Banach J. Math. Anal. 10 (2016), no. 3, 523–546
http://dx.doi.org/10.1215/17358787-3607222
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

BOUNDARY VALUES OF VECTOR-VALUED HARDY SPACES ON NONSMOOTH DOMAINS AND THE RADON–NIKODYM PROPERTY

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Dedicated to Professor S. Pérez-Esteva on the occasion of his 60th birthday

Communicated by R. E. Curto

Abstract. We define Hardy spaces of functions taking values on a Banach space $\mathcal{X}$ over nonsmooth domains. The types of functions we consider are harmonic functions on a starlike Lipschitz domain and solutions to the heat equation on a time-varying domain. Our purpose is twofold: (a) to characterize the Radon–Nikodym property of the Banach space $\mathcal{X}$ in terms of the existence of nontangential limits of $\mathcal{X}$-valued functions u in the corresponding Hardy space with index $p \geq 1$, (b) to identify the function of the boundary values of u in the Hardy space with index $p > 1$ with an element in the space $\mathcal{V}^p_{\mathcal{X}}$ of measures of p-bounded variation in the absence of the Radon–Nikodym property of $\mathcal{X}$. This extends similar results already known on the unit disk of $\mathbb{C}$ and the semispace $\mathbb{R}^n \times (0, \infty)$.

1. Introductory remarks

In this work we continue the study started in [28] in which we began to generalize some well-known results on the connection of the Radon–Nikodym property of a Banach space $\mathcal{X}$ and the boundary behavior of harmonic functions $u : \mathbb{D} \to \mathcal{X}$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$ (see, e.g., [6], [2], [15]). The issue to address is to

Copyright 2016 by the Tusi Mathematical Research Group.
Received Mar. 18, 2015; Accepted Nov. 7, 2015.
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2010 *Mathematics Subject Classification.* Primary 46B22; Secondary 31A20, 46G10, 31A15.
Keywords. vector-valued Hardy spaces, Radon–Nikodym property, Fatou-type theorems, Lipschitz domains, noncylindrical domains.

consider more general $\mathcal{X}$-valued functions on nonsmooth domains of Euclidean spaces.

Recall that a Banach space $\mathcal{X}$ has the *Radon–Nikodym property* ($\mathcal{X} \in RNP$) if, for every probability space $(\Omega, \Sigma, \lambda)$, and every λ-continuous measure μ defined on Σ with values in $\mathcal{X}$, one can find a Bochner λ-integrable function $f : \Omega \to \mathcal{X}$ such that $\mu(E) = \int_E f \, d\lambda$ for every $E \in \Sigma$. Also, recall that μ is λ-*continuous* if $\mu(E) = 0$ whenever $\lambda(E) = 0$, $E \in \Sigma$. A fundamental point is that the Radon–Nikodym property is independent of the (nonatomic) probability space $(\Omega, \Sigma, \lambda)$ as established in [8, Theorem 2]. Therefore, we may consider the Radon–Nikodym property with respect to the *harmonic* or *caloric measure* on the boundary of the nonsmooth domain in question.

The first type of question that occurs has already been addressed for harmonic functions in [28]. Indeed, in that work, it is established that the equivalence of $\mathcal{X} \in RNP$ with the existence of boundary values for $\mathcal{X}$-valued harmonic functions in a suitable defined Hardy space (of index $1 \leq p \leq \infty$) defined on a starlike Lipschitz domain.

In the present article, we prove the analogue result for solutions of the heat equation *(caloric functions)* defined on a noncylindrical domain in $\mathbb{R}^{n+1}$. The only related result, to our knowledge, is that of [26], in which caloric functions of one space variable on a rectangle are considered. The technique in [26] uses an expansion of the heat kernel in series and certain *parabolic coordinates* on a rectangle introduced in [1]. Here we use a representation formula for caloric functions using the *caloric measure* along with some ideas from [28]. We have included a thorough description of the noncylindrical setting on which there has been some relatively recent developments on Dirichlet-type problems for the heat equation.

A second type of question arises when considering the boundary values of elements in any Hardy space with index $p > 1$. In the classical scalar-valued case of the unit disk $\mathbb{D}$ or the semispace $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$, it is well known that this function of boundary values is identified with the corresponding Lebesgue spaces through the Poisson integral. In the vector-valued setting, it turns out that, in the absence of the Radon–Nikodym property of $\mathcal{X}$, the identification with Lebesgue spaces of a classical setting does not prevail as observed originally in [2] and later extended to $\mathbb{R}^{n+1}_+$ in [27].

In this work, we prove that the result from [2] in the setting of the unit disk $\mathbb{D}$ can be extended to both of our settings over nonsmooth domains. A feature of the proofs we present is that some well-known estimates for the harmonic or the caloric measures provide the essential tools for the argumentations. In fact, for harmonic functions over Lipschitz domains, or caloric functions over noncylindrical domains, the right *Poisson formula* is given in terms of the harmonic or the caloric measure, respectively. We provide some details of this adaptation in both the *elliptic setting* of the Laplace's equation as well as the *parabolic setting* of the heat equation.

Finally, we note that [3, Theorem 5.9] allows us to obtain similar results for appropriate Hardy spaces of solutions to the stationary Schrödinger equation as those for harmonic functions, and so we state and prove results for harmonic functions bearing in mind this generalization. We are aware of a different definition of Hardy spaces associated to the Schrödinger equation considered, for instance, in [13]. It may be of interest to establish similar results as those reported in this note for functions in that type of Hardy space.

Plan of the paper. In Section 2, we introduce some of the basic concepts and notation from Banach space theory that we will use throughout this work. In Section 3, we provide precise descriptions of the elliptic setting and we prove the new result for harmonic functions. Sections 4 and 5 contain basic detailed definitions of the parabolic setting and the statements of the main theorems related to the caloric Hardy spaces. We also include the adaptation to caloric functions of the proof of the new result from Section 3. In Section 6, we have included the adaptation to caloric functions of [28, Theorem 1.1].

2. Notation and definitions from Banach space theory

As observed in the introductory remarks, we will be considering the Radon–Nikodym property of a Banach space $\mathcal{X}$ with respect to a measure space associated to the boundary of a domain. These measure spaces turn out to be *spaces of a homogeneous type*, and so we initiate our descriptions in this generality.

We consider the triple $(\mathcal{M}, \mu, d)$, where $\mathcal{M}$ is a compact topological space, μ a regular finite measure defined on $\mathfrak{B} \equiv \mathfrak{B}(\mathcal{M})$, the Borel sets of $\mathcal{M}$, and $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is a *quasidistance* on $\mathcal{M}$; this means that d satisfies the following properties:

(1) $d(x, y) = d(y, x)$ for every x, y in $\mathcal{M}$ *(symmetry property)*,
(2) $d(x, y) = 0$ if and only if $x = y$,
(3) there exists a constant $K > 0$ such that $d(x, y) \le K(d(x, z) + d(z, y))$ for every x, y, z in $\mathcal{M}$ *(quasidistance property)*.

Additional features of the measure μ are that it is *nonatomic*, and it satisfies the following *doubling property*: there exist constants $r_0 > 0$ and $c_0 > 0$ such that, for $0 < r < r_0$ and $x \in M$, we have

$$\mu\big(B_{2r}(x)\big) \le c_0 \mu\big(B_r(x)\big), \tag{2.1}$$

where, as usual, $B_r(x) = \{y : d(x, y) < r\}$ denotes the open ball of radius r centered at x.

Given any positive measure μ defined on Borel sets of $\mathcal{M}$ and $1 \le p < \infty$, $L^p_{\mathcal{X}}(\mathcal{M}, d\mu)$ will denote the Banach space of all Bochner measurable $\mathcal{X}$-valued functions such that

$$\|f\|_{L^p_{\mathcal{X}}(\mathcal{M},d\mu)} = \Big(\int_{\mathcal{M}} \big\|f(Q)\big\|^p_{\mathcal{X}}\, d\mu(Q)\Big)^{1/p} < \infty.$$

The notation $L^p(\mathcal{M}, d\mu)$ is adopted for the Lebesgue spaces of scalar-valued functions, $1 \le p < \infty$. The space of $\mathcal{X}$-valued continuous functions defined on $\mathcal{M}$ is denoted by either $C_{\mathcal{X}}(\mathcal{M})$ or $C(\mathcal{M}, \mathcal{X})$. For scalar-valued continuous functions, we use analogous notation, dropping the dependance on $\mathcal{X}$.

Define the *Hardy–Littlewood maximal function with respect to* μ for a Borel measure ν defined on $\mathfrak{B}$, with total variation $|\nu|$, as

$$M_\mu \nu(x) = \sup_{r>0} \frac{\nu(B_r(x))}{\mu(B_r(x))}. \tag{2.2}$$

If $f \in L^1(\mathcal{M}, d\mu)$, then we set $M_\mu f(x) = M_\mu \nu(x)$ with $d\nu(x) = f(x)\,dx$. It is well known that the following version of the Hardy–Littlewood theorem holds.

Theorem 2.1. *There exist a constant* $c_0 > 0$ *such that, for* $1 < p \le \infty$ *and* $f \in L^p(\mathcal{M}, d\mu)$,

$$\mu\big\{x \in \mathcal{M} : M_\mu \nu(x) > t\big\} \le \frac{c_0}{t} |\nu|(\mathcal{M}),$$
$$\|M_\mu f\|_{L^p(\mathcal{M},d\mu)} \le c_0 \|f\|_{L^p(\mathcal{M},d\mu)}.$$

We call *vector measures on* $(\mathcal{M}, \mathfrak{B})$ the finitely additive $\mathcal{X}$-valued functions ν defined on the Borel sets of $\mathcal{M}$. If such a function is countably additive, then we call it a *countably additive vector measure*.

Let $p \in (1, \infty]$, and let ν be a vector measure. We recall the *variation* and *p-variation with respect to* μ of ν on $A \in \mathfrak{B}$ defined, respectively, as

$$|\nu|(A) = \sup_\pi \sum_{E\in\pi} \big\|\nu(E)\big\|_{\mathcal{X}},$$
$$|\nu|_p(A) = \sup_\pi \Big\{ \sum_{E\in\pi} \frac{\|\nu(E)\|_{\mathcal{X}}^p}{\mu(E)^{p-1}} \Big\}^{1/p}, \quad p \in (1, \infty),$$
$$|\nu|_\infty(A) = \sup\big\{C > 0 : \big\|\nu(A)\big\| \le C\mu(A)\big\}, \quad p = \infty.$$

In the first two cases, the supremum is taken over all finite partitions π of the set A. There are scalar versions of these spaces, obtained by taking $\mathcal{X}$ equal to $\mathbb{C}$ or $\mathbb{R}$, and, as before, we will drop the dependence on $\mathcal{X}$ from the notation.

We say that ν has *bounded variation* or *bounded p-variation with respect to* μ if $|\nu|(\mathcal{M}) < \infty$, or $|\nu|_p(\mathcal{M}) < \infty$, for $p \in (1, \infty]$, respectively. We denote the space of *measures of bounded p-variation with respect to the measure* μ with $V^p_{\mathcal{X}}(\mathcal{M}, \mu)$, and when the measure μ is clear from the context, we use only the term *measures of bounded p-variation*. If ν is a countably additive vector measure with bounded variation, then there exists a finite positive measure denoted by $|\nu|$ such that

$$\big\|\nu(A)\big\|_{\mathcal{X}} \le |\nu|(A)$$

for all Borel sets A, and $|\nu|$ is minimal with this property (see [11, Chapter 1]). In this case we say that ν is *regular* if $|\nu|$ is regular. This measure is the *variation of* ν.

The space of Borel $\mathcal{X}$-valued regular measures of bounded variation is denoted by $M_{\mathcal{X}}(\mathcal{M})$. Once again, for scalar-valued measures over $\mathcal{M}$, we drop the subscript $\mathcal{X}$ from this notation. By Singer's representation theorem, it turns out that $C_{\mathcal{X}}(\mathcal{M})^* = M_{\mathcal{X}^*}(\mathcal{M})$, where in this case the equal sign means that there is an isometry between the spaces involved.

At this point we state some results that we will use in the remainder of this article.

Lemma 2.2 ([2, Remark 1.1]). *For $p \in (1,\infty]$, let ν be a vector measure of bounded p-variation with respect to μ. Then ν is a countably additive vector measure, which is μ-continuous and has bounded variation. Moreover, the variation $|\nu|$ belongs to the scalar space $V^p(\mathcal{M},\mu)$.*

Lemma 2.3 ([12, Theorem 12.7]). *Let ν be a vector measure, and suppose that there exists a $\mathcal{X}$-valued function f such that $\nu(E) = \int_E f\,d\mu$ for every $E \in \mathfrak{B}$. Let $p \in (1,\infty]$. Then ν has bounded p-variation with respect to μ if and only if $f \in L^p_{\mathcal{X}}(\mathcal{M}, d\mu)$.*

Lemma 2.2 and the Radon–Nikodym theorem imply the existence of an integrable function f such that $d|\nu| = f\,d\mu$, and Lemma 2.3 implies that $f \in L^p(\mathcal{M}, d\mu)$ since $|\nu| \in V^p(\mathcal{M}, d\mu)$.

For measures ν in $V^p_{\mathcal{X}}(\mathcal{M},\mu)$, it is not difficult to see that the integral

$$\int_{\mathcal{M}} f\,d\nu \tag{2.3}$$

is well defined for every function $f \in L^q_{\mathbb{B}}(\mathcal{M}, d\mu)$ with $1/p + 1/q = 1$, and $\mathbb{B}$ is a Banach space in duality with $\mathcal{X}$ or $\mathbb{B} = \mathbb{R}$, the real numbers. Moreover, we have the next result.

Theorem 2.4 ([12, Corollary 13.1.1]). *Let $1 < p \le \infty$ and let $1/p + 1/q = 1$. Then there is an isomorphism between $V^p_{\mathcal{X}^*}(\partial D,\mu)$ and $L^q_{\mathcal{X}}(\partial D, d\mu)^*$ given by the integral (2.3).*

Before continuing, we make some notational comments. For $\xi^* \in \mathcal{X}^*$ and $x \in \mathcal{X}$, we adopt the notation $\langle \xi^*, x\rangle$ for the action of ξ^* on x, also denoted by $\xi^*(x)$. In several of the estimates in the next paragraphs and sections, we may use a generic constant that may change from line to line but still is denoted with the same letter as long as the dependence of this constant on other parameters does not interfere with the essence of the argumentation. In particular, we adopt the standard notation $A \lesssim B$ to mean that there is a constant k that may depend on the geometric features of the domain in question, or the dimension n, or with the dependence just described such that $A \le kB$. Likewise, $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$ hold simultaneously.

3. The result for harmonic functions

Harmonic measures on a starlike Lipschitz domain. A bounded open set $D \subset \mathbb{R}^n$ is a *starlike Lipschitz domain* centered at the origin with character $M > 0$ if, letting $S^{n-1} = \{X \in \mathbb{R}^n : |X| = 1\}$, there exist $\delta_0 > 0$ and a function $\varphi : S^{n-1} \longrightarrow \mathbb{R}$ satisfying $|\varphi(t) - \varphi(s)| \le M|t-s|$ and $\varphi(t) \ge \delta_0 > 0$, and such that in spherical coordinates $D = \{(\rho, s) : 0 \le \rho < \varphi(s), s \in S^{n-1}\}$. On ∂D there is a well-defined surface measure that we denote by σ, and in fact it is equivalent to the $(n-1)$-dimensional Hausdorff measure.

When working in the setting of starlike Lipschitz domains, it is customary to use letters X, Y, and Z to denote points in D, and to use P and Q to denote points in ∂D. For points in $\mathbb{R}^{n-1}$, we use lowercase letters x, y, z, and so forth.

With this convention, for $0 < r < 1$, set $D_r = \{(\rho, s) : 0 \le \rho \le r\varphi(s)\}$, and for $Q \in \partial D$, $Q = (\varphi(s_0), s_0)$, we let $rQ \in D_r$ be the point $rQ = (r\varphi(s_0), s_0)$. We also define the *diameter of* D as $\operatorname{diam} D = \sup\{|X - Y| : X, Y \in D\}$.

A well-established fact is that the boundary of a starlike Lipschitz domain can be covered, after translations and rotations, by a finite collection of *basic Lipschitz domains* of the form $\Omega(\psi_j) = \{(x, \psi_j(x)) : x \in U_j \subset \mathbb{R}^{n-1}\}$, with $\psi_j : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $|\psi_j(x) - \psi_j(y)| \le M_D|x - y|$ uniformly for $x, y \in \mathbb{R}^{n-1}$ and $U_j \subset \mathbb{R}^{n-1}$ open bounded sets. The constant $M_D = \max_j \|\nabla\psi_j\|_\infty$ is proportional to the constant M of D.

The size of the local neighborhoods U_j used to describe ∂D locally will determine a constant $0 < r_0 < \operatorname{diam} D/8$ which will be fixed all throughout this section. Given $Q \in \partial D$ and $0 < r < r_0$, define $\mathcal{O}_r(Q)$ the *cylinder around* Q *of radius* r as the infinite cylinder with axis coinciding with the line emerging from the origin that crosses Q. The *surface ball centered at* Q *and radius* $r > 0$ is defined as $\Delta_r(Q) = \mathcal{O}_r(Q) \cap \partial D$.

Similarly, $\Gamma_\alpha(Q)$ is defined as the right cone with axis joining Q with the origin, with vertex at Q, and with an aperture $\alpha > 0$ such that $\Gamma_\alpha(Q) \subset D$. We stress that these cones are always truncated at a height such that the upper part of the cone reaches the origin. The aperture constant $\alpha > 0$ is chosen and fixed in such a way that $\Gamma_\alpha(Q) \subset D$ for every $Q \in \partial D$.

We can also define the *Carleson region* of diameter r around $Q \in \partial D$ as $T_r(Q) = \mathcal{O}_r(Q) \cap D \cap B_{s_0 r}(Q)$, where $s_0 > 0$ is chosen and fixed so that $\mathcal{O}_r(Q) \cap \partial B_{s_0 r}(Q) \subset \overline{\Gamma(Q)}$. With the constant $s_0 > 0$ already fixed, we can define the *corkscrew point* associated to Q and $r > 0$, denoted by $A_r(Q)$, as the point in $D \cap \partial B_{s_0 r}$ which lies in the line joining Q with the origin.

We have observed above that there is a fundamental result that allows us to translate the results we have for harmonic functions to solutions of a stationary Schrödinger equation. Thus, throughout the following definitions and descriptions, we denote momentarily by L either the Laplace operator Δ or the stationary Schrödinger operator $\Delta - V$ with V a positive potential in $L^p(D)$, $p \ge n/2$. (For a basic review of the potential theory associated to $\Delta - V$ from the viewpoint of harmonic measure techniques, see, for example, [3].)

Given $X \in D$, consider the linear mapping $f \mapsto u(X)$ defined on the space $C(\partial D)$ of continuous functions on the boundary ∂D of D. Here $u(X)$ denotes the solution of the *continuous Dirichlet problem* $Lu = 0$ on D, $u|_{\partial D} = f$. By the maximum principle, the mapping is positive and bounded, and so by the Riesz representation theorem there exists a unique Borel measure, denoted by ω^X and supported on ∂D, such that

$$u(X) = \int_{\partial D} f(Q)\, d\omega^X(Q). \tag{3.1}$$

The *harmonic measure with pole in* X is precisely ω^X. From now on, it will become handy to denote by $\omega \equiv \omega^{\vec{0}}$ the harmonic measure with the pole at the origin, which is the starlike center of D.

Define an *adapted Hardy–Littlewood maximal function* of a Borel measure ν on ∂D as

$$M_\omega \nu(Q) = \sup_{r>0} \frac{\nu(\Delta_r(x))}{\omega(\Delta_r(x))},$$

adopting the notation $M_\omega f$ for functions in $f \in L^1(\partial D, d\omega)$ described in (2.2). The *nontangential maximal function* of $u : D \to \mathcal{X}$ is defined as

$$N_\alpha u(P) \equiv Nu(P) = \sup\big\{ \big|u(X)\big| : X \in \Gamma(P) \big\}.$$

A well-known comparison between these maximal functions is addressed in (vi) of Theorem 3.2 below. We collect some basic properties of the harmonic measure in the following theorem.

Theorem 3.1 (see [22, Lemma 1.2.7, Theorem 1.4.1], [17, p. 311], [9, Theorem 1])**.** *Let D be a starlike Lipschitz domain centered at the origin, and retain notation previously introduced.*

(i) *For any Borel set $E \subset \partial D$, the measure $\omega^X(E)$ is a solution to $Lu = 0$ as a function of the variable $X \in D$ with boundary datum χ_E.*

(ii) *ω^X is mutually absolutely continuous with respect to $\omega = \omega^{\vec{0}}$ for every $X \in D$.*

(iii) *ω is a doubling measure in the sense that there exists $C_0 > 0$ such that for $0 < r < r_0$ one has*

$$\omega^X\big(\Delta_{2r}(Q)\big) \le C_0 \omega^X\big(\Delta_r(Q)\big) \quad \textit{for } X \in D \setminus T_{3r}(Q).$$

(iv) *ω and σ are mutually absolutely continuous measures.*

(v) *Theorem 2.1 holds for $M_\omega \nu$ in the setting of ∂D equipped with ω and the Euclidean distance.*

The kernel function. By (ii) of Theorem 3.1, the Radon–Nikodym derivative

$$K(X, P) = \frac{d\omega^X}{d\omega}(P), \quad X \in D, P \in \partial D$$

is well defined as an element in $L^1(\partial D, d\omega)$, and we refer to it as the *kernel function* associated to L on D. Some basic properties of the kernel function are summarized in the following theorem (see [22, pp. 11–14], [17], [18], [3]).

Theorem 3.2. *Let D be a starlike Lipschitz domain centered at the origin, and retain notation previously introduced.*

(i) *For $P \in \partial D$, the kernel function $K(-, P)$ is a positive solution to $Lu = 0$ on D with $K(\vec{0}, P) = 1$.*

(ii) *For every $X \in D$ and $Q \in \partial D$,*

$$K(X, Q) = \lim_{r \to 0} \frac{\omega^X(\Delta_r(Q))}{\omega(\Delta_r(Q))}.$$

(iii) *For $X \in D$, $K(X, -) \in C(\partial D)$, and if $P \in \partial D$, then for $Q \in \partial D \setminus \{P\}$ one has $\lim_{X \to Q} K(X, P) = 0$.*

(iv) *More precisely, for* $0 < r < r_0$ *and* $Q_0 \in \partial D$,

$$\lim_{X \to Q_0} \Big(\sup_{Q \in \partial D \setminus \Delta_r(Q_0)} K(X,Q) \Big) = 0.$$

(v) *For* $Q_0 \in \partial D$ *and* $0 < r < r_0$, *let* $\Delta_j = \Delta_{2^j r}(Q_0)$, $R_j = \Delta_j \setminus \Delta_{j-1}$, $j = 0, 1, 2, \dots$. *Then there exist constants* $C_1 > 0$ *and* $\alpha > 0$ *depending at most on* D *and* n *such that*

$$\sup_{Q \in R_j} K\big(A_r(Q_0), Q\big) \le \frac{C_1}{2^{\alpha j} \omega(\Delta_j)}.$$

(vi) *If* ν *is a finite Borel measure on* ∂D *and*

$$u(X) = \int_{\partial D} K(X,Q)\, d\nu(Q),$$

then there exists a constant C_2 *such that for each* $P \in \partial D$

$$Nu(P) \le C_2 M_\omega \nu(P).$$

An important remark about the harmonic measures associated to the Laplacian and the stationary Schrödinger operators is contained in the following result, which is actually valid for more general potentials V taken in the so-called *Kato–Stummel class* (see details in [3]).

Theorem 3.3 ([3, Theorem 5.9]). *Denote by* ω_0 *and* ω_1 *the harmonic measures associated to the Laplace operator and the stationary Schrödinger operator, respectively. There exist constants* $c_1, c_2 > 0$ *such that, for any Borel set* $E \subset \partial D$ *and* $X \in D$,

$$c_1 \omega_0^X(E) \le \omega_1^X(E) \le c_2 \omega_0^X(E).$$

In view of this theorem, the results we provide for harmonic functions have an immediate corollary for solutions to the stationary Schrödinger equation; hence, we state only results for harmonic functions, bearing in mind the corresponding result for solutions to the stationary Schrödinger equation.

The main result. In this paragraph we consider $\mathcal{X}$-valued functions $u : D \to \mathcal{X}$. For such a type of function we can define the *nontangential maximal function* as $N_\alpha u(P) \equiv Nu(P) = \sup\{\|u(X)\|_{\mathcal{X}} : X \in \Gamma(P)\}$, and we can define the *vector Hardy spaces* $\mathcal{H}^p_{\mathcal{X}}(D)$ with $0 < p \le \infty$ as the space of $\mathcal{X}$-valued harmonic functions u in D such that the nontangential maximal function $N_\alpha u \in L^p(\partial D, d\omega)$. We equip this space with the norm $\|u\|_{\mathcal{H}^p_{\mathcal{X}}(D)} = \|N_\alpha u\|_{L^p(\partial D, d\omega)}$.

Now recall a result already proved in [28] which takes up the question of characterizing the Radon–Nikodym property of $\mathcal{X}$ in terms of the validity of a Fatou-type theorem for harmonic functions defined on a starlike Lipchitz domain.

Theorem 3.4 ([28, Theorem 1.1]). *Let* D *be a starlike Lipschitz domain centered at the origin, and let* $\mathcal{X}$ *be a Banach space. Then it is equivalent that* $\mathcal{X}$ *has the Radon–Nikodym property with the property that, for some* $1 \le p \le \infty$ *and* $u \in \mathcal{H}^p_{\mathcal{X}}(D)$, *the limit*

$$\lim_{\substack{X \to P \\ X \in \Gamma(P)}} u(X) = u(P)$$

exists for σ-almost every $P \in \partial D$. In fact, this is equivalent to the same statement for all $1 \leq p \leq \infty$.

Next we present the main theorem of this section. This result identifies the boundary values of elements in $\mathcal{H}^p_{\mathcal{X}}(D)$, $p > 1$, with elements in $V^p_{\mathcal{X}}(\partial D, \omega)$.

Theorem 3.5. *For $p > 1$ there is an isomorphism between $V^p_{\mathcal{X}}(\partial D, \omega)$ and $\mathcal{H}^p_{\mathcal{X}}(D)$ given through integration with the kernel function. More precisely, for any vector measure $\nu \in V^p_{\mathcal{X}}(\partial D, \omega)$, one can assign the function u defined as*

$$u(X) = \int_{\partial D} K(X, P)\, d\nu(P), \tag{3.2}$$

and this mapping defines an isomorphism.

Proof. Starting with a vector measure $\nu \in V^p_{\mathcal{X}}(\partial D, \omega)$, define the function u by integration of the kernel function with respect to ν as in (3.2).

By (iii) of Theorem 3.2, the kernel function $K(X, -)$ is continuous in ∂D, a compact space, and so it belongs to $L^q(\partial D, d\omega)$; hence, the function u is well defined. To prove that u is harmonic, one actually proves that it is a weakly harmonic function as follows.

Let $\xi^* \in \mathcal{X}^*$ be fixed. Define $u_*(X) \equiv \langle \xi^*, u(X)\rangle$, and consider the set function ν_* defined as $\nu_*(A) = \langle \xi^*, \nu(A)\rangle$. Notice that $|\nu_*(A)| \leq \|\xi^*\|_{\mathcal{X}^*}|\nu|(A)$; therefore, by Lemma 2.2 we have $\nu_* \in V^p(\partial D, d\omega)$. Moreover, ν_* has bounded variation and is ω-continuous.

The Radon–Nikodym theorem assures the existence of a scalar function f_* in $L^1(\partial D, d\omega)$ such that $d\nu_* = f_* d\omega$, and because of Lemma 2.3 $f_* \in L^p(\partial D, d\omega)$. Therefore, we have

$$u_*(X) = \int_{\partial D} K(X, P)\, d\nu_*(P) = \int_{\partial D} K(X, P) f_*(P)\, d\omega(P).$$

Finally, notice that the last integral represents a scalar-valued harmonic function in D with boundary value f_* (see [17, p. 310]).

Let us see now that the nontangential maximal function of u belongs to $L^p(\partial D, d\omega)$. Since ν is of bounded variation, we have the following inequality (see [12, Proposition 8.4]):

$$\Big\| \int_{\partial D} K(X, P)\, d\nu(P) \Big\|_{\mathcal{X}} \leq \int_{\partial D} K(X, P)\, d|\nu|(P).$$

With the same token used above, and using Lemmas 2.2 and 2.3, we obtain a function f in $L^p(\partial D, d\omega)$ such that $d|\nu| = f\, d\omega$. And from (vi) of Theorem 3.2,

$$\sup_{X \in \Gamma_\alpha(Q)} \Big\| \int_{\partial D} K(X, P)\, d\nu(P) \Big\|_{\mathcal{X}} \leq \sup_{X \in \Gamma_\alpha(Q)} \int_{\partial D} K(X, P) f(P)\, d\omega(P) \leq M_\omega f(Q).$$

By Theorem 2.1 we conclude that $Nu \in L^p(\partial D, d\omega)$, and in fact $\|Nu\|_{L^p(\partial D, d\omega)} \lesssim \|f\|_{L^p(\partial D, d\omega)}$.

The above argument proves the inclusion $V^p_{\mathcal{X}}(\partial D, \omega) \hookrightarrow \mathcal{H}^p_{\mathcal{X}}(D)$ via integration with the kernel function. Now we explore the other inclusion. Let u be a function in $\mathcal{H}^p_{\mathcal{X}}(D)$. First, we recall that a standard argument using Banach–Alaouglu's theorem provides the existence of a measure $\mu \in V^p_{\mathcal{X}^{**}}(\partial D, d\omega)$ such that

$$\lim_{\rho \to 1} u_\rho = \mu \quad \text{in the weak}^* \text{ sense,} \tag{3.3}$$

where for $P \in \partial D$ we have set $u_\rho(P) = u(\rho P)$ (see [2], [28], [27]). By Theorem 2.4 we have $V^p_{\mathcal{X}^{**}}(\partial D, d\omega) = (L^q_{\mathcal{X}^*}(\partial D, d\omega))^*$ with $1/p + 1/q = 1$. Therefore, (3.3) means that for every $\mathcal{X}^*$-valued function $g \in L^q_{\mathcal{X}^*}(\partial D, d\omega)$ we have the convergence $\langle u_\rho, g\rangle \to \langle \mu, g\rangle$ as $\rho \to 1$. Since for every $\xi^* \in \mathcal{X}^*$ fixed we have $K(X, -)\xi^* \in L^q_{\mathcal{X}^*}(\partial D, d\omega)$, this in particular implies that

$$\big\langle \xi^*, u(X)\big\rangle = \Big\langle \xi^*, \int_{\partial D} K(X, P)\, d\mu(P)\Big\rangle \tag{3.4}$$

for every $X \in D$.

The weak* argument gives a limit measure that is $\mathcal{X}^{**}$-valued. Nevertheless, we can follow the proof in [28, Lemma 2.2] or [27, Proposition 2.2] to see that the measure μ is actually $\mathcal{X}$-valued. This way we obtain the stronger equality

$$u(X) = \int_{\partial D} K(X, P)\, d\mu(P); \tag{3.5}$$

hence the arbitrary element $u \in \mathcal{H}^p_{\mathcal{X}}(D)$ is associated to $\mu \in V^p_{\mathcal{X}}(\partial D, \omega)$.

Suppose now that the original $\mathcal{X}$-valued harmonic function u is given by the kernel integral of a vector measure with bounded p-variation, say

$$u(X) = \int_{\partial D} K(X, P)\, d\nu(P)$$

for some $\nu \in V^p_{\mathcal{X}}(\partial D, \omega)$. In order to establish uniqueness of the representation, we wish to prove that $\nu = \mu$, where μ is the measure just obtained, which amounts to showing that $\nu(F) = \mu(F)$ for all closed sets $F \in \mathfrak{B}$. For this purpose, we will need a preliminary construction and an auxiliary result.

Let F be a closed set in ∂D, and let $\{G_k\}$ be a decreasing sequence of open sets of $\mathbb{R}^n$ such that $F = \bigcap_{k \geq 1} G_k$. We may assume that the domains $D_k = D \setminus \overline{G_k}$ are regular for the Dirichlet problem associated to Laplace's equation, and through the argument in and around (3.1) we can consider ω_k, the harmonic measure with respect to D_k with the pole at the origin.

For $Q \in F$, notice that by (iii) of Theorem 3.2 the function $K(X, Q)$ vanishes for $X \in \partial D_k \cap \partial D$, and by (i) of the same theorem it is harmonic in the variable X; hence,

$$\int_{\partial D_k \cap D} K(X, Q)\, d\omega_k(X) = K(0, Q) = 1$$

because this integral represents the value at the origin of the harmonic function on D_k with boundary datum given by the function $K(X, Q)$. For $Q \notin F$, there is a large k_0 such that $Q \notin \partial D_k \cap D$ for $k > k_0$. Then, by (iii) of Theorem 3.2, we have

$$\lim_{k \to \infty} \int_{\partial D_k \cap D} K(X, Q)\, d\omega_k(X) = 0.$$

All in all, k

$$\lim_{k\to\infty}\int_{\partial D_k\cap D} K(X,Q)\,d\omega_k(X)=\chi_F(Q). \tag{3.6}$$

Armed with this identity, we now note that, for $\xi^*\in\mathcal{X}^*$, the very definition of ν and Fubini's theorem imply that

$$\lim_{k\to\infty}\Big\langle\xi^*,\int_{\partial D_k\cap D} u(X)\,d\omega_k(X)\Big\rangle=\big\langle\xi^*,\nu(F)\big\rangle. \tag{3.7}$$

On the other hand, by the construction of μ and (3.5) we have

$$\begin{aligned}\Big\langle\xi^*,\int_{\partial D_k\cap D} u(X)\,d\omega_k(X)\Big\rangle &= \int_{\partial D_k\cap D}\big\langle\xi^*,u(X)\big\rangle\,d\omega_k(X)\\ &= \int_{\partial D_k\cap D}\Big\langle\xi^*,\int_{\partial D} K(X,Q)\,d\mu(Q)\Big\rangle\,d\omega_k(X);\end{aligned}$$

however, the last integral is equal to

$$\int_{\partial D}\int_{\partial D_k\cap D} K(X,Q)\,d\omega_k(X)\big\langle\xi^*,d\mu(Q)\big\rangle.$$

Continuity of ξ^* and the bounded p-variation of μ make the measure $\mu_*=\langle\xi^*,\mu\rangle$ a scalar-valued measure of bounded p-variation. From the above argumentation and (3.6), we conclude that

$$\lim_{k\to\infty}\int_{\partial D}\int_{\partial D_k\cap D} K(X,Q)\,d\omega_k(X)\,d\mu_*(Q)=\int_{\partial D}\chi_F(Q)\,d\mu_*(Q)=\mu_*(F).$$

Therefore, by (3.7) we get $\langle\xi^*,\mu(F)\rangle=\langle\xi^*,\nu(F)\rangle$ for every ξ^* in $\mathcal{X}^*$, which implies that $\mu(F)=\nu(F)$ for $F\in\mathfrak{B}$ a closed set.

In this way we have proved that, for every element $u\in\mathcal{H}^p_{\mathcal{X}}$, there exists a unique vector measure μ in $V^p_{\mathcal{X}}(\partial D,\omega)$ such that it represents the original function u via the integral of the kernel function. Moreover, the boundary value of u is this vector measure μ in the weak* sense. □

4. Noncylindrical domains for the heat equation

The contents of this section are rather technical, and provide the appropriate framework that generalizes for the heat equation the environment already described for harmonic functions. After these geometric notions are introduced, we leave until the next section the definition of the caloric measure and the caloric Hardy spaces.

A first idea to consider is the introduction of a *time variable* in Euclidean spaces that leads us to consider $\mathbb{R}^{n+1}\equiv\mathbb{R}^n\times\mathbb{R}$ as the environment space. This way, points in $\mathbb{R}^{n+1}$ are denoted by (X,t), with $X\in\mathbb{R}^n$ recalled as the *space variable* and with $t\in\mathbb{R}$ recalled as the *time variable*.

Along with this new framework, there is an underlying *change of homogeneity* of the space $\mathbb{R}^{n+1}$. There are some fundamental works providing a thorough description of the *parabolic homogeneity* associated to $\mathbb{R}^{n+1}$ (see, e.g., [7] and references therein). Here we just record some notions used at several stages of this paper.

With a slight abuse of language, within this work we call the *parabolic distance* between $(X,t),(Y,s) \in \mathbb{R}^{n+1}$ the expression $\operatorname{dist}(X,t;Y,s) = |X-Y| + |t-s|^{1/2}$, and the *parabolic norm* of $(X,t) \in \mathbb{R}^{n+1}$ is defined as $\|X,t\| \equiv |X| + |t|^{1/2}$. This *norm* may also be applied to points $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, as they still have dependance on the variable t.

Given $(X,t) \in \mathbb{R}^{n+1}$, denote by $C_r(X,t)$ the *parabolic cylinder* $\{(Y,s) \in \mathbb{R}^{n+1} : |X-Y| < r, |t-s| < r^2\}$. We also introduce the *lower parabolic cylinder* of radius $r > 0$ and centered at (X,t), defined as $\mathcal{C}_r(X,t) = \{(Y,s) \in \mathbb{R}^{n+1} : |X-Y| < r, 0 < t-s < r^2\}$.

The *parabolic boundary* of an open connected set $\Omega \in \mathbb{R}^{n+1}$, denoted by $\partial_p\Omega$, consists of points $(Q,s) \in \partial\Omega$ (the topological boundary of Ω) such that for every $r > 0$ one has $\mathcal{C}_r(Q,s) \setminus \Omega \neq \emptyset$.

We now define the regularity of functions that will describe locally the boundary of the noncylindrical domains on which we work. For this description we adopt the convention that points in $\mathbb{R}^{n+1}$ may be denoted by $(x_0,x,t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ to stress that in graph coordinates x_0 is the variable depending on (x,t). This particular way to denote the graph coordinates for problems associated to the heat equation goes back at least to the work of [23]. We also adopt the notation $\mathbf{X}$, $\mathbf{Y}$, etc. for points in $\mathbb{R}^{n+1}$ whenever both the time variable t and the graph variable x_0 are irrelevant for the argumentation.

A function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a $\operatorname{Lip}(1,1/2)$ *function with constant* $A_1 > 0$ if, for $(x,t),(y,s) \in \mathbb{R}^n$, $|\psi(x,t) - \psi(y,s)| \leq A_1\|x-y,t-s\|$. The function ψ is called a *parabolic Lipschitz function with constant* A_1 if it satisfies the following two conditions:

- ψ satisfies a *Lipschitz condition in the space variable*
$$\big|\psi(x,t) - \psi(y,t)\big| \leq A_1|x-y| \quad \text{uniformly on } t \in \mathbb{R}, \tag{4.1}$$
- for every interval $I \subseteq \mathbb{R}$ and every $x \in \mathbb{R}^n$,
$$\frac{1}{|I|}\int_I\int_I \frac{|\psi(x,t)-\psi(x,s)|^2}{|s-t|^2}\,dt\,ds \leq A_1 < \infty. \tag{4.2}$$

This last condition can be recalled as a *BMO-Sobolev scale* in the t-variable by results in [30], and it roughly states that a half-order derivative of $\psi(x,t)$ with respect to the t variable is in $BMO(\mathbb{R})$. Intuitively, the smaller the constant A_1 is, the *flatter* the graph of ψ is (more details appear in [16]).

The notion of parabolic Lipschitz functions was motivated by a question posed by R. Hunt. Indeed, according to [16], he had asked to find adequate regularity conditions on functions that describe locally the boundary of a domain where one can solve initial L^p-Dirichlet problems $(1 < p < \infty)$ associated to the heat equation. The fact that the $\operatorname{Lip}(1,1/2)$ condition is not the correct type of regularity was settled by a 1-dimensional example in [20], where it is proved the singularity of caloric measure (as defined right above (4.4)) with respect to graph measure of a $\operatorname{Lip}(1/2)$ function.

On the other hand, it has been established that parabolic Lipschitz functions are slightly more regular than $\mathrm{Lip}(1,1/2)$ functions, and, in particular, every parabolic Lipschitz function is a $\mathrm{Lip}(1,1/2)$ function; moreover, on domains with the boundary described locally by parabolic Lipschitz graphs one can solve this type of L^p-Dirichlet problem (see [16]). One can also obtain the mutual absolute continuity of caloric measure and the surface measure defined for any Borel set $F \subset \mathbb{R}^{n+1}$ by

$$\sigma(F) = \int_F d\sigma_t\, dt, \tag{4.3}$$

where σ_t is the $(n-1)$-dimensional Hausdorff measure of $F_t \equiv F \cap \mathbb{R}^n \times \{t\}$, and dt denotes integration with respect to the 1-dimensional Hausdorff measure (see (4.4) below). This is proved in [23] and [16] for instance (see also [29]).

To describe domains whose boundary is given locally by parabolic Lipschitz graphs, we adapt some definitions from [4]. Given $A_1, r_0 > 0$, define the *local cylinder with constants* A_1*,* r_0 as

$$\mathcal{Z} = \big\{(x_0, x, t) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R} : |x_i| < r_0, i = 1, 2, \dots, n-1, |x_0| < 2nA_1 r_0, t \in \mathbb{R}\big\}.$$

Here, $x \in \mathbb{R}^{n-1}$ is viewed as the $(n-1)$-tuple $x = (x_1, x_2, \dots, x_{n-1})$. We denote by $2\mathcal{Z}$ the concentric double of $\mathcal{Z}$.

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open connected set such that $\partial_p\Omega = \partial\Omega$. We say that Ω is an *infinite noncylindrical region with constants* A_1*,* r_0 if there exist local cylinders $\{\mathcal{Z}_i : i = 1, 2, \dots, N\}$, with constants A_1, r_0, which are obtained from $\mathcal{Z}$ through rigid motions in the space variables and parabolic Lipschitz functions $\{\psi_i : i = 1, 2, \dots, N\}$ with constant A_1, defined on the transformation of $\mathbb{R}^n$ through the same rigid motion defining $\mathcal{Z}_i$, and such that the following conditions hold:

- $2\mathcal{Z}_i \cap \partial\Omega = \{(x_0, x, t) : x_0 = \psi_i(x, t)\} \cap 2\mathcal{Z}_i, i = 1, 2, \dots, N$;
- $2\mathcal{Z}_i \cap \Omega = \{(x_0, x, t) : x_0 > \psi_i(x, t)\} \cap 2\mathcal{Z}_i, i = 1, 2, \dots, N$;
- $\partial\Omega$ is covered by $\bigcup_{i=1}^N \mathcal{Z}_i$;
- for every $s \in \mathbb{R}$, the set $\Omega(s) = \{(X, t) \in \Omega : t = s\}$ is a bounded starlike Lipschitz domain centered at the origin with fixed constants $M > 0$ and $\delta_0 > 0$, and with M depending on the constant A_1.

Given an infinite noncylindrical region Ω and $(X, t) \in \Omega$, we write $\delta(X, t) = \operatorname{dist}(X, t; \partial\Omega)$, and we set $\operatorname{diam}\Omega = \sup\{\operatorname{diam}\Omega(s) : s \in \mathbb{R}\}$.

If Ω is an infinite noncylindrical region, for $T > 0$ we define the *bounded noncylindrical region of height* T as $\Omega_T = \{(X, t) \in \Omega : 0 < t < T\}$. The *lateral boundary of* Ω_T is defined by $S_T \equiv \partial_p\Omega_T \cap \partial\Omega$. Also, set $\mathbf{\Xi} = (\vec{0}, T + 1)$, which, even though it is not in Ω_T, will be recalled as the *parabolic center of* Ω_T in the sense that will be explained in a later paragraph.

Define the *surface cubes* $\Delta_\rho(Q, s) \equiv C_\rho(Q, s) \cap S$ for $0 < \rho < r_0/4$ and $(Q, s) \in S$. If Δ is any surface cube such that its closure satisfies $\overline{\Delta} \subset S_T$, then, to shorten notation, we write $\Delta \Subset S_T$.

Denote by H the *heat operator* $Hu = \Delta u - \partial u/\partial t$. For Ω an infinite noncylindrical region as described above, the *caloric measure*, denoted by $\omega^{(X,t)}(\cdot)$

for $(X,t) \in \Omega$, is the unique Borel measure supported on S_T such that

$$u_f(X,t) = \int_{S_T} f(Y,s)\, d\omega^{(X,t)}(Y,s) \tag{4.4}$$

is the solution of the Dirichlet-type of problem $Hu = 0$ on Ω_T, $u|_{S_T} = f$ for f continuous and *supported on* S_T. This solution exists because of the *regularity* (in the sense of Perron–Wiener–Brelot, see [24, Chapter III, Section 4]) of Ω_T or in general domains with lateral boundaries given by $\mathrm{Lip}(1,1/2)$ functions (by Wiener's criterion, for instance; see [14]).

Observe that in particular $u(X,s) = 0$ for every $X \in \Omega(s)$ and every $s \leq 0$. Hence u_f may be thought of as a solution to an *initial-Dirichlet problem* on Ω_T. This is the reason why $\omega^{\mathbf{X}}$ is actually a measure supported on S_T although it is well defined for $\mathbf{X} \in \Omega_+ = \{(Y,s) \in \Omega : s > 0\}$. We denote by $\omega(\cdot)$ the caloric measure $\omega^{\boldsymbol{\Xi}}(\cdot)$.

Given $(Q,s) \in S_T$, consider the *parabolic path* (Q_r, s_r) joining $\boldsymbol{\Xi}$ with (Q,s) given by

$$Q_r = rQ, \qquad s_r = (1-r^2)(T+1) + r^2 s \quad \text{for } 0 < r < 1. \tag{4.5}$$

This way $\boldsymbol{\Xi}$ may be thought of as a parabolic center of Ω_T, and the parabolic path may be viewed as a *parabolic radial trajectory* joining points in S_T with $\boldsymbol{\Xi}$. This is an analogue of the 2-dimensional *parabolic coordinates* from [1].

Recalling x_0 as the coordinate depending on (x,t), we define a *parabolic cone fixed at the origin* as $\widetilde{\Gamma}_\alpha = \{(x_0,x,t) : \|x,t\| < \alpha x_0\}$. For $(Q,s) \in \partial\Omega$, we use a translation and a rotation not involving the t variable in order to move $\widetilde{\Gamma}_\alpha$ from the origin to (Q,s), keeping the axis of the cone in the direction of $\overline{OQ}$. This way we can define infinite parabolic cones $\widetilde{\Gamma}_\alpha(Q,s)$ pointing toward the interior of Ω with the vertex on (Q,s), accordingly.

We define appropriate *bounded nontangential approach regions* adapted to an infinite parabolic cylinder Ω as follows. First, for $(Q,s) \in \Omega$ we set $\rho(Q,s) \equiv |Q|$ and define $\rho_0 = \inf\{\rho(Q,s) : (Q,s) \in \partial\Omega\}$ in such a way that $\rho_0 \geq \delta_0$, the constant of the starlike Lipschitz domains $\Omega(s)$, $s \in \mathbb{R}$. Then we define $\Gamma_\alpha(Q,s)$ as the truncation of $\widetilde{\Gamma}_\alpha(Q,s)$ at height ρ_0.

If Ω is an infinite noncylindrical region such that

(i) for every $(Q,s) \in S_T$ one has $(Q_r,s_r) \in \Omega_{T+1}$ for every $0 < r < 1$,

(ii) there is an aperture $\alpha > 0$ such that $\Gamma_\alpha(Q,s) \subset \Omega$ and $(Q_r,s_r) \in \widetilde{\Gamma}_\alpha(Q,s)$ for every $(Q,s) \in S_T$ and every $0 < r < 1$,

then we say that Ω_T is a *special bounded starlike region.* Observe that any *Lipschitz cylinder* $D \times (0,T)$ within the infinite cylinder $D \times \mathbb{R}$, with $D \subset \mathbb{R}^n$ a starlike Lipschitz domain, is an example of a special bounded starlike region. Also, a bounded noncylindrical region of height T within an infinite noncylindrical region is a special bounded starlike region, provided that the constant A_1 in the definition of local cylinders is suitably small.

Fixing $\alpha > 0$ as described above, we can drop the subscript α from the notation in the cones and nontangential approach regions. Also, when working on the special bounded starlike region we will restrict the radii of the surface balls to be strictly smaller than $\min\{r_0, \rho_0/2\}$, and, in order to disregard degenerate situations, from now on we always take $T > 5\rho_0$.

As for harmonic functions on starlike Lipschitz domains, we can now define for $u : \Omega_T \to \mathbb{R}$ a *nontangential maximal function* as $N_\alpha u(Q,s) \equiv Nu(Q,s) = \sup\{|u(X,t)| : (X,t) \in \Gamma(Q,s)\}$.

For Ω_T a special bounded starlike region, choose $N_0 \geq 1$ such that both of the following points,

$$\overline{\mathcal{A}}(\Delta) = \overline{\mathcal{A}}_r(Q,s) \equiv (q_0 + N_0 r, q, s + r^2),$$
$$\underline{\mathcal{A}}(\Delta) = \underline{\mathcal{A}}_r(Q,s) \equiv (q_0 + N_0 r, q, s - r^2),$$

are contained in Ω for every $(Q,s) = (q_0,q,s) \in \partial\Omega$ and $0 < r < \rho_0$ such that $\Delta_r(Q,s) \Subset S \cap \{(Y,s) : -1 < s < T+1\}$. This choice of N_0 depends only on the aperture α of the definition of special bounded starlike regions, which in turn depends on *how flat* the lateral boundary S is.

5. Statement of results for Hardy spaces of caloric functions

Caloric measure and kernel function. Let Ω_T be a special bounded starlike region. As observed above, the caloric measure on S_T provides the representation formula (4.4) for solutions of a Dirichlet-type problem for the heat equation on Ω_T. As for harmonic functions, a fundamental property of this caloric measure over the time-varying domains Ω_T is the *doubling property*, which can be roughly described as follows: there exists a constant $C > 0$ such that for any *surface ball* $\Delta_r(\mathbf{Q}) \subset S$ of radius $0 < r < r_0$, centered at $\mathbf{Q} \in S_T$, the estimate

$$\omega^{\mathbf{X}}\big(\Delta_{2r}(\mathbf{Q})\big) \leq C\omega^{\mathbf{X}}\big(\Delta_r(\mathbf{Q})\big) \tag{5.1}$$

holds for any $\mathbf{X} \in \Omega$ such that $\|\mathbf{X} - \mathbf{Q}\| \geq 3r$. The proof of this property is in [25, Theorem 3.2].

At this point we recall that the interior Harnack inequality for positive solutions of the heat equation has a *time-lag* by the evolutive nature of solutions. This result is proved as an interior estimate, for instance, in [10, p. 163], and we state it here with sufficient generality to be used within this work.

Theorem 5.1. *Let Ω_T be a special bounded starlike region within the infinite noncylindrical region Ω, and let $\delta > 0$. Suppose that δ is chosen small enough in such a way that $\Omega_T^\delta \equiv \{(Y,s) \in \Omega_T : \delta < s < T+2, \delta(Y,s) > \delta\}$ is connected, and let u be a nonnegative caloric function on Ω_{T+2}. Then there exists a constant $C > 0$ depending on the geometric constants of Ω, T, n, and δ such that $u(X,t) \leq Cu(Y,s)$ whenever $(X,t),(Y,s) \in \Omega_T^\delta$ satisfy $s - t > \delta^2$.*

Recall that $\omega = \omega^{\mathbf{\Xi}}$ with $\mathbf{\Xi} = (\vec{0}, T+1)$. Now, with $\delta > 0$ appropriately small, $\mathbf{\Xi} \in \Omega_T^\delta$ so that, as an application of Theorem 5.1, for every $\mathbf{X} \in \Omega_T^\delta$ the measure $\omega^{\mathbf{X}}$ is absolutely continuous with respect to ω. This is true because $\omega^{\mathbf{X}}(E)$ is a nonnegative caloric function of $\mathbf{X} \in \Omega$ for every Borel set $E \subset S_T$. In fact, the analogue of (i) of Theorem 3.1 holds in this setting (see, e.g., [24]).

From now on, for $\mathbf{X} \in \Omega_T$ and $\mathbf{Q} \in S_T$, we denote by $K(\mathbf{X};\mathbf{Q})$ the Radon–Nykodým derivative $K(\mathbf{X};\mathbf{Q}) = (d\omega^{\mathbf{X}}/d\omega)(\mathbf{Q})$. This function is referred to as the *kernel function* associated to the heat equation on Ω_T.

Note that since the caloric measures are supported on S_T, the maximum principle for caloric functions implies that $K(X, 0; \mathbf{Q}) = 0$ for $(X, 0) \in \Omega(0)$; that is, the kernel function satisfies the vanishing initial condition on Ω_T (see, e.g., [24]). Moreover, the following representation formula for functions $f \in C(S_T)$ is immediate from (4.4):

$$u(\mathbf{X}) = \int_{S_T} K(\mathbf{X};\mathbf{Q}) f(\mathbf{Q})\, d\omega(\mathbf{Q}). \tag{5.2}$$

Instead of gathering some of the important properties of caloric measure over special bounded starlike regions in the form of a theorem as we did for harmonic functions in Theorems 3.1 and 3.2, we will state the corresponding results when needed.

It will be convenient, though, to record the following general properties of the caloric measure and the caloric kernel function over the special bounded starlike region Ω_T.

(i) Theorem 2.1 holds for the Hardy–Littlewood maximal function of a Borel measure ν with respect to ω on S_T defined as

$$M_\omega \nu(\mathbf{Q}) = \sup_{r>0} \frac{\nu(\Delta_r(\mathbf{Q}))}{\omega(\Delta_r(\mathbf{Q}))}$$

with a similar definition for $M_\omega f$.

(ii) $K(-, \mathbf{P})$ is a positive caloric function on Ω_T with $K(\mathbf{\Xi}; \mathbf{P}) = 1$.

(iii) For every $\mathbf{Q} \in S_T$,

$$K(\mathbf{X}, \mathbf{Q}) = \lim_{r\to 0} \frac{\omega^{\mathbf{X}}(\Delta_r(\mathbf{Q}))}{\omega(\Delta_r(\mathbf{Q}))}.$$

(iv) If ν is a finite Borel measure on S_T and

$$u(\mathbf{X}) = \int_{S_T} K(\mathbf{X}, \mathbf{Q})\, d\nu(\mathbf{Q}),$$

then there exists a constant C_2 such that for each $\mathbf{P} \in S_T$

$$Nu(\mathbf{P}) \le C_2 M_\omega \nu(\mathbf{P}).$$

Caloric Hardy spaces. Because of the definition of the caloric measure, we want to consider Hardy spaces of caloric functions taking a vanishing initial value. Let $H_0(\Omega)$ denote the class of caloric scalar functions u defined on Ω and satisfying $u(x, t) = 0$ for $t \le 0$. Whenever the function takes values on a Banach space $\mathcal{X}$, we adopt the notation $H_0(\Omega, \mathcal{X})$ for the class defined in a similar fashion. Now for $1 \le p \le \infty$, the symbol $\mathcal{H}^p_{\mathcal{X}}(\Omega)$ denotes the Banach space of caloric functions $u \in H_0(\Omega, \mathcal{X})$ with $Nu \in L^p(S_T, d\omega)$. As in the classical scalar theory and the vector-valued case of harmonic functions, we endow $\mathcal{H}^p_{\mathcal{X}}(\Omega)$ with the norm

$$\|u\|_{\mathcal{H}^p_{\mathcal{X}}} = \|Nu\|_{L^p_{\mathcal{X}}(S_T, d\omega)}.$$

And, again, a similar definition is adopted for scalar-valued caloric functions, dropping the dependance of $\mathcal{X}$ from the notation.

The use of the harmonic measure as a weight in the definition of Hardy spaces of harmonic functions is a well-established feature, and it has actually been adopted in Section 3 in this work (see, e.g., [19]). For caloric functions, to our knowledge, this type of definition has not been considered yet, although an unweighted parabolic Hardy space was considered in [5]. Our first result for caloric Hardy spaces is the analogue of Theorem 3.4 in Section 3. It is proved in the next section following through the proof in [28] for the case of harmonic functions.

Theorem 5.2. *Let $\mathcal{X}$ be a Banach space, and let Ω_T be a special bounded starlike region. Then $\mathcal{X} \in RNP$ if and only if, for some $1 \leq p \leq \infty$ (or, equivalently, for all $1 \leq p \leq \infty$) and for every $u \in \mathcal{H}^p_{\mathcal{X}}(\Omega)$, the limit*

$$\lim_{\substack{\mathbf{X}\to\mathbf{P} \\ \mathbf{X}\in\Gamma(\mathbf{P})}} u(\mathbf{X}) = u(\mathbf{P})$$

exists for σ-almost every $P \in S_T$.

As in Theorem 3.4, the existence of nontangential limits for σ-almost every $P \in S_T$ is a consequence of the mutual absolute continuity between these measures over the class of time-varying domains that we consider in this work (see [23], [16]). Also, as in the case for harmonic functions in Theorem 3.5, we can identify the boundary values of elements in $\mathcal{H}^p_{\mathcal{X}}(\Omega)$, $p > 1$, with elements in $V^p_{\mathcal{X}}(S_T, \omega)$.

Theorem 5.3. *For $p > 1$, we have the isomorphism $V^p_{\mathcal{X}}(S_T, \omega) = \mathcal{H}^p_{\mathcal{X}}(\Omega)$ through integration with the kernel function.*

Sketch of Proof. We will only give a sketch of the proof, and for further details we refer the reader to the proof of Theorem 3.5 above.

For $\nu \in V^p_{\mathcal{X}}(S_T, \omega)$, one defines

$$u(\mathbf{X}) = \int_{S_T} K(\mathbf{X}, \mathbf{P})\, d\nu(\mathbf{P}).$$

The kernel function $K(\mathbf{X}, -)$ is continuous on the compact set $\overline{\partial_p \Omega_T}$ so that $K(\mathbf{X}, \mathbf{P})$ (as a function of P) belongs to $L^q(S_T, d\omega)$, and hence u is well defined. Since $K(\mathbf{X}, \mathbf{P})$ as a function of $\mathbf{X}$ is a solution of the heat equation, we can use the same device in the proof of Theorem 3.5 to establish that u is a weak solution of the heat equation, and hence u is actually a caloric function in $H_0(\Omega, \mathcal{X})$. In this case, however, our use of the scalar theory requires not only Lemmas 2.2 and 2.3, but also Lemma 5.4, proved below.

To prove that $Nu \in L^p(S_T, d\omega)$, again we observe that if ν is of bounded variation, then

$$\Big\| \int_{S_T} K(\mathbf{X}, \mathbf{P})\, d\nu(\mathbf{P}) \Big\|_{\mathcal{X}} \leq \int_{S_T} K(\mathbf{X}, \mathbf{P})\, d|\nu|(\mathbf{P}).$$

Using Lemmas 2.2 and 2.3, we obtain a function f in $L^p(S_T, d\omega)$ such that $d|\nu| = f\, d\omega$, and by property (iv) above we get

$$\sup_{\mathbf{X}\in\Gamma(\mathbf{Q})}\Big\|\int_{S_T} K(\mathbf{X},\mathbf{P})\, d\nu(\mathbf{P})\Big\|_{\mathcal{X}} \le \sup_{\mathbf{X}\in\Gamma(\mathbf{Q})}\int_{S_T} K(\mathbf{X},\mathbf{P}) f(\mathbf{P})\, d\omega(\mathbf{P}) \le C_2 M_\omega f(\mathbf{Q}).$$

By Theorem 2.1 our assertion is proved, and thus we have proved the inclusion $V^p_{\mathcal{X}}(S_T,\omega) \hookrightarrow \mathcal{H}^p_{\mathcal{X}}(\Omega)$.

For the proof of the converse inclusion, we start with $u \in \mathcal{H}^p_{\mathcal{X}}(\Omega)$ and apply Banach–Alaouglu's theorem to obtain a measure $\mu \in V^p_{\mathcal{X}^{**}}(S_T,\omega)$ such that

$$\lim_{\rho\to 1} u_\rho = \mu \quad \text{in the weak}^* \text{ sense}, \tag{5.3}$$

where for $(Q,s) \in S_T$ this time we have set $u_\rho(Q,s) = u(Q_\rho, s_\rho)$ as defined in (4.5). It turns out, as in Theorem 3.5, that μ is $\mathcal{X}$-valued and we can associate μ with u via integration with the kernel function. The uniqueness of ν can be proved following the same procedure as in Theorem 3.5, and we omit the details. □

A Fatou-type theorem for caloric functions. We have already used the following scalar parabolic version of Fatou's theorem, and we will have another use for it in a subsequent section. As will be shown, it follows from standard techniques, and we include the proof for completeness.

Lemma 5.4 (Fatou-type theorem)**.** *Let Ω_T be a special bounded starlike region, and suppose that $u \in \mathcal{H}^1(\Omega)$. Then there exists a unique Borel measure ν defined on S_T such that*

$$u(\mathbf{X}) = \int_{S_T} K(\mathbf{X};\mathbf{Q})\, d\nu(\mathbf{Q}),$$

and in fact ν is absolutely continuous with respect to ω with Radon–Nikodym derivative $f \in L^1(S_T, d\omega)$. Moreover, u converges nontangentially to f for ω-almost every point in S_T.

For the proof of this lemma and for easy future reference, we now state the following results on estimates between the caloric measure and the caloric kernel function, stated with adaptations to our noncylindrical setting and with the pertinent references provided.

Lemma 5.5 ([21, Lemma 2.1])**.** *Suppose that Ω_T is a special bounded starlike region. Let $\mathbf{Q} \in S_T$, and let $\Delta = \Delta_r(\mathbf{Q})$ denote a surface ball of radius $0 < r < r_0$. Then, for $r > 0$ sufficiently small,*

$$\sup_{\mathbf{Q}'\in S_T\setminus\Delta} K(\mathbf{X};\mathbf{Q}') \to 0 \quad \textit{as } \mathbf{X}\to\mathbf{Q} \textit{ within } \Omega_T. \tag{5.4}$$

Lemma 5.6 ([21, Lemma 2.4])**.** *Let Ω_T be a special bounded starlike region, and for $\mathbf{Q_0} \in S_T$ let $0 < r < r_0$ be such that $\Delta_{2r}(\mathbf{Q_0}) \Subset S_T$. For $0 < s < r$ sufficiently small and $\mathbf{Q} \in \Delta_r(\mathbf{Q_0})$,*

$$\sup_{\mathbf{Q}'\in\Delta_s(\mathbf{Q})} K\big(\overline{\mathcal{A}}_s(\mathbf{Q});\mathbf{Q}'\big) \le \frac{c}{\omega^{\mathbf{\Xi}}(\Delta_s(\mathbf{Q}))}.$$

Here $c > 0$ is a constant depending only on Ω.

Proof of Lemma 5.4. For $0 < r < 1$ and $(X,t) \in \Omega_T \cup S_T$, define $u_r(X,t) = u(rX,(1-r^2)(T+1)+r^2t)$. Note that this simply extends the parabolic path in (4.5) to points in the interior of Ω_T. As observed before, by Harnack's inequality $\omega^{(X,t)}$ is absolutely continuous with respect to $\omega^{\mathbf{\Xi}}$ for $(X,t) \in \Omega_T$. Therefore, $K(X,t;Q,s)$ is essentially bounded and positive for σ almost every $(Q,s) \in S_T$ for each fixed $(X,t) \in \Omega_T$. Moreover,

$$\int_{S_T} K(X,t;Q,s)\,d\omega(Q,s) = \int_{S_T} d\omega^{(X,t)}(Q,s) = 1. \tag{5.5}$$

On the other hand, we also have

$$u_r(X,t) = \int_{S_T} K(X,t;Q,s)u_r(Q,s)\,d\omega(Q,s). \tag{5.6}$$

Since $Nu \in L^1(S_T,d\omega)$, the family of measures $\{d\mu_r(Q,s) \equiv u_r(Q,s)\,d\omega(Q,s) : 0<r<1\}$ is a bounded set on $C(S_T)^*$, and each of these measures is absolutely continuous with respect to ω. By Banach–Alaoglu's theorem there is a sequence $\{r_n\}$ with $r_n \to 1$ and a measure μ such that μ_{r_n} converges in the weak* sense to $\mu \in C(S_T)^* = M(S_T)$; that is,

$$\lim_{r\to 1}\int_{S_T} h(\mathbf{Q})\,d\mu_r(\mathbf{Q}) = \int_{S_T} h(\mathbf{Q})\,d\mu(\mathbf{Q}) \quad \text{for every } h \in C(S_T). \tag{5.7}$$

Now, taking $E \subset S_T$ and a sequence $\{g_m\} \subset C(S_T)$ approaching to χ_E in $L^1(S_T,d\omega)$-norm, one can use (5.7) along with the absolute continuity $\mu_r \ll \omega$ to prove that μ is absolutely continuous with respect to ω. Therefore, we can restate (5.7) by writing

$$\lim_{r\to 1}\int_{S_T} h(\mathbf{Q})\,d\mu_r(\mathbf{Q}) = \int_{S_T} h(\mathbf{Q})f(\mathbf{Q})\,d\omega(\mathbf{Q}) \quad \text{for every } h \in C(S_T) \tag{5.8}$$

for certain $f \in L^1(S_T,d\omega)$. If $g \in L^\infty(S_T,d\omega)$, then by Lusin's theorem, $g = g_1 + g_2$, where g_1 is continuous and g_2 has arbitrarily small support; hence,

$$\lim_{r\to 1}\Big|\int_{S_T} g(\mathbf{Q})u_r(\mathbf{Q})\,d\omega(\mathbf{Q}) - \int_{S_T} g(\mathbf{Q})f(\mathbf{Q})\,d\omega(\mathbf{Q})\Big| = 0.$$

Since $K(\mathbf{X};\mathbf{Q})$ is bounded in the variable $\mathbf{Q} \in S_T$ for fixed $\mathbf{X} \in \Omega_T$, letting $r_n \to 1$ from (5.6), we get

$$u(\mathbf{X}) = \int_{S_T} K(\mathbf{X};\mathbf{Q})f(\mathbf{Q})\,d\omega(\mathbf{Q}).$$

Now we prove the ω-almost everywhere nontangential convergence. Let $\mathbf{Q_0} \in S_T$ be a Lebesgue point of f with respect to ω, and let $\mathbf{X} \in \Gamma(\mathbf{Q_0})$ with $\|\mathbf{X} - \mathbf{Q_0}\| = r$. If $\Delta = \Delta_r(\mathbf{Q_0})$, then

$$\begin{aligned}\big|u(\mathbf{X}) - f(\mathbf{Q_0})\big| &\le \int_{\Delta} K(\mathbf{X};\mathbf{Q})\big|f(\mathbf{Q}) - f(\mathbf{Q_0})\big|\,d\omega(\mathbf{Q}) \\ &\quad + \int_{S_T\setminus\Delta} K(\mathbf{X};\mathbf{Q})\big|f(\mathbf{Q}) - f(\mathbf{Q_0})\big|\,d\omega(\mathbf{Q}) \\ &= I + II. \end{aligned} \tag{5.9}$$

By Lemma 5.6 and the Lebesgue differentiation theorem (recall that ω is a doubling measure), we deduce that I tends to 0 as $\mathbf{X} \to \mathbf{Q_0}$ in $\Gamma(\mathbf{Q_0})$. On the other hand, the term II in (5.9) is majorized by

$$\big(\|f\|_1 + f(\mathbf{Q_0})\big) \sup_{\mathbf{Q} \in S_T \setminus \Delta} K(\mathbf{X}; \mathbf{Q}).$$

By Lemma 5.5 we can see that II tends to 0 as $\mathbf{X} \to \mathbf{Q_0}$ in $\Gamma(\mathbf{Q_0})$. This proves the lemma. ☐

6. Proof of Theorem 5.2

In this section, we basically adapt an argument from [28] to caloric functions in our noncylindrical domain. So we will keep the notation from the previous section while introducing new terminology and definitions as needed. In particular, throughout this section we assume that Ω_T is a special bounded starlike region within an infinite noncylindrical region Ω. For some arguments in this section, we will use fundamental properties of the caloric measure and the kernel function, some of which we already quoted above, all of them contained, for instance, in [21]. We will refer to those results providing the pertinent reference.

The next two lemmas prepare the ground for the proof of Theorem 5.2, which will be explained afterward.

Lemma 6.1. *If $f \in C_{\mathcal{X}}(S_T)$, then the vector-valued function $v(\mathbf{X})$ defined for $\mathbf{X} \in \Omega_T$ as*

$$v(\mathbf{X}) = \int_{\partial_p \Omega_T} f(\mathbf{Q})\, d\omega^{\mathbf{X}}(\mathbf{Q})$$

is a solution to the heat equation satisfying

$$\lim_{\substack{\mathbf{X} \to \mathbf{P} \\ \mathbf{X} \in \Gamma(\mathbf{P})}} v(\mathbf{X}) = f(\mathbf{P})$$

for every $\mathbf{P} \in S_T$.

Proof. To prove that $v(\mathbf{X})$ is a solution to the heat equation, it suffices to prove that it is a weak solution to that heat equation.

In this case we simply observe that, by well-known properties of Bochner integrals, the following identity holds:

$$\Big\langle \xi^*, \int_{S_T} f(\mathbf{Q})\, d\omega^{\mathbf{X}}(\mathbf{Q}) \Big\rangle = \int_{S_T} \big\langle \xi^*, f(\mathbf{Q}) \big\rangle\, d\omega^{\mathbf{X}}(\mathbf{Q});$$

also, by (5.2), the scalar-valued function in the left-hand side is a solution to the heat equation on Ω_T with continuous datum over S_T given by $\langle \xi^*, f(\mathbf{Q}) \rangle$. Now, the proof for nontangential convergence is similar to that in (5.9). ☐

Lemma 6.2. *Let $u \in \mathcal{H}^1_{\mathcal{X}}(\Omega)$. Then there exists $\mu \in M_{\mathcal{X}}(S_T)$ an ω-continuous $\mathcal{X}$-valued measure such that*

$$u(\mathbf{X}) = \int_{S_T} K(\mathbf{X}; \mathbf{Q})\, d\mu(\mathbf{Q}). \tag{6.1}$$

Proof. This is simply the vector-valued case of Lemma 5.4, and its proof is a reprise of that scalar case. In this case we invoke Singer's representation theorem in order to use the isomorphism $M_{\mathcal{X}^{**}}(S_T) = C_{\mathcal{X}^*}(S_T)^*$. □

Proof of Theorem 5.2. At this point, it only remains for us to provide the following two main blocks to construct the proof of Theorem 5.2.

Lemma 6.3. *If $\mathcal{X} \in RNP$, then every function $u \in \mathcal{H}^1_{\mathcal{X}}(\Omega)$ has nontangential limits for ω-almost every $\mathbf{Q} \in S_T$ (and, equivalently, σ-almost every $\mathbf{Q} \in S_T$).*

Proof. According to Lemma 6.2 and the Radon–Nikodym property of $\mathcal{X}$, we can represent $u \in \mathcal{H}^1_{\mathcal{X}}(\Omega_T)$ by

$$u(\mathbf{X}) = \int_{\partial_p \Omega_T} K(\mathbf{X}; \mathbf{Q}) f(\mathbf{Q})\, d\omega(\mathbf{Q})$$

with $f \in L^1_{\mathcal{X}}(S_T, d\omega)$. Recalling that caloric measure is a doubling measure as observed in (5.1), it will suffice to prove that the nontangential limits exist in every ω-Lebesgue point of f.

To prove this assertion, and also that f is the function of nontangential (ω-almost everywhere) limits of u, we proceed again as in the proof of Lemma 5.4, taking $\mathbf{P} \in S_T$ as an ω-Lebesgue point of f. Given $\varepsilon > 0$, we choose $\delta > 0$ such that

$$\frac{1}{\omega(\Delta)} \int_\Delta \big\| f(\mathbf{Q}) - f(\mathbf{P}) \big\|_{\mathcal{X}}\, d\omega(\mathbf{Q}) < \varepsilon$$

whenever $\|\mathbf{Q} - \mathbf{P}\| < \delta$, where $\Delta \equiv \Delta_\delta(\mathbf{P})$. Using Lemmas 5.5 and 5.6. we have the ingredients to finish the proof again as in the estimate (5.9). □

Since $\mathcal{H}^p_{\mathcal{X}}(\Omega_T) \subseteq \mathcal{H}^1_{\mathcal{X}}(\Omega_T)$ for $p > 1$, then the previous lemma implies one part of Theorem 5.2. The other part is contained in the next lemma because $\mathcal{H}^\infty_{\mathcal{X}}(\Omega_T) \subseteq \mathcal{H}^p_{\mathcal{X}}(\Omega_T)$ for $1 \le p < \infty$.

Lemma 6.4. *If every function in $\mathcal{H}^\infty_{\mathcal{X}}(\Omega_T)$ has nontangential limits ω-almost everywhere, then $\mathcal{X} \in RNP$.*

Proof. We will prove that every continuous linear operator $T : L^1(S_T, d\omega) \to \mathcal{X}$ is representable by a function $f \in L^\infty_{\mathcal{X}}(S_T, d\omega)$ in the sense that

$$T(g) = \int_{S_T} f(\mathbf{Q}) g(\mathbf{Q})\, d\omega(\mathbf{Q}).$$

(By [11, Chapter III, Section 1, Theorem 5], this already implies that $\mathcal{X} \in RNP$.) Define for $\mathbf{X} \in \Omega_T$ the vector-valued function $v(\mathbf{X}) = T[K(\mathbf{X}; \cdot)]$. Then

$$\big\| v(\mathbf{X}) \big\|_{\mathcal{X}} \le \|T\| \big\| K(\mathbf{X}, \cdot) \big\|_{L^1(S_T, d\omega)} = \|T\|.$$

Moreover, since T is continuous, then v is caloric, and, as observed right above (5.2), the kernel function satisfies the initial condition $K(\mathbf{X}, \mathbf{Q}) = 0$ for $\mathbf{X} \in \Omega(0)$. Therefore, $v \in \mathcal{H}^\infty_{\mathcal{X}}(\Omega)$.

Let $f \in L^\infty_{\mathcal{X}}(S_T, d\omega)$ be the nontangential limit of v, which we know exists by assumption. We claim that f represents the operator T. By a standard density argument it suffices to prove that

$$T(\chi_A) = \int_A f\,d\omega$$

for every Borel set A in S_T, where χ_A denotes the characteristic function of A. Now, by linearity of T,

$$\begin{aligned}\int_A f\,d\omega &= \lim_{r\to 1}\int_A v(Q_r, s_r)\,d\omega(Q,s) = \lim_{r\to 1}\int_{S_T} T\big[K(Q_r, s_r; -)\big]\chi_A(Q,s)\,d\omega(Q,s)\\ &= \lim_{r\to 1}\int_{S_T} T\big[K(Q_r, s_r; -)\chi_A(Q,s)\big]\,d\omega(Q,s), \end{aligned} \tag{6.2}$$

where the last integral is interpreted as a Bochner integral. The continuity of K on $\Omega \times S_T$ implies that

$$\int_{S_T} T\big(K(Q_r, s_r; -)\chi_A(Q,s)\big)\,d\omega(Q,s) = T\Big(\int_{S_T} K(Q_r, s_r; -)\chi_A(Q,s)\,d\omega(Q,s)\Big).$$

We now claim that

$$\lim_{r\to 1}\int_{S_T} K(Q_r, s_r; -)\chi_A(Q,s)\,d\omega(Q,s) = \chi_A(-)$$

in the weak topology of $L^1_{\mathcal{X}}(S_T, d\omega)$. Indeed, for every $g \in L^\infty_{\mathcal{X}}(S_T, d\omega)$, by Fubini's theorem we have

$$\begin{aligned}&\int_{S_T}\Big(\int_A K(Q_r, s_r; \mathbf{P})\,d\omega(Q,s)\Big)g(\mathbf{P})\,d\omega(\mathbf{P})\\ &\qquad = \int_A\Big(\int_{S_T} K(Q_r, s_r; \mathbf{P})g(\mathbf{P})\,d\omega(\mathbf{P})\Big)\,d\omega(Q,s).\end{aligned}$$

But the interior integral is uniformly bounded and converges ω-almost everywhere to g; hence,

$$\lim_{r\to 1}\int_{S_T}\Big(\int_A K(Q_r, s_r; \mathbf{P})\,d\omega(Q,s)\Big)g(\mathbf{P})\,d\omega(\mathbf{P}) = \int_A g(\mathbf{P})\,d\omega(\mathbf{P}),$$

and our claim follows.

The continuity of T implies the continuity of T when $L^1(S_T, d\omega)$ and $\mathcal{X}$ are endowed with the weak topology. This in turn implies that for every $\xi^* \in \mathcal{X}^*$, we have

$$\big\langle \xi^*, T(\chi_A)\big\rangle = \lim_{r\to 1}\Big\langle \xi^*, T\Big[\int_{\partial D} K(Q_r, s_r; -)\chi_A(Q,s)\,d\omega(Q,s)\Big]\Big\rangle. \tag{6.3}$$

From (6.2) and (6.3) we conclude that

$$\big\langle \xi^*, T(\chi_A)\big\rangle = \Big\langle \xi^*, \int_A f\,d\omega\Big\rangle$$

for all $\xi^* \in \mathcal{X}^*$ and, therefore, $T(\chi_A) = \int_A f\,d\omega$. The lemma follows. ☐

Acknowledgments. The authors thank the referees for useful comments that improved the presentation of the paper.

Ocampo-Salgado's work was partially supported by a PROMEP-México Postdoctoral Grant from the Universidad Autónoma del Estado de Morelos. Rivera-Noriega was part of the program "Programa para el Mejoramiento del Profesorado (PROMEP) Red de Cuerpos Académicos" during preparation of this work. San Martin's work was partially supported by a Consejo Nacional de Ciencia y Tecnología (CONACYT)-México scholarship.

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