the C*-algebra generated by $S(A)$, show that $S(e)\in\partial_e(B_1)$.

Finally, given $e\in\partial_e(A_1)$, we know that $S(e)\in\partial_e(B_1)$, and hence

$$
\begin{aligned}
\big(1-T(e)T(e)^*\big)B\big(1-T(e)^*T(e)\big)&=\big(1-vS(e)S(e)^*v\big)B\big(1-S(e^*)v^*vS(e)\big)\\
&=v\big(1-S(e)S(e)^*\big)v^*B\big(1-S(e^*)S(e)\big)\\
&\subseteq v\big(1-S(e)S(e)^*\big)B\big(1-S(e^*)S(e)\big)\\
&=\{0\}
\end{aligned}
$$

because $S(e)\in\partial_e(B_1)$. We have therefore shown that $T(e)\in\partial_e(B_1)$. □

When M is an infinite von Neumann factor, a linear map $T:M\to M$ preserves extreme points if and only if there exist a unitary u in M and a linear map $\Phi:M\to M$ which is either a unital *-homomorphism or a unital *-anti-homomorphism such that $T(a)=u\Phi(a)$ $(a\in A)$ (see [24, Theorem 1]). When M is a finite von Neumann algebra, a linear map T on M preserves extreme points if and only if there exist a unitary u in M and a Jordan *-homomorphism $\Phi:M\to M$ satisfying $T(a)=u\Phi(a)$ $(a\in A)$ (see [24, Theorem 2]). Motivated by these results, it is natural to ask whether a similar conclusion remains true for operators preserving extreme points between unital C*-algebras. The next simple examples show that the answer is, in general, negative.

Remark 5.9. Let H be an infinite-dimensional complex Hilbert space. Suppose v is a maximal partial isometry in $B(H)$ which is not a unitary. The operator $T:\mathbb{C}\to B(H)$, $\lambda\mapsto\lambda v$, preserves extreme points, but we cannot write T in the form $T=u\Phi$, where u is a unitary in $B(H)$ and Φ is a unital Jordan *-homomorphism.

Remark 5.10. Under the assumptions of Remark 5.9, let $v,w\in\partial_e(B(H)_1)$ such that $v^*v=1=w^*w$ and $vv^*\perp ww^*$. Let $A=\mathbb{C}\oplus^\infty\mathbb{C}$. We consider the following operator:

$$
T:A\to B(H)
$$
$$
T(\lambda,\mu)=\frac{\lambda}{2}(v+w)+\frac{\mu}{2}(v-w).
$$

Clearly, $T(1,1)=v$. Furthermore, every extreme point of the closed unit ball of A can be written in the form (λ_0,μ_0) with $|\lambda_0|=|\mu_0|=1$. Therefore, $T(\lambda_0,\mu_0)=\frac{\lambda_0}{2}(v+w)+\frac{\mu_0}{2}(v-w)=\frac{\lambda_0+\mu_0}{2}v+\frac{\lambda_0-\mu_0}{2}w$ satisfies

$$
\begin{aligned}
T(\lambda_0,\mu_0)^*T(\lambda_0,\mu_0)&=\Big(\frac{\lambda_0+\mu_0}{2}v+\frac{\lambda_0-\mu_0}{2}w\Big)^*\Big(\frac{\lambda_0+\mu_0}{2}v+\frac{\lambda_0-\mu_0}{2}w\Big)\\
&=\frac{|\lambda_0+\mu_0|^2}{4}v^*v+\frac{|\lambda_0-\mu_0|^2}{4}w^*w
\end{aligned}
$$

$$= \Big(\frac{|\lambda_0+\mu_0|^2}{4} + \frac{|\lambda_0-\mu_0|^2}{4} \Big) 1$$
$$= \frac{2(|\lambda_0|^2+|\mu_0|^2)}{4} 1 = 1,$$

which proves that $T(\lambda_0,\mu_0) \in \partial_e(B(H)_1)$, and hence T preserves extreme points.

The mapping T satisfies a stronger property. Elements a and b in the C*-algebra A satisfy $B(a,b)=0$ if and only if $ab^*=1$. We observe that $A_q^{-1}=A^{-1}$, and hence an element $(\lambda,\mu)\in A_q^{-1}$ if and only if $\lambda\mu\neq 0$. Let us pick $a=(\lambda_0,\mu_0)\in A_q^{-1}$ (with $\lambda_0\mu_0\neq 0$). Clearly, $a^\wedge = (\overline{\lambda_0^{-1}}, \overline{\mu_0^{-1}})$. It is easy to check that

$$\begin{aligned} T(a^\wedge)^* T(a) &= \Big(\frac{\overline{\lambda_0}^{-1}+\overline{\mu_0}^{-1}}{2} v + \frac{\overline{\lambda_0}^{-1}-\overline{\mu_0}^{-1}}{2} w \Big)^* \Big(\frac{\lambda_0+\mu_0}{2} v + \frac{\lambda_0-\mu_0}{2} w \Big) \\ &= \Big(\frac{{\lambda_0}^{-1}+{\mu_0}^{-1}}{2} v^* + \frac{{\lambda_0}^{-1}-{\mu_0}^{-1}}{2} w^* \Big) \Big(\frac{\lambda_0+\mu_0}{2} v + \frac{\lambda_0-\mu_0}{2} w \Big) \\ &= \frac{1}{4} \frac{(\lambda_0+\mu_0)^2-(\lambda_0-\mu_0)^2}{\lambda_0\mu_0} 1 = 1, \end{aligned}$$

and hence $B(T(a),T(a^\wedge))=0$, which shows that T preserves Bergmann-zero pairs.

It is easy to check that $T(1,-1)=w$, and hence $v^*T(1,-1)=v^*w=0$, and $vv^*T(1,-1)=0$. For $S=v^*T$ we have $S(1,-1)^2=0$, but $S((1,-1)^2)=S(1,1)=v$; that is, S is not a Jordan homomorphism. We can further check that T is not a triple homomorphism; for example, $(1,0)$ is a tripotent in A but $\|T(1,0)\|=\frac{1}{\sqrt{2}}$, and hence $T(1,0)$ is not a tripotent in $B(H)$.

Finally, for $a=(\lambda_0,\mu_0)\in A_q^{-1}$ (with $\lambda_0\mu_0\neq 0$), $T(a^\wedge) = \frac{\overline{\lambda_0}^{-1}}{2}(v+w) + \frac{\overline{\mu_0}^{-1}}{2}(v-w)$ need not coincide with $T(a)^\wedge = (\frac{\lambda_0}{2}(v+w)+\frac{\mu_0}{2}(v-w))^\wedge$. Indeed, $T(2,1)=\frac{3}{2}v+\frac{1}{2}w=\frac{\sqrt{10}}{2}r$, where $r=\frac{3}{\sqrt{10}}v+\frac{1}{\sqrt{10}}w$ is the range tripotent of $T(2,1)$, and thus $T(2,1)^\wedge=\frac{2}{\sqrt{10}}r=\frac{3}{5}v+\frac{1}{5}w$. Clearly, $T((2,1)^\wedge)=T(1/2,1)=\frac{3}{4}v-\frac{1}{4}w$.

The counterexamples provided by Remark 5.10 point out that the conclusions found by Mascioni and Molnár for linear maps preserving extreme points on the infinite von Neumann factor are not expectable for general C*-algebras (cf. [24]). We shall show that a more tractable description is possible for linear maps strongly preserving Brown–Pedersen quasi-invertibility. The proofs are based on the JB*-triple structure underlying every C*-algebra.

The following variant of Proposition 3.1 follows with similar arguments; its proof is outlined here.

Proposition 5.11. *Let E and F be JB*-triples, and let $T: E\to F$ be a linear map strongly preserving Brown–Pedersen quasi-invertible elements; that is, $T(x^\wedge)=T(x)^\wedge$ for every $x\in E_q^{-1}$. Then*

$$T(x^{[3]}) = T(x)^{[3]}$$

for every $x\in E_q^{-1}$.

Proof. Let x be an element in E_q^{-1}, and let $e = r(x) \in \partial_e(E_1)$ denote its range tripotent. For each $0 < \lambda < \|x^\wedge\|^{-2}$ the element $\lambda x^\wedge - x$ is Brown–Pedersen quasi-invertible in E. Indeed, if we regard $\lambda x^\wedge - x$ as an element in $E_x \equiv C(\mathrm{Sp}(x))$, the JB*-subtriple of E generated by x (see p. 551), then $x - \lambda x^\wedge$ is invertible and positive in E_x, and its range tripotent is $r(x - \lambda x^\wedge) = e \in \partial_e(E_1)$. By Hua's identity, we have

$$x - \lambda^{-1} x^{[3]} = \big(x^\wedge - (x - \lambda x^\wedge)^\wedge\big)^\wedge$$

(cf. (3.1)).

Given $0 < \lambda < \mathrm{Min}\{\|x^\wedge\|^{-2}, \|T(x)^\wedge\|^{-2}\}$, since T strongly preserves Brown–Pedersen quasi-invertible elements, and $x, \lambda x^\wedge - x, T(x)$ and $T(\lambda x^\wedge - x)$ are Brown–Pedersen quasi-invertible, we deduce that

$$\begin{aligned} T(x) - \lambda^{-1} T(x)^{[3]} &= \big(T(x)^\wedge - \big(T(x) - \lambda T(x)^\wedge\big)^\wedge\big)^\wedge \\ &= \big(T(x^\wedge) - \big(T(x) - \lambda T(x^\wedge)\big)^\wedge\big)^\wedge \\ &= T\big(\big(x^\wedge - (x - \lambda x^\wedge)^\wedge\big)^\wedge\big) \\ &= T(x) - \lambda^{-1} T(x^{[3]}) \end{aligned}$$

for every $0 < \lambda$ as above, which proves the desired statement. □

The full meaning of Theorem 3.2 (and the role played by [17, Lemma 2.2] in its proof) is more explicit in the following result, whose proof follows the lines we gave in the aforementioned theorem but replaces Proposition 3.1 with Proposition 5.11.

Theorem 5.12. *Let E and F be JB*-triples with $\partial_e(E_1) \neq \emptyset$. Suppose $T : E \to F$ is a linear map strongly preserving Brown–Pedersen quasi-invertible elements. Then T is a triple homomorphism.*

We can state now our conclusions on linear maps strongly preserving Brown–Pedersen quasi-invertibility.

Theorem 5.13. *Let A and B be unital C*-algebras. Let $T : A \to B$ be a linear map strongly preserving Brown–Pedersen quasi-invertible elements. Then there exists a Jordan *-homomorphism $S : A \to B$ satisfying $T(x) = T(1)S(x)$ for every $x \in A$.*

We further know that

$$T(A) \subseteq T(1)T(1)^* B T(1)^* T(1), \qquad S(A) \subseteq T(1)^* T(1) B T(1)^* T(1),$$

and $S : A \to T(1)^* T(1) B T(1)^* T(1)$ is a unital Jordan *-homomorphism.

Proof. Since T preserves extreme points, $v = T(1) \in \partial_e(B_1)$ is a partial isometry with

$$(1 - vv^*) T(x) (1 - v^* v) = 0 \tag{5.7}$$

for every $x \in A$. It follows from (5.5) that $vT(a)^* v = vv^* T(a) v^* v$ for every $a = a^* \in A$.

Now, Theorem 5.12 assures that T is a triple homomorphism. Thus, we have

$$T(x) = T\{x,1,1\} = \{T(x),v,v\} = \frac{1}{2}\big(T(x)v^*v + vv^*T(x)\big) \tag{5.8}$$

and

$$T(x^*) = T\{1,x,1\} = \{v,T(x),v\} = vT(x)^*v \tag{5.9}$$

for every $x \in A$. Identities (5.7) and (5.8) give

$$T(x) = vv^*T(x)v^*v = vv^*T(x) = T(x)v^*v \tag{5.10}$$

for every $x \in A$. Multiplying on the left by v^*, we get

$$v^*T(x) = v^*T(x)v^*v = \big(\text{by } (5.9)\big) = T(x^*)^*v$$

for every $x \in A$, which proves that $S = v^*T : A \to B$ is a symmetric operator. Furthermore, since T is a triple homomorphism, we have

$$S(x^2) = v^*T\{x,1,x\} = v^*\{T(x),v,T(x)\} = v^*T(x)v^*T(x) = S(x)^2$$

for all $x \in A$, which guarantees that S is a Jordan *-homomorphism. The identity in (5.10) gives $T(x) = vv^*T(x) = vS(x)$ for every $x \in A$. The rest is clear. □

Remark 5.14. Under the hypothesis of Theorem 5.13 we can similarly prove that the mapping $S_1 : A \to B$, $S_1(x) = T(x)T(1)^*$ is a Jordan *-homomorphism and $T(x) = S_1(x)v$ for every x in A.

If v is an extreme point of the closed unit ball of a prime unital C*-algebra B, then $1 = vv^*$ or $v^*v = 1$. Therefore, the next result is a straight consequence of the previous Theorem 5.13.

Corollary 5.15. *Let A and B be unital C*-algebras with B prime. Let $T : A \to B$ be a linear map strongly preserving Brown–Pedersen quasi-invertible elements. Then one of the following statements holds:*

(a) $T(1)^*T(1) = 1$, $T(1)T(1)^*T(a) = T(a)$ *for every* $a \in A$, *and there exists a unital Jordan* **-homomorphism* $S : A \to B$ *satisfying* $T(a) = T(1)S(a)$ *for every* $a \in A$;

(b) $T(1)T(1)^* = 1$, $T(a)T(1)^*T(1) = T(a)$ *for every* $a \in A$, *and there exists a unital Jordan* **-homomorphism* $S : A \to B$ *satisfying* $T(a) = S(a)T(1)$ *for every* $a \in A$. □

Acknowledgments. The authors are partially supported by the Spanish Ministry of Economy and Competitiveness and European Regional Development Fund project no. MTM2014-58984-P and Junta de Andalucía grants FQM375 and FQM194. The second author is supported by a grant from Plan Propio de Investigación, University of Almería. The fourth author acknowledges support from the Deanship of Scientific Research at King Saud University (Saudi Arabia) research group no. RG-1435-020.

We would like to express our gratitude to the anonymous referee for the insightful comments and suggestions which improved the final form of this article.

References

1. T. Barton and R. M. Timoney, *Weak*-continuity of Jordan triple products and applications*, Math. Scand. **59** (1986), no. 2, 177–191. Zbl 0621.46044. MR0884654. 551
2. L. G. Brown and G. K. Pedersen, *On the geometry of the unit ball of a C^*-algebra*, J. Reine Angew. Math. **469** (1995), 113–147. Zbl 0834.46041. MR1363827. DOI 10.1515/crll.1995.469.113. 549
3. L. J. Bunce and C.-H. Chu, *Compact operations, multipliers and Radon–Nikodym property in JB^*-triples*, Pacific J. Math. **153** (1992), no. 2, 249–265. Zbl 0801.46057. MR1151560. 554, 555
4. M. Burgos, F. J. Fernández-Polo, J. J. Garcés, J. Martínez Moreno, and A. M. Peralta. *Orthogonality preservers in C^*-algebras, JB^*-algebras and JB^*-triples*, J. Math. Anal. Appl. **348** (2008), no. 1, 220–233. Zbl 1156.46045. MR2449340. DOI 10.1016/j.jmaa.2008.07.020. 553
5. M. Burgos, A. Kaidi, A. Morales, A. M. Peralta, and M. Ramírez, *Von Neumann regularity and quadratic conorms in JB^*-triples and C^*-algebras*, Acta Math. Sin. (Engl. Ser.) **24** (2008), no. 2, 185–200. Zbl 1157.46042. MR2383347. DOI 10.1007/s10114-007-0985-x. 552
6. M. Burgos, A. C. Márquez-García, and A. Morales-Campoy, *Linear maps strongly preserving Moore–Penrose invertibility*, Oper. Matrices **6** (2012), no. 4, 819–831. Zbl 1282.47055. MR3025038. DOI 10.7153/oam-06-53. 549, 553
7. M. Burgos, A. C. Márquez-García, and A. Morales-Campoy, *Strongly preserver problems in Banach algebras and C^*-algebras*, Linear Algebra Appl. **437** (2012), no. 5, 1183–1193. Zbl 1259.47049. MR2942341. DOI 10.1016/j.laa.2012.04.019. 552
8. C.-H. Chu, *Jordan Structures in Geometry and Analysis*, Cambridge Tracts in Math. **190**, Cambridge Univ. Press, Cambridge, 2012. Zbl 1238.17001. MR2885059. 554
9. S. Dineen, "The second dual of a JB*-triple system" in *Complex Analysis, Functional Analysis and Approximation Theory* (ed. by J. Múgica), North-Holland Math. Stud. **125**, North-Holland, Amsterdam, 1986, 67–69. Zbl 0653.46053. MR0893410. 551
10. C. M. Edwards and G. T. Rüttimann, *Compact tripotents in bi-dual JB^*-triples*, Math. Proc. Cambridge Philos. Soc. **120** (1996), no. 1, 155–174. Zbl 0853.46070. MR1373355. DOI 10.1017/S0305004100074740. 552
11. A. Férnandez López, E. García Rus, E. Sánchez Campos, and M. Siles Molina, *Strong regularity and generalized inverses in Jordan systems*, Comm. Algebra **20** (1992), no. 7, 1917–1936. Zbl 0759.17019. MR1167082. DOI 10.1080/00927879208824440. 552
12. F. J. Fernández-Polo, J. Martínez Moreno, and A. M. Peralta, *Contractive Perturbations in JB^*-triples*, J. Lond. Math. Soc. (2) **85** (2012), no. 2, 349–364. Zbl 1244.46034. MR2901068. DOI 10.1112/jlms/jdr048. 555, 556
13. Y. Friedman and B. Russo, *Structure of the predual of a JBW^*-triple*, J. Reine Angew. Math. **356** (1985), 67–89. Zbl 0547.46049. MR0779376. DOI 10.1515/crll.1985.356.67. 552
14. R. Harte and M. Mbekhta, *On generalized inverses in C^*-algebras*, Studia Math. **103** (1992), no. 1, 71–77. Zbl 0810.46062. MR1184103. 549
15. L. K. Hua, *On the automorphisms of a sfield*, Proc. Natl. Acad. Sci. USA **35** (1949), 386–389. Zbl 0033.10402. MR0029886. 549
16. N. Jacobson, *Structure and Representations of Jordan Algebras*. Amer. Math. Soc. Colloq. Publ. **39**, Amer. Math. Soc., Providence, 1968. Zbl 0218.17010. MR0251099. 552, 553
17. F. Jamjoom, A. Peralta, A. Siddiqui, and H. Tahlawi, *Approximation and convex decomposition by extremals and the λ-function in JBW^*-triples*, Q. J. Math. **66** (2015), no. 2, 583–603. Zbl 06450232. MR3356839. DOI 10.1093/qmath/hau036. 554, 556, 562
18. F. B. Jamjoom, A. A. Siddiqui, and H. M. Tahlawi, *On the geometry of the unit ball of a JB^*-triple*, Abstr. Appl. Anal. **2013**, art. ID 891249. MR3064400. DOI 10.1155/2013/891249. 549
19. R. V. Kadison, *A generalized Schwarz inequality and algebraic invariants for operator algebras*, Ann. of Math. (2) **56** (1952), 494–503. Zbl 0047.35703. MR0051442. 559

20. W. Kaup, *A Riemann mapping theorem for bounded symmentric domains in complex Banach spaces*, Math. Z. **183** (1983), no. 4, 503–529. Zbl 0519.32024. MR0710768. DOI 10.1007/BF01173928. 550, 551, 552
21. W. Kaup, *On spectral and singular values in JB^*-triples*, Proc. Roy. Irish Acad. Sect. A **96** (1996), no. 1, 95–103. Zbl 0904.46039. MR1644656. 552
22. L. E. Labuschagne and V. Mascioni, *Linear maps between C^*-algebras whose adjoint preserve extreme points of the dual ball*, Adv. Math. **138** (1998), no. 1, 15–45. Zbl 0944.46054. MR1645056. DOI 10.1006/aima.1998.1738. 548
23. O. Loos, *Jordan pairs and bounded symmetric domains*, math lectures at Univ. of California, Irvine, 1977, http://molle.fernuni-hagen.de/~loos/jordan/archive/irvine. Zbl 0228.32012. 551
24. V. Mascioni and L. Molnár, *Linear maps on factors which preserve the extreme points of the unit ball*, Canad. Math. Bull. **41** (1998), no. 4, 434–441. Zbl 0919.47024. MR1658239. DOI 10.4153/CMB-1998-057-7. 548, 557, 560, 561
25. M. Mbekhta, *A Hua type theorem and linear preserver problems*, Math. Proc. R. Ir. Acad. **109** (2009), no. 2, 109–121. Zbl 1215.47031. MR2511235. DOI 10.3318/PRIA.2008.109.2.109. 549
26. K. Meyberg, *Von Neumann regularity in Jordan triple systems*. Arch Math. (Basel) **23** (1972), 589–593. Zbl 0253.17017. MR0332901. 555
27. B. Russo and H. A. Dye, *A note on unitary operators in C^*-algebras*, Duke Math. J. **33** (1966), 413–416. Zbl 0171.11503. MR0193530. 548, 557, 559
28. E. Størmer, *On the Jordan structure of C^*-algebras*, Trans. Amer. Math. Soc. **120** (1965), no. 3, 438–447. Zbl 0136.11401. MR0185463. 559
29. M. Takesaki, *Theory of Operator Algebras, I*, Springer, New York, 1979. Zbl 0436.46043. MR0548728. 548, 557, 559

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