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LINEAR AND NONLINEAR DEGENERATE ABSTRACT DIFFERENTIAL EQUATIONS WITH SMALL PARAMETER

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Abstract. The boundary value problems for linear and nonlinear regular degenerate abstract differential equations are studied. The equations have the principal variable coefficients and a small parameter. The linear problem is considered on a parameter-dependent domain (i.e., on a moving domain). The maximal regularity properties of linear problems and the optimal regularity of the nonlinear problem are obtained. In application, the well-posedness of the Cauchy problem for degenerate parabolic equations and boundary value problems for degenerate anisotropic differential equations are established.

1. Introduction, notation, and background

Boundary value problems (BVPs) for abstract differential equations (ADEs) have been studied extensively by many researchers (see [1], [3], [4], [8]–[21], and [24] and the references therein). A comprehensive introduction to ADEs and historical references may be found in [13] and [24]. The maximal regularity properties of ADEs have been studied in [1], [3]–[5], [8], [9], [16]–[20], and [23], for example. The main objective of the present article is to discuss the BVP for degenerate linear ADEs with variable coefficients

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$$(-1)^m ta(x)u^{[2m]}(x) + A(x)u(x) + \sum_{j=1}^{2m-1} t^{\frac{j}{2m}} A_j(x)u^{[j]}(x) = f(x),$$

and the BVP for the following nonlinear degenerate equation,

$$\begin{aligned}&(-1)^m a(x)u^{[2m]}(x) + O(x, u, u^{[1]}, \ldots, u^{[2m-1]})u(x)\\ &\quad = F(x, u, u^{[1]}, \ldots, u^{[2m-1]}), \quad x \in (0,1),\end{aligned}$$

where t is a small parameter, m is an integer number, $a(x)$ is a complex valued $A(x)$, $A_j(x)$ are linear, and $O(x, \ldots,)$, $F(x, \ldots,)$ are nonlinear operators in a Banach space E for $x \in (0,1)$ and

$$u^{[i]}(x) = \left[x^{\gamma_1}(1-x)^{\gamma_2}\frac{d}{dx}\right]^i u(x), \quad \gamma_k \geq 0, k = 1, 2.$$

The uniform separability, resolvent estimates, and the Fredholm properties of the linear problem are obtained in abstract L_p-spaces. In particular, we prove that the corresponding linear differential operator is both R-positive and a generator of the analytic semigroup.

Then, by using the separability properties of linear problems, the existence and uniqueness of the maximal regular solution of the nonlinear problem is proved in E-valued L_p spaces. Moreover, the well-posedness of the Cauchy problem for the degenerate parabolic equation

$$\frac{\partial u}{\partial y} + (-1)^m tau_x^{[2m]} + Au = f, \quad y \in R_+, x \in (0,1)$$

is established, where $u = u(y,x)$, $f = f(y,x)$, $a = a(x)$, $A = A(x)$, and

$$u_x^{[i]} = \left[x^{\gamma_1}(1-x)^{\gamma_2}\frac{\partial}{\partial x}\right]^i u(y,x).$$

In application, the BVP for degenerate partial differential equations on a cylindrical domain is studied. Some of the important characteristics of these problems are the following: (1) here the main and boundary equations are degenerated on all boundaries; (2) the principal parts of the problems are non-self-adjoint; (3) the equations possess variable coefficients with small parameter; and (4) the equations are degenerated with different speeds at boundary lines, in general. Note that the maximal regularity properties of ADEs considered in Banach spaces were treated in, for example, [1], [8], [11], [16]–[20], and [23]. Nonlinear BVPs for ADEs are studied in, for instance, [3], [17], and [20].

Let $\gamma = \gamma(x)$ be a positive measurable function on a domain $\Omega \subset R^n$. And $L_{p,\gamma}(\Omega; E)$ denotes the space of strongly measurable E-valued functions that are defined on Ω with the norm

$$\|f\|_{p,\gamma} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int \|f(x)\|_E^p \gamma(x)\, dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

For $\gamma(x) \equiv 1$, the space $L_{p,\gamma}(\Omega; E)$ will be denoted by $L_p = L_p(\Omega; E)$.

The weight function γ is said to be of *Muckenhopt type*, that is, $\gamma \in A_p$, $1 < p < \infty$, if there is a positive constant C such that

$$\Big(\frac{1}{|Q|}\int_Q \gamma(x)\,dx\Big)\Big(\frac{1}{|Q|}\int_Q \gamma^{-\frac{1}{p-1}}(x)\,dx\Big)^{p-1} \le C$$

for all cubes $Q \subset R^n$.

The Banach space E is called a *UMD-space* if the Hilbert operator $(Hf)(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}\,dy$ is bounded in $L_p(R,E)$, $p \in (1,\infty)$ (see, e.g., [7, Theorem 1]). UMD spaces include, for example, L_p- and l_p-spaces and Lorentz spaces L_{pq}, p, $q \in (1,\infty)$.

Let $\mathbb{C}$ be the set of complex numbers, and let

$$S_\varphi = \big\{\lambda; \lambda \in \mathbb{C}, |\arg\lambda| \le \varphi\big\} \cup \{0\}, \quad 0 \le \varphi < \pi.$$

A linear operator A is said to be *φ-positive* in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and $\|(A+\lambda I)^{-1}\|_{B(E)} \le M(1+|\lambda|)^{-1}$ for any $\lambda \in S_\varphi$, where I is the identity operator in E and $B(E)$ denotes the space of bounded linear operators in E. Sometimes $A+\lambda I$ will be written as $A+\lambda$ and will be denoted by A_λ. It is known (see [22, Section 1.15.1]) that a positive operator A has well-defined fractional powers A^θ. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with norm

$$\|u\|_{E(A^\theta)} = \big(\|u\|^p + \|A^\theta u\|^p\big)^{\frac{1}{p}}, \quad 1 \le p < \infty, 0 < \theta < \infty.$$

Let $S(R^n;E)$ denote the Schwartz class, that is, the space of all E-valued rapidly decreasing smooth functions on R^n. Let F denote the Fourier transformation. A function $\Psi \in C(R^n;B(E))$ is called a *Fourier multiplier* in $L_{p,\gamma}(R^n;E)$ if the map

$$u \to \Phi u = F^{-1}\Psi(\xi)Fu, \quad u \in S(R^n;E)$$

is well defined and extends to a bounded linear operator in $L_{p,\gamma}(R^n;E)$. The set of all multipliers in $L_{p,\gamma}(R^n;E)$ will be denoted by $M_{p,\gamma}(E)$. Let

$$W_h = \big\{\Psi_h \in M_{p,\gamma}(E), h \in Q \subset \mathbb{C}\big\}$$

be a collection of multipliers in $M_{p,\gamma}(E)$. We say that W_h is a *uniform collection of multipliers* if there exists a positive constant C independent of h such that

$$\|F^{-1}\Psi_h Fu\|_{L_{p,\gamma}(R^n;E)} \le C\|u\|_{L_{p,\gamma}(R^n;E)}$$

for all $h \in Q$ and $u \in S(R^n;E)$.

Definition 1.1. A Banach space E is said to be a space *satisfying the multiplier condition with respect to* $p \in (1,\infty)$ *and weighted function* γ, if, for any $\Psi \in C^{(1)}(R;B(E))$, the R-boundedness (see, e.g., [8, Section 4.1]) of the set

$$\big\{\xi^j\Psi^{(j)}(\xi) : \xi \in \mathbb{R}\backslash\{0\}, j = 0,1\big\}$$

implies $\Psi \in M_{p,\gamma}(E)$.

Let E_1 and E_2 be two Banach spaces continuously embedded in a locally convex space.

The operator $A(x)$ is said to be *φ-positive in E uniformly with respect to $x \in G$* if $D(A(x))$ is independent of x, $D(A(x))$ is dense in E, and $\|(A(x)+\lambda)^{-1}\| \leq \frac{M}{1+|\lambda|}$ for all $\lambda \in S(\varphi)$, $0 \leq \varphi < \pi$, where the constant M is independent of x and λ.

The φ-positive operator $A(x)$, $x \in G$ is said to be *uniformly R-positive* in a Banach space E if there exists $\varphi \in [0, \pi)$ such that the set

$$\left\{A(x)\left(A(x)+\xi I\right)^{-1} : \xi \in S_\varphi\right\}$$

is uniformly R-bounded, that is,

$$\sup_{x \in G} R\left(\left\{\left[A(x)\left(A(x)+\xi I\right)^{-1}\right] : \xi \in S_\varphi\right\}\right) \leq M.$$

Let $\sigma_\infty(E)$ denote the space of all compact operators in E. Let E_0 and E be two Banach spaces, and let E_0 be continuously and densely embedded into E. Let us consider the Sobolev–Lions-type space $W^m_{p,\gamma}(a,b;E_0,E)$, consisting of all functions $u \in L_{p,\gamma}(a,b;E_0)$ that have generalized derivatives $u^{(m)} \in L_{p,\gamma}(a,b;E)$ with the norm

$$\|u\|_{W^m_{p,\gamma}} = \|u\|_{W^m_{p,\gamma}(a,b;E_0,E)} = \|u\|_{L_{p,\gamma}(a,b;E_0)} + \|u^{(m)}\|_{L_{p,\gamma}(a,b;E)} < \infty.$$

Let

$$\begin{aligned} W^{[m]}_{p,\gamma} &= W^{[m]}_{p,\gamma}(0,1;E_0,E) \\ &= \{u; u \in L_p(0,1;E_0), u^{[m]} \in L_p(0,1;E), \\ &\qquad \|u\|_{W^{[m]}_{p,\gamma}} = \|u\|_{L_p(0,1;E_0)} + \|u^{[m]}\|_{L_p(0,1;E)} < \infty\}. \end{aligned}$$

Let t be a positive parameter. We define the following parameterized norm in $W^m_{p,\gamma}(a,b;E_0,E)$:

$$\|u\|_{W^m_{p,\gamma,t}} = \|u\|_{W^m_{p,\gamma,t}(a,b;E_0,E)} = \|u\|_{L_{p,\gamma}(a,b;E_0)} + \|tu^{(m)}\|_{L_{p,\gamma}(a,b;E)} < \infty.$$

The embedding theorems play a key role in the perturbation theory of DOEs. For estimating lower-order derivatives, we use the following embedding theorems from [18].

Theorem 1.2 ([18, Theorem 2.3])**.** *Assume that the following conditions are satisfied:*

1. *E is a Banach space satisfying the multiplier condition with respect to p and weighted function γ.*
2. *A is an R-positive operator in E.*
3. *We have that $0 \leq j \leq m$, $0 \leq \mu \leq 1 - \frac{j}{m}$, and $1 < p < \infty$; and h and t are positive parameters, that is, $0 < h < h_0 < \infty$, $0 < t < 1$.*
4. *There exists a bounded linear extension operator from $W^m_{p,\gamma}(a,b;E(A),E)$ to $W^m_{p,\gamma}(R^n;E(A),E)$.*

Then the embedding $D^j W^m_{p,\gamma}(a,b;E(A),E) \subset L_{p,\gamma}(a,b;E(A^{1-\frac{j}{m}-\mu}))$ *is continuous. Moreover, for* $u \in W^m_{p,\gamma}(a,b;E(A),E)$ *the following estimate holds:*

$$\|u^{(j)}\|_{L_{p,\gamma}(a,b;E(A^{1-\frac{j}{m}-\mu}))} \leq h^{\mu}\|u\|_{W^m_{p,\gamma,t}(a,b;E(A),E)} + h^{-(1-\mu)}\|u\|_{L_{p,\gamma}(a,b;E)}.$$

Theorem 1.3 ([18, Theorem 2.4])**.** *Suppose that all conditions of Theorem 1.2 are satisfied. Moreover, let* $\gamma \in A_p$, Ω *be a bounded region, and let* $A^{-1} \in \sigma_\infty(E)$. *Then the embedding*

$$W^m_{p,\gamma}\big(a,b;E(A),E\big) \subset L_{p,\gamma}(a,b;E)$$

is compact.

Consider the BVP for the ADE with constant coefficients

$$\begin{gathered}(-1)^m t u^{[2m]}(x) + (A+\lambda)u(x) = f(x), \quad x \in (0,1),\\ \sum_{i=0}^{\nu_{0k}} t^{\sigma_i}\Big[\alpha_{ki}u^{[i]}(0) + \sum_{j=1}^{N}\delta_{kij}u^{[i]}(x_{kj})\Big] = f_{0k}, \quad k = 1,2,\ldots,m,\\ \sum_{i=0}^{\nu_{1k}} t^{\sigma_i}\Big[\beta_{ki}u^{[i]}(1) + \sum_{j=1}^{N}\eta_{kij}u^{[i]}(x_{kj})\Big] = f_{1k}, \quad k = 1,2,\ldots,m,\end{gathered} \tag{1.1}$$

where $u^{[i]}(x) = (x^\gamma \frac{d}{dx})^i u(x)$, $\nu_{jk} \in \{0, 2m-1\}$; $f_{jk} \in E_k = (E(A),E)_{\theta_{jk},p}$, $\theta_{jk} = \frac{\nu_{jk}+\frac{1}{p(1-\gamma)}}{2m}$, $j = 0,1$, $\sigma_i = \frac{i}{2m} + \frac{1}{2mp}$, α_{ki}, β_{ki}, δ_{kij}, η_{kij} are complex numbers, and $x_{kj} \in (0,1)$; and A is a linear operator in a Banach space E.

In a similar way as [18, Theorem 5.1] we obtain the following.

Theorem 1.4. *Suppose that the following conditions are satisfied:*

(1) t *is a small positive parameter and* $\alpha_{k\nu_{0k}}, \beta_{k\nu_{1k}} \neq 0$;
(2) E *is a Banach space satisfying the multiplier condition with respect to* p *and weighted function* $\gamma(x) = x^\gamma$, $0 \leq \gamma < 1 - \frac{1}{p}, 1 < p < \infty$;
(3) A *is an* R*-positive operator in* E.

Then problem (1.1) has a unique solution $u \in W^{[2m]}_{p,\gamma}(0,1;E(A),E)$ *for* $f \in L_p(0,1;E)$ *and* $f_{jk} \in E_k$. *Moreover, for* $|\arg\lambda| \leq \varphi$ *and sufficiently large* $|\lambda|$, *the following uniform coercive estimate holds:*

$$\begin{aligned}&\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\frac{i}{2}} \|u^{[i]}\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)}\\ &\quad \leq C\Big[\|f\|_{L_p(0,1;E)} + \sum_{k=1}^{m}\big(\|f_{0k}\|_{E_k} + \|f_{1k}\|_{E_k}\big)\Big].\end{aligned}$$

2. Degenerate equation with variable coefficients

Consider the BVP for the parameter-dependent ADE with variable coefficients

$$\begin{aligned}(L+\lambda)u &= (-1)^m ta(x)u^{[2m]}(x) + A(x)u(x) + \sum_{i=1}^{2m-1} t^{\frac{i}{2m}} A_i(x)u^{[i]}(x) + \lambda u(x)\\ &= f(x),\\ L_{0k}u &= \sum_{i=0}^{\nu_{0k}} t^{\sigma_i}\Big[\alpha_{ki}u^{[i]}(0) + \sum_{j=1}^{N}\delta_{kij}u^{[i]}(x_{kj})\Big] = 0, \quad k=1,2,\dots,m,\\ L_{1k}u &= \sum_{i=0}^{\nu_{1k}} t^{\sigma_i}\Big[\beta_{ki}u^{[i]}(1) + \sum_{j=1}^{N}\eta_{kij}u^{[i]}(x_{kj})\Big] = 0, \quad k=1,2,\dots,m,\end{aligned} \tag{2.1}$$

where $u^{[i]} = [x^{\gamma_1}(1-x)^{\gamma_2}\frac{d}{dx}]^i u(x)$, $x\in(0,1)$, $0\le\gamma_1,\gamma_2<1$; $\sigma_i = \frac{i}{2m}+\frac{1}{2mp}$; ν_{0k}, $\nu_{1k}\in\{0,2m-1\}$; $k=1,2,\dots,m$; α_{ki}, β_{ki}, δ_{kij}, and η_{kij} are complex numbers; t is a small positive parameter; λ is a complex parameter; and $A(x)$ and $A_i(x)$ are linear operators in a Banach space E. Note that

$$\int_0^x z^{-\gamma_1}(1-z)^{-\gamma_2}\,dz<\infty.$$

A function $u\in W_{p,\gamma}^{[2m]}(0,1;E(A),E)$ satisfying the equation (2.1) a.e. on $(0,1)$ is said to be the solution of the equation (2.1) on $(0,1)$.

Remark 2.1. Let

$$y=\int_0^x z^{-\gamma_1}(1-z)^{-\gamma_2}\,dz. \tag{2.2}$$

Under the substitution (2.2), the spaces $L_p(0,1;E)$ and $W_{p,\gamma}^{[2m]}(0,1;E(A),E)$ are mapped isomorphically onto the weighted spaces

$$L_{p,\tilde\gamma}(0,b;E), \qquad W_{p,\tilde\gamma}^{2}\big(0,b;E(A),E\big),$$

respectively, where

$$\tilde\gamma=\tilde\gamma\big(x(y)\big), \qquad b=\int_0^1 z^{-\gamma_1}(1-z)^{-\gamma_2}\,dz.$$

Under the substitution (2.2), the problem (2.1) is transformed into the following nondegenerate problem:

$$(-1)^m ta(y)u^{(2m)}(y) + A(y)u(y) + \sum_{i=1}^{2m-1} t^{\frac{i}{2m}} A_i(y)u^{(i)}(y) + \lambda u(y) = f(y), \tag{2.3}$$

$$L_{0k}u = \sum_{i=0}^{\nu_{0k}} t^{\sigma_i}\Big[\alpha_{ki}u^{(i)}(0) + \sum_{j=1}^{N}\delta_{kij}u^{(i)}(y_{kj})\Big] = 0, \quad k=1,2,\dots,m, \tag{2.4}$$

$$L_{1k}u = \sum_{i=0}^{\nu_{1k}} t^{\sigma_i}\Big[\beta_{ki}u^{(i)}(b) + \sum_{j=1}^{N}\eta_{kij}u^{(i)}(y_{kj})\Big] = 0, \quad k=1,2,\dots,m$$

considered in the weighted space $L_{p,\gamma}(0,b;E)$, where y, $\tilde{a}(y)=a(x(y))$, $\tilde{A}(y)=A(x(y))$, $\tilde{A}_i(y)=A_i(x(y))$, and $\tilde{\gamma}(y)=\gamma(x(y))$ will be denoted by $a(y)$, $A(y)$, $A_i(y)$, and γ, respectively.

3. Degenerate ADEs with parameter

Consider the problem (2.3)–(2.4). Let $X=L_{p,\gamma}(0,b;E)$ and

$$Y=W^{2m}_{p,\gamma}\big(0,b;E(A),E\big).$$

By $(E_1,E_2)_{\theta,p}$, $0<\theta<1, 1\le p\le\infty$, we will denote the interpolation spaces obtained from $\{E_1,E_2\}$ by the K-method (see [22, Section 1.3.2]).

Theorem 3.1. *Assume that the following conditions are satisfied:*

(1) $\alpha_{k\nu_{0k}}$, $\beta_{k\nu_{1k}}\neq 0$ *and* $a(y)$ *is a positive continuous function on* $[0,b]$.
(2) E *is a Banach space satisfying the multiplier condition with respect to* p *and weighted function* $\gamma(y)=y^{\gamma_1}(b-y)^{\gamma_2}$, $0\le\gamma_1,\gamma_2<1-\frac{1}{p}, 1<p<\infty$.
(3) $A(y)$ *is an R-positive operator in* E *uniformly with respect to* $y\in[0,b]$ *and* $A(y)A^{-1}(y_0)\in C([0,b];B(E))$, $y_0\in(0,b)$.
(4) *For any* $\varepsilon>0$ *there is a positive constant* $C(\varepsilon)$ *such that* $\|A_i(y)u\|\le\varepsilon\|u\|_{(E(A),E)_{\frac{i}{2m},\infty}}+C(\varepsilon)\|u\|$ *for* $u\in(E(A),E)_{\frac{i}{2m},\infty}$.

Then, the problem (2.3)–(2.4) has a unique solution $u\in Y$ *for* $f\in X$ *and* $|\arg\lambda|\le\varphi$ *with sufficiently large* $|\lambda|$. *Moreover, the uniform coercive estimate holds:*

$$\sum_{i=0}^{2m}|\lambda|^{1-\frac{i}{2m}}t^{\frac{i}{2m}}\|u^{(i)}\|_{p,\gamma}+\|Au\|_{p,\gamma}\le C\|f\|_{p,\gamma}. \tag{3.1}$$

Proof. First, we will show the uniqueness of the solution. Let $G_1,G_2,\dots,G_N$ be regions in $\mathbb{R}$, and let $\varphi_1,\varphi_2,\dots,\varphi_N$ correspond to a partition of a unit in which the functions φ_j are smooth on R, $\operatorname{supp}\varphi_j\subset G_j$, and $\sum_{j=1}^N\varphi_j(y)=1$. Then, for all $u\in Y$, we have $u(y)=\sum_{j=1}^N u_j(y)$, where $u_j(y)=u(y)\varphi_j(y)$. Let $u\in Y$ be a solution of (2.3)–(2.4). Then from (2.3)–(2.4) we obtain

$$(L+\lambda)u_j=(-1)^m tau_j^{(2m)}+(A+\lambda)u_j=f_j, \tag{3.2}$$

where

$$\begin{gathered} f_j=f\varphi_j+(-1)^m ta\sum_{\nu=0}^{2m-1}C_{2m}u^{(\nu)}\varphi_j^{(2m-\nu)}+\sum_{i=1}^{2m-1}\sum_{\nu=0}^{i}t^{\frac{i}{2m}}C_i^{\nu}u^{(\nu)}\varphi_j^{(i-\nu)},\\ L_{0k}u_j=0,\qquad L_{1k}u_j=0,\quad j=1,2,\dots,N, k=1,2,\dots,m. \end{gathered} \tag{3.3}$$

By freezing coefficients in (3.2) we obtain

$$(-1)^m ta(y_{0j})u_j^{(2m)}(y)+\big(A(y_{0j})+\lambda\big)u_j(y)=F_j(y), \tag{3.4}$$

where

$$F_j=f_j+\big[A(y_{0j})-A(y)\big]u_j+\big[a(y)-a(y_{0j})\big]u_j^{(2m)}. \tag{3.5}$$

Since functions $u_j(x)$ have compact supports, by extending $u_j(x)$ on the outsides of $\operatorname{supp}\varphi_j$ we obtain BVPs for ADEs with constant coefficients:

$$\begin{gathered}(-1)^m ta(y_{0j})u_j^{(2m)} + \big(A(y_{0j}) + \lambda\big)u_j = F_j,\\ L_{ik}u_j = 0, \quad i = 0,1, k = 1,2,\dots,m.\end{gathered} \tag{3.6}$$

Let $\|\cdot\|_{G_j,p,\gamma_k}$ denote E-valued weighted L_p-norms with respect to weighted functions x^{γ_k} on G_j. Let φ_j be such that $0 \in \operatorname{supp}\varphi_j$. Then, by virtue of Theorem 1.4, we obtain that problem (3.6) has a unique solution u_j and that the coercive uniform estimates hold:

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\frac{i}{2m}} \|u_j^{(i)}\|_{G_j,p,\gamma_1} + \|Au_j\|_{G_j,p,\gamma_1} \le C\|F_j\|_{G_j,p,\gamma_1}. \tag{3.7}$$

In a similar way, Theorem 1.4 implies the following estimates:

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\frac{i}{2m}} \|u_j^{(i)}\|_{G_j,p,\gamma_2} + \|Au_j\|_{G_j,p,\gamma_2} \le C\|F_j\|_{G_j,p,\gamma_2} \tag{3.8}$$

on domains G_j adjoin the boundary point b. Similarly, these estimates are derived for domains G_j, which do not intersect with boundary points. Hence, by using the properties of the smoothness of coefficients of equations (3.3) and (3.5) and by choosing diameters of $\operatorname{supp}\varphi_j$ sufficiently small, we get

$$\|F_j\|_{G_j,p,\gamma} \le \varepsilon\|u_j\|_{W^{2m}_{p,\gamma,t}(G_j;E(A),E)} + C(\varepsilon)\|f_j\|_{G_j,p,\gamma}, \quad j = 1,2,\dots,N \tag{3.9}$$

for the sufficiently small positive ε and the continuous function $C(\varepsilon)$. Consequently, from (3.7) and (3.8) and Theorem 1.2 we get

$$\begin{aligned}\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\frac{i}{2m}} \|u_j^{(i)}\|_{G_j,p,\gamma} &+ \|Au_j\|_{G_j,p,\gamma}\\ &\le C\|f\|_{G_j,p,\gamma} + \varepsilon\|u_j\|_{W^2_{p,\gamma}} + C(\varepsilon)\|u_j\|_{G_j,p,\gamma}.\end{aligned}$$

Choosing $\varepsilon < 1$ from the above inequality, we have

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\frac{i}{2m}} \|u_j^{(i)}\|_{G_j,p,\gamma} + \|Au_j\|_{G_j,p,\gamma} \le C\big[\|f\|_{G_j,p,\gamma} + \|u_j\|_{G_j,p,\gamma}\big]. \tag{3.10}$$

It is clear that $\gamma(x) \sim x^{\gamma_1}$ on domains G_j in neighborhoods of 0, $\gamma(x) \sim (1-x)^{\gamma_2}$ on domains G_j around the point b and that $\gamma(x)$ is equivalent to the constant for others domains G_j. Then using the equality $u(y) = \sum_{j=1}^N u_j(y)$ and the estimate (3.10) for $u \in Y$ we have

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\frac{i}{2m}} \|u^{(i)}\|_{p,\gamma} + \|Au_j\|_{p,\gamma} \le C\big[\big\|(L+\lambda)u\big\|_{p,\gamma} + \|u\|_{p,\gamma}\big]. \tag{3.11}$$

Let $u \in Y$ be the solution of problem (2.3)–(2.4). Then, for $|\arg\lambda| \le \varphi$, we have

$$\|u\|_X = \big\|(L+\lambda)u - Lu\big\|_X \le \frac{1}{\lambda}\big[\big\|(L+\lambda)u\big\|_X + \|u\|_Y\big]. \tag{3.12}$$

Then by Theorem 1.2 and by virtue of (3.10) and (3.12) for sufficiently large $|\lambda|$, we have

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\frac{i}{2m}} \|u^{(i)}\|_X + \|Au\|_X \leq C\big\|(L+\lambda)u\big\|_X. \tag{3.13}$$

Consider the operator O in $L_{p,\gamma}(0,b;E)$ generated by problem (2.3)–(2.4) for $\lambda = 0$, that is,

$$\begin{aligned} D(O) &= W_{p,\gamma}^{2m}\big(0,b;E(A),E,L_{ik}\big), \\ Ou &= (-1)^m ta(x)u^{[2m]}(x) + A(x)u(x) + \sum_{i=1}^{2m-1} t^{\frac{i}{2m}} A_i(x)u^{[i]}(x). \end{aligned}$$

The estimate (3.13) implies that problem (2.3)–(2.4) has only one unique solution and that the operator O_λ has an invertible operator in its rank space. We need to show that this rank space coincides with the space $L_{p,\gamma}(0,b;E)$. We consider the smooth functions $g_j = g_j(y)$ with respect to the partition of the unique $\varphi_j = \varphi_j(y)$ on $(0,b)$ that equal one on $\operatorname{supp}\varphi_j$, where $\operatorname{supp} g_j \subset G_j$ and $|g_j(y)| < 1$. Let us construct the function u_j for all j that are defined on $\Omega_j = (0,b) \cap G_j$ and satisfying problem (2.3)–(2.4). Problem (2.3)–(2.4) can be expressed as

$$\begin{aligned} &(-1)^m ta(y_{0j})u_j^{(2m)} + \big(A(y_{0j}) + \lambda\big)u_j \\ &\quad = g_j\{F_j + \big[A(y_{0j}) - A(y)\big]u_j + \big[a(y) - a(y_{0j})\big]u_j\}, \\ &L_{ik}u_j = 0, \quad j = 1,2,\ldots,N, i = 0,1. \end{aligned} \tag{3.14}$$

Consider operators $O_{j\lambda t}$ in $L_{p,\gamma}(G_j;E)$ generated by BVPs (3.14). By virtue of Theorem 1.2 for $f \in L_{p,\gamma}(G_j;E)$, $|\arg\lambda| \leq \varphi$, and sufficiently large $|\lambda|$, we have

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\frac{i}{2m}} \left\| \frac{d^i}{dy^i} O_{j\lambda t}^{-1} f \right\|_{p,\gamma} + \|AO_{j\lambda t}^{-1} f\|_{p,\gamma} \leq C\|f\|_{p,\gamma}. \tag{3.15}$$

Extending u_j to zero on the outside of $\operatorname{supp}\varphi_j$ and passing substitutions $u_j = O_{j\lambda}^{-1}\upsilon_j$ in (3.15), we obtain operator equations with respect to υ_j:

$$\upsilon_j = K_{j\lambda t}\upsilon_j + g_j f, \quad j = 1,2,\ldots,N. \tag{3.16}$$

By virtue of Theorem 1.2 and estimate (3.15), in view of the smoothness of the coefficients of the expression $K_{j\lambda t}$, for sufficiently large $|\lambda|$ we have $\|K_{j\lambda t}\| < \varepsilon$, where ε is sufficiently small. Consequently, equations (3.16) have unique solutions $\upsilon_j = [I - K_{j\lambda t}]^{-1} g_j f$. Moreover,

$$\|\upsilon_j\|_{p,\gamma} = \big\|[I - K_{j\lambda t}]^{-1} g_j f\big\|_{p,\gamma} \leq \|f\|_{p,\gamma}.$$

Hence, $[I - K_{j\lambda t}]^{-1} g_j$ are bounded linear operators from X to $L_{p,\gamma}(G_j;E)$. Thus, we obtain that

$$u_j = U_{j\lambda t} f = O_{j\lambda t}^{-1}[I - K_{j\lambda t}]^{-1} g_j f$$

are solutions of (3.16). Consider the linear operator $(U_t + \lambda)$ in X such that

$$(U_t + \lambda)f = \sum_{j=1}^{N} \varphi_j(y) U_{j\lambda t} f.$$

It is clear from the constructions $U_{jt}$and the estimate (3.15) that operators $U_{j\lambda t}$ are bounded linear from X to Y and

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\frac{i}{2m}} \left\| \frac{d^i}{d^i y} U_{j\lambda t}^{-1} f \right\|_{p,\gamma} + \|A U_{j\lambda t}^{-1} f\|_{p,\gamma} \leq C \|f\|_{p,\gamma}. \tag{3.17}$$

Therefore, $(U_t + \lambda)$ is a bounded linear operator from $L_{p,\gamma}$ to $L_{p,\gamma}$. Let $\mathbf{L}_t$ denote the operator in X generated by BVPs (3.4)–(3.5). The act of $(\mathbf{L}_t + \lambda)$ to $u = \sum_{j=1}^{N} \varphi_j U_{j\lambda t} f$ gives $(\mathbf{L}_t + \lambda)u = f + \sum_{j=1}^{N} \Phi_{j\lambda t} f$, where the $\Phi_{j\lambda t}$ are linear combinations of $U_{j\lambda t}$ and $\frac{d}{dy} U_{j\lambda t}$. By virtue of embedding Theorem 1.2 and the estimate (3.17) and from the expression $\Phi_{j\lambda t}$, we obtain that operators $\Phi_{j\lambda t}$ are bounded linear from X to $L_{p,\gamma}(G_j; E)$ and that $\|\Phi_{j\lambda t}\| < \varepsilon$. Therefore, there exists a bounded linear invertible operator $(I + \sum_{j=1}^{N} \Phi_{j\lambda t})^{-1}$. Hence, we obtain that the BVP (2.3)–(2.4) for $f \in X$ has a unique solution

$$\begin{aligned} u(y) = (\mathbf{L}_t + \lambda)^{-1} f &= (U_t + \lambda)\Big(I + \sum_{j=1}^{N} \Phi_{j\lambda t}\Big)^{-1} f \\ &= \sum_{j=1}^{N} \varphi_j(y) O_{j\lambda t}^{-1} [I - K_{j\lambda t}]^{-1} \Big(I + \sum_{j=1}^{N} \Phi_{j\lambda t}\Big)^{-1} f. \end{aligned} \tag{3.18}$$

Then by using the above representation and in view of Theorem 1.2 we obtain the assertion. □

Conclusion 3.2. Theorem 3.1 implies that the differential operator $\mathbf{L}_t$ has a resolvent $(\mathbf{L}_t + \lambda)^{-1}$ for $|\arg \lambda| \leq \varphi$, and the estimate holds:

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\frac{i}{2m}} \left\| D^i (\mathbf{L}_t + \lambda)^{-1} \right\|_{B(X)} + \left\| A(\mathbf{L}_t + \lambda)^{-1} \right\|_{B(X)} \leq C.$$

Theorem 3.3. *Let all conditions of Theorem 3.1 hold, and let $A^{-1} \in \sigma_\infty(E)$. Then the operator $\mathbf{L}_t$ is Fredholm from Y into X.*

Proof. Theorem 3.1 implies that the operator $\mathbf{L}_t + \lambda$ has a bounded inverse, say $(\mathbf{L}_t + \lambda)^{-1}$, from X to Y for sufficiently large $|\lambda|$; that is, the operator $\mathbf{L}_t + \lambda$ is Fredholm from Y into X. Moreover, by virtue of Theorem 1.3, the embedding $Y \subset X$ is compact. Then we obtain that the operator $\mathbf{L}_t$ is Fredholm from Y into X. □

Let $\mathbf{G}_t$ denote the operator in $L_p(0, 1; E)$ generated by BVP (2.1) for $\lambda = 0$. By virtue of Theorem 3.1 and Remark 2.1 we obtain the following.

Conclusion 3.4. Let all conditions of Theorem 3.1 be satisfied. Then we have the following:

(a) Problem (2.1) has a unique solution $u \in W_{p,\gamma}^{[2m]}(0,1;E(A),E)$ for $f \in L_p(0,1;E)$, $|\arg\lambda| \leq \varphi$, and sufficiently large $|\lambda|$. Moreover, the uniform coercive estimate holds:

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\frac{i}{2m}} \|u^{[i]}\|_p + \|Au\|_p \leq C\|f\|_p.$$

(b) The operator $\mathbf{G}_t$ is Fredholm from $W_{p,\gamma}^{[2m]}(0,1;E(A),E)$ into $L_p(0,1;E)$.
(c) $\mathbf{G}_t$ has a resolvent operator $(\mathbf{G}_t+\lambda)^{-1}$ for $|\arg\lambda| \leq \varphi$, and

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} t^{\frac{i}{2m}} \left\|D^{[i]}(\mathbf{G}_t+\lambda)^{-1}\right\|_{B(L_p(0,1;E))} + \left\|A(\mathbf{G}_t+\lambda)^{-1}\right\|_{B(L_p(0,1;E))} \leq C.$$

4. R-positive properties of the degenerate ADE with parameter

Conclusion 3.4 implies that the operator $\mathbf{G}_t$ is positive in $L_p(0,1;E)$. In the following theorem we prove that this operator is R-positive in $L_p(0,1;E)$.

Theorem 4.1. *Let all conditions of Theorem 1.4 be satisfied. Then the operator $\mathbf{G}_t$ is R-positive in $L_p(0,1;E)$.*

Proof. First consider the BVP with constant coefficients

$$(-1)^m t u^{(2m)}(x) + (A+\lambda)u(x) = f(x), \quad x \in (0,1), \tag{4.1}$$

$$L_{ik}u = 0, \quad i = 0,1, k = 1,2,\ldots,m, \tag{4.2}$$

where boundary conditions $L_{ik}u$ are defined as in (2.4) and A is a linear operator in E. Let B_t denote the operator in $X = L_{p,\gamma}(0,b;E)$ generated by problem (4.1)–(4.2), where $\gamma(x) = x^{\frac{\gamma}{1-\gamma}}$. Since A is a positive operator in E, in view of [8, Lemma 2.6] there exists the semigroup $U_{j\lambda t}(x) = e^{x\omega_j t^{-\frac{1}{2m}} A_\lambda^{\frac{1}{2m}}}$, $Re\omega_j < 0$, $j = 1,2,\ldots,m$, $U_{j\lambda t}(x) = e^{-(b-x)\omega_j t^{-\frac{1}{2m}} A_\lambda^{\frac{1}{2m}}}$, $Re\omega_j > 0$, $j = m+1, m+2, \ldots, 2m$, that is holomorphic for $x > 0$ and strongly continuous for $x \geq 0$, where ω_j are roots of the equation $(-1)^m\omega^{2m} + 1 = 0$. By using a technique similar to that applied in [24, Lemma 5. 3. 2(1)], we have that for $f \in D(0,b;E(A))$ the solution of the equation (4.1) is represented as

$$u(x) = \sum_{j=1}^{2m} U_{j\lambda t}(x) g_k + \int_0^b U_{0\lambda t}(x-y) f(y)\,dy, \quad g_k \in E, \tag{4.3}$$

where

$$U_{0\lambda t}(x-y) = \begin{cases} -t^{-\frac{1}{2m}} \sum_{k=1}^m A_\lambda^{-\frac{1}{2m}} U_{k\lambda t}(x-y), & x \geq y, \\ t^{-\frac{1}{2m}} \sum_{k=1}^m A_\lambda^{-\frac{1}{2m}} U_{k\lambda t}(y-x), & x \leq y. \end{cases}$$

By taking into account the boundary conditions (4.2), we obtain the following equation with respect to $g_1, g_2, \ldots, g_{2m}$:

$$\sum_{k=1}^{2m} L_k(U_{j\lambda t}) g_1 = L_k(\Phi_\lambda), \quad j = 1,2,\ldots,2m, \Phi_\lambda = \int_0^b U_{0\lambda t}(x-y) f(y)\,dy.$$

By solving the above system and substituting it into (4.3), we obtain the representation of the solution for problem (4.1)–(4.2):

$$u(x) = \left[B(t) + \lambda\right]^{-1} f = \int_0^b G_t(\lambda, x, y) f(y)\, dy,$$
$$G_t(\lambda, x, y) = \sum_{k=1}^{2m} \sum_{j=1}^{2m} t^{\frac{1}{2m}} A_\lambda^{-\frac{1}{2m}} B_{kjt}(\lambda) U_{j\lambda t}(x) \tilde{U}_{kj\lambda t}(x-y) + U_{0\lambda t}(x-y), \tag{4.4}$$

where $B_{kjt}(\lambda)$ are uniformly bounded operators in E and

$$\tilde{U}_{kj\lambda t}(x-y) = \begin{cases} b_{kj} U_{k\lambda t}(x-y), & x \geq y, \\ \beta_{kj} U_{k\lambda t}(y-x), & x \leq y, \end{cases} \quad b_{kj}, \beta_{kj} \in \mathbf{C}.$$

Let us to show that the set $\{\Phi_t(\lambda, x, y); \lambda \in S(\varphi)\}$ is uniformly R-bounded. By using the generalized Minkowski and Young inequalities, and by semigroup estimates from (4.3), we have the uniform estimate

$$\begin{aligned} \left\|\Phi_t(\lambda, x, y) f\right\|_X &\leq C \sum_{k=1}^{2m} \sum_{j=1}^{2m} \{\|A_\lambda^{-\frac{1}{2m}}\| \left\|B_{kjt}(\lambda)\right\| \left\|\tilde{U}_{kj\lambda t}(x) f\right\|_X + \left\|U_{0\lambda t}(x) f\right\|_X\} \\ &\leq C\|f\|_X. \end{aligned}$$

Due to the R-positivity of A and the uniform boundedness of operators $B_{kjt}(\lambda)$, in view of Kahane's contraction principle, and from the product properties of the collection of R-bounded operators (see [8, Lemma 3.5, Proposition 3.4]), we get that the sets

$$\begin{aligned} b_{kjt}(\lambda, x, y) &= \left\{B_{kjt}(\lambda) A_\lambda^{-\frac{1}{2m}} U_{j\lambda t}(x) \left[U_{k\lambda t}(1-y) + U_{\nu\lambda t}(y)\right] : \lambda \in S_\varphi\right\}, \\ b_{0t}(\lambda, x, y) &= \left\{U_{0\lambda t}(x-y) : \lambda \in S_\varphi\right\} \end{aligned}$$

are uniformly R-bounded. Then, by using Kahane's contraction principle and product and additional properties of the collection of R-bounded operators, and in view of the R-boundedness of the sets b_{kj}, b_0, for all $u_1, u_2, \ldots, u_\mu \in F$, $\lambda_1, \lambda_2, \ldots, \lambda_\mu \in S(\varphi)$, and independent symmetric $\{-1, 1\}$-valued random variables $r_i(y)$, $i = 1, 2, \ldots, \mu$, $\mu \in N$, we have the uniform estimate

$$\begin{aligned} &\int_\Omega \left\|\sum_{i=1}^{\mu} r_i(y) \Phi_t(\lambda_i, x, y) u_i\right\|_X d\tau \\ &\leq C \sum_{k,j=1}^{2m} \int_\Omega \left\|\sum_{i=1}^{\mu} r_i(y) b_{kjt}(\lambda_i, x, y) u_i\right\|_X d\tau + \int_\Omega \left\|\sum_{i=1}^{\mu} r_i(y) b_{0t}(\lambda_i, x, y) u_i\right\|_X d\tau \\ &\leq C e^{\beta|\lambda t^{-1}|^{\frac{1}{2}}|x-y|} \int_\Omega \left\|\sum_{i=1}^{\mu} r_i(y) u_i\right\|_X d\tau. \end{aligned}$$

This implies that

$$R\left\{\Phi_t(\lambda, x, y) : \lambda \in S_\varphi\right\} \leq C e^{\beta|\lambda t^{-1}|^{\frac{1}{2m}}|x-y|}, \quad \beta < 0, x, y \in (0, b).$$

By applying the R-boundedness property of kernel operators (see, e.g., [8, Proposition 4.12]), and due to the density of $D(0,b;E(A))$ in X (see, e.g., [14, Section 2.2]), we get that the operator B_t is uniformly R-positive in X. From the representation solution (4.4) of problem (4.1)–(4.2), and in view of Remark 2.1, it is easy to see that the operator $(\mathbf{L}_t+\lambda)^{-1}$ can be expressed as a linear combination of operators $O^{-1}_{j\lambda t}$ like $(B_t+\lambda)^{-1}$. The hypothesis is therefore validated by the representation (4.4) and by virtue of Kahane's contraction principle, by the product and additional properties of the collection of R-bounded operators, and by Remark 2.1. ☐

5. The Cauchy problem for degenerate abstract parabolic equations with parameter

Consider the mixed problem for the parabolic ADE with parameter:

$$\begin{gathered}\frac{\partial u(x,y)}{\partial y}+(-1)^m ta(x)u_x^{[2m]}(x,y)+\big[A(x)+d\big]u(x,y)=f(x,y),\\ \sum_{i=0}^{\nu_{0k}} t^{\sigma_i}\Big[\alpha_{ki}u^{[i]}(0,y)+\sum_{j=1}^{N}\delta_{kij}u^{[i]}(x_{kj},y)\Big]=0,\quad k=1,2,\dots,m,\\ \sum_{i=0}^{\nu_{1k}} t^{\sigma_i}\Big[\beta_{ki}u^{[i]}(1,y)+\sum_{j=1}^{N}\eta_{kij}u^{[i]}(x_{kj},y)\Big]=0,\quad k=1,2,\dots,m,\\ u(x,0)=0,\quad x\in(0,1), y\in R_+.\end{gathered}\tag{5.1}$$

Here, α_{ki}, β_{ki}, δ_{kij}, η_{kij} are complex numbers; a is a complex-valued function on $(0,1)$, ν_{0k}, $\nu_{1k}\in\{0,2m-1\}$, $d>0$; $A(x)$ is a linear operator in a Banach space E for $x\in(0,1)$; $\sigma_i=\frac{i}{2m}+\frac{1}{2mp}$; and $0\le\gamma_k<1$.

For $\mathbf{p}=(p,p_1)$, $\Delta_+=R_+\times(0,1)$, $L_{\mathbf{p},\boldsymbol{\gamma}}(\Delta_+;E)$ denotes the space of all E-valued weighted $\mathbf{p}$-summable functions with mixed norm (see, e.g., [6, Section 8]), that is, the space of all measurable functions f defined on Δ_+ for which

$$\|f\|_{L_{\mathbf{p},\boldsymbol{\gamma}}(\Delta_+)}=\Big(\int_{R_+}\Big(\int_0^1\big\|f(x,y)\big\|^p\gamma(x)\,dx\Big)^{\frac{p_1}{p}}\,dy\Big)^{\frac{1}{p_1}}<\infty.$$

Analogously, $W^m_{\mathbf{p},\boldsymbol{\gamma}}(\Delta_+,E(A),E)$ denotes the Sobolev space with corresponding mixed norm (see [6, Section 8] for the scalar case).

In this section, we prove the following result.

Theorem 5.1. *Assume that all conditions of Theorem 3.1 are satisfied for $\varphi>\frac{\pi}{2}$. Then, for $f\in L_{\mathbf{p}}(\Delta_+;E)$ and sufficiently large $d>0$, problem (5.1) has a unique solution belonging to $W^{1,[2m]}_{\mathbf{p},\gamma}(\Delta_+;E(A),E)$ and the following coercive estimate holds:*

$$\Big\|\frac{\partial u}{\partial y}\Big\|_{L_{\mathbf{p}}(\Delta_+;E)}+t\|D_x^{[2m]}u\|_{L_{\mathbf{p}}(\Delta_+;E)}+\|Au\|_{L_{\mathbf{p}}(\Delta_+;E)}\le C\|f\|_{L_{\mathbf{p}}(\Delta_+;E)}.$$

Proof. The problem (5.1) can be express as the following Cauchy problem:

$$\frac{du}{dy} + (\mathbf{G}_t + d)u = f, \quad u(0) = 0. \tag{5.2}$$

Here $\mathbf{G}_t$ denotes the operator generated by (5.2) for $\lambda = 0$. Theorem 4.1 implies that the operator $\mathbf{G}_t$ is R-positive in $F = L_p(0,1;E)$. By virtue of [22, Section 1.14], G_t is a generator of an analytic semigroup in F. Then applying [23, Theorem 4.2] we obtain that, for $f \in L_{p_1}(R_+;F)$ and sufficiently large $d > 0$, problem (5.2) has a unique solution belonging to $W^1_{p_1,\gamma}(R_+;D(\mathbf{G}_t),F)$ and the estimate holds:

$$\Big\|\frac{du}{dy}\Big\|_{L_{p_1}(R_+;F)} + \big\|(\mathbf{G}_t + a)u\big\|_{L_{p_1}(R_+;F)} \le C\|f\|_{L_{p_1}(R_+;F)}.$$

Since $L_{p_1}(R_+;F) = L_{\mathbf{p}}(\Delta_+;E)$ by Theorem 3.1, we have

$$\big\|(\mathbf{G}_t + d)u\big\|_F = \|u\|_{W^{[2m]}_{p,\gamma}(0,1;E(A),E)}.$$

These relations and the above estimate prove the hypothesis to be true. □

6. Degenerate ADEs in moving domains

Consider at first the inhomogeneous BVP for an ADE with constant coefficients on a moving domain $(0,b(s))$:

$$\begin{gathered}(-1)^m u^{[2m]}(x) + (A+\lambda)u(x) = f(x),\\ \sum_{i=0}^{\nu_{0k}} t^{\sigma_i}\Big[\alpha_{ki}u^{[i]}(0) + \sum_{j=1}^{N}\delta_{kij}u^{[i]}(x_{kj})\Big] = f_k, \quad k = 1,2,\dots,m,\\ \sum_{i=0}^{\nu_{1k}} t^{\sigma_i}\Big[\beta_{ki}u^{[i]}(b) + \sum_{j=1}^{N}\eta_{kij}u^{[i]}(x_{kj})\Big] = f_k, \quad k = 1,2,\dots,m,\end{gathered} \tag{6.1}$$

where $u^{[i]} = (x^\gamma \frac{d}{dx})^i u$, $\alpha_{ki}, \beta_{ki}, \delta_{kij}, \eta_{kij}$ are complex numbers, $x_{kj} \in (0,b(s))$; A is a linear operator in E; t is positive; and λ is a complex parameter,

$$\sigma_i = \frac{i}{2m} + \frac{1}{2mp(1-\gamma)}.$$

In a similar way as Theorem 1.4 we obtain the following.

Conclusion 6.1. Let all conditions of Theorem 1.4 be satisfied for $t = 1$, and let $b = b(s)$ be a continuous function on $[c,d]$. Then problem (6.1) for all $f \in L_p(0,b;E)$, $f \in E_k$, has a unique solution $u \in W^{[2m]}_{p,\gamma}(0,b;E(A),E)$. Moreover, for $|\arg\lambda| \le \varphi$ and sufficiently large $|\lambda|$, the following uniform coercive estimate holds:

$$\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}}\|u^{[i]}\|_{L_p(0,b;E)} + \|Au\|_{L_p(0,b;E)} \le C\Big[\|f\|_{L_p(0,b;E)} + \sum_{k=1}^{m}\|f_k\|_{E_k}\Big].$$

Consider the BVP (2.1) in the moving domain $(0, b(s))$; that is,

$$
\begin{gathered}
(-1)^m t a(x) u^{[2m]}(x) + A(x)u(x) + \sum_{i=1}^{2m-1} t^{\frac{i}{2m}} A_i(x) u^{[i]}(x) = f(x), \\
L_{0k}u = 0, \qquad L_{1k}u = 0, \quad k = 1, 2, \ldots, m,
\end{gathered}
\tag{6.2}
$$

where

$$
u^{[i]} = \left[x^{\gamma_1}(b-x)^{\gamma_2} \frac{d}{dx}\right]^i u(x), \quad 0 \le \gamma_1, \gamma_2 < 1.
$$

Then Theorem 3.1 implies the following.

Conclusion 6.2. Assume that all conditions of Theorem 3.1 are satisfied and that $b = b(s)$ is a continuous function on $[c, d]$. Then problem (6.2) has a unique solution $u \in W_{p,\gamma}^{[2m]}(0, b; E(A), E)$ for $f \in L_p(0, b(s); E)$, $p \in (1, \infty)$, and $\lambda \in S_\varphi$ with sufficiently large $|\lambda|$, and the coercive uniform estimate holds:

$$
\sum_{i=0}^{2m} |\lambda|^{1-\frac{i}{2m}} \|u^{[i]}\|_{L_p(0,b;E)} + \|Au\|_{L_p(0,b;E)} \le \|f\|_{L_p(0,b;E)}.
$$

Proof. Really, under the substitution $\tau = xb^{-1}(s)$, the moving BVP (6.2) is transformed into the following problem with a parameter in the fixed domain $(0, 1)$:

$$
\begin{gathered}
(-1)^m b^{-2m}(s) a(\tau) u^{[2m]}(\tau) + A(\tau)u(\tau) + \sum_{i=1}^{2m-1} t^{\frac{i}{2m}} A_i(\tau) u^{[i]}(\tau) = f(\tau), \\
\sum_{i=0}^{\nu_{0k}} b^{-\sigma_i}(s) \Big[\alpha_{ki} u^{[i]}(0) + \sum_{j=1}^{N} \delta_{kij} u^{[i]}(x_{kj})\Big] = 0, \quad k = 1, 2, \ldots, m, \\
\sum_{i=0}^{\nu_{1k}} b^{-\sigma_i}(s) \Big[\beta_{ki} u^{[i]}(1) + \sum_{j=1}^{N} \eta_{kij} u^{[i]}(x_{kj})\Big] = 0, \quad k = 1, 2, \ldots, m.
\end{gathered}
$$

Then, by virtue of Theorem 3.1, we obtain the assertion. □

7. Nonlinear degenerate ADEs

Consider now the following nonlinear problem:

$$
\begin{aligned}
&(-1)^m a(x) u^{[2m]}(x) + O(x, u, u^{[1]}, \ldots, u^{[2m-1]}) u(x) \\
&\quad = F(x, u, u^{[1]}, \ldots, u^{[2m-1]}), \quad x \in (0, a),
\end{aligned}
\tag{7.1}
$$

$$
\sum_{i=0}^{\nu_{0k}} t^{\sigma_i} \Big[\alpha_{ki} u^{[i]}(0) + \sum_{j=1}^{N} \delta_{kij} u^{[i]}(x_{kj})\Big] = 0, \quad k = 1, 2, \ldots, m, \tag{7.2}
$$

$$
\sum_{i=0}^{\nu_{1k}} t^{\sigma_i} \Big[\beta_{ki} u^{[i]}(a) + \sum_{j=1}^{N} \eta_{kij} u^{[i]}(x_{kj})\Big] = 0, \quad k = 1, 2, \ldots, m,
$$

where $u^{[i]}(x) = (x^\gamma \frac{d}{dx})^i u(x)$, $m_k \in \{0, 2m-1\}$; $\sigma_i = \frac{i}{2m} + \frac{1}{2mp(1-\gamma)}$; α_{ki}, β_{ki}, δ_{kij}, η_{kij} are complex numbers; and $x_{kj} \in (0, 1)$, ν_{0k}, $\nu_{1k} \in \{0, 2m-1\}$.

In this section we will prove the existence and uniqueness of the maximal regular solution for the nonlinear problem (7.1)–(7.2). Let

$$U=(u_0,u_1,\dots,u_{2m-1}),\qquad X=L_p(0,a;E),\qquad Y=W_{p,\gamma}^{[2m]}\big(0,a;E(A),E\big),$$
$$E_i=\big(E(A),E\big)_{\theta i,p},\qquad \theta_i=\frac{i+\frac{1}{p(1-\gamma)}}{2m},\qquad X_0=\prod_{i=0}^{2m-1}E_i,$$

Remark 7.1. By using a result of J. Lions and I. Petree (see, e.g., [22, Section 1.8]), we obtain that the embedding $D^iY\in E_i$ is continuous and there is a constant C_1 such that for $w\in Y$, $W=\{w_i\}$, $w_i=D^iw(\cdot)$, $i=1,2,\dots,2m-1$,

$$\|u\|_{\infty,X_0}=\prod_{i=0}^{2m-1}\|D^iw\|_{C([0,a],E_j)}=\sup_{x\in[0,a]}\prod_{i=0}^{2m-1}\big\|D^iw(x)\big\|_{E_j}\le C_1\|w\|_Y.$$

For $r>0$ denote by O_r the closed ball in X_0 of radios r; that is,

$$O_r=\big\{u\in X_0,\|u\|_{X_0}\le r\big\}.$$

Consider the linear problem

$$\begin{gathered}(-1)^mw^{[2m]}(x)+\big(A(x)+d\big)w(x)=f(x),\\ L_{ik}w=0,\quad i=0,1,k=1,2,\dots,m,\end{gathered}\tag{7.3}$$

where $A(x)$ is a linear operator in a Banach space E for $x\in(0,a)$, L_{ik} are boundary conditions defined by (7.2), and $d>0$.

Assume that E is a Banach space satisfying the multiplier condition with respect to p and the weighted function $x^{\frac{\gamma}{1-\gamma}}$, $0\le\gamma<1-\frac{1}{p}$, $p\in(1,\infty)$, and assume that $A(x)$ is uniformly R-positive in E. By virtue of Conclusion 6.2, the problem (7.3) is maximal regular in X uniformly with respect to $a\in(0,a_0]$; that is, there is a unique solution $w\in Y$ of the problem (7.3) for all $f\in X$ and for sufficiently large $d>0$. Moreover, it has the following coercive estimate:

$$\|w\|_Y\le C_0\|f\|_X,$$

where the constant C_0 is independent of $f\in X$ and $a\in(0,a_0]$.

Condition 1. Assume that the following are satisfied:

(1) $\alpha_{k\nu_{0k}}$, $\beta_{k\nu_{1k}}\ne 0$, $a(x)$ is a positive continuous function on $[0,a]$;
(2) E is a Banach space satisfying the multiplier condition with respect to p and weighted function $x^{\frac{\gamma}{1-\gamma}}$, $0\le\gamma<1-\frac{1}{p}$, $p\in(1,\infty)$;
(3) $F:[0,a]\times X_0\to E$ is a measurable function for each $\upsilon_i\in E_i$, $i=0,1,\dots,2m-1$; $F(x,u_0,u_1,\dots,u_{2m-1})$ is continuous with respect to $x\in[0,a]$ and $f(x)=F(x,0)\in X$. Moreover, for each $r>0$ there exists the positive functions $h_k(x)$ such that

$$\begin{gathered}\big\|F(x,U)\big\|_E\le h_1(x)\|U\|_{X_0},\\ \big\|F(x,U)-F(x,\bar U)\big\|_E\le h_2(x)\|U-\bar U\|_{X_0},\end{gathered}$$

where $h_k\in L_p(0,a)$ with

$$\|h_k\|_{L_p(0,a)}<C_0^{-1},\quad k=1,2;$$

and $U = \{u_0, u_1, \dots, u_{2m} - 1\}$, $\bar{U} = \{\bar{u}_0, \bar{u}_1, \dots, \bar{u}_{2m-1}\}$, $u_i, \bar{u}_i \in E_i$, and $U, \bar{U} \in O_r$.

(4) There exist $\Phi_i \in E_i$, such that the operator $O(x, \Phi)$ for $\Phi = \{\Phi_i\}$ is R-positive in E uniformly with respect to $x \in [0, a]$; $O(x, \Phi)O^{-1}(x^0, \Phi) \in C([0, a]; B(E))$; $O(x, 0) = A(x)$.

(5) $O(x, U)$ for $x \in (0, a)$ is a uniform positive operator in a Banach space E, where the domain definition $D(O(x, U))$ does not depend on x, U, and $O : (0, a) \times X_0 \to B(E(A), E)$ is continuous. Moreover, for each $r > 0$, there is a positive constant $L(r)$ such that $\|[O(x, U) - O(x, \bar{U})]\upsilon\|_E \leq L(r)\|U - \bar{U}\|_{X_0}\|A\upsilon\|_E$ for $x \in (0, a)$, $U, \bar{U} \in O_r$ and $\upsilon \in D(O(x, U))$.

Theorem 7.2. *Let Condition 1 hold. Then there is $a \in (0\ a_0]$ such that problem (7.1)–(7.2) has a unique solution that belongs to the space $W^{[2m]}_{p,\gamma}(0, a; E(A), E)$.*

Proof. We want to solve the problem (7.1)–(7.2) locally by means of maximal regularity of the linear problem (7.3) via the contraction mapping theorem. For this purpose, let w be a solution of the linear problem (7.3). Consider a ball

$$B_r = \big\{\upsilon \in Y, L_{ik}(\upsilon - w) = 0, \|\upsilon - w\|_Y \leq r\big\}.$$

Let $w \in Y$ be a solution of the problem (7.3), and let

$$W = \big(w(0), w^{[1]}(0), \dots, w^{[2m-1]}(0)\big).$$

Given $\upsilon \in B_r$, solve the linear problem

$$\begin{aligned} &(-1)^m a(x)u^{[2m]}(x) + A(x)u(x) + du(x) \\ &\quad = F(x, \upsilon) + \big[O(x, 0) - O(x, \upsilon)\big]\upsilon(x), \\ &L_{ik}u = 0, \quad i = 0, 1, k = 1, 2, \dots, m, \end{aligned} \tag{7.4}$$

where

$$V = (\upsilon, \upsilon^{[1]}, \dots, \upsilon^{[2m-1]}), \quad \upsilon \in Y.$$

Consider the function

$$\Phi(x) = F(x, \upsilon) + \big[O(x, 0) - O(x, \upsilon)\big]\upsilon(x).$$

First of all, we show that $\Phi \in X$ and $\|\Phi\|_X \leq C_0^{-1}r$ for

$$\upsilon \in Y, \quad \|\upsilon\|_Y \leq r.$$

Indeed, by Remark 7.1, $\upsilon \in C([0, a]; E_0)$, and one has

$$O(x, 0) - O(x, \upsilon) \in C\big([0, a]; B\big(E(A), E\big)\big).$$

Hence, by assumption (3), Φ is measurable and

$$\|\Phi\|_X \leq L(r)\|\upsilon\|_{X_0}\|A\upsilon\|_X + h_1(x)\|\upsilon\|_{X_0}.$$

Then, by using Remark 7.1, we obtain

$$\|\Phi\|_X \leq rL(r)\|\upsilon\|_X + r\|h_1\|_{L_p} \leq r^2L(r) + r\|h_1\|_{L_p} \leq r.$$

Define a map Q on B_r by $Q\upsilon = u$, where u is a solution of the problem (7.4). We want to show that $Q(B_r) \subset B_r$ and that Q is a contraction operator provided that

a is sufficiently small and r is chosen properly. For this aim, by using maximal regularity properties of the problem (7.3), we have

$$\|Q\upsilon - w\|_Y = \|u-w\|_Y \le C_0\{\big\|F(x,\upsilon)-F(x,0)\big\|_X + \big\|\big[O(x,0)-O(x,V)\big]\upsilon\big\|_X\}.$$

By assumption (3) for $\upsilon \in O_r$ we get

$$\big\|F(x,\upsilon)-F(x,0)\big\|_X \le \|h_2\|_{L_p(0,a)}\|\upsilon\|_{X_0}.$$

By assumptions (4) and (5) and Remark 7.1, for $\upsilon \in O_r$, we have

$$\begin{aligned}
&\big\|\big[O(x,0)\upsilon - O(x,V)\big]\upsilon\big\|_X \\
&\quad\le \sup_{x\in[0,a]}\{\big\|\big[O(x,0)-O(x,W)\big]\upsilon\big\|_{L(X_0,X)} \\
&\qquad + \big\|\big[O(x,W)-O(x,V)\big]\upsilon\big\|_{B(X_0,X)}\|\upsilon\|_Y\} \\
&\quad\le L(r)\big[\|W\|_{X_0}\|A\upsilon\|_X + \|\upsilon-w\|_{\infty,X_0}\big]\big[\|\upsilon-w\|_Y+\|w\|_Y\big] \\
&\quad\le rL(r)\big\{\big[\|W\|_{X_0}\|\upsilon\|_Y + C_1\|\upsilon-w\|_Y\big] + L(r)\|w\|_Y\big\}.
\end{aligned}$$

Choosing r and $a \in (0\ a_0]$ so that $\|w\|_Y < \delta_a$, by assumptions (3)–(5) we obtain from the above inequalities that

$$\|Q\upsilon - w\|_Y \le r + r^2L(r)\|W\|_{X_0} + r^2L(r)C_1 + rL(r)\|w\|_Y < r.$$

That is, the operator Q maps B_r into itself:

$$Q(B_r) \subset B_r.$$

Let $u_1 = Q(\upsilon_1)$ and $u_2 = Q(\upsilon_2)$. Then $u_1 - u_2$ is a solution of the problem

$$\begin{aligned}
&(-1)^m a(x)u^{[2m]}(x) + A(x)u(x) + du(x) \\
&\quad = F(x,\upsilon_1) - F(x,\upsilon_1) + \big[O(x,\upsilon_2)-O(x,0)\big]\big[\upsilon_1(x)-\upsilon_2(x)\big] \\
&\qquad - \big[O(x,\upsilon_1)-O(x,\upsilon_2)\big]\upsilon_1(x), \\
&\qquad\qquad L_{ik}u = 0, \quad i=0,1, k=1,2,\dots,m.
\end{aligned}$$

In a similar way, by using assumption (5), we obtain

$$\begin{aligned}
\|u_1-u_2\|_Y &\le C_0\{rL(r)\|\upsilon_1-\upsilon_2\|_X + L(r)\|\upsilon_1-\upsilon_2\|_Y\|\upsilon_1\|_X + \|h_2\|_{L_p}\|\upsilon_1-\upsilon_2\|_Y\} \\
&\le C_0\big[2rL(r) + \|h_2\|_{L_p}\big]\|\upsilon_1-\upsilon_2\|_Y.
\end{aligned}$$

Thus Q is a strict contraction. Eventually, the contraction mapping principle implies a unique fixed point of Q in B_r, which is the unique strong solution

$$u \in Y = W^{[2m]}_{p,\gamma}\big(0,a;E(A),E\big). \qquad \square$$

8. The BVP for degenerate anisotropic equations

The Fredholm property of the BVP for elliptic equations was studied in, for example, [2], [8], and [24]. Let $\Omega = (0,1)\times G$, where $G \subset R^n$, $n \geq 2$, is a bounded domain with $(n-1)$-dimensional boundary ∂G. Let us consider the BVP for the degenerate elliptic equation

$$\begin{aligned}(-1)^m ta(x)D_x^{[2m]}u(x,y) + \sum_{|\alpha|\leq 2l} b_\alpha(x)a_\alpha(y)D_y^\alpha u(x,y)\\ + \sum_{i=1}^{2m-1}\sum_{|\beta|\leq\mu_i} t^{\frac{i}{2m}}a_{i\beta}(x,y)D_x^i D_y^\beta u(x,y) + du(x,y) = f(x,y),\end{aligned} \tag{8.1}$$

$$\sum_{i=0}^{\nu_{0k}} t^{\sigma_i}\left[\alpha_{ki}u^{[i]}(0,y) + \sum_{j=1}^N \delta_{kij}u^{[i]}(x_{kj},y)\right] = 0, \quad k = 1,2,\ldots,m, \tag{8.2}$$

$$\sum_{i=0}^{\nu_{1k}} t^{\sigma_i}\left[\beta_{ki}u^{[i]}(b,y) + \sum_{j=1}^N \eta_{kij}u^{[i]}(x_{kj},y)\right] = 0, \quad k = 1,2,\ldots,m,$$

$$B_j u = \sum_{|\beta|\leq m_j} b_{j\beta}(y)D_y^\beta u(x,y) = 0, \quad x\in G, y\in\partial\Omega, j = 1,2,\ldots,l, \tag{8.3}$$

where $u^{[i]} = [x^{\gamma_1}(1-x)^{\gamma_2}\frac{d}{dx}]^i u(x)$, $0 \leq \gamma_1,\gamma_2 < 1$, $\sigma_i = \frac{i}{2m} + \frac{1}{2mp}$, ν_{0k}, $\nu_{1k} \in \{0, 2m-1\}$, $k = 1,2,\ldots,m$; α_{ki}, β_{ki}, δ_{kij}, η_{kij} are complex numbers; $d > 0$; t is a small positive parameter;

$$D_j = -i\frac{\partial}{\partial y_j}, \quad D_y = (D_1,\ldots,D_n), y = (y_1,\ldots,y_n)$$

a, a_α, b_α, $a_{i\beta}$, $b_{j\beta}$ are complex valued functions; and $\mu_i < 2l$. Let $\mathbf{p} = (p_1, p)$.

Let Q denote the differential operator in $L_{\mathbf{p}}(\Omega)$ generated by the BVP (8.1)–(8.3).

Theorem 8.1. *Let the following conditions be satisfied:*

1. $a, b_\alpha \in C([0,1])$, $a_\alpha \in C(\bar{\Omega})$ *for each* $|\alpha| = 2l$ *and* $a_\alpha \in L_\infty(\Omega)$ *for each* $|\alpha| < 2l$.
2. $b_{j\beta} \in C^{2l-m_j}(\partial\Omega)$ *for each* j, β, *and* $m_j < 2l$, $\sum_{j=1}^l b_{j\beta}(y')\sigma_j \neq 0$, *for* $|\beta| = m_j$, $y' \in \partial G$, *where* $\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_n) \in R^n$ *is a normal to* ∂G.
3. *For* $y \in \bar{\Omega}$, $\xi \in R^n$, $\lambda \in S(\varphi_0)$, $|\xi| + |\lambda| \neq 0$, *let* $\lambda + \sum_{|\alpha|=2l} a_\alpha(y)\xi^\alpha \neq 0$.
4. *For each* $y_0 \in \partial\Omega$, *a local BVP in local coordinates corresponding to* y_0,

$$\lambda + \sum_{|\alpha|=2l} a_\alpha(y_0)D^\alpha\vartheta(y) = 0,$$

$$B_{j0}\vartheta = \sum_{|\beta|=m_j} b_{j\beta}(y_0)D^\beta u(y) = h_j, \quad j = 1,2,\ldots,\mu,$$

has a unique solution $\vartheta \in C_0(R_+)$ *for all* $h = (h_1,h_2,\ldots,h_n) \in R^n$, *and for* $\xi' \in R^{n-1}$ *with*

(6) $0 \leq \gamma_k < 1 - \frac{1}{p}$, $p \in (1, \infty)$, $k = 1, 2, \ldots, 2m$, *and* $\alpha_{k\nu_{ik}} \neq 0$, $\beta_{k\nu_{ik}} \neq 0$, $i = 0, 1$, $k = 1, 2, \ldots, m$.

Then we have the following:

(a) *Problem* (8.1)–(8.3) *has a unique solution* $u \in W^{[2m],2l}_{\mathbf{p},\gamma}(\Omega)$ *for* $f \in L_{\mathbf{p}}(\Omega)$ *and sufficiently large* $d > 0$. *Moreover, the uniform coercive estimate holds:*

$$\|tD_x^{[2m]}u\|_{L_{\mathbf{p}}(\Omega)} + \sum_{|\alpha| \leq 2l} \|D^\alpha u\|_{L_{\mathbf{p}}(\Omega)} \leq C\|f\|_{L_{\mathbf{p}}(\Omega)}.$$

(b) *The operator* $u \to Qu$ *is Fredholm from* $W^{[2m],2l}_{\mathbf{p},\gamma}(\Omega)$ *into* $L_{\mathbf{p}}(\Omega)$.

(c) *The operator* Q *is* R*-positive in* $L_{\mathbf{p}}(\Omega)$.

Proof. Let us consider the operators $A(x)$ and $A_i(x)$ in $E = L_{p_1}(G)$ that are defined by the equalities

$$D(A) = \{u \in W^{2l}_{p_1}(G), B_j u = 0, j = 1, 2, \ldots, m\}, \qquad Au = \sum_{|\alpha| \leq 2l} b_\alpha a_\alpha D_y^\alpha u,$$

$$A_i u = \sum_{|\beta| \leq \mu_i} a_{i\beta} D_y^\beta u, \quad i = 0, 1, \ldots, 2m - 1.$$

Then problem (8.1)–(8.3) can be rewritten as problem (2.1), where $u(x) = u(x, \cdot)$, $f(x) = f(x, \cdot)$, $x \in (0, 1)$ are the functions with values in $E = L_{p_1}(G)$. By virtue of [3, Theorem 4.5.2] the space $E = L_{p_1}(G)$, $p_1 \in (1, \infty)$, satisfies the multiplier condition. By virtue of [8, Theorem 8.2], the operator $A + \mu$ for sufficiently large $\mu > 0$ is R-positive in L_{p_1}. Moreover, (1) and (2) imply condition (3) of Theorem 3.1; that is, conditions (1)–(3) of Theorem 3.1 are fulfilled. It is known that the embedding $W^{2l}_{p_1}(G) \subset L_{p_1}(G)$ is compact (see, e.g., [22, Section 3]). Using the interpolation properties of Sobolev spaces (see [22, Section 4]), we obtain that condition (4) of Theorem 3.1 is satisfied. Hence, all hypotheses of Theorem 3.1 are valid. Then, by using Conclusions 3.4 and 6.1, we obtain assertions (a) and (b). Then assertion (c) is obtained from Theorem 4.1. □

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