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APPROXIMATION BY COMPLEX LUPAŞ–DURRMEYER POLYNOMIALS BASED ON POLYA DISTRIBUTION

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ABSTRACT. The study of the generalization of the Bernstein and Bernstein–Durrmeyer polynomials attached to a continuous function $f : [0,1] \to \mathbb{R}$ and based on the Polya distribution were considered in some recent publications. The aim of the present note is to study the approximation properties of the complex version of these Durrmeyer-type operators, attached to analytic functions in a disk $D_R = \{z \in \mathbb{C}; |z| < R\}$ with $R > 1$.

1. INTRODUCTION

Stancu in [14] introduced a sequence of positive linear operators $P_n^{(\alpha)} : C[0,1] \to C[0,1]$, depending on a nonnegative parameter α given by

$$\begin{aligned} P_n^{(\alpha)}(f,x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{(\alpha)}(x) \\ &= \sum_{i=0}^{n} \binom{n}{i} \frac{z(z+\alpha) \times \cdots \times (z+(i-1)\alpha)}{(1+\alpha) \times \cdots \times (1+(i-1)\alpha)} \Delta_{1/n}^{i} f(0), \end{aligned} \tag{1.1}$$

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where $\Delta^i_{(1/n)}f(0)$ is the finite difference of order i, with the step $1/n$,

$$\Delta^i_{(1/n)}f(0) = \sum_{\nu=0}^{i}(-1)^\nu \binom{i}{\nu} f\Big(\frac{i-\nu}{n}\Big),$$

and where $p^{(\alpha)}_{n,k}(x)$ is the Polya distribution with density function given by

$$p^{(\alpha)}_{n,k}(x) = \binom{n}{k} \frac{\prod_{\nu=0}^{k-1}(x+\nu\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{\prod_{\lambda=0}^{n-1}(1+\lambda\alpha)}, \quad x \in [0,1],$$

and where the fraction in the last sum in (1.1) is by convention equal to 1 for $i = 0$.

In the case where $\alpha = 0$, these operators reduce to the classical Bernstein polynomials. For $\alpha = 1/n$, a special case of the operators (1.1) was considered by Lupaş and Lupaş in [6], which can be represented in an alternate form as

$$P^{(1/n)}_n(f,x) = \frac{2(n!)}{(2n)!}\sum_{k=0}^{n}\binom{n}{k} f\Big(\frac{k}{n}\Big)(nx)_k(n-nx)_{n-k}, \tag{1.2}$$

where the rising factorial is given as $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$ and $(x)_0 = 1$. Recently, Gupta and Rassias in [5] proposed the Durrmeyer-type integral modification of the operators (1.2), which in the complex domain is defined as

$$D^{(1/n)}_n(f,z) = (n+1)\sum_{k=0}^{n} p^{(1/n)}_{n,k}(z)\int_0^1 p_{n,k}(t)f(t)\,dt, \tag{1.3}$$

where

$$p^{(1/n)}_{n,k}(z) = \frac{2(n!)}{(2n)!}\binom{n}{k}(nz)_k(n-nz)_{n-k}$$

and

$$p_{n,k}(t) = \binom{n}{k} t^k(1-t)^{n-k}.$$

Recent quantitative estimates for the usual Durrmeyer-type operators and their variants in complex domain have been discussed in, for example, [8], [12], [3, Chapter 1], and [4, Chapter 6]. For detailed studies on other operators, we mention the papers [1], [7], [9]–[16] and the recent books [2], [3], and [4].

Here, we study the complex Lupaş–Durrmeyer polynomials, which are based on Polya distribution. Upper-estimate, Voronovskaja-type asymptotic formula and exact order are obtained for the approximation of analytic functions in a disk $D_R = \{z \in \mathbb{C}; |z| < R\}$ with $R > 1$.

2. Auxiliary results

We start this section with the following useful lemmas, which will be used later in the paper.

Lemma 2.1. *If we denote the mth-order moment as*

$$T_{n,m}(z) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(1/n)}(z)\int_0^1 p_{n,k}(t)t^m\,dt,$$

then we have

$$\begin{aligned} T_{n,m+1}(z) = {} & \frac{[n(n-m)z + (n+m)(2m+1) + nm]}{(m+n)(m+n+2)} T_{n,m}(z) \\ & - \frac{m^2(m+2n-nz)}{(n+m)(n+m+1)(m+n+2)} T_{n,m-1}(z). \end{aligned}$$

Remark 2.2. By simple computation, we have

$$\begin{aligned} \int_0^1 p_{n,k}(t)t^r\,dt &= \binom{n}{k}\int_0^1 t^{k+r}(1-t)^{n-k}\,dt \\ &= \binom{n}{k} B(k+r+1, n-k+1) \\ &= \frac{n!(k+r)!}{k!(n+r+1)!}. \end{aligned}$$

Hence with $e_k(z) = z^k$, $k = 0, 1, 2, \ldots$, we have

$$D_n^{(1/n)}(e_0, z) = 1, \qquad D_n^{(1/n)}(e_1, z) = \frac{nz+1}{n+2}$$

and

$$D_n^{(1/n)}(e_2, z) = \frac{n^3z^2 + 5n^2z - n^2z^2 + 3nz + 2n + 2}{(n+1)(n+2)(n+3)}.$$

Remark 2.3. By simple applications of Lemma 2.1, we have

$$D_n^{(1/n)}(t-z, z) = \frac{nz+1}{n+2} - z = \frac{1-2z}{n+2}$$

and

$$D_n^{(1/n)}\big((t-z)^2, z\big) = \frac{(z-z^2)(3n^2-5n-6) + 2(n+1)}{(n+1)(n+2)(n+3)}.$$

Throughout the present article, we denote $F_m(u) = \prod_{j=1}^{m}(u+j)$, $m \in \mathbb{N}$ and $F_0(u) = 1$.

Lemma 2.4.

(i) *For all $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$, we have $D_n^{(1/n)}(e_m, 1) \leq 1$.*

(ii) *For all $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}$, we have*

$$D_n^{(1/n)}(e_m, z) = \frac{(n+1)!}{(n+m+1)!}\sum_{k=0}^{n} p_{n,k}^{(1/n)}(z) F_m(k).$$

Proof. (i) For $m = 0$, we have $D_n^{(1/n)}(e_0, 1) = 1$. Let $m \geq 1$. By definition, we have

$$\begin{aligned}
D_n^{(1/n)}(e_m, z) &= (n+1) \sum_{k=0}^{n} p_{n,k}^{(1/n)}(z) \int_0^1 p_{n,k}(t) t^m \, dt \\
&= \sum_{k=0}^{n} p_{n,k}^{(1/n)}(z) \frac{(n+1)!(k+m)!}{k!(n+m+1)!} \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{\prod_{\nu=0}^{k-1}(z+\nu/n) \prod_{\mu=0}^{n-k-1}(1-z+\mu/n)}{\prod_{\lambda=0}^{n-1}(1+\lambda/n)} \frac{(n+1)!(k+m)!}{n!(n+m+1)!}.
\end{aligned}$$

Thus, when denoting $F_{n,k}(z) = \prod_{\mu=0}^{n-k-1}(1-z+\mu/n)$, we have $F_{n,k}(1) = 0$ for $1 \leq k \leq n-1$ and $F_{n,n}(1) = 1$; it follows that the above sum reduces to the term for $k = n$, which immediately implies that $D_n^{(1/n)}(e_m, 1) = \frac{n+1}{n+m+1} \leq 1$.

(ii) By using Remark 2.2, we get

$$\int_0^1 p_{n,k}(t) t^m \, dt = \frac{n!(k+m)!}{k!(n+m+1)!} = \frac{n!}{(n+m+1)!} F_m(k),$$

where $F_m(u) = (u+1)_m$. Therefore,

$$D_n^{(1/n)}(e_m, z) = \frac{(n+1)!}{(n+m+1)!} \sum_{k=0}^{n} p_{n,k}^{(1/n)}(z) F_m(k).$$

Note that obviously $\Delta_1^k F_m(0) \geq 0$ for all k and m. □

3. Main results

The first main result refers to upper estimates.

Theorem 3.1. *Let $r \geq 1$.*

(i) *For all $m \in \mathbb{N} \cup \{0\}$ and $|z| \leq r$, we have $|D_n^{(1/n)}(e_m, z)| \leq r^m$.*

(ii) *Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $|z| < R$ and take $1 \leq r < R$. For all $|z| \leq r$ and $n \in \mathbb{N}$, we have*

$$\left| D_n^{(1/n)}(f, z) - f(z) \right| \leq \frac{C_r(f)}{n},$$

where $C_r(f) = 4 \sum_{p=1}^{\infty} |c_p| p^2 r^p < \infty$.

Proof. (i) By using the second equality in (1.1) and Lemma 2.4, it follows that

$$\begin{aligned}
D_n^{(1/n)}(e_m, z) = {} & \frac{(n+1)!}{(n+m+1)!} \\
& \times \sum_{k=0}^{\min\{n,m\}} \binom{n}{k} \frac{z(z+1/n) \times \cdots \times (z+(k-1)/n)}{(1+1/n) \times \cdots \times (1+(k-1)/n)} \Delta_1^k F_m(0)
\end{aligned} \tag{3.1}$$

and

$$D_n^{(1/n)}(e_m, 1) = \frac{(n+1)!}{(n+m+1)!} \sum_{k=0}^{n} \binom{n}{k} \Delta_1^k F_m(0) \le 1.$$

By taking into account that for $|z| \le r$ we evidently have

$$\Big| \frac{z(z+1/n) \times \cdots \times (z+(k-1)/n)}{(1+1/n) \times \cdots \times (1+(k-1)/n)} \Big| \le \frac{r(r+1/n) \times \cdots \times (r+(k-1)/n)}{(1+1/n) \times \cdots \times (1+(k-1)/n)} \le r^k,$$

we immediately get

$$\big|D_n^{(1/n)}(e_m, z)\big| \le \frac{(n+1)!}{(n+m+1)!} \sum_{k=0}^{n} \binom{n}{k} \Delta_1^k F_m(0) r^k \le r^m.$$

(ii) First we prove that $D_n^{(1/n)}(f, z) = \sum_{k=0}^{\infty} c_k D_n^{(1/n)}(e_k, z)$. Indeed, denoting $f_m(z) = \sum_{j=0}^{m} c_j z^j$, $|z| \le r$ with $m \in \mathbb{N}$, by the linearity of $D_n^{(1/n)}$ we have

$$D_n^{(1/n)}(f_m, z) = \sum_{k=0}^{m} c_k D_n^{(1/n)}(e_k, z),$$

and it is sufficient to show that, for any fixed $n \in \mathbb{N}$ and $|z| \le r$ with $r \ge 1$, we have $\lim_{m\to\infty} D_n^{(1/n)}(f_m, z) = D_n^{(1/n)}(f, z)$. But this is immediate from $\lim_{m\to\infty} \|f_m - f\|_r = 0$, the norm being defined as $\|f\|_r = \max\{|f(z)| : |z| \le r\}$, and from the inequality

$$\begin{aligned} \big|D_n^{(1/n)}(f_m, z) - D_n^{(1/n)}(f, z)\big| &\le (n+1) \sum_{k=0}^{n} \big|p_{n,k}^{(1/n)}(z)\big| \int_0^1 p_{n,k}(t) \big|f_m(t) - f(t)\big| \, dt \\ &\le C_{r,n} \|f_m - f\|_r, \end{aligned}$$

valid for all $|z| \le r$. Therefore, we get

$$\big|D_n^{(1/n)}(f, z) - f(z)\big| \le \sum_{m=1}^{\infty} |c_m| \times \big|D_n^{(1/n)}(e_m, z) - e_m(z)\big|,$$

as $D_n^{(1/n)}(e_0, z) = e_0(z) = 1$.

We have these two cases: (a) $1 \le m \le n$, and (b) $m > n$.

Case (a): Let us denote $E_{n,m}(z) = \frac{z(z+1/n)\times\cdots\times(z+(m-1)/n)}{(1+1/n)\times\cdots\times(1+(m-1)/n)}$ and

$$G_{n,m}(z) = \frac{(n+1)!}{(n+m+1)!} \binom{n}{m} \Delta_1^m F_m(0).$$

By the formula (3.1) in the proof of Theorem 3.1(i), we have

$$\begin{aligned} D_n^{(1/n)}(e_m, z) - e_m(z) &= E_{n,m}(z) \times \big[G_{n,m}(z) - 1\big] + \big[E_{n,m}(z) - z^m\big] \\ &\quad + \frac{(n+1)!}{(n+m+1)!} \sum_{k=0}^{m-1} \binom{n}{k} \Delta_1^k F_m(0) z^k, \end{aligned}$$

and therefore we get

$$\begin{aligned}
&\left|D_n^{(1/n)}(e_m,z)-e_m(z)\right| \\
&\quad \le r^m\left|1-G_{n,m}(z)\right|+r^m\left|1-G_{n,m}(z)\right|+\left|E_{n,m}(z)-z^m\right| \\
&\quad = 2r^m\left[1-G_{n,m}(z)\right]+\left|E_{n,m}(z)-z^m\right|.
\end{aligned}$$

But by mathematical induction after $m\in\mathbb{N}$, we get

$$\left|E_{n,m}(z)-z^m\right|\le\frac{2(m-1)^2r^m}{n}.$$

Now, by the obvious formula

$$E_{n,m+1}(z)-z^{m+1}=\frac{z+m/n}{1+m/n}\cdot\left[E_{n,m}(z)-z^m\right]+z^m\left[\frac{z+m/n}{1+m/n}-z\right],$$

for all $|z|\le r$ it follows that

$$\begin{aligned}
\left|E_{n,m+1}(z)-z^{m+1}\right| &\le \frac{r+m/n}{1+m/n}\left|E_{n,m}(z)-z^m\right|+\frac{2r^{m+1}m}{n+m} \\
&\le r\cdot\left|E_{n,m}(z)-z^m\right|+\frac{2r^{m+1}m}{n+m}.
\end{aligned}$$

Since $E_{n,1}(z)-z=0$ for all z and n, taking in the above recurrence inequality step by step $m=1,2,\ldots$, we easily arrive at

$$\left|E_{n,m}(z)-z^m\right|\le 2r^m\cdot\sum_{j=1}^{m-1}\frac{j}{n+j}\le\frac{2(m-1)^2r^m}{n}.$$

Also, we can write

$$\frac{(n+1)!}{(n+m+1)!}\binom{n}{m}\Delta_1^mF_m(0)=\frac{(n+1)!}{(n+m+1)!}\binom{n}{m}m!=\prod_{j=1}^m\frac{n+j-m}{n+j+1}.$$

By using the formula

$$1-\prod_{j=1}^k x_j\le\sum_{j=1}^k(1-x_j),\quad 0\le x_j\le 1, j=1,2,\ldots,k,$$

with $x_j=\frac{n+j-m}{n+j+1}$ and $k=m$, we obtain

$$\begin{aligned}
1-\prod_{j=1}^m\frac{n+j-m}{n+j+1} &\le \sum_{j=1}^m\left(1-\frac{n+j-m}{n+j+1}\right)=(m+1)\sum_{j=1}^m\frac{1}{n+j+1} \\
&\le\frac{m(m+1)}{n}.
\end{aligned}$$

Therefore it follows that

$$\left|D_n^{(1/n)}(e_m,z)-e_m(z)\right|\le\frac{2m(m+1)r^m}{n}+\frac{2(m-1)^2r^m}{n}\le\frac{4m^2r^m}{n}.$$

Case (b): By (i) and for $m > n \geq 1$, we obtain

$$\left|D_n^{(1/n)}(e_m, z) - e_m(z)\right| \leq \left|D_n^{(1/n)}(e_m, z)\right| + \left|e_m(z)\right| \leq 2r^m < 2\frac{mr^m}{n} \leq \frac{2m^2r^m}{n}.$$

By the cases (a) and (b), we conclude that, for all $m, n \in \mathbb{N}$, one has

$$\left|D_n^{(1/n)}(e_m, z) - e_m(z)\right| \leq \frac{4m^2r^m}{n}.$$

Hence, we get

$$\left|D_n^{(1/n)}(f, z) - f(z)\right| \leq \frac{4}{n}\sum_{m=1}^{\infty}|c_m|m^2r^m,$$

which proves the theorem. □

The following Voronovskaja-type result with a quantitative estimate holds.

Theorem 3.2. *Let* $\rho_0 = \frac{1+\sqrt{5}}{2}$ *be the golden ratio,* $\rho > 5$ *and* $R > (\rho+1)\rho_0$. *Suppose that* $f : D_R \to \mathbb{C}$ *is analytic in* $D_R = \{z \in \mathbb{C} : |z| < R\}$*; that is, we can write* $f(z) = \sum_{k=0}^{\infty} c_k z^k$ *for all* $z \in D_R$. *For any fixed* $r \in [1, \min\{\frac{\rho-2}{3}, \frac{R}{\rho_0} - \rho\})$ *and for all* $n \in \mathbb{N}$, $|z| \leq r$, *we have*

$$\left|D_n^{(1/n)}(f, z) - f(z) - \frac{1.5z(1-z)f''(z) + (1-2z)f'(z)}{n}\right| \leq \frac{M_r(f)}{n^2},$$

where $M_r(f) = \frac{2}{\sqrt{5}} \cdot \sum_{k=1}^{\infty}|c_k| \cdot [1 + kA_{k,r}] \cdot [(r+\rho)\rho_0]^k$ *and where*

$$\begin{aligned} A_{k,r} &= r^3(3k^4 + 10k^3 + 8k^2 + k + 2) + r^2(8k^3 + 7k^2 + 10k + 8) \\ &\quad + r(3k^4 + 4k^3 + 10k^2 + 19k + 4) + (6k^4 + 38k^3 + 86k^2 + 82k + 28). \end{aligned}$$

Proof. We denote $\pi_{k,n}(z) = D_n^{(1/n)}(e_k, z)$. By the proof of Theorem 3.1(ii), we can write $D_n^{(1/n)}(f, z) = \sum_{k=0}^{\infty} c_k \pi_{k,n}(z)$. Also since

$$\begin{aligned} &\frac{3z(1-z)f''(z) + (2-4z)f'(z)}{2n} \\ &= \frac{3z(1-z)}{2n}\sum_{k=2}^{\infty} c_k k(k-1)z^{k-2} + \frac{2-4z}{2n}\sum_{k=1}^{\infty} c_k k z^{k-1} \\ &= \frac{1}{2n}\sum_{k=1}^{\infty} c_k k\big[(3k-1) - (3k+1)z\big]z^{k-1}. \end{aligned}$$

Thus

$$\begin{aligned} &\left|D_n^{(1/n)}(f, z) - f(z) - \frac{1.5z(1-z)f''(z) + (1-2z)f'(z)}{n}\right| \\ &\leq \sum_{k=1}^{\infty}|c_k|\left|\pi_{k,n}(z) - e_k(z) - \frac{k[(3k-1) - (3k+1)z]z^{k-1}}{2n}\right|, \end{aligned}$$

for all $z \in D_R$, $n \in \mathbb{N}$.

By Lemma 2.1, for all $n \in \mathbb{N}$, $z \in \mathbb{C}$, and $k = 0, 1, 2, \dots$, we have

$$\pi_{n,k+1}(z) = \frac{[n(n-k)z + (n+k)(2k+1) + nk]}{(k+n)(k+n+2)}\pi_{n,k}(z) + \frac{k^2(nz-k-2n)}{(n+k)(n+k+1)(k+n+2)}\pi_{n,k-1}(z).$$

If we denote

$$H_{k,n}(z) = \pi_{k,n}(z) - e_k(z) - \frac{k[(3k-1)-(3k+1)z]z^{k-1}}{2n},$$

then it is obvious that $H_{k,n}(z)$ is a polynomial of degree less than or equal to k and by simple computation and the use of the above recurrence relation, we are led to

$$H_{k,n}(z) = \frac{n(n-k+1)z + (n+k-1)(2k-1) + n(k-1)}{(n+k-1)(n+k+1)}H_{k-1,n}(z) + \frac{(k-1)^2(nz-k+1-2n)}{(n+k-1)(n+k)(n+k+1)}H_{k-2,n}(z) + X_{k,n}(z),$$

where after simple computation, we have

$$\begin{aligned}
X_{k,n}(z) = z^k\Big[&\frac{9k^3-5k^2+2k+2}{2(n+k-1)(n+k+1)} + \frac{3k^4+k^3-3k^2-k}{2n(n+k-1)(n+k+1)}\Big]\\
&+ z^{k-1}\Big[-\frac{k(k+1)(3k-1)}{2n(n+k+1)} - \frac{3k(k-1)}{2(n+k+1)(n+k-1)}\\
&- \frac{(2k-1)(3k-2)}{2n(n+k+1)} + \frac{(k-1)^2}{(n+k)(n+k+1)}\\
&- \frac{(k-1)^2(5k-7)}{2(n+k)(n+k-1)(n+k+1)}\Big]\\
&+ z^{k-2}\Big[\frac{(2k-1)(k-1)(3k-4)}{2(n+k+1)n} + \frac{(k-1)^2(3k-4)}{2(n+k-1)(n+k+1)}\\
&- \frac{4(k-1)}{(n+k)(n+k+1)} + \frac{k(k-1)^2(3k-7)}{(n+k-1)(n+k)(n+k+1)}\\
&+ \frac{(k-1)^2(k-2)(3k-5)}{2n(n+k)(n+k+1)}\Big]\\
&+ z^{k-3}\Big[\frac{(k-1)^2(k-2)(3k-7)}{2(n+k)(n+k+1)n} + \frac{(k-1)^2(k-2)(3k-7)}{2(n+k-1)(n+k)(n+k+1)}\Big]
\end{aligned}$$

for all $k \geq 1$, $n \in \mathbb{N}$, and $|z| \geq r$.

Using the estimate in the proof of Theorem 3.1(ii), we have

$$\left|\pi_{k,n}(z) - e_k(z)\right| \leq \frac{4k^2 r^k}{n},$$

for all $k, n \in \mathbb{N}$, $|z| \leq r$, with $1 \leq r$.

For all $k, n \in \mathbb{N}$, $k \geq 1$, and $|z| \leq r$, it follows that

$$\begin{aligned}\big|H_{k,n}(z)\big| &\leq (r+3)\big|H_{k-1,n}(z)\big| + (r+2)\big|H_{k-2,n}(z)\big| + \big|X_{k,n}(z)\big| \\ &\leq (r+\rho)\big|H_{k-1,n}(z)\big| + (r+\rho)\big|H_{k-2,n}(z)\big| + \big|X_{k,n}(z)\big|,\end{aligned}$$

where

$$\begin{aligned}\big|X_{k,n}(z)\big| \leq \frac{r^{k-3}}{n^2}\big[&r^3(3k^4+10k^3+8k^2+k+2) + r^2(8k^3+7k^2+10k+8) \\ &+ r(3k^4+4k^3+10k^2+19k+4) + (6k^4+38k^3+86k^2+82k+28)\big] \\ \leq \frac{r^k}{n^2}&A_{k,r} \leq \frac{(r+\rho)^k}{n^2}A_{k,r},\end{aligned}$$

for all $|z| \leq r$, $k \geq 1$, $n \in \mathbb{N}$, with

$$\begin{aligned}A_{k,r} = {}& r^3(3k^4+10k^3+8k^2+k+2) + r^2(8k^3+7k^2+10k+8) \\ &+ r(3k^4+4k^3+10k^2+19k+4) + (6k^4+38k^3+86k^2+82k+28).\end{aligned}$$

Thus for all $|z| \leq r$, $k \geq 1$, $n \in \mathbb{N}$, we have

$$\big|H_{k,n}(z)\big| \leq (r+\rho)\big|H_{k-1,n}(z)\big| + (r+\rho)\big|H_{k-2,n}(z)\big| + \frac{(r+\rho)^k}{n^2}A_{k,r},$$

where $A_{k,r}$ is a polynomial of degree 4 in k.

Consider the Fibonacci numbers $F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, for all $n \in \mathbb{N}$, $n \geq 2$.

We have $H_{0,n}(z) = 0$, for any $z \in \mathbb{C}$, and since $4r + 2 < r + \rho$ for $\rho > 5$ and $r < (\rho - 2)/3$, it follows that

$$\big|H_{1,n}(z)\big| = \Big|\frac{nz+1}{n+2} - z - \frac{1-2z}{n}\Big| = \Big|\frac{4z-2}{n(n+2)}\Big| \leq \frac{4r+2}{n^2} < \frac{r+\rho}{n^2}.$$

Now, by writing the last inequality for $k = 2, 3, \ldots$, we easily obtain step by step the following estimates:

- for $k = 2$,

$$\big|H_{2,n}(z)\big| \leq \frac{(r+\rho)^2}{n^2} + \frac{(r+\rho)^2}{n^2}A_{2,r} = \frac{(r+\rho)^2}{n^2}[1 + A_{2,r}];$$

- for $k = 3$,

$$\begin{aligned}\big|H_{3,n}(z)\big| &\leq \frac{(r+\rho)^3}{n^2}[1 + A_{2,r}] + \frac{(r+\rho)^2}{n^2} + \frac{(r+\rho)^3}{n^2}A_{3,r} \\ &\leq \frac{(r+\rho)^3}{n^2}\big[2 + 1\cdot(A_{2,r} + A_{3,r})\big] \\ &= \frac{(r+\rho)^3}{n^2}\big[F_2 + F_1(A_{2,r} + A_{3,r})\big];\end{aligned}$$

- for $k = 4$,

$$\big|H_{4,n}(z)\big| \leq \frac{(r+\rho)^4}{n^2}\big[F_3 + F_2(A_{2,r} + A_{3,r} + A_{4,n})\big];$$

- for $k=5$,

$$\left|H_{5,n}(z)\right| \le \frac{(r+\rho)^5}{n^2}\left[F_4+F_3(A_{2,r}+A_{3,r}+A_{4,r}+A_{5,r})\right];$$

- and in general for $k>5$ we obtain

$$\left|H_{k,n}(z)\right| \le \frac{(r+\rho)^k}{n^2}\left[F_{k-1}+F_{k-2}(A_{2,r}+A_{3,r}+A_{4,r}+A_{5,r}+\cdots+A_{k,r})\right],$$

  for all $k\in\mathbb{N}$, $k\ge 2$.

But from the well-known Binet's formula for the Fibonacci numbers (see, e.g., [17]), we have

$$F_k=\frac{1}{\sqrt{5}}\left(\rho_0^k-(1-\rho_0)^k\right)\le\frac{2}{\sqrt{5}}\rho_0^k,\quad k=1,2,\ldots,$$

where $\rho_0=\frac{1+\sqrt{5}}{2}=1.6180\ldots$ is the golden ratio.

Therefore, we obtain

$$\left|H_{k,n}(z)\right| \le \frac{2}{\sqrt{5}}\cdot\frac{[(r+\rho)\rho_0]^k}{n^2}\left[1+\sum_{j=2}^{k}A_{j,r}\right]\le\frac{2}{\sqrt{5}}\cdot\frac{[(r+\rho)\rho_0]^k}{n^2}[1+k\cdot A_{k,r}],$$

taking into account that $A_{j,r}$ is evidently strictly increasing with respect to j.

We conclude that

$$\begin{aligned}&\left|D_n^{(1/n)}(f,z)-f(z)-\frac{1.5z(1-z)f''(z)+(1-2z)f'(z)}{n}\right|\\&\le\sum_{k=1}^{\infty}|c_k|\cdot|H_{k,n}|\\&\le\frac{1}{n^2}\cdot\frac{2}{\sqrt{5}}\cdot\sum_{k=1}^{\infty}|c_k|[1+kA_{k,r}]\cdot\left[(r+\rho)\rho_0\right]^k.\end{aligned}$$

As $f^{(5)}(z)=\sum_{k=5}^{\infty}c_k k(k-1)(k-2)(k-3)(k-4)z^{k-5}$ and the series is absolutely convergent in $|z|\le(r+\rho)\rho_0<R$, it easily follows that $\sum_{k=5}^{\infty}|c_k|k(k-1)(k-2)(k-3)(k-4)[(r+\rho)\rho_0]^{k-5}<\infty$, which implies that $\sum_{k=1}^{\infty}|c_k|kA_{k,r}[(r+\rho)\rho_0]^k<\infty$. This completes the proof of theorem. □

In the following, we will obtain below the exact order in approximation by this type of complex operator.

Theorem 3.3. *Under the hypothesis in Theorem 3.2, if f is not a constant function, then for any $r\in[1,\min\{\frac{\rho-2}{3},\frac{R}{\rho_0}-\rho\})$, we have*

$$\left\|D_n^{(1/n)}(f,\cdot)-f\right\|_r\ge\frac{C_r(f)}{n},\quad n\in\mathbb{N},$$

where $C_r(f)$ depends only on f and r.

Proof. For all $f \in D_r$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} &D_n^{(1/n)}(f,z) - f(z) \\ &= \frac{1}{n}\Big[1.5z(1-z)f''(z) + (1-2z)f'(z) \\ &\quad + \frac{1}{n}\Big\{n^2\Big(D_n^{(1/n)}(f,z) - f(z) - \frac{1.5z(1-z)f''(z) + (1-2z)f'(z)}{n}\Big)\Big\}\Big]. \end{aligned}$$

Also, we have

$$\|F+G\|_r \geq \big|\|F\|_r - \|G\|_r\big| \geq \|F\|_r - \|G\|_r.$$

It follows that

$$\begin{aligned} &\big\|D_n^{(1/n)}(f,\cdot) - f\big\|_r \\ &\geq \frac{1}{n}\Big[\big\|1.5e_1(1-e_1)f'' + (1-2e_1)f'\big\|_r \\ &\quad - \frac{1}{n}\Big\{n^2\Big\|D_n^{(1/n)}(f,\cdot) - f - \frac{1.5e_1(1-e_1)f'' + (1-2e_1)f'}{n}\Big\|_r\Big\}\Big]. \end{aligned}$$

Taking into account that by hypothesis f is not a constant function in D_R, we get $\|3e_1(1-e_1)f'' + (2-4e_1)f'\|_r > 0$.

Indeed, supposing the contrary, it follows that $3z(1-z)f''(z) + (2-4z)f'(z) = 0$ for all $|z| \leq r$. The last equality easily implies that $f'(z) = C \cdot (z(1-z))^{(-2/3)}$ for all $|z| \leq r$, with C a constant.

But since f is analytic in $\overline{D_r}$ and $r \geq 1$, we necessarily have $C = 0$ (contrariwise, we would get that $f'(z)$ is not differentiable at $z = 0$ and $z = 1$, which is impossible because $f'(z)$ too is analytic on $\overline{D_r}$, with $r \geq 1$), which implies that $f'(z) = 0$ and $f(z) = c$ for all $z \in \overline{D_r}$, a contradiction with the hypothesis.

Now by Theorem 3.2, we have

$$n^2\Big\|D_n^{(1/n)}(f,z) - f(z) - \frac{1.5z(1-z)f''(z) + (1-2z)f'(z)}{n}\Big\|_r \leq M_r(f).$$

Therefore there exists an index n_0 depending only on f and r such that, for all $n \geq n_0$, we have

$$\begin{aligned} &\big\|1.5e_1(1-e_1)f'' + (1-2e_1)f'\big\|_r \\ &\qquad - \frac{1}{n}\Big\{n^2\Big\|D_n^{(1/n)}(f,z) - f(z) - \frac{1.5z(1-z)f''(z) + (1-2z)f'(z)}{n}\Big\|_r\Big\} \\ &\geq \frac{1}{2}\big\|1.5e_1(1-e_1)f'' + (1-2e_1)f'\big\|_r, \end{aligned}$$

which immediately implies that

$$\big\|D_n^{(1/n)}(f,\cdot) - f\big\|_r \geq \frac{1}{2n}\big\|1.5e_1(1-e_1)f'' + (1-2e_1)f'\big\|_r, \quad \forall n \geq n_0.$$

For $n \in \{1, 2, \ldots, n_0 - 1\}$ we obviously have $\|D_n^{(1/n)}(f,\cdot) - f\|_r \geq \frac{M_{r,n}(f)}{n}$ with $M_{r,n}(f) = n\|D_n^{(1/n)}(f,\cdot) - f\|_r > 0$. Indeed, if we should have $\|D_n^{(1/n)}(f,\cdot) - f\|_r = 0$, then it would follow that $D_n^{(1/n)}(f,z) = f(z)$ for all $|z| \leq r$, which is valid only for f a constant function, contracting the hypothesis on f. Therefore, finally, we obtain $\|D_n^{(1/n)}(f,\cdot)-f\|_r \geq \frac{C_r(f)}{n}$ for all n, where

$$C_r(f)=\min\Big\{M_{r,1}(f), M_{r,2}(f),\dots,M_{r,n_0-1}(f),\frac{1}{2}\big\|1.5e_1(1-e_1)f''+(1-2e_1)f'\big\|_r\Big\},$$

which completes the proof. □

As a consequence of Theorem 3.1 and Theorem 3.3, we have the following.

Corollary 3.4. *Under the hypothesis in Theorem 3.2, if f is not a constant function, then for any $r\in[1,\min\{\frac{\rho-2}{3},\frac{R}{\rho_0}-\rho\})$, we have*

$$\big\|D_n^{(1/n)}(f,\cdot)-f\big\|_r \sim \frac{1}{n}, \quad n\to\infty,$$

where the constants in the equivalence depend only on f and r.

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