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CHARACTERIZATIONS OF JORDAN LEFT DERIVATIONS ON SOME ALGEBRAS

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Abstract. A linear mapping δ from an algebra $\mathcal{A}$ into a left $\mathcal{A}$-module $\mathcal{M}$ is called a *Jordan left derivation* if $\delta(A^2) = 2A\delta(A)$ for every $A \in \mathcal{A}$. We prove that if an algebra $\mathcal{A}$ and a left $\mathcal{A}$-module $\mathcal{M}$ satisfy one of the following conditions—(1) $\mathcal{A}$ is a C^*-algebra and $\mathcal{M}$ is a Banach left $\mathcal{A}$-module; (2) $\mathcal{A} = \operatorname{Alg}\mathcal{L}$ with $\cap\{L_- : L \in \mathcal{J}_{\mathcal{L}}\} = (0)$ and $\mathcal{M} = B(X)$; and (3) $\mathcal{A}$ is a commutative subspace lattice algebra of a von Neumann algebra $\mathcal{B}$ and $\mathcal{M} = B(\mathcal{H})$—then every Jordan left derivation from $\mathcal{A}$ into $\mathcal{M}$ is zero. δ is called *left derivable* at $G \in \mathcal{A}$ if $\delta(AB) = A\delta(B) + B\delta(A)$ for each $A, B \in \mathcal{A}$ with $AB = G$. We show that if $\mathcal{A}$ is a factor von Neumann algebra, G is a left separating point of $\mathcal{A}$ or a nonzero self-adjoint element in $\mathcal{A}$, and δ is left derivable at G, then $\delta \equiv 0$.

1. Introduction

Let $\mathcal{R}$ be an associative ring. For an integer $n \geqslant 2$, $\mathcal{R}$ is said to be *n-torsion-free* if $nA = 0$ implies $A = 0$ for every A in $\mathcal{R}$. Recall that a ring $\mathcal{R}$ is *prime* if $A\mathcal{R}B = (0)$ implies that either $A = 0$ or $B = 0$ for each A, B in $\mathcal{R}$, and it is *semiprime* if $A\mathcal{R}A = (0)$ implies $A = 0$ for every A in $\mathcal{R}$.

Suppose that $\mathcal{M}$ is an $\mathcal{R}$-bimodule. An additive mapping δ from $\mathcal{R}$ into $\mathcal{M}$ is called a *derivation* if $\delta(AB) = \delta(A)B + A\delta(B)$ for each A, B in $\mathcal{R}$, and δ is called a *Jordan derivation* if $\delta(A^2) = \delta(A)A + A\delta(A)$ for every A in $\mathcal{R}$. Obviously, every derivation is a Jordan derivation. The converse is, in general, not true. A classi-

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cal result of Herstein [13] asserts that every Jordan derivation on a 2-torsion-free prime ring is a derivation. In [4], Brešar and Vukman give a brief proof of [13, Theorem 3.1]. In [7], Cusack generalizes [13, Theorem 3.1] to 2-torsion-free semiprime rings. In [3], Brešar gives an alternative proof of [7, Corollary 5].

In [5], Brešar and Vukman introduce the concepts of left derivations and Jordan left derivations. In [24], Vukman introduces the concept of (m,n)-Jordan derivations.

Let $\mathcal{M}$ be a left $\mathcal{R}$-module. An additive mapping δ from $\mathcal{R}$ into $\mathcal{M}$ is called a *left derivation* if $\delta(AB) = A\delta(B) + B\delta(A)$ for each A, B in $\mathcal{R}$, and δ is called a *Jordan left derivation* if $\delta(A^2) = 2A\delta(A)$ for every A in $\mathcal{R}$. Let $m \geqslant 0$ and $n \geqslant 0$ be two fixed integers with $m + n \neq 0$; δ is called an (m,n)*-Jordan derivation* if $(m + n)\delta(A^2) = 2mA\delta(A) + 2n\delta(A)A$ for every A in $\mathcal{R}$. The concept of (m,n)-Jordan derivations covers the concept of Jordan derivations, as well as the concept of Jordan left derivations.

In [5], Brešar and Vukman prove that if there exists a nonzero Jordan left derivation from a prime ring $\mathcal{R}$ into a left $\mathcal{R}$-module $\mathcal{M}$ of characteristic not 2 and 3, then $\mathcal{R}$ is commutative. In [10], Deng shows that [5, Theorem 2.1] is still true when $\mathcal{M}$ is only characteristic not 2. In [23], Vukman shows that every Jordan left derivation from a complex semisimple Banach algebra into itself is zero. In [15], Kosi-Ulbl and Vukman prove that if $m \geqslant 1$ and $n \geqslant 1$ are two integers with $m \neq n$, then every (m,n)-Jordan derivation from a complex semisimple Banach algebra into itself is zero.

Throughout this paper, $\mathcal{A}$ denotes an algebra over the complex field $\mathbb{C}$, and $\mathcal{M}$ denotes a left $\mathcal{A}$-module. In the paper, we assume that all mappings from $\mathcal{A}$ into $\mathcal{M}$ are linear.

This paper is organized as follows. In Section 2, we show that every Jordan left derivation from a C^*-algebra $\mathcal{A}$ into its Banach left module $\mathcal{M}$ is zero.

In Section 3, we show that if $\mathcal{L}$ is a subspace lattice on a complex Banach space X with $\cap\{L_- : L \in \mathcal{J}_\mathcal{L}\} = (0)$, then every Jordan left derivation from $\operatorname{Alg}\mathcal{L}$ into $B(X)$ is zero. The class of reflexive algebras $\operatorname{Alg}\mathcal{L}$ with $\cap\{L_- : L \in \mathcal{J}_\mathcal{L}\} = (0)$ is very large, and it includes the following:

(1) $\mathcal{P}$-subspace lattice algebras;
(2) completely distributive subspace lattice algebras;
(3) reflexive algebras $\operatorname{Alg}\mathcal{L}$ such that $(0)_+ \neq (0)$.

In Section 4, we show that if $\mathcal{B}$ is a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\mathcal{L} \subseteq \mathcal{B}$ is a commutative subspace lattice (CSL) on $\mathcal{H}$, then every Jordan left derivation from $\mathcal{B} \cap \operatorname{Alg}\mathcal{L}$ into $B(\mathcal{H})$ is zero.

A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called *left derivable at* $G \in \mathcal{A}$ if $\delta(AB) = A\delta(B) + B\delta(A)$ for each $A, B \in \mathcal{A}$ with $AB = G$. In [16], Li and Zhou show that if $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice, then every left derivable mapping at a unit on $\operatorname{Alg}\mathcal{L}$ is zero.

For a unital algebra $\mathcal{A}$ and a unital left $\mathcal{A}$-module $\mathcal{M}$, we call an element $W \in \mathcal{A}$ a *left separating point* of $\mathcal{M}$ if $WM = 0$ implies $M = 0$ for every $M \in \mathcal{M}$. It is easy to see that every left invertible element in $\mathcal{A}$ is a left separating point of $\mathcal{M}$.

In Section 5, we prove that if $\mathcal{A}$ is a factor von Neumann algebra, then every left derivable mapping at a left separating point or a nonzero self-adjoint element is zero.

Let X be a complex Banach space, and let $B(X)$ be the set of all bounded linear operators on X. We denote by X^* and X^{**} the dual space and the double dual space of X, respectively. In this paper, every subspace of X is a closed linear manifold. By a *subspace lattice* on X, we mean a collection $\mathcal{L}$ of subspaces of X with (0) and X in $\mathcal{L}$ such that, for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\cap M_r$ and $\vee M_r$ belong to $\mathcal{L}$, where $\vee M_r$ denotes the closed linear span of $\{M_r\}$.

Let $\mathcal{L}$ be a subspace lattice on X. Define

$$\mathcal{J}_{\mathcal{L}} = \{E \in \mathcal{L} : E \neq (0) \text{ and } E_- \neq X\} \qquad \text{and} \qquad \mathcal{P}_{\mathcal{L}} = \{E \in \mathcal{L} : E_- \not\supseteq E\},$$

where $E_- = \vee\{F \in \mathcal{L} : F \not\supseteq E\}$. $\mathcal{L}$ is called a $\mathcal{J}$*-subspace lattice* on X if it satisfies $E \vee E_- = X$ and $E \cap E_- = (0)$ for every E in $\mathcal{J}_{\mathcal{L}}$; $\vee\{E : E \in \mathcal{J}_{\mathcal{L}}\} = X$ and $\cap\{E_- : E \in \mathcal{J}_{\mathcal{L}}\} = (0)$. $\mathcal{L}$ is called a $\mathcal{P}$*-subspace lattice* on X if it satisfies $\vee\{E : E \in \mathcal{P}_{\mathcal{L}}\} = X$ or $\cap\{E_- : E \in \mathcal{P}_{\mathcal{L}}\} = (0)$.

$\mathcal{L}$ is said to be *completely distributive* if its subspaces satisfy the identity

$$\bigwedge_{a\in I}\bigvee_{b\in J} L_{a,b} = \bigvee_{f\in J^I}\bigwedge_{a\in I} L_{a,f(a)},$$

where J^I denotes the set of all $f : I \to J$. For some properties of completely distributive subspace lattices and $\mathcal{J}$-subspace lattices, see [17] and [18].

For every subspace lattice $\mathcal{L}$ on X, we use $\operatorname{Alg}\mathcal{L}$ to denote the algebra of all operators in $B(X)$ that leave members of $\mathcal{L}$ invariant; and for a subalgebra $\mathcal{A}$ of $B(X)$, we use $\operatorname{Lat}\mathcal{A}$ to denote the lattice of all subspaces of X that are invariant under all operators in $\mathcal{A}$. An algebra $\mathcal{A}$ is called *reflexive* if $\mathcal{A} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}$.

The following two lemmas will be used repeatedly.

Lemma 1.1 ([16, Lemma 2.1])**.** *Let $\mathcal{A}$ be an algebra, let $\mathcal{M}$ be a left $\mathcal{A}$-module, and let δ be a Jordan left derivation from $\mathcal{A}$ into $\mathcal{M}$. Then for each A, B in $\mathcal{A}$, the following two statements hold:*

(1) $\delta(AB + BA) = 2A\delta(B) + 2B\delta(A)$;
(2) $\delta(ABA) = A^2\delta(B) + 3AB\delta(A) - BA\delta(A)$.

Lemma 1.2 ([16, Lemma 2.2])**.** *Let $\mathcal{A}$ be an algebra, let $\mathcal{M}$ be a left $\mathcal{A}$-module, and let δ be a Jordan left derivation from $\mathcal{A}$ into $\mathcal{M}$. Then for every A and every idempotent P in $\mathcal{A}$, the following two statements hold:*

(1) $\delta(P) = 0$;
(2) $\delta(PA) = \delta(AP) = \delta(PAP) = P\delta(A)$.

2. Jordan left derivations on C^*-algebras

In this section, we study Jordan left derivations from a C^*-algebra into its Banach left module and prove that these Jordan left derivations are zero.

Proposition 2.1. *Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{M}$ be a Banach left $\mathcal{A}$-module. If δ is a Jordan left derivation from $\mathcal{A}$ into $\mathcal{M}$, then δ is automatically continuous.*

To prove Proposition 2.1, we need the following lemma. The proof of Lemma 2.2 is similar to the proof of [22, Theorem 2], but, for the sake of completeness, we give it here.

Lemma 2.2. *Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{M}$ be a Banach left $\mathcal{A}$-module. If δ is a left derivation from $\mathcal{A}$ into $\mathcal{M}$, then δ is automatically continuous.*

Proof. Let $\mathcal{J} = \{J \in \mathcal{A} : D_J(T) = \delta(JT)$ is continuous for every T in $\mathcal{A}\}$. Since δ is a left derivation from $\mathcal{A}$ into $\mathcal{M}$, we have that

$$J\delta(T) = \delta(JT) - T\delta(J)$$

for every T in $\mathcal{A}$ and every J in $\mathcal{J}$. Then

$$\mathcal{J} = \big\{J \in \mathcal{A} : S_J(T) = J\delta(T) \text{ is continuous every every } T \text{ in } \mathcal{A}\big\}.$$

We divide the proof into three steps.

First, we show that $\mathcal{J}$ is a closed two-sided ideal in $\mathcal{A}$. Clearly, $\mathcal{J}$ is a right ideal in $\mathcal{A}$. Moreover, for each A, T in $\mathcal{A}$ and every J in $\mathcal{J}$, we have that

$$\delta(AJT) = A\delta(JT) + JT\delta(A);$$

thus $D_{AJ}(T)$ is continuous for every T in $\mathcal{A}$ and $\mathcal{J}$ is also a left ideal in $\mathcal{A}$.

Suppose that $\{J_n\} \subseteq \mathcal{J}$ and $J \in \mathcal{A}$ such that $\lim_{n\to\infty} J_n = J$. Then every S_{J_n} is a continuous linear operator; hence we obtain

$$S_J(T) = J\delta(T) = \lim_{n\to\infty} J_n\delta(T) = \lim_{n\to\infty} S_{J_n}(T)$$

for every T in $\mathcal{A}$. By the principle of uniform boundedness, we have that S_J is norm continuous and $J \in \mathcal{A}$. Thus, $\mathcal{J}$ is a closed two-sided ideal in $\mathcal{A}$.

Next, we show that the restriction $\delta|_{\mathcal{J}}$ is norm continuous. Suppose the contrary. We can choose $\{J_n\} \subseteq \mathcal{J}$ such that

$$\sum_{n=1}^{\infty} \|J_n\|^2 \leqslant 1 \qquad \text{and} \qquad \big\|\delta(J_n)\big\| \to \infty \quad \text{when } n \to \infty.$$

Let $B = (\sum_{n=1}^{\infty} J_n J_n^*)^{1/4}$. Then B is a positive element in $\mathcal{J}$ with $\|B\| \leqslant 1$. By [22, Lemma 1] it follows that $J_n = BC_n$ for some $\{C_n\} \subseteq \mathcal{J}$ with $\|C_n\| \leqslant 1$, and

$$\big\|D_B(C_n)\big\| = \big\|\delta(BC_n)\big\| = \big\|\delta(J_n)\big\| \to \infty \quad \text{when } n \to \infty.$$

This leads to a contradiction; hence $\delta|_{\mathcal{J}}$ is norm continuous.

Finally, we show that the C^*-algebra $\mathcal{A}/\mathcal{J}$ is finite-dimensional. Otherwise, by [19] we know that $\mathcal{A}/\mathcal{J}$ has an infinite-dimensional abelian C^*-subalgebra $\tilde{\mathcal{A}}$. Since the carrier space X of $\tilde{\mathcal{A}}$ is infinite, it follows easily from the isomorphism between $\tilde{\mathcal{A}}$ and $C_0(X)$ that there is a positive element H in $\tilde{\mathcal{A}}$ whose spectrum is infinite; hence we can choose nonnegative continuous functions $f_1, f_2, \ldots$, defined on the positive real axis such that

$$f_j f_k = 0 \quad \text{if } j \neq k \qquad \text{and} \qquad f_j(H) \neq 0 \quad (j = 1, 2, \ldots).$$

Let φ be a natural mapping from $\mathcal{A}$ into $\mathcal{A}/\mathcal{J}$. Then there exists a positive element K in $\mathcal{A}$ such that $\varphi(K) = H$. Denote $A_j = f_j(K)$ for each j. Then we have that $A_j \in \mathcal{A}$ and

$$\varphi(A_j^2) = \varphi\big(f_j(K)\big)^2 = \big[f_j\big(\varphi(K)\big)\big]^2 = f_j(H)^2 \neq 0.$$

It follows that $A_j^2 \notin \mathcal{J}$ and $A_jA_k = 0$ if $j \neq k$. If we replace A_j by an appropriate scalar multiple, we may suppose that $\|A_j\| \leqslant 1$. By $A_j^2 \notin \mathcal{J}$, we have that $D_{A_j^2}$ is unbounded. Thus, we can choose $T_j \in \mathcal{A}$ such that

$$\|T_j\| \leqslant 2^{-j} \qquad \text{and} \qquad \big\|\delta(A_j^2T_j)\big\| \geqslant M\big\|\delta(A_j)\big\| + j,$$

where M is the bound of the linear mapping

$$(T, x) \to xT : \mathcal{A} \times \mathcal{M} \to \mathcal{A}.$$

Let $C = \sum_{j\geqslant 1} A_jT_j$. Then we have that $\|C\| \leqslant 1$ and $A_jC = A_j^2T_j$, and so

$$\begin{aligned}\big\|A_j\delta(C)\big\| &= \big\|\delta(A_jC) - C\delta(A_j)\big\| \\ &\geqslant \big\|\delta(A_j^2T_j)\big\| - M\|C\|\big\|\delta(A_j)\big\| \\ &\geqslant M\big\|\delta(A_j)\big\| + j - M\big\|\delta(A_j)\big\| = j.\end{aligned}$$

However, this is impossible because, in fact, $\|A_j\| \leqslant 1$ and the linear mapping

$$T \to T\delta(C) : \mathcal{A} \to \mathcal{M}$$

is bounded; hence we prove that $\mathcal{A}/\mathcal{J}$ is finite-dimensional.

Since $\delta|_{\mathcal{J}}$ is continuous and $\mathcal{A}/\mathcal{J}$ is finite-dimensional, it follows that δ is automatically continuous. □

Given an element A of the algebra $B(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$, we denote by $\mathcal{G}(A)$ the C^*-algebra generated by A. For any self-adjoint subalgebra $\mathcal{A}$ of $B(\mathcal{H})$, if $\mathcal{G}(B) \subseteq \mathcal{A}$ for every self-adjoint element $B \in \mathcal{A}$, then we call $\mathcal{A}$ *locally closed*. Obviously, every C^*-algebra is locally closed.

Lemma 2.3 ([6, Corollary 1.2])**.** *Let $\mathcal{A}$ be a locally closed subalgebra of $B(\mathcal{H})$, let Y be a locally convex linear space, and let ψ be a linear mapping from $\mathcal{A}$ into Y. If ψ is continuous from every commutative self-adjoint subalgebra of $\mathcal{A}$ into Y, then ψ is continuous.*

Proof of Proposition 2.1. By Lemma 2.3, it is sufficient to prove that δ is continuous from each commutative self-adjoint subalgebra $\mathcal{B}$ of A into $\mathcal{M}$. It is clear that the norm closure $\bar{\mathcal{B}}$ of $\mathcal{B}$ is an abelian C^*-algebra. Thus, we only need to show that the restriction $\delta|_{\bar{\mathcal{B}}}$ is continuous.

In fact, for each A, B in $\bar{\mathcal{B}}$, we have that

$$\delta(AB + BA) = \delta(2AB) = 2A\delta(B) + 2B\delta(A).$$

This means that $\delta|_{\bar{\mathcal{B}}}$ is a left derivation. By Lemma 2.2 we know that $\delta|_{\bar{\mathcal{B}}}$ is automatically continuous; hence δ is continuous on $\mathcal{B}$. □

By Lemma 1.2(1) and Proposition 2.1, we can easily show the following result.

Corollary 2.4. *Let $\mathcal{A}$ be a von Neumann algebra, and let $\mathcal{M}$ be a Banach left $\mathcal{A}$-module. If δ is a Jordan left derivation from $\mathcal{A}$ into $\mathcal{M}$, then $\delta \equiv 0$.*

Applying some techniques from [1], [8], and [9], we can obtain the following result.

Theorem 2.5. *Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{M}$ be a Banach left $\mathcal{A}$-module. If δ is a Jordan left derivation from $\mathcal{A}$ into $\mathcal{M}$, then $\delta \equiv 0$.*

Proof. By [9, p. 26], we can define a product $\diamond$ in $\mathcal{A}^{**}$ by $a^{**} \diamond b^{**} = \lim_\lambda \lim_\mu \alpha_\lambda \beta_\mu$ for each a^{**}, b^{**} in $\mathcal{A}^{**}$, where (α_λ) and (β_μ) are two nets in $\mathcal{A}$ with $\|a_\lambda\| \leqslant \|a^{**}\|$ and $\|b_\mu\| \leqslant \|b^{**}\|$ such that $\alpha_\lambda \to a^{**}$ and $\beta_\mu \to b^{**}$ in the weak*-topology $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$. By [14, p. 726], we know that $\mathcal{A}^{**}$ is $*$-isomorphic to a von Neumann algebra, and so we may assume that $(\mathcal{A}^{**},\diamond)$ is a von Neumann algebra.

It is well known that $\mathcal{M}^{**}$ turns into a Banach left $(\mathcal{A}^{**}, \diamond)$-module with the operation defined by

$$a^{**} \cdot m^{**} = \lim_\lambda \lim_\mu a_\lambda m_\mu$$

for every a^{**} in $\mathcal{A}^{**}$ and every m^{**} in $\mathcal{M}^{**}$, where (a_λ) is a net in $\mathcal{A}$ with $\|a_\lambda\| \leqslant \|a^{**}\|$ and $(a_\lambda) \to a^{**}$ in $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$, (m_μ) is a net in $\mathcal{M}$ with $\|m_\mu\| \leqslant \|m^{**}\|$, and $(m_\mu) \to m^{**}$ in $\sigma(\mathcal{M}^{**}, \mathcal{M}^*)$.

By Proposition 2.1 we have that $\delta^{**} : (\mathcal{A}^{**}, \diamond) \to \mathcal{M}^{**}$ is the weak*-continuous extension of δ to the double duals of $\mathcal{A}$ and $\mathcal{M}$.

Let a^{**}, b^{**} be in $\mathcal{A}^{**}$, and let (a_λ), (b_μ) be two nets in $\mathcal{A}$ with $\|a_\lambda\| \leqslant \|a^{**}\|$ and $\|b_\mu\| \leqslant \|b^{**}\|$ such that $a^{**} = \lim_\lambda a_\lambda$ and $b^{**} = \lim_\mu b_\mu$ in $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$. We have that

$$\delta^{**}(a^{**} \diamond b^{**} + b^{**} \diamond a^{**}) = \delta^{**}\Big(\lim_\lambda \lim_\mu a_\lambda b_\mu + \lim_\mu \lim_\lambda b_\mu a_\lambda\Big).$$

By [1, p. 553], we know that every continuous bilinear map φ from $\mathcal{A} \times \mathcal{M}$ into $\mathcal{M}$ is Arens regular, which means that

$$\lim_\lambda \lim_\mu \varphi(a_\lambda, m_\mu) = \lim_\mu \lim_\lambda \varphi(a_\lambda, m_\mu)$$

for every $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-convergent net (a_λ) in $\mathcal{A}$ and every $\sigma(\mathcal{M}^{**}, \mathcal{M}^*)$-convergent net (m_μ) in $\mathcal{M}$. It follows that

$$\begin{aligned}
\delta^{**}\Big(\lim_\lambda \lim_\mu a_\lambda b_\mu + \lim_\mu \lim_\lambda b_\mu a_\lambda\Big) &= \delta^{**}\Big(\lim_\lambda \lim_\mu a_\lambda b_\mu + \lim_\lambda \lim_\mu b_\mu a_\lambda\Big) \\
&= \lim_\lambda \lim_\mu \delta(a_\lambda b_\mu + b_\mu a_\lambda) \\
&= \lim_\lambda \lim_\mu 2a_\lambda \delta(b_\mu) + \lim_\lambda \lim_\mu 2b_\mu \delta(a_\lambda) \\
&= \lim_\lambda \lim_\mu 2a_\lambda \delta(b_\mu) + \lim_\mu \lim_\lambda 2b_\mu \delta(a_\lambda) \\
&= 2a^{**}\delta^{**}(b^{**}) + 2b^{**}\delta^{**}(a^{**}).
\end{aligned}$$

It means that δ^{**} is a Jordan left derivation from $\mathcal{A}^{**}$ into $\mathcal{M}^{**}$. Thus, by Corollary 2.4 we obtain

$$\delta^{**}(a^{**}) = 0$$

for every a^{**} in $\mathcal{A}^{**}$; hence $\delta(a) = 0$ for every a in $\mathcal{A}$. $\square$

3. Jordan left derivations on reflexive algebras

Let X be a complex Banach space. For any nonzero elements x in X and f in X^*, the *rank* 1 *operator* $x \otimes f \in B(X)$ is defined by $(x \otimes f)y = f(y)x$ for every y in X. For every nonempty subset E of X, let $E^\perp = \{f \in X^* : f(x) = 0 \text{ for every } x \text{ in } E\}$, and let $E_-^\perp = (E_-)^\perp$.

The main result in this section is Theorem 3.1.

Theorem 3.1. *Let $\mathcal{L}$ be a subspace lattice on X such that $\cap\{L_- : L \in \mathcal{J}_\mathcal{L}\} = (0)$. If δ is a Jordan left derivation from $\operatorname{Alg}\mathcal{L}$ into $B(X)$, then $\delta \equiv 0$.*

In order to prove Theorem 3.1, we need the following two lemmas.

Lemma 3.2 ([12, Lemma 3.2])**.** *Let X be a Banach space, let $E \subseteq X$ and $F \subseteq X^*$, and let ϕ be a bilinear mapping from $E \times F$ into $B(X)$. If $\phi(x, f)X \subseteq \mathbb{C}x$ for every $x \in E$ and $f \in F$, then there exists a linear mapping S^* from F into X^* such that $\phi(x, f) = x \otimes S^* f$.*

Lemma 3.3. *Let $\mathcal{L}$ be a subspace lattice on X, and let E and L be in $\mathcal{J}_\mathcal{L}$ such that $E_- \not\supseteq L$. If δ is a Jordan left derivation from $\operatorname{Alg}\mathcal{L}$ into $B(X)$, then $\delta(x \otimes f) \subseteq \mathbb{C}x$ for every $x \in E$ and $f \in L_-^\perp$.*

Proof. Since $E_- \not\supseteq L$, it follows that $E \subseteq L$ and $x \otimes f \in \operatorname{Alg}\mathcal{L}$. Obviously, we can choose an element z in L and an element g in $E_-^\perp$ such that $g(z) = 1$. In the following proof, we let $x \in E$ and $f \in F_-^\perp$; then $x \in F$.

Case 1: Suppose that $f(x) \neq 0$.

It is easy to show that

$$\Big(\frac{1}{f(x)}(x \otimes f)\Big)^2 = \frac{1}{f(x)}(x \otimes f)$$

and that $\frac{1}{f(x)}(x \otimes f)$ is an idempotent in $\operatorname{Alg}\mathcal{L}$. By Lemma 1.2(1) we obtain

$$\delta(x \otimes f) = f(x)\delta\Big(\frac{1}{f(x)}(x \otimes f)\Big) = 0.$$

Case 2: Suppose that $f(x) = 0$.

If $g(x) \neq 0$, then $(g + f)(x) \neq 0$. Hence

$$\delta\big(x \otimes (g + f)\big) = \delta(x \otimes g) = 0.$$

Thus $\delta(x \otimes f) = 0$.

If $g(x) = 0$, then since $g(z) = 1$, by Lemma 1.1 we have

$$\begin{aligned} \delta(x \otimes f) &= \delta\big((x \otimes g)(z \otimes f) + (z \otimes f)(x \otimes g)\big) \\ &= 2(x \otimes g)\delta(z \otimes f) + 2(z \otimes f)\delta(x \otimes g) \end{aligned} \tag{3.1}$$

and

$$0 = \delta\big((x \otimes g)(z \otimes f)(x \otimes g)\big) = 3(x \otimes f)\delta(x \otimes g). \tag{3.2}$$

By (3.2) we have

$$\big(\delta(x \otimes g)\big)^* f = 0.$$

Hence

$$(z \otimes f)\delta(x \otimes g) = z \otimes \big(\delta(x \otimes g)^* f\big) = 0.$$

By (3.1) we know that

$$\delta(x \otimes f) = 2(x \otimes g)\delta(z \otimes f). \tag{3.3}$$

By (3.3) we have that $\delta(x \otimes f) \subseteq \mathbb{C}x$. □

Proof of Theorem 3.1. Let $F \in \mathcal{J}(\mathcal{L})$. There exists an element $E \in \mathcal{J}(\mathcal{L})$ such that $E_- \not\supseteq F$. Let $x \in E$, $f \in F_-^\perp$. First we prove that $\delta(x \otimes f) = 0$.

If $f(x) \neq 0$, then we have $\delta(x \otimes f) = 0$. In the following we assume that $f(x) = 0$.

Case 1: Suppose that $\dim(E) = 1$.

Since $\cap\{L_- : L \in \mathcal{J}_\mathcal{L}\} = (0)$, there exists an element $L \in \mathcal{J}(\mathcal{L})$ such that $L_- \not\supseteq E$, and so $L \subseteq E$. Since $\dim(E) = 1$, we have that $L = E$ and $E_- \not\supseteq E$. Hence there exists an element $g \in E_-^\perp$ such that $g(x) \neq 0$. Since $f(x) = 0$, we have that $(f+g)(x) \neq 0$ and

$$\delta\big(x \otimes (f+g)\big) = \delta(x \otimes g) = 0,$$

Thus $\delta(x \otimes f) = 0$.

Case 2: Suppose that $\dim(E) \geqslant 2$.

By Lemma 3.3 we know that $\delta(x \otimes f) \subseteq \mathbb{C}x$ for every $x \in E$ and $f \in F_-^\perp$. By Lemma 3.2 there exists a linear mapping S^* from $F_-^\perp$ to X^* such that

$$\delta(x \otimes f) = x \otimes S^* f.$$

We only need to prove that $S^* f = 0$.

Let $A \in \operatorname{Alg}\mathcal{L}$. We have that $Ax \in E$, $A^* f \in F_-^\perp$. By Lemma 1.1(1) it follows that

$$\begin{aligned}\delta\big(A(x \otimes f) + (x \otimes f)A\big) &= 2A\delta(x \otimes f) + 2(x \otimes f)\delta(A) \\ &= Ax \otimes S^* f + x \otimes S^* A^* f \\ &= 2Ax \otimes S^* f + 2x \otimes \big(\delta(A)\big)^* f.\end{aligned} \tag{3.4}$$

By (3.4) we have that

$$Ax \otimes S^* f = x \otimes \Big(S^* A^* f - 2\big(\delta(A)\big)^* f\Big). \tag{3.5}$$

If $S^* f \neq 0$, then there exists an element $z \in X$ such that $(S^* f)(z) \neq 0$. By (3.5),

$$(S^* f)(z)Ax = \Big(S^* A^* f - 2\big(\delta(A)\big)^* f\Big)(z)x.$$

Hence there exists a number λ_A such that $Ax = \lambda_A x$ for every $x \in E$.

Since $\cap\{L_- : L \in \mathcal{J}_\mathcal{L}\} = (0)$, there exists an element $L \in \mathcal{J}_\mathcal{L}$ such that $L_- \not\supseteq E$, and we can choose an element $x_1 \in E$ and $\eta \in L_-^\perp$ such that $\eta(x_1) = 1$. Let $0 \neq y \in L$. We have that

$$(y \otimes \eta)x_1 = \lambda_{y \otimes \eta} x_1,$$

and thus

$$y = \lambda_{y \otimes \eta} x_1. \tag{3.6}$$

Since $y \neq 0$, $\lambda_{y\otimes\eta} \neq 0$. Because $\dim(E) \geqslant 2$, there exists an $x_2 \in E$ such that x_1 and x_2 are linearly independent. If $\eta(x_2) = 0$, then $\eta(x_1 + x_2) = 1$, and it follows that

$$(y \otimes \eta)(x_1 + x_2) = \lambda_{y\otimes\eta}(x_1 + x_2);$$

thus

$$y = \lambda_{y\otimes\eta}(x_1 + x_2).$$

By (3.6) we know that $x_2 = 0$. If $\eta(x_2) \neq 0$, then

$$(y \otimes \eta)(x_2) = \lambda_{y\otimes\eta} x_2;$$

thus

$$y = \frac{\lambda_{y\otimes\eta}}{\eta(x_2)} x_2.$$

By (3.6) we know that x_1 and x_2 are linearly dependent. Hence $S^* f = 0$ and $\delta(x \otimes f) = 0$.

In the following we prove that $\delta(A)^* = 0$ for every $A \in \operatorname{Alg}\mathcal{L}$. Let $A \in \operatorname{Alg}\mathcal{L}$, $x \in E$, and $f \in F_-^{\perp}$. We have that

$$\delta\big((x \otimes f)A + A(x \otimes f)\big) = 2(x \otimes f)\delta(A) = 0;$$

thus,

$$\delta(A)^* f = 0.$$

Since $\cap\{L_- : L \in \mathcal{J}_{\mathcal{L}}\} = (0)$, we obtain $\vee\{L_-^{\perp} : L \in \mathcal{J}_{\mathcal{L}}\} = X^*$. It implies that

$$\delta(A)^* = 0$$

for every $A \in \operatorname{Alg}\mathcal{L}$. Since $\|\delta(A)\| = \|\delta(A)^*\|$, we have $\delta(A) = 0$ for every $A \in \operatorname{Alg}\mathcal{L}$. □

Remark. Similarly to the definition of Jordan left derivations, we can define a *Jordan right derivation*. Similarly to the proof of Theorem 3.1, we can show that every Jordan right derivation from $\operatorname{Alg}\mathcal{L}$ with $\vee\{L : L \in \mathcal{J}_{\mathcal{L}}\} = X$ into $B(X)$ is zero.

4. Jordan left derivations on CSL subalgebras of von Neumann algebras

For a Hilbert space $\mathcal{H}$, we disregard the distinction between a closed subspace and the orthogonal projection onto it. Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. $\mathcal{L}$ is called a *CSL* if it consists of mutually commuting projections. Let $\mathcal{B}$ be a von Neumann algebra on $\mathcal{H}$, and let $\mathcal{L} \subseteq \mathcal{B}$ be a CSL on $\mathcal{H}$. Then $\mathcal{A} = \mathcal{B} \cap \operatorname{Alg}\mathcal{L}$ is said to be a *CSL subalgebra of a von Neumann algebra* $\mathcal{B}$.

Theorem 4.1. *If δ is a Jordan left derivation from $\mathcal{A}$ into $B(\mathcal{H})$, then $\delta \equiv 0$.*

To prove Theorem 4.1, we need the following lemma.

Lemma 4.2. *Let $\mathcal{L}$ be a CSL in a von Neumann algebra $\mathcal{B}$ on $\mathcal{H}$. Define*

$$Q = \{P^{\perp} A^* P x : P \in \mathcal{L}, A \in \mathcal{A}, x \in \mathcal{H}\}.$$

Then we have

(1) $Q \in \mathcal{L}' \cap \mathcal{B} \subseteq \mathcal{A}$;
(2) $Q^\perp \mathcal{A} Q^\perp$ *is a von Neumann algebra on* $Q^\perp \mathcal{H}$ *when* $Q \neq I$.

Proof. (1) Since $\mathcal{L}$ is a CSL in $\mathcal{B}$, it is easy to show that $\mathcal{L}' \cap \mathcal{B} \subseteq \mathcal{A}$. Then we only need to prove that $Q \in \mathcal{L}' \cap \mathcal{B}$.

For every T in $\mathcal{B} \cap \operatorname{Alg} \mathcal{L}^\perp$, it means that $PTP^\perp = 0$ for every P in $\mathcal{L}$. Hence by the definition of Q we have that $Q^\perp TQ = 0$ and $Q \in \operatorname{Lat}(\mathcal{B} \cap \operatorname{Alg} \mathcal{L}^\perp)$. It follows that

$$PQ = QPQ \qquad \text{and} \qquad QP = QPQ$$

for every $P \in \mathcal{L}$, and so $Q \in \mathcal{L}'$.

Letting $P \in \mathcal{L}$, $A \in \mathcal{A} \subseteq \mathcal{B}$, $B \in \mathcal{B}'$, and $x \in \mathcal{H}$, we have that $P^\perp A^* P \in \mathcal{B}$. It follows that

$$QBP^\perp A^* Px = QP^\perp A^* PBx = P^\perp A^* PBx = BP^\perp A^* Px.$$

By the definition of Q we obtain $QBQ = BQ$.

Similarly, since $B^* \in \mathcal{B}'$ for every $B \in \mathcal{B}'$, we have that $QB^*Q = B^*Q$. It follows that $QB = BQ$ for every $B \in \mathcal{B}'$. This means that $Q \in \mathcal{B}'' = \mathcal{B}$ and $Q \in \mathcal{L}' \cap \mathcal{B} \subseteq \mathcal{A}$.

(2) It is obvious that $Q^\perp \mathcal{A} Q^\perp$ is a weakly closed operator algebra with an identity $Q^\perp$ on $Q^\perp \mathcal{H}$; hence it is sufficient to prove that $Q^\perp \mathcal{A} Q^\perp$ is a self-adjoint algebra.

Fix an element $A \in \mathcal{A}$ and $P \in \mathcal{L}$. By the fact that Q commutes with P and the definition of Q, we have that

$$P(AQ^\perp)P^\perp = (Q^\perp P^\perp A^* P)^* = 0.$$

This means that $AQ^\perp \in \mathcal{B} \cap \operatorname{Alg} \mathcal{L}^\perp$. Then we obtain

$$AQ^\perp \in \operatorname{Alg} \mathcal{L}^\perp \cap \operatorname{Alg} \mathcal{L} \cap \mathcal{B} = \mathcal{L}' \cap \mathcal{B} \subseteq \mathcal{A}.$$

It follows that $Q^\perp A^* \in \mathcal{A}$; thus $Q^\perp A^* Q^\perp \in Q^\perp \mathcal{A} Q^\perp$ for every $A \in \mathcal{A}$, which tells us that $Q^\perp \mathcal{A} Q^\perp$ is a von Neumann algebra on $Q^\perp \mathcal{H}$. $\square$

Proof of Theorem 4.1. Letting Q be as in Lemma 4.2, it is obvious that if $Q = I$, then $\delta(A) = Q\delta(A)$. We suppose that $Q \neq I$. Let $Q_1 = Q$, $Q_2 = I - Q$, and $\mathcal{A}_{ij} = Q_i \mathcal{A} Q_j$. Then we have the Peirce decomposition of $\mathcal{A}$ as follows:

$$\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}.$$

By Lemma 1.2(2), we have that

$$\delta(\mathcal{A}_{12}) = \delta(\mathcal{A}_{21}) = 0.$$

Moreover, by Lemma 4.2 we know that $\mathcal{A}_{22}$ is a von Neumann algebra on $Q^\perp \mathcal{H}$; hence by Corollary 2.4 we obtain

$$\delta(\mathcal{A}_{22}) = 0.$$

It follows that

$$\delta(A)=\delta(QAQ)=Q\delta(A)$$

for every A in $\mathcal{A}$.

In the following, we show that $Q\delta(A)\equiv 0$ for every A in $\mathcal{A}$. Let P be in $\mathcal{L}$, and let A, B be in $\mathcal{A}$. By Lemmas 1.1(1) and 1.2(2) we have that

$$\begin{aligned}0&=\delta(PBP^{\perp}P^{\perp}AP^{\perp})\\&=\delta\big((PBP^{\perp}P^{\perp}AP^{\perp})+(P^{\perp}AP^{\perp}PBP^{\perp})\big)\\&=2PBP^{\perp}\delta(P^{\perp}AP^{\perp})\\&=2PBP^{\perp}\delta(A).\end{aligned}$$

It implies that $\delta(A)^*P^{\perp}B^*P=0$; that is, $\delta(A)^*Q=0$. Thus $Q\delta(A)\equiv 0$ for every A in $\mathcal{A}$. □

Remark. In [21, pp. 741–742], Park introduces the concept of Jordan higher left derivations as follows.

Let $\mathcal{A}$ be a unital algebra, and let $\mathbb{N}=\mathbb{N}^*\cup\{0\}$ be the set of all nonnegative integers. $\Delta=(\delta_i)_{i\in\mathbb{N}}$ is a sequence of linear mappings on $\mathcal{A}$, where $\delta_0=id_{\mathcal{A}}$. Suppose that $c_{ij}=1$ if $i=j$ and $c_{ij}=0$ if $i\neq j$. Δ is called a *Jordan higher left derivation* if

$$\delta_n(A^2)=\sum_{\substack{i+j=n\\ i\leqslant j}}\big[(c_{ij}+1)\delta_i(A)\delta_j(A)\big]$$

for every A in $\mathcal{A}$, n in $\mathbb{N}^*$, and i, j in $\mathbb{N}$. It is clear that δ_1 is a Jordan left derivation on $\mathcal{A}$.

By the definition of Jordan higher left derivations, it is easy to show that each Jordan higher left derivation on these algebras, which are studied in Sections 2 to 4, is zero.

5. Left derivable mappings at some points

In this section, we consider left derivable mappings on factor von Neumann algebras at every left separating point or every nonzero self-adjoint element.

Lemma 5.1. *Let $\mathcal{A}$ be a unital algebra, let $\mathcal{M}$ be a unital left $\mathcal{A}$-module, and let δ be a linear mapping from $\mathcal{A}$ into $\mathcal{M}$. If δ is left derivable at a left separating point W, then $\delta(P)=0$ for every idempotent P in $\mathcal{A}$.*

Proof. It is clear that $\delta(W)=W\delta(I)+\delta(W)$. Then $W\delta(I)=0$. Since W is a left separating point of $\mathcal{M}$, it follows that $\delta(I)=0$.

For every idempotent $P\in\mathcal{A}$ and $t\in\mathbb{R}$ with $t\neq 1$, it is easy to show

$$I=(I-tP)\Big(I-\frac{t}{t-1}P\Big).$$

Thus, we have that

$$W=(I-tP)\Big(W-\frac{t}{t-1}PW\Big)$$

and

$$\delta(W) = (I - tP)\delta\Big(W - \frac{t}{t-1}PW\Big) + \Big(W - \frac{t}{t-1}PW\Big)\delta(I - tP);$$

that is,

$$\begin{aligned}\delta(W) = \delta(W) - \frac{t}{t-1}\delta(PW) - tP\delta(W) \\ + \frac{t^2}{t-1}P\delta(PW) - tW\delta(P) + \frac{t^2}{t-1}PW\delta(P).\end{aligned}$$

Hence, for any $t \neq 0, 1$, we obtain

$$0 = -\delta(PW) - (t-1)P\delta(W) + tP\delta(PW) - (t-1)W\delta(P) + tPW\delta(P);$$

that is,

$$\begin{aligned}0 = t\big(PW\delta(P) + P\delta(PW) - P\delta(W) - W\delta(P)\big) \\ - \big(\delta(PW) - P\delta(W) - W\delta(P)\big).\end{aligned}$$

Thus,

$$PW\delta(P) + P\delta(PW) - P\delta(W) - W\delta(P) = 0 \tag{5.1}$$

and

$$\delta(PW) - P\delta(W) - W\delta(P) = 0. \tag{5.2}$$

Multiplying P from the left-hand sides of (5.1) and (5.2), we have that

$$PW\delta(P) + P\delta(PW) - P\delta(W) - PW\delta(P) = 0 \tag{5.3}$$

and

$$P\delta(PW) - P\delta(W) - PW\delta(P) = 0. \tag{5.4}$$

Comparing (5.3) and (5.4), we have that $PW\delta(P) = 0$ and $P\delta(PW) - P\delta(W) = 0$. Thus, by (5.1), we have that $W\delta(P) = 0$. Since W is a left separating point of $\mathcal{M}$, we obtain $\delta(P) = 0$ for every idempotent P in $\mathcal{A}$. □

By Lemma 5.1 and [16, Proposition 4.4], we have the following result.

Corollary 5.2. *Let $\mathcal{A}$ be a weakly closed unital algebra of $B(\mathcal{H})$ of infinite multiplicity, and let δ be a linear mapping from $\mathcal{A}$ into a unital left $\mathcal{A}$-module $\mathcal{M}$. If δ is left derivable at a left separating point W, then $\delta \equiv 0$.*

Lemma 5.3 ([11, Theorem 3])**.** *Let $\mathcal{A}$ be a von Neumann algebra. Then any self-adjoint operator in $\mathcal{A}$ can be written as a linear combination of* 12 *projections with* 4 *central and* 8 *real coefficients.*

By Lemmas 5.1 and 5.3, it is easy to prove the following result.

Theorem 5.4. *Let $\mathcal{A}$ be a factor von Neumann algebra, let $\mathcal{M}$ be a unital left $\mathcal{A}$-module, and let δ be a linear mapping from $\mathcal{A}$ into $\mathcal{M}$. If δ is left derivable at a left separating point W, then $\delta \equiv 0$.*

Lemma 5.5 ([2, Lemma 5]). *Let $\mathcal{A}$ be a von Neumann algebra, and $\mathcal{A}$ has no direct summands of finite type* I. *Then each invertible operator $A \in \mathcal{A}^+$ can be written as a linear combination of projections in $\mathcal{A}$ with positive coefficients, where $\mathcal{A}^+$ denotes the set of all positive operators in $\mathcal{A}$.*

By Lemmas 5.1 and 5.5, we have the following corollary.

Corollary 5.6. *Let $\mathcal{A}$ be a von Neumann algebra, and $\mathcal{A}$ has no direct summands of finite type* I. *Let $\mathcal{M}$ be a unital left $\mathcal{A}$-module, and let δ be a linear mapping from $\mathcal{A}$ into $\mathcal{M}$. If δ is left derivable at a left separating point W, then $\delta \equiv 0$.*

By [16, Lemma 3.1] and Lemma 5.3, we know that if $\mathcal{A}$ is a factor von Neumann algebra and δ is a left derivable mapping at zero from $\mathcal{A}$ into any unital left $\mathcal{A}$-module $\mathcal{M}$ with $\delta(I) = 0$, then $\delta \equiv 0$. Now we consider left derivable mappings at every nonzero self-adjoint element of factor von Neumann algebras.

Theorem 5.7. *Let $\mathcal{A}$ be a factor von Neumann algebra, let C in $\mathcal{A}$ be a nonzero self-adjoint element, and let δ be a linear mapping from $\mathcal{A}$ into itself. If δ is left derivable at C, then $\delta \equiv 0$.*

Proof. If $\ker C = 0$, then C is a left separating point of $\mathcal{A}$. By Theorem 5.4 we know the conclusion holds.

In the following, we suppose that $\ker C \neq 0$.

Since $\mathcal{A}$ is a factor von Neumann algebra, it is well known that $\mathcal{A}$ is a prime algebra; that is,

$$A\mathcal{A}B = (0) \quad \text{implies} \quad A = 0 \text{ or } B = 0 \tag{5.5}$$

for each A, B in $\mathcal{A}$.

Let $P = \overline{\operatorname{ran} C}$, and let $Q = I - P$. By assumption, we know $P \neq 0$ and $Q \neq 0$. For every M in $\mathcal{A}$, by $C = C^*$, we have that $MC = 0$ implies $MP = 0$ and $CM = 0$ implies $PM = 0$.

Let $\mathcal{A}_{11} = P\mathcal{A}P$, $\mathcal{A}_{12} = P\mathcal{A}Q$, $\mathcal{A}_{21} = Q\mathcal{A}P$, and $\mathcal{A}_{22} = Q\mathcal{A}Q$. It follows that $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$. Since $CQ = QC = 0$, we have that $C = C_{11} \in \mathcal{A}_{11}$. We divide the proof into two steps.

First we show that $\delta(\mathcal{A}_{22}) = \delta(\mathcal{A}_{12}) = P\delta(\mathcal{A}_{21}) = Q\delta(\mathcal{A}_{11}) = 0$.

Since $CI = C$ and δ is left derivable at C, it is easy to show that $C\delta(I) = P\delta(I) = 0$.

Letting $A_{11} \in \mathcal{A}_{11}$ be invertible, $A_{12} \in \mathcal{A}_{12}$, $A_{22} \in \mathcal{A}_{22}$, and $0 \neq t \in \mathbb{R}$. By a simple computation, we have that

$$A_{11}(A_{11}^{-1}C) = C$$

and

$$(A_{11} + tA_{11}A_{12})(A_{11}^{-1}C - A_{12}A_{22} + t^{-1}A_{22}) = C.$$

It follows that

$$\delta(C) = A_{11}^{-1}C\delta(A_{11}) + A_{11}\delta(A_{11}^{-1}C) \tag{5.6}$$

and

$$\begin{aligned}\delta(C) &= (A_{11}^{-1}C - A_{12}A_{22} + t^{-1}A_{22})\delta(A_{11} + tA_{11}A_{12}) \\ &\quad + (A_{11} + tA_{11}A_{12})\delta(A_{11}^{-1}C - A_{12}A_{22} + t^{-1}A_{22}) \\ &= \big[(A_{11}^{-1}C - A_{12}A_{22})\delta(A_{11}) + A_{11}\delta(A_{11}^{-1}C - A_{12}A_{22}) \\ &\quad + A_{22}\delta(A_{11}A_{12}) + A_{11}A_{12}\delta(A_{22})\big] \\ &\quad + t\big[(A_{11}^{-1}C - A_{12}A_{22})\delta(A_{11}A_{12}) + A_{11}A_{12}\delta(A_{11}^{-1}C - A_{12}A_{22})\big] \\ &\quad + t^{-1}\big[A_{22}\delta(A_{11}) + A_{11}\delta(A_{22})\big]. \end{aligned} \tag{5.7}$$

Since t is an arbitrary nonzero number in $\mathbb{R}$, by (5.7) and [20, Proposition 2.0] it is easy to obtain some identities as follows:

$$(A_{11}^{-1}C - A_{12}A_{22})\delta(A_{11}A_{12}) + A_{11}A_{12}\delta(A_{11}^{-1}C - A_{12}A_{22}) = 0, \tag{5.8}$$

$$A_{22}\delta(A_{11}) + A_{11}\delta(A_{22}) = 0, \tag{5.9}$$

and

$$\begin{aligned}\delta(C) &= (A_{11}^{-1}C - A_{12}A_{22})\delta(A_{11}) + A_{11}\delta(A_{11}^{-1}C - A_{12}A_{22}) \\ &\quad + A_{22}\delta(A_{11}A_{12}) + A_{11}A_{12}\delta(A_{22}) \\ &= A_{11}^{-1}C\delta(A_{11}) + A_{11}\delta(A_{11}^{-1}C) \\ &\quad - A_{12}A_{22}\delta(A_{11}) - A_{11}\delta(A_{12}A_{22}) \\ &\quad + A_{22}\delta(A_{11}A_{12}) + A_{11}A_{12}\delta(A_{22}). \end{aligned} \tag{5.10}$$

By (5.6) and (5.10) we have that

$$-A_{12}A_{22}\delta(A_{11}) - A_{11}\delta(A_{12}A_{22}) + A_{22}\delta(A_{11}A_{12}) + A_{11}A_{12}\delta(A_{22}) = 0. \tag{5.11}$$

Multiplying Q from the left of (5.9) and taking $A_{22} = Q$ in it, we have that

$$Q\delta(A_{11}) = 0.$$

Since $\mathcal{A}_{11}$ is a von Neumann algebra, it can be linearly generated by its invertible elements. Since δ is linear, we have

$$Q\delta(\mathcal{A}_{11}) = 0.$$

It follows that $Q\delta(C) = 0$ and $\delta(C) = P\delta(C)$. Similarly, we have that $P\delta(\mathcal{A}_{22}) = 0$.

Multiplying P from the left of (5.9) and taking $A_{11} = P$ and $A_{22} = Q$ in (5.9), we have that

$$P\delta(Q) = Q\delta(P) = 0.$$

It follows that $\delta(P) = P\delta(P) = P\delta(I) = 0$.

Multiplying Q from the left of (5.11) and taking $A_{11} = P$ and $A_{22} = Q$ in it, we have that

$$Q\delta(A_{12}) = 0.$$

Taking $A_{11} = P$ and $A_{22} = Q$ in (5.8), we obtain

$$(C - A_{12})\delta(A_{12}) + A_{12}\delta(C - A_{12}) = 0. \tag{5.12}$$

By $Q\delta(A_{12}) = 0$ and $Q\delta(C) = 0$, we have that $C\delta(A_{12}) = P\delta(A_{12}) = 0$; thus

$$\delta(A_{12}) = 0$$

for every A_{12} in $\mathcal{A}_{12}$, which means that $\delta(\mathcal{A}_{12}) = 0$.

By $Q\delta(\mathcal{A}_{11}) = 0$ and taking $A_{11} = P$ in (5.11), we have that

$$A_{12}\delta(A_{22}) = 0. \tag{5.13}$$

Since A_{12} is arbitrary, it follows that $P\mathcal{A}Q\delta(A_{22}) = 0$. By (5.5) and $P \neq 0$ we obtain

$$Q\delta(A_{22}) = 0$$

for every A_{22} in $\mathcal{A}_{22}$, which means that $Q\delta(\mathcal{A}_{22}) = 0$. Using $P\delta(\mathcal{A}_{22}) = 0$ and $Q\delta(\mathcal{A}_{22}) = 0$, we have that $\delta(\mathcal{A}_{22}) = 0$.

Taking $A_{22} = Q$ in (5.13), we obtain

$$A_{12}\delta(Q) = 0.$$

Similarly, by (5.5) it follows that $Q\delta(Q) = 0$; that is, $\delta(Q) = 0$ by $P\delta(Q) = 0$.

By $P(C + A_{21}) = C$, we have that

$$(C + A_{21})\delta(P) + P\delta(C + A_{21}) = \delta(C).$$

Since $\delta(P) = 0$, it follows that $P\delta(C+A_{21}) = \delta(C)$; hence we obtain $P\delta(A_{21}) = 0$ for every A_{21} in $\mathcal{A}_{21}$. Thus $P\delta(\mathcal{A}_{21}) = 0$.

Similarly, letting $A_{11} \in \mathcal{A}_{11}$ be invertible, $A_{21} \in \mathcal{A}_{21}$, $A_{22} \in \mathcal{A}_{22}$, and $0 \neq t \in \mathbb{R}$. We have that

$$(CA_{11}^{-1} - A_{22}A_{21} + t^{-1}A_{22})(A_{11} + tA_{21}A_{11}) = C.$$

Thus, applying the same technique as in the previous proof, we can prove that $Q\delta(\mathcal{A}_{21}) = P\delta(\mathcal{A}_{11}) = 0$. Hence $\delta(\mathcal{A}_{21}) = \delta(\mathcal{A}_{11}) = 0$. $\square$

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