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ON UNIFORM CONNECTIVITY OF ALGEBRAIC MATRIX SETS

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Abstract. We study the uniform path connectivity of sets of matrix tuples that satisfy some additional constraints, and more specifically, given $\varepsilon > 0$, a fixed metric $\eth$ in $M_n(\mathbb{C})^m$ induced by the operator norm $\|\cdot\|$, any collection of r nonconstant polynomials $p_1(x_1,\ldots,x_m),\ldots,p_r(x_1,\ldots,x_m)$ over $\mathbb{C}$ with finite zero set $\mathbf{Z}(p_1,\ldots,p_r) \subset \mathbb{C}^m$ and any m-tuple $\mathbf{X} = (X_1,\ldots,X_m)$ in the set $\mathbb{ZD}_n^m(p_1,\ldots,p_r) \subseteq M_n^m(\mathbb{C})$ of commuting normal matrix contractions such that $\|p_j(Y_1,\ldots,Y_m)\| = 0$ for each $(Y_1,\ldots,Y_m) \in \mathbb{ZD}_n^m(p_1,\ldots,p_r)$ and each $1 \leq j \leq r$. The author proves the existence of paths between arbitrary m-tuples that belong to the intersection of $\mathbb{ZD}_n^m(p_1,\ldots,p_r)$ and the open δ-ball $B_\eth(\mathbf{X},\delta)$ centered at $\mathbf{X}$ for some $\delta > 0$ that can be chosen independently of n. In addition, the author proves that the aforementioned paths are contained in the intersection of $B_\eth(\mathbf{X},\varepsilon)$ and $\mathbb{ZD}_n^m(p_1,\ldots,p_r)$. Some connections of the main results with structure-preserving perturbation theory and preconditioning techniques are outlined.

1. Introduction

In this document we study the uniform local path connectivity of sets of matrix tuples of commuting normal matrices with some additional geometric and algebraic constraints in their joint spectrum.

Let $\varepsilon > 0$ be given, along with a fixed metric $\eth$ in $M_n(\mathbb{C})^m$ induced by the operator norm $\|\cdot\|$, any collection of r nonconstant polynomials $p_1(x_1,\ldots,x_m),\ldots,p_r(x_1,\ldots,x_m)$ of m complex variables with coefficients over $\mathbb{C}$ and finite zero set

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$$\mathbf{Z}(p_1,\ldots,p_r) = \left\{\mathbf{x}\in\mathbb{C}^m \;\middle|\; p_j(\mathbf{x}) = 0, 1\le j\le r\right\}\subset\mathbb{C}^m$$

and any m-tuple $\mathbf{X} = (X_1,\ldots,X_m)\in M_n(\mathbb{C})^m$ of pairwise commuting normal matrix contractions such that $\|p_j(X_1,\ldots,X_m)\| = 0$ for each $1\le j\le r$. We prove the existence of paths between arbitrary m-tuples that belong to the intersection of the set of m-tuples of pairwise commuting normal matrix contractions in $M_n(\mathbb{C})^m$ and the δ-ball

$$B_{\eth}(\mathbf{X},\delta) = \left\{\mathbf{Y}\in M_n(\mathbb{C})^m \;\middle|\; \eth(\mathbf{X},\mathbf{Y})<\delta\right\}$$

centered at $\mathbf{X}$ for some $\delta>0$ that can be chosen independently of n. In addition, we prove that the aforementioned paths are contained in the intersection of $B_{\eth}(\mathbf{X},\varepsilon)$, and the set $\mathbb{ZD}_n^m(p_1,\ldots,p_r)$ of m-tuples of pairwise commuting normal matrix contractions such that $\|p_j(Y_1,\ldots,Y_m)\| = 0$ for each $(Y_1,\ldots,Y_m)\in\mathbb{ZD}_n^m(p_1,\ldots,p_r)$ and each $1\le j\le r$.

Let $\epsilon>0$ and $m\in\mathbb{Z}^+$ be given. The reason why independence on matrix size n (uniformity) is important in this study is that, in many applications, one needs to perform computations such as matrix inversion or matrix decomposition/factorization, with some approximations $\{\mathbf{X}_k\}_{k\ge1}$ of sequences of matrix m-tuples $\{\mathbf{Y}_k\}_{k\ge1}$ such that $\mathbf{X}_k,\mathbf{Y}_k\in M_{n_k}(\mathbb{C})^m$ and $\|\mathbf{X}_k-\mathbf{Y}_k\|\le\epsilon$ for $k\ge1$, and $\{n_k\}_{k\ge1}\subseteq\mathbb{Z}^+$ is an increasing sequence of positive integers. In these circumstances, one needs some properties of $\{\mathbf{X}_k\}_{k\ge1}$ and $\{\mathbf{Y}_k\}_{k\ge1}$ to be (uniform) independent of the matrix size n_k.

A common technique, implemented in order to achieve uniformity in the approximation of the sequences already mentioned, consists in *preconditioning* each pair of m-tuples $\mathbf{X}_k,\mathbf{Y}_k\in M_{n_k}(\mathbb{C})^m$ of the original sequences $\{\mathbf{X}_k\}_{k\ge1}$, $\{\mathbf{Y}_k\}_{k\ge1}$ in order to obtain two sequences $\{\hat{\mathbf{X}}_k\}_{k\ge1}$, $\{\hat{\mathbf{Y}}_k\}_{k\ge1}$ with some desirable *artificial* properties that, in addition, satisfy the constraints $\|\hat{\mathbf{X}}_k-\hat{\mathbf{Y}}_k\|\le\nu(\epsilon)$ for each $k\ge1$, and with some function $\nu:\mathbb{R}\to\mathbb{R}$ determined by the preconditioning/preprocessing technique.

In this document, we focus on some artificial spectral properties, more specifically on *eigenvalue clustering* in the sense of [1], [9], [18]. It is worth mentioning that, on occasion, eigenvalue clustering appears naturally in computational models, before any preconditioning has been performed; as an example, one can consider lossless Drude dispersive metallic photonic crystals in the sense of [9].

The main motivation for the research reported in this document came from *structure-preserving perturbation theory* in the sense of [16], especially by the effect that preconditioning (in the sense of [6], [7], [11], [17]–[19]) has on the numerical solution of linear systems of equations and eigenvalue/diagonalization problems, which are two of the main problems in numerical linear algebra. Two other sources of motivation include the *perturbation theory of matrix polynomials* in the sense of [10], and the *simultaneous block-diagonalization of matrices* in the sense of [14].

In this document, we combine the duality between matrix paths and numerical linear algebra algorithms studied in [5] with the geometric approach to matrix perturbation theory presented in [8] in order to derive a topological approach to the solution of normal matrix approximation problems. We do this by interpreting structure-preserving (and almost-structure-preserving) numerical approximation/refinement problems as *algebraically constrained* topological conectivity problems in the metric matrix space $(M_n(\mathbb{C})^m, \eth)$.

Matrix sets like $\mathbb{ZD}_n^m(p_1, \ldots, p_r)$ are called *algebraic matrix sets* in this document. The path connectivity properties of some important families of algebraic matrix sets are studied in Section 3.3. In order to extend the applicability of these connectivity results, some connections with the approximation theory for matrix functions of several normal (*preconditioned*) matrix variables are studied in Section 3.4.

One of the reasons for extending the results in Section 3.3 to those in Section 3.4 comes from the study of the notion of *approximate solvability* and stability of (Krylov-type) iterative and direct methods implemented to solve linear algebra problems numerically, especially when the computation is performed with finite precision in the sense of [15].

2. Preliminaries and notation

Given r polynomials $p_1(x_1, \ldots, x_m), \ldots, p_r(x_1, \ldots, x_m)$ of m complex variables, with coefficients over $\mathbb{C}$, we denote by $\mathbf{Z}(p_1, \ldots, p_r)$ the subset of $\mathbb{C}^m$ determined by the expression.

$$\mathbf{Z}(p_1, \ldots, p_r) = \big\{(x_1, \ldots, x_m) \in \mathbb{C}^m \;\big|\; p_j(x_1, \ldots, x_m) = 0, 1 \leq j \leq r\big\}. \tag{2.1}$$

We write $M_{m,n}$ to denote the set $M_{m,n}(\mathbb{C})$ of $(m \times n)$ complex matrices; if $m = n$, then we write M_n; we write M_n^m to denote the set $M_n(\mathbb{C})^m$ of m-tuples of $n \times n$ complex matrices. The symbols $\mathbf{1}_n$ and $\mathbf{0}_{m,n}$ are used to denote the identity matrix and the zero matrix in M_n and $M_{m,n}$, respectively; if $m = n$, then we write $\mathbf{0}_n$. Given a matrix $A \in M_n$, we write A^* to denote the conjugate transpose $(\bar{A})^\top$ of A.

A matrix $X \in M_n$ is said to be *normal* if $XX^* = X^*X$; a matrix $H \in M_n$ is said to be *Hermitian* if $H^* = H$; a matrix $K \in M_n$ is said to be *skew-Hermitian* if $K^* = -K$; and a matrix $U \in M_n$ such that $U^*U = UU^* = \mathbf{1}_n$ is called *unitary*. The set of all unitary matrices in M_n is denoted by $\mathbb{U}(n)$. We write $\mathbf{i}$ to denote the number $\sqrt{-1}$. Given any set S, we write $|S|$ to denote the number of elements of S, counted without multiplicity.

Let (X, d) be a metric space. We say that $\tilde{X}_\delta \subset X$ is a δ-*dense subset* of X if, for all $x \in X$, there exists $\tilde{x} \in \tilde{X}_\delta$ such that $d(x, \tilde{x}) \leq \delta$. Given two locally compact Hausdorff spaces X, Y, we write $C(X, Y)$ and $C^1(X, Y)$ to denote the sets of continuous and (differentiable) C^1-functions between X and Y, respectively.

For the remainder of this article, $\|\cdot\|$ denotes the operator norm defined for any $A \in M_n$ by $\|A\| := \sup_{\|x\|_2 = 1} \|Ax\|_2$, where $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{C}^n$. Let us denote by $\eth$ the metric in M_n^m defined by $\eth : M_n^m \times M_n^m \to \mathbb{R}_0^+$, $(\mathbf{S}, \mathbf{T}) \mapsto \max_j \|S_j - T_j\|$. We write $\mathbb{D}^2$ to denote the closed unit disk in $\mathbb{C}$ defined by $\mathbb{D}^2 := \{Z \in \mathbb{C} \mid |z| \leq 1\}$. Given any $x_0 \in \mathbb{C}^m$, we write $B(x_0, r)$ to denote the r-ball $\{x \in \mathbb{C}^m \mid \|x - x_0\|_2 < r\}$ in $\mathbb{C}^m$.

We say throughout this document that by *matrix contraction* we mean a matrix X in M_n such that $\|X\| \leq 1$. A matrix $P \in M_n$ such that $P^* = P = P^2$ is called a *projector* or a *projection*. Given two projectors P and Q, if $PQ = QP = \mathbf{0}_n$, then we consider P and Q to be orthogonal. By an *orthogonal partition of unity (OPU)* in M_n, we mean a finite set of pairwise orthogonal projectors $\{P_j\}$ in $M_n \backslash \{\mathbf{0}_n\}$ such that $\sum_j P_j = \mathbf{1}_n$ (we omit the explicit reference to M_n when it is clear from the context).

Given m mutually orthogonal projections $P_1, \ldots, P_m \in M_n$, let us consider the m linear subspaces $U_1, \ldots, U_m$ of $\mathbb{C}^n$, determined by the expressions $U_j = P_j\mathbb{C}^n = \{P_j z | z \in \mathbb{C}^n\}$. We write $U_1 \oplus U_2 \oplus \cdots \oplus U_m$ to denote the direct sum of the normed linear spaces $U_1, \ldots, U_m$, defined in the usual way.

Given any two matrices $X, Y \in M_n$ we write $[X, Y]$ and $\mathrm{Ad}[X](Y)$ to denote the operations $[X, Y] := XY - YX$ and $\mathrm{Ad}[X](Y) := XYX^*$.

A $*$-*homomorphism* $\varphi : M_n \to M_n$ is a linear and multiplicative map that satisfies $\varphi(X^*) = \varphi(X)^*$ for all X in M_n. Given $U \in \mathbb{U}(n)$, it can be easily verified that the map $\psi : M_n \to M_n$ defined by $\psi := \mathrm{Ad}[U]$ is a $*$-homomorphism; any $*$-homomorphism of this form is called an *inner $*$-homomorphism*.

Given any map $\Psi : M_n \to M_n$ in M_n, we write $\breve{\Psi}$ to denote the extended map $\breve{\Psi} : M_n^m \to M_n^m$ in M_n^m determined by the assignment $\breve{\Psi} : (X_1, \ldots, X_m) \mapsto (\Psi(X_1), \ldots, \Psi(X_m))$ for all $(X_1, \ldots, X_m)$ in M_n^m.

We write $\mathbb{GL}_n$ to denote the set of invertible elements in M_n. Given a matrix $A \in M_n$, we write $\sigma(X)$ to denote the set $\{\lambda \in \mathbb{C} \,|\, A - \lambda \mathbf{1}_n \notin \mathbb{GL}_n\}$ of eigenvalues of A; the set $\sigma(A)$ is called the *spectrum* of A. Given a matrix $X \in M_n$, we write $X \geq 0$ if X is Hermitian and $\sigma(X) \subseteq \mathbb{R}_0^+$.

Definition 2.1 ($\circledast$ operation). Given two matrix paths $\alpha, \beta \in C([0,1], M_n^m)$, we write $\alpha \circledast \beta$ to denote the concatenation of α and β, which is the matrix path defined in terms of α and β by the expression,

$$\alpha \circledast \beta(s) := \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2}, \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Given a matrix $A \in M_n$, we write $\mathscr{D}(A)$ to denote the diagonal matrix defined by the following operation.

$$\begin{aligned} \mathscr{D}(A) &:= \mathrm{diag}[a_{11}, a_{22}, \ldots, a_{nn}] \\ &= \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}. \end{aligned}$$

It can be seen that $\mathscr{D}(\mathscr{D}(A)) = \mathscr{D}(A)$ for any $A \in M_n$; the map $\mathscr{D}$ is called a *full pinching*.

Remark 2.1. By "pinching inequalities" (used here in the sense of [3]), we have $\|\mathscr{D}(A) - \mathscr{D}(B)\| = \|\mathscr{D}(A - B)\| \leq \|A - B\|$ for any two matrices $A, B \in M_n$.

Given m-matrices $A_1, \ldots, A_m$ such that $A_j \in M_{n_j}$ for each $1 \leq j \leq m$ and some $n_j \in \mathbb{Z}^+$, we write $A_1 \oplus A_2 \oplus \cdots \oplus A_m$ to denote the block-diagonal matrix in $M_{n_1+\cdots+n_m}$, determined by the following expression:

$$\begin{aligned} A_1 \oplus A_2 \oplus \cdots \oplus A_m &:= \mathrm{diag}[A_1, A_2, \ldots, A_m] \\ &= \begin{pmatrix} A_1 & \mathbf{0}_{n_1,n_2} & \cdots & \mathbf{0}_{n_1,n_m} \\ \mathbf{0}_{n_2,n_1} & A_2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0}_{n_{m-1},n_m} \\ \mathbf{0}_{n_m,n_1} & \cdots & \mathbf{0}_{n_m,n_{m-1}} & A_m \end{pmatrix}. \end{aligned}$$

Given any matrix X in M_n, we write $\mathrm{Re}(X)$ and $\mathrm{Im}(X)$ to denote the Hermitian matrices defined by the equations

$$\begin{cases} \mathrm{Re}(X) = (X + X^*)/2, \\ \mathrm{Im}(X) = (X - X^*)/(2\mathbf{i}). \end{cases} \tag{2.2}$$

Remark 2.2. It is important here to recall that, for any X in M_n, it holds that $X^*X = XX^*$ if and only if $\mathrm{Re}(X)\,\mathrm{Im}(X) = \mathrm{Im}(X)\,\mathrm{Re}(X)$.

It is often convenient to have N-tuples (or $2N$-tuples) of matrices with real spectra. For this purpose, we use the following construction. If $\mathbf{X} = (X_1, \ldots, X_N)$ is a N-tuple of n by n matrices, then we can always decompose X_j in the form $X_j = \mathrm{Re}(X_j) + \mathbf{i}\,\mathrm{Im}(X_j)$ for each $1 \leq j \leq N$.

We write $\hat{\pi}(\mathbf{X})$ to denote the operation $\hat{\pi}(\mathbf{X}) := (\mathrm{Re}(X_1), \ldots, \mathrm{Re}(X_N), \mathrm{Im}(X_1), \ldots, \mathrm{Im}(X_N))$; we call $\hat{\pi}(\mathbf{X})$ a *partition* of $\mathbf{X}$. If all the matrix coordinates of $\hat{\pi}(\mathbf{X})$ commute, then we say that $\hat{\pi}(\mathbf{X})$ is a *commuting partition*, and if all the matrix coordinates of $\hat{\pi}(\mathbf{X})$ are simultaneously triangularizable, then $\hat{\pi}(\mathbf{X})$ is called a *triangularizable partition*. If all the matrix coordinates of $\hat{\pi}(\mathbf{X})$ are semisimple (diagonalizable), then $\hat{\pi}(\mathbf{X})$ is called a *semisimple partition*. Given a $2m$-tuple $\mathbf{X} = (X_{11}, \ldots, X_{1m}, X_{21}, \ldots, X_{2m})$ in M_n^{2m}, the m-tuple obtained through the operation $\upsilon(\mathbf{X}) := (X_{11} + \mathbf{i}X_{21}, \ldots, X_{1m} + \mathbf{i}X_{2m}) \in M_n^m$ is called a *juncture* of $\mathbf{X}$.

We say that N normal matrices $X_1, \ldots, X_N \in M_n$ are *simultaneously diagonalizable* if there is a unitary matrix $Q \in M_n$ such that Q^*X_jQ is diagonal for each $j = 1, \ldots, N$. In this case, for $1 \leq k \leq n$, let $\Lambda^{(k)}(X_j) := (Q^*X_jQ)_{kk}$ the (k,k) element of Q^*X_jQ, and set $\Lambda^{(k)}(X_1, \ldots, X_N) := (\Lambda^{(k)}(X_1), \ldots, \Lambda^{(k)}(X_N))$ in $\mathbb{C}^N$. The set

$$\Lambda(X_1, \ldots, X_N) := \left\{\Lambda^{(k)}(X_1, \ldots, X_N)\right\}_{1 \leq k \leq N}$$

is called the *joint spectrum of* $X_1, \ldots, X_N$ *with respect to* Q, or just the *joint spectrum of* $X_1, \ldots, X_N$ for short (we omit the explicit reference to Q when it is clear from the context). The unitary matrix Q is called a *joint diagonalizer* of $X_1, \ldots, X_N$ in this document.

Given a set $S \subseteq M_n^m$ of m-tuples of pairwise commuting normal matrices, we write $\Lambda(S)$ to denote the set $\{\Lambda(X) \mid X \in S\}$; the set $\Lambda(S)$ is called the *joint spectra* of S. We write $\Lambda(X_j)$ to denote the diagonal matrix representation of the j-component of $\Lambda(X_1, \ldots, X_N)$; in other words, we have

$$\Lambda(X_j) = \operatorname{diag}\left[\Lambda^{(1)}(X_j), \ldots, \Lambda^{(n)}(X_j)\right].$$

Given a m-tuple $\mathbf{X} = (X_1, \ldots, X_m) \in M_n^m$ of commuting normal matrices, any orthogonal projection $P \in M_n$ such that $X_j P = P X_j = \Lambda^{(r)}(X_j) P$ for each $1 \leq j \leq m$ and some $1 \leq r \leq n$ is called a *joint spectral projector* of $\mathbf{X}$.

3. Path connectivity of algebraic normal matrix sets

3.1. Algebraic Hermitian matrix sets. For any $n \in \mathbb{Z}^+$, we write $\mathbb{I}_n^m$ to denote the subset of M_n^m determined by the following expression,

$$\mathbb{I}_n^m = \left\{ (X_1, \ldots, X_m) \in M_n^m \middle| \begin{array}{l} X_j X_k - X_k X_j = \mathbf{0}_n, \\ X_j - X_j^* = \mathbf{0}_n, \qquad 1 \leq j, k \leq m \\ \|X_j\| \leq 1 \end{array} \right\}. \tag{3.1}$$

The set $\mathbb{I}_n^m(p_1, \ldots, p_r)$ is called a *matrix m-cube* in this document.

Given any $n \in \mathbb{Z}^+$ and any r nonconstant polynomials $p_1(x_1, \ldots, x_m), \ldots, p_r(x_1, \ldots, x_m)$ of m complex variables with coefficients over $\mathbb{R}$, we write $\mathbb{Z}\mathbb{I}_n^m(p_1 , \ldots , p_r)$ to denote the subset of $\mathbb{I}_n^m$ determined by the expression

$$\begin{aligned} &\mathbb{Z}\mathbb{I}_n^m(p_1, \ldots, p_r) \\ &\qquad = \left\{ (X_1, \ldots, X_m) \in \mathbb{I}_n^m \,\middle|\, p_j(X_1, \ldots, X_m) = \mathbf{0}_n, 1 \leq j \leq r \right\}. \end{aligned} \tag{3.2}$$

The algebraic Hermitian matrix set $\mathbb{Z}\mathbb{I}_n^m(p_1, \ldots, p_r)$ is called an *algebraic matrix m-cube* in this document.

3.2. Algebraic normal matrix sets. For any $n \in \mathbb{Z}^+$, we write $\mathbb{D}_n^m$ to denote the subset of M_n^m determined by the following expression.

$$\mathbb{D}_n^m = \left\{ (X_1, \ldots, X_m) \in M_n^m \middle| \begin{array}{l} X_j X_k - X_k X_j = \mathbf{0}_n, \\ X_j X_j^* - X_j^* X_j = \mathbf{0}_n, 1 \leq j, k \leq m \\ \|X_j\| \leq 1 \end{array} \right\}. \tag{3.3}$$

The set $\mathbb{Z}\mathbb{D}_n^m(p_1 , \ldots , p_r)$ is called a *matrix m-disk* in this document.

Given any $n \in \mathbb{Z}^+$ and any r nonconstant polynomials $p_1(z_1, \ldots, z_m), \ldots, p_r(z_1, \ldots, z_m)$ of m complex variables with coefficients over $\mathbb{C}$, we write $\mathbb{Z}\mathbb{D}_n^m(p_1 , \ldots , p_r)$ to denote the subset of $\mathbb{D}_n^m$ determined by the expression

$$\begin{aligned} &\mathbb{Z}\mathbb{D}_n^m(p_1, \ldots, p_r) \\ &\qquad = \left\{ (X_1, \ldots, X_m) \in \mathbb{D}_n^m \,\middle|\, p_j(X_1, \ldots, X_m) = \mathbf{0}_n, 1 \leq j \leq r \right\}, \end{aligned} \tag{3.4}$$

the algebraic normal matrix set $\mathbb{Z}\mathbb{D}_n^m(p_1, \ldots, p_r)$ is called an *algebraic matrix m-disk* in this document.

3.3. Uniform path connectivity of algebraic normal contractions.

Lemma 3.1. *Given any 2 matrix m-tuples $(X_1, \ldots, X_m), (Y_1, \ldots, Y_m) \in \mathbb{I}_n^m$ such that $X_j Y_k = Y_k X_j$ for each $1 \leq j, k \leq m$, there is a path $\gamma \in C^1([0,1], \mathbb{I}_n^m)$ that satisfies the conditions*

$$\begin{cases} \gamma(0) = (X_1, \ldots, X_m), \\ \gamma(1) = (Y_1, \ldots, Y_m), \end{cases}$$

together with the constraints,

$$\eth\big(\gamma(t),(Y_1,\ldots,Y_m)\big) \leq \eth\big((X_1,\ldots,X_m),(Y_1,\ldots,Y_m)\big)$$

for each $0 \leq t \leq 1$.

Proof. Given $(X_1,\ldots,X_m),(Y_1,\ldots,Y_m) \in \mathbb{I}_n^m$ such that $X_jY_k = Y_kX_j$ for each $1 \leq j,k \leq m$, we have that, for each $1 \leq j \leq m$, each matrix path of the form $\gamma_j(t) = X_j + t(Y_j - X_j)$ satisfies the interpolating conditions $\gamma_j(0) = X_j$ and $\gamma_j(1) = Y_j$, together with the constraints

$$\big\|\gamma_j(t) - Y_j\big\| \leq (1-t)\|X_j - Y_j\| \leq \|X_j - Y_j\| \tag{3.5}$$

for each $0 \leq t \leq 1$.

Let us set $\gamma(t) = (\gamma_j(t))$, $0 \leq t \leq 1$. It can be easily verified that $\gamma(t) \in \mathbb{I}_n^m$ for each $0 \leq t \leq 1$. By the definition of $\eth$ and as a consequence of (3.5), we can derive the following estimate,

$$\begin{aligned}\eth\big(\gamma(t),(Y_1,\ldots,Y_m)\big) &= \max_j\big\|\gamma_j(t) - Y_j\big\| \\ &\leq \max_j \|X_j - Y_j\| \\ &= \eth\big((X_1,\ldots,X_m),(Y_1,\ldots,Y_m)\big)\end{aligned} \tag{3.6}$$

for each $0 \leq t \leq 1$. This completes the proof. □

Lemma 3.2. *Given any two simultaneously commuting OPU* $\mathcal{P} := \{P_1,\ldots,P_r\}$ *and* $\mathcal{Q} := \{Q_1,\ldots,Q_s\}$ *in* M_n, *there is an OPU* $\mathcal{R} := \{R_1,\ldots,R_t\}$ *of* M_n *such that* $\operatorname{span}\mathcal{P}\cup\mathcal{Q} \subseteq \operatorname{span}\mathcal{R}$ *and* $|\mathcal{R}| \leq |\mathcal{P}||\mathcal{Q}|$.

Proof. Since the elements of $\mathcal{P}$ and $\mathcal{Q}$ simultaneously commute, by setting $R_{j,k} := P_jQ_k$ and $\mathcal{R} := \{R_{j,k}\}\backslash\{\mathbf{0}\}$ it can be easily verified that $\mathcal{P},\mathcal{Q} \subseteq \operatorname{span}\{R_{j,k}\}$. Let us set $\mathcal{R} := \{R_{j,k}\}$. It can be seen that $|\mathcal{R}| \leq |\mathcal{P}||\mathcal{Q}|$ and $\operatorname{span}\{\mathcal{P},\mathcal{Q}\} \subseteq \operatorname{span}\mathcal{R}$. This completes the proof. □

Definition 3.1 (Projective Refinement). Given any collection of OPU $\mathcal{P}_1 = \{P_{1,j_1}\}_{j_1=1}^{r_1},\ldots,\mathcal{P}_s = \{P_{s,j_s}\}_{j_s=1}^{r_s}$ such that $P_{k,j_k}P_{l,j_l} = P_{l,j_l}P_{k,j_k}$ for any $1 \leq k,l \leq s$, each $P_{k,j_k} \in \mathcal{P}_k$ and each $P_{l,j_l} \in \mathcal{P}_l$. The set $\mathcal{R}(\mathcal{P}_1,\ldots,\mathcal{P}_s)$ defined by the expression

$$\mathcal{R}(\mathcal{P}_1,\ldots,\mathcal{P}_s) = \{P_{1,j_1}P_{2,j_2}\cdots P_{s,j_s} \mid P_{k,j_k} \in \mathcal{P}_k\}\backslash\{\mathbf{0}\}, \tag{3.7}$$

is called a *projective refinement* of $\mathcal{P}_1,\ldots,\mathcal{P}_s$.

By iterating on Lemma 3.2, we can obtain the following corollary.

Corollary 3.1. *For any collection of simultaneously commuting OPU* $\mathcal{P}_1,\ldots,\mathcal{P}_s$, *we have that* $\mathcal{R}(\mathcal{P}_1,\ldots,\mathcal{P}_s)$ *is an OPU.*

Lemma 3.3 (Projective Polar Decomposition)**.** *Given an OPU* $\mathcal{P} = \{P_j\}_{j=1}^r$ *in* M_n, *and given any matrix* $X \in M_n$, *there is a polar decomposition* $X_{j,j} = V_jR_j$ *of the matrix* $X_{j,j} = P_jXP_j$ *that satisfies the conditions* $V_jV_j^* = V_j^*V_j = P_j$, $R_j \geq \mathbf{0}_n$, $P_jV_j = V_jP_j = V_j$, $P_jR_j = R_jP_j = R_j$ *and* $P_kV_j = P_kR_j = R_jP_k = V_jP_k = \mathbf{0}_n$, *for* $1 \leq k,j \leq r$ *with* $k \neq j$.

Proof. By changing basis, if necessary, we can assume that the elements of $\mathcal{P}$ are diagonal matrices. We have that, for each $j = 1, \ldots, r$, there is a unitary (permutation) matrix $S_j \in M_n$, such that $P_j = S_j \hat{P}_j S_j^*$ with $\hat{P}_j = \mathbb{1}_{m_j} \oplus \mathbf{0}_{n-m_j}$, for some $1 \leq m_j \leq n$. If we set $\psi_j := \mathrm{Ad}[S_j]$, then each ψ_j is a $*$-homomorphism. We have, for each j,

$$\hat{P}_j S_j^* X S_j \hat{P}_j = \hat{X}_{j,j} \hat{P}_j = \hat{X}_{j,j} \tag{3.8}$$

with $\hat{X}_{j,j} = \tilde{X}_{j,j} \oplus \mathbf{0}_{n-m_j}$ for some $\tilde{X}_{j,j} \in M_{m_j}$. Let $\tilde{X}_{j,j} = \tilde{U}_j \tilde{\Sigma}_j \tilde{W}_j$ be a singular value decomposition of $\tilde{X}_{j,j}$, with $\tilde{V}_j, \tilde{W}_j \in M_{m_j}$ unitary and $\tilde{\Sigma}_j \geq 0$. It can be seen that if we set $\tilde{V}_j = \tilde{U}_j \tilde{W}_j$ and $\tilde{R}_j = \tilde{W}_j^* \tilde{\Sigma}_j \tilde{W}_j$, then $\tilde{V}_j$ is unitary and $\tilde{R}_j \geq 0$. Moreover, we have that $\tilde{V}_j \tilde{R}_j = \tilde{U}_j \tilde{\Sigma}_j \tilde{W}_j = \tilde{X}_{j,j}$ determines a representation of the polar decomposition of $\tilde{X}_{j,j}$, with $\tilde{V}_j$ unitary and $\tilde{R}_j \geq 0$, for each $1 \leq j \leq r$.

Let us set $\hat{V}_j = \tilde{V}_j \oplus \mathbf{0}_{n-m_j}$ and $\hat{R}_j = \tilde{R}_j \oplus \mathbf{0}_{n-m_j}$. It can be seen that, for $1 \leq k, j \leq r, k \neq j$,

$$\begin{cases} \hat{X}_{j,j} = \hat{V}_j \hat{R}_j, \\ \hat{V}_j \hat{V}_j^* = \hat{V}_j^* \hat{V}_j = \hat{P}_j, \\ \hat{R}_j \geq \mathbf{0}_n, \\ \hat{P}_j \hat{V}_j = \hat{V}_j \hat{P}_j = \hat{V}_j, \\ \hat{P}_j \hat{R}_j = \hat{R}_j \hat{P}_j = \hat{R}_j, \\ \hat{P}_k \hat{V}_j = \hat{P}_k \hat{R}_j = \hat{R}_j \hat{P}_k = \hat{V}_j \hat{P}_k = \mathbf{0}_n. \end{cases} \tag{3.9}$$

We can use ψ_j together with (3.8) and (3.9) in order to obtain the following decomposition:

$$\begin{aligned} X_{j,j} &= P_j X P_j \\ &= \psi_j(\hat{P}_j) X \psi_j(\hat{P}_j) \\ &= \psi_j(\hat{P}_j S_j^* X S_j \hat{P}_j) \\ &= \psi_j(\hat{X}_{j,j} \hat{P}_j) \\ &= \psi_j(\hat{V}_{j,j} \hat{R}_{j,j} \hat{P}_j) \\ &= \psi_j(\hat{V}_{j,j} \hat{R}_{j,j}) \\ &= \psi_j(\hat{V}_{j,j}) \psi_j(\hat{R}_{j,j}). \end{aligned}$$

Let us set $V_j = \psi_j(\hat{V}_j)$ and $R_j = \psi_j(\hat{R}_j)$. Since each ψ_j preserves commutativity and positivity, by (3.9) we have $V_j V_j^* = V_j^* V_j = P_j$, $R_j \geq \mathbf{0}_n$, $P_j V_j = V_j P_j = V_j$, $P_j R_j = R_j P_j = R_j$ for each $1 \leq j \leq r$. By the previous commutativity relations, we have $V_j = P_j V_j P_j$ and $R_j = P_j R_j P_j$ for each j. In addition, since the set $\{P_j\}_{j=1}^r$ is an OPU, then for $1 \leq k, j \leq r$ with $k \neq j$, one can derive the following relations:

$$\begin{cases} P_k V_j = P_k P_j V_j = \mathbf{0}_n V_j = \mathbf{0}_n \\ V_j P_k = V_j P_j P_k = V_j \mathbf{0}_n = \mathbf{0}_n \\ P_k R_j = P_k P_j R_j = \mathbf{0}_n R_j = \mathbf{0}_n \\ R_j P_k = R_j P_j P_k = R_j \mathbf{0}_n = \mathbf{0}_n. \end{cases}$$

This completes the proof. $\square$

Lemma 3.4. *Given an OPU $\{P_1,\dots,P_m\}$ in M_n, let us consider m matrices $A_1,\dots,A_m\in M_n$ such that*

$$A_jP_j = P_jA_j = A_j. \tag{3.10}$$

If we set $A=\sum_{j=1}^m A_j$, then $\|A\| = \max_{1\leq j\leq m}\|A_j\|$.

Proof. Given an OPU $\{P_1,\dots,P_m\}$ in M_n, let us consider m matrices $A_1,\dots,A_m\in M_n$ that satisfy (3.10) for $1\leq j\leq m$. Then (3.10) implies that

$$P_jA_j^* = (A_jP_j)^* = (P_jA_j)^* = A_j^*P_j, \tag{3.11}$$

for $1\leq j\leq m$. By (3.10) and (3.11), we have that the matrix $A:=\sum_{j=1}^m A_j$ is a direct sum of operators on the direct sum of normed linear spaces $P_1\mathbb{C}^n\oplus P_2\mathbb{C}^n\oplus\cdots\oplus P_m\mathbb{C}^n$. We also have

$$\begin{aligned}A^*A &= \Big(\sum_{j=1}^m A_j\Big)^*\Big(\sum_{j=1}^m A_j\Big) = \Big(\sum_{j=1}^m A_j^*P_j\Big)\Big(\sum_{j=1}^m P_jA_j\Big)\\ &= \sum_{j=1}^m A_j^*P_jP_jA_j = \sum_{j=1}^m A_j^*A_j.\end{aligned}$$

By elementary spectral theory, since A^*A is normal, there is an orthonormal basis $\{v_1,v_2,\dots,v_n\}$ of $\mathbb{C}^n$ consisting of eigenvectors of A^*A. This implies that for each v_j there is $\lambda_j\geq 0$ in $\sigma(A^*A)$ such that

$$A^*Av_j = \lambda_jv_j. \tag{3.12}$$

By (3.12) we have, for each $1\leq j\leq n$ and each $1\leq k\leq m$,

$$A_k^*A_kP_kv_j = P_kA_k^*A_kv_j = P_kA^*Av_j = \lambda_jP_kv_j. \tag{3.13}$$

Since each $x\in\mathbb{C}^n$ has a representation

$$x = \sum_{j=1}^n (v_j^*x)v_j, \tag{3.14}$$

then (3.14) implies that

$$A^*Ax = \sum_{j=1}^n (v_j^*x)A^*Av_j = \sum_{j=1}^n (v_j^*x)\lambda_jv_j. \tag{3.15}$$

Let us set $\mu=\max_{\lambda\in\sigma(A^*A)}\sqrt{\lambda}$. Since $\sigma(A^*A)\subseteq[0,\infty)$, by orthonormality of the eigenvectors $\{v_1,\dots,v_n\}$ and by (3.15), we have

$$\begin{aligned}\|Ax\|_2^2 &= x^*A^*Ax\\ &= \Big(\sum_{j=1}^n \overline{(v_j^*x)}v_j^*\Big)\Big(\sum_{j=1}^n (v_j^*x)\lambda_jv_j\Big)\\ &= \sum_{j=1}^n |v_j^*x|^2\lambda_j \leq \mu^2\|x\|_2^2.\end{aligned} \tag{3.16}$$

By taking $x = x_k$, where x_k is an eigenvector such that

$$A^*Ax_k = \mu^2 x_k, \tag{3.17}$$

we have that (3.16) is tight. From (3.16) we have, for any $x \in \mathbb{C}^n$ such that $\|x\|_2 = 1$,

$$\|Ax\|_2 \leq \mu \tag{3.18}$$

By (3.18), by the compactness of $\mathbb{S}^n(\mathbb{C}) = \{x \in \mathbb{C}^n | \|x\|_2 = 1\}$, and by the continuity of $\|A \cdot\|_2$ on $\mathbb{S}^n(\mathbb{C})$, we have that

$$\|A\| = \sup_{\|x\|_2=1} \|Ax\|_2 = \max_{\lambda\in\sigma(A^*A)} \sqrt{\lambda}.$$

Elementary matrix theory implies that

$$\|A_j\| = \|P_jA\| \leq \|A\|. \tag{3.19}$$

Let us consider again the eigenvector x_k that satisfies (3.17). We have that if $A_j^*A_jx_k = 0$ for each $1 \leq j \leq m$, then, on one hand,

$$0 = \|A_jx_k\|_2 \leq \mu = \|A\| = \|Ax_k\|_2 = \left\|\sum_j A_jx_k\right\|_2 = 0. \tag{3.20}$$

On the other hand, if there is an A_j such that $A_j^*A_jx_k \neq 0$, then if we set $\hat{x}_k = 1/(\|P_jx_k\|_2)P_jx_k$ we have $\|\hat{x}_k\|_2 = 1$, and by (3.13) we have that

$$\|A\hat{x}_k\|_2 \leq \|A\| = \mu = \|A_j\hat{x}_k\|_2 \leq \|A_j\|. \tag{3.21}$$

By (3.19), (3.20), and (3.21), we have

$$\max_{1\leq j\leq m} \|A_j\| \leq \|A\| \leq \max_{1\leq j\leq m} \|A_j\|. \tag{3.22}$$

Now (3.22) implies that $\|A\| = \max_{1\leq j\leq m} \|A_j\|$. This completes the proof. $\square$

We can generalize the proof of [2, VI.6.6] to obtain the following lemma.

Lemma 3.5. *Given a unitary W and a normal contraction D in M_n for $n \geq 2$, if $D = \sum_{j=1}^r \alpha_jP_j$ is diagonal for $2 \leq r \in \mathbb{Z}$, and if $\alpha_1, \ldots, \alpha_r \in \mathbb{D}^2$, $\{P_j\}$ is an OPU of diagonal matrices in M_n, and $\alpha_j \neq \alpha_k$ whenever $k \neq j$, then there is a unitary matrix $Z \in M_n$ and a constant C depending on r and $\sigma(D)$ such that $[Z, P_j] = [Z, D] = 0$ for $1 \leq j \leq r$, and $\|\mathbb{1}_n - WZ\| \leq C\|WDW^* - D\|$.*

Proof. We have r mutually orthogonal projections $\mathbf{0}_n \leq P_1, \ldots, P_r \leq \mathbb{1}_n$ in M_n such that $\sum_j P_j = \mathbb{1}_n$ and $D := \sum_j \alpha_jP_j$ with $\alpha_j \in \mathbb{D}^2$. By setting $W_{j,k} := P_jWP_k$, we have that W has a decomposition $W = \sum_{j,k} W_{j,k}$, and it can be seen that

$$\begin{aligned}\|WDW^* - D\| &= \|WD - DW\| \\ &= \left\|\sum_{j,k}(\alpha_jP_jW_{j,k} - \alpha_kW_{j,k}P_k\right\| \\ &= \left\|\sum_{j,k}(\alpha_j - \alpha_k)W_{j,k}\right\|.\end{aligned}$$

Hence, for $j \neq k$,

$$\begin{aligned}\|W_{j,k}\| &\leq \frac{1}{|\alpha_j - \alpha_k|}\|WDW^* - D\| \\ &\leq \max_{j,k}\Big\{\frac{1}{|\alpha_j - \alpha_k|}\Big\}\|WDW^* - D\|.\end{aligned}$$

Hence, by setting $s = \min_{j,k,j\neq k}|\alpha_j - \alpha_k|$, we have

$$\Big\|W - \sum_j W_{j,j}\Big\| \leq \frac{r(r-1)}{s}\|WDW^* - D\|.$$

By Lemma 3.3 for $1 \leq j,k \leq r$, there is a projective polar decomposition $W_{j,j} := V_jR_j$ of $W_{j,j}$ that satifies the constraints $V_jV_j^* = V_j^*V_j = P_j$, $R_j \geq \mathbf{0}_n$, $P_jV_j = V_jP_j = V_j$, $P_jR_j = R_jP_j = R_j$, and $P_kV_j = P_kR_j = R_jP_k = V_jP_k = \mathbf{0}_n$, if $k \neq j$.

Let $X := \sum_j W_{j,j} = \sum_j P_jWP_j$, $R := \sum_j R_j$ and $V := \sum_j V_j$. Let us set $\mathbf{R}_{P_j} = P_j\mathbb{C}^n = \{P_jz | z \in \mathbb{C}^n\}$. Building on the proof of Lemma 3.3, the above relations imply that X, V and R are a direct sums of operators in the direct sum $\mathbf{R}_{P_1} \oplus \mathbf{R}_{P_2} \oplus \cdots \oplus \mathbf{R}_{P_r}$ of the normed linear spaces $\mathbf{R}_{P_j}$, $1 \leq j \leq r$. Moreover, it can be seen that

$$X = VR.$$

By Lemma 3.4, we also have $\|X\| = \max_{1\leq j\leq r}\|W_{j,j}\| \leq \|W\| = 1$.

It can be easily verified that each R_j satisfies the contraint $\|R_j\| \leq 1$. We also have

$$\|W_{j,j} - V_j\| = \|R_j - P_j\| \leq \|R_j^2 - P_j\|,$$

since R_j is a contraction. It can be verified that $V = \sum_j V_j \in \mathbb{U}n$, and from the above inequality and by Lemma 3.4, we see that

$$\|X - V\| = \max_{1\leq j\leq r}\|W_{j,j} - V_j\| \leq \max_{1\leq j\leq r}\|R_j^2 - P_j\| = \|X^*X - \mathbb{1}_n\| = \|X^*X - W^*W\|.$$

Hence,

$$\begin{aligned}\|V - W\| &\leq \|V - X\| + \|X - W\| \leq \|W - X\| + \|X^*X - W^*W\| \\ &\leq \|W - X\| + \big\|(X^* - W^*)X\big\| + \big\|W^*(X - W)\big\| \\ &\leq 3\|W - X\| \leq \frac{3r(r-1)}{s}\|WDW^* - D\|.\end{aligned}$$

By setting $Z := V^*$ and $C := \frac{3r(r-1)}{s}$, it can be seen that $\|\mathbb{1}_n - WZ\| = \|V - W\| \leq C\|WDW^* - D\|$, and by definition of V we have

$$VP_j = \Big(\sum_k V_k\Big)P_j = \Big(\sum_k V_kP_k\Big)P_j = V_jP_j^2 = V_jP_j = V_j \tag{3.23}$$

and

$$P_jV = P_j\Big(\sum_k V_k\Big) = P_j\Big(\sum_k P_kV_k\Big) = P_j^2V_j = P_jV_j = V_j. \tag{3.24}$$

By (3.23) and (3.24), we have $VP_j = P_jV$ for $1 \leq j \leq r$, and this implies that

$$ZP_j = V^*P_j = (P_jV)^* = (VP_j)^* = P_jV^* = P_jZ, \tag{3.25}$$

for each $1 \le j \le r$. Then (3.25) implies that

$$ZD = Z\Big(\sum_j \alpha_j P_j\Big) = \sum_j \alpha_j Z P_j = \sum_j \alpha_j P_j Z = \Big(\sum_j \alpha_j P_j\Big) Z = DZ.$$

This completes the proof. ∎

Remark 3.1. Given any normal contraction D such that $p(D) = \mathbf{0}_n$ for some $p \in \mathbb{C}[z]$ with $\deg(p) \le r$, we have that there are integer $r' \le r$, r' complex numbers $\alpha_1, \dots, \alpha_{r'} \in \mathbb{D}^2$, and r' pairwise orthogonal projections $P_1, \dots, P_{r'}$ such that $p(\alpha_j) = 0$, $\sum_j P_j = \mathbb{1}_n$, and $D = \sum_j \alpha_j P_j$.

Lemma 3.6. *Given a unitary W and a collection of normal contractions $D_1, \dots, D_m$ in M_n for $n \ge 2$, if each $D_k = \sum_{j=1}^{r_k} \alpha_{k,j} P_{k,j}$ is diagonal for $2 \le r_k \in \mathbb{Z}$ and $\{\alpha_{k,j}\} \subseteq \mathbb{D}^2$, each set $\{P_{k,j}\}$ is a diagonal OPU in M_n, and $\alpha_{k,j} \ne \alpha_{k,l}$ whenever $l \ne j$, then there is a unitary matrix $Z \in M_n$ and a constant C depending on $m, r_1, \dots, r_m$ and the spectra $\sigma(D_1), \dots, \sigma(D_m)$ such that $[Z, D_k] = 0$ for each $1 \le k \le m$, and $\|\mathbb{1}_n - WZ\| \le C \max_{1 \le k \le m} \|WD_kW^* - D_k\|$.*

Proof. We can apply Lemma 3.5 to W and each D_k to obtain for each k a unitary matrix Z_k that satisfies the conditions

$$\begin{cases} [Z_k, D_k] = \mathbf{0}_n, \\ \|\mathbb{1}_n - WZ\| \le C_k \|WDW^* - D\|, \end{cases} \tag{3.26}$$

where C_k is a constant that depends on r and $\sigma(D_k)$. By (3.26), we have, for each $P_{k,j}$,

$$\begin{aligned} \|WP_{k,j}W^* - P_{k,j}\| &= \|WP_{k,j} - P_{k,j}W\| \\ &\le \|WP_{k,j} - Z_kP_{k,j}\| + \|P_{k,j}Z_k - P_{k,j}W\| \\ &\le 2\|W - Z_k\| \\ &\le 2C_k\|WD_kW^* - D_k\|. \end{aligned} \tag{3.27}$$

Let us consider a fixed but arbitrary element P in the projective refinement $\mathcal{R}(\{P_{1,j_1}\}, \dots, \{P_{m,j_m}\})$. We have $P = P_{1,j'_1}, \dots, P_{m,j'_m}$ with $P_{k,j'_k} \in \{P_{k,j_k}\}$ for each $1 \le k \le m$. This implies that

$$\begin{aligned} \|WPW^* - P\| &= \|WP_{1,j'_1} \cdots P_{m,j'_m} W^* - P_{1,j'_1} \cdots P_{m,j'_m}\| \\ &\le \sum_{k=1}^{m} \|WP_{k,j'_k}W^* - P_{k,j'_k}\|. \end{aligned} \tag{3.28}$$

Combining (3.27) and (3.28), we obtain the following estimate:

$$\|WPW^* - P\| \le 2m \max_{1 \le k \le m} C_k \max_{1 \le k \le m} \|WD_kW^* - D_k\|. \tag{3.29}$$

Let us set $\nu = \max_{1 \le k \le m} \|WD_kW^* - D_k\|$. If $\nu < 1/(2m \max_{1 \le k \le m} C_k)$, then by [12, Lemma 2.5.1] it holds that (3.29) implies that, for each P in the projective refinement $\mathcal{R}(\{P_{1,j_1}\}, \dots, \{P_{m,j_m}\})$, there is a unitary $W_P \in M_n$ such that

$$
\begin{cases} WPW^* = W_P^* P W_P, \\ \|\mathbf{1}_n - W_P\| \leq \sqrt{2}\|WPW^* - P\|. \end{cases} \tag{3.30}
$$

Now the matrix $Z_P = W_P W \in M_n$ is a unitary satisfying the following commutation relation.

$$
Z_P P = P Z_P. \tag{3.31}
$$

Let us list $\mathcal{R}(\{P_{1,j_1}\}, \ldots, \{P_{m,j_m}\})$ in the form

$$
\mathcal{R}\big(\{P_{1,j_1}\}, \ldots, \{P_{m,j_m}\}\big) = \{P_1, \ldots, P_N\}.
$$

By (3.30) and (3.29), we have that, for each $P_j \in \mathcal{R}(\{P_{1,j_1}\}, \ldots, \{P_{m,j_m}\})$, there is a unitary $Z_j = W_{P_j} W \in M_n$ such that

$$
\begin{cases} Z_j P_j = P_j Z_j, \\ \|W - Z_j\| = \|\mathbf{1}_n - W_{P_j}\| \leq \sqrt{2}\|WP_jW^* - P_j\|. \end{cases} \tag{3.32}
$$

As a consequence of (3.32) and Corollary 3.1, it can be easily verified that $\hat{Z} = \sum_{j=1}^N Z_j P_j$ is unitary. Moreover, by (3.29) we obtain the following estimate:

$$
\begin{aligned}
\|W - \hat{Z}\| &= \Big\|(W - \hat{Z})\Big(\sum_{j=1}^N P_j\Big)\Big\| \\
&= \Big\|\sum_{j=1}^N (WP_j - Z_jP_j)\Big\| \\
&\leq \sum_{j=1}^N \|(\mathbf{1}_n - W_{P_j})WP_j\| \\
&\leq \sum_{j=1}^N \|\mathbf{1}_n - W_{P_j}\| \\
&\leq 2\sqrt{2}mN \max_{1\leq k\leq m} C_k \nu.
\end{aligned} \tag{3.33}
$$

Let us set

$$
\begin{aligned}
C &= 2\sqrt{2}mN \max_{1\leq k\leq m} C_k \\
&= 6\sqrt{2}mN \frac{\max_{1\leq k\leq m} r_k(r_k - 1)}{\min_{1\leq k\leq m} \min_{1\leq j,l\leq r_k, j\neq l} |\alpha_{k,j} - \alpha_{k,l}|}
\end{aligned} \tag{3.34}
$$

and

$$
Z = \hat{Z}^*. \tag{3.35}
$$

Since, by Lemma 3.2, it holds that each $D_k \in \operatorname{span}\{P_1, \ldots, P_N\}$, we then have that $ZD_k = D_k Z$ for $1 \leq k \leq m$, and

$$
\|\mathbb{1}_n - ZW\| = \|\hat{Z} - W\| \leq C \max_{1\leq k\leq m} \|WD_kW^* - D_k\|.
$$

This completes the proof. ∎

The following result was proved in [13].

Lemma 3.7 (Existence of isospectral approximants)**.** *Given $\varepsilon > 0$, there is $\delta > 0$ such that, for any 2 families of m pairwise commuting normal matrices $X_1, \dots, X_m$ and $Y_1, \dots, Y_m$ which satisfy the constraints $\|X_j - Y_j\| \leq \delta$ for each $1 \leq j \leq N$, there is a constant K_m and a unitary $W \in \mathbb{U}(n)$ such that the inner $*$-homomorphism $\Psi = \mathrm{Ad}[W]$ satisfies the conditions: $\sigma(\Psi(X_j)) = \sigma(X_j)$, $[\Psi(X_j), Y_j] = 0$, and $\max\{\|\Psi(X_j) - Y_j\|, \|\Psi(X_j) - X_j\|\} \leq K_m\delta$ for every $1 \leq j \leq N$.*

Remark 3.2. The constant K_m in the statement of Lemma 3.7 depends only on m.

Lemma 3.8. *Given any $\varepsilon \geq 0$ and m nonconstant polynomials $p_1(x), \dots, p_m(x)$ over $\mathbb{C}$, there is $\delta \geq 0$ such that, for any integer $n \geq 1$ and any $2m$-tuples $(X_1, \dots, X_m)$, $(Y_1, \dots, Y_m)$ in $\mathbb{I}_n^m$ which satisfy the relations*

$$\begin{cases} p_j(X_j) = p_j(Y_j) = \mathbf{0}_n, & 1 \leq j \leq m \\ \eth((X_1, \dots, X_m), (Y_1, \dots, Y_m)) \leq \delta, \end{cases}$$

there is a path $\{\Psi_t\}_{t\in[0,1]}$ $*$-homomorphisms $\Psi_t : M_n \to M_n$ such that the extended maps $\breve{\Psi}_t : M_n^m \to M_n^m$ satisfy the following relations,

$$\begin{cases} \breve{\Psi}_0(Y_1, \dots, Y_m) = (X_1, \dots, X_m), \\ \breve{\Psi}_1(Y_1, \dots, Y_m) = (Y_1, \dots, Y_m), \end{cases}$$

together with the constraint

$$\eth\big(\breve{\Psi}_t(Y_1, \dots, Y_m), (Y_1, \dots, Y_m)\big) \leq \varepsilon,$$

for each $0 \leq t \leq 1$.

Proof. By changing basis if necessary, we can assume that $Y_1, \dots, Y_m$ are diagonal matrices. By Lemma 3.1, the result is clear when $n = 1$ or $|\sigma(X_j)| = |\sigma(Y_j)| = \deg(p_j) = 1$, for each $1 \leq j \leq m$. Without loss of generality, let us assume that $\max_{1\leq j\leq m} \deg(p_j) \geq 2$, $\max_{1\leq j\leq m} |\sigma(X_j)| \geq 2$, $\max_{1\leq j\leq m} |\sigma(Y_j)| \geq 2$, and $n \geq 2$. Then let us set $K := \prod_{j=1}^m \deg(p_j)$ and $L := \max_{1\leq j\leq m} \deg(p_j)$, and let us consider the sets $\mathbf{Z}(p_j) = \{z \in \mathbb{D}^2 \mid p_j(z) = 0\}$, $1 \leq j \leq m$.

By Lemma 3.7 there are a constant K_m, a unitary $\hat{W} \in M_n$, and an inner $*$-homomorphism $\Psi = \mathrm{Ad}[\hat{W}] : M_n \to M_n$ such that $[\Psi(X_j), Y_j] = 0$ and $\|\Psi(X_j) - Y_j\| \leq K_m\delta$. Let $\varepsilon > 0$ be given. It is enough to consider the case $\varepsilon < 4\sin(1/8) < 1/2$. Since $\max\{\|X_j\|, \|Y_j\|\} \leq 1$ for each $1 \leq j \leq m$, for the rest of the proof we consider only the sets $\mathbf{Z}(p_j) \cap [-1, 1]$, $1 \leq j \leq r$. Let $h_p > 0$ be a number chosen so that

$$h_p \leq \frac{1}{3K_m} \min_{1\leq j\leq N} \Big\{ \min_{x,y\in\mathbf{Z}(p_j)\cap[-1,1]} \big\{ |x - y| \bigm| x \neq y \big\} \Big\};$$

since $\mathbf{Z}(p_j) \cap [-1, 1] \subset [-1, 1]$ for each $1 \leq j \leq r$, it holds that $h_p \leq 2$. We have that there is $\delta > 0$ that can be chosen so that

$$\begin{aligned}\delta &\leq \frac{2h_p \arcsin(\varepsilon/4)}{3\pi\sqrt{2}mK_mKL(L-1)}\\ &< \frac{h_p\varepsilon}{2} \leq \min\Big\{\varepsilon, \frac{1}{3K_m}\min_{1\leq j\leq N}\Big\{\min_{x,y\in\mathbf{Z}(p_j)\cap[-1,1]}\big\{|x-y| \,\big|\, x\neq y\big\}\Big\}\Big\}. \end{aligned}\tag{3.36}$$

Since $\delta < \frac{1}{3K_m}\min_{1\leq j\leq N}\{\min_{x,y\in\mathbf{Z}(p_j)\cap[-1,1]}\{|x-y|\,\mathrm{diag}\, x\neq y\}\}$ and also since $p_j(X_j) = p_j(\Psi(X_j)) = p_j(Y_j) = \mathbf{0}_n$, we have $Y_j = \hat{W}X_j\hat{W}^* = \Psi(X_j)$, since otherwise we get a contradiction.

By Remark 3.1, for each $1 \leq j \leq m$ there is an OPU $\{P_{j,k_j}\}$ such that $Y_j \in \operatorname{span}\{P_{j,k}\}$, and by Corollary 3.1 we have that the projective refinement $\mathcal{P} := \mathcal{R}(\{P_{1,k_1}\},\ldots,\{P_{m,k_m}\}) = \{P_1,\ldots,P_{K'}\}$ is an OPU with $|\mathcal{P}| \leq K' \leq K$ such that $Y_j \in \operatorname{span}\mathcal{P}$ for each $1 \leq j \leq m$.

By (3.34), (3.36), and Lemma 3.6, there is a unitary Z that satisfies the constraint $\|Z - \hat{W}\| \leq \frac{4}{\pi}\arcsin(\varepsilon/4)$, together with the relations $[Z, \Psi(X_j)] = [Z, Y_j] = 0$, $1 \leq j \leq N$. If we set $W := \hat{W}^*Z$, then

$$WY_jW^* = \hat{W}^*Y_j\hat{W} = \Psi^{-1}(Y_j) = X_j \tag{3.37}$$

for each $1 \leq j \leq m$. Moreover, as a consequence of the proof of [4, Theorem 5.2], there is a skew Hermitian matrix $K \in M_n$ that satisfies the relations.

$$\begin{cases} e^K = W, \\ \|K\| \leq \frac{\pi}{2}\|\mathbf{1}_n - W\| = \frac{\pi}{2}\|Z-\hat{W}\| \leq \frac{\pi}{2}\frac{4}{\pi}\arcsin(\varepsilon/2) = 2\arcsin(\varepsilon/4). \end{cases}\tag{3.38}$$

For any $t \in [0,1]$, we get

$$|1 - e^{\mathbf{i}t}| = 2\sin\Big(\frac{t}{2}\Big). \tag{3.39}$$

As a consequence of (3.38) and (3.39), if we set $W(t) = e^{tK}$ with $0 \leq t \leq 1$, then $W(t) \in \mathbb{U}(n)$ for each $t \in [0,1]$, $W(0) = \mathbf{1}_n$, $W(1) = W$, and we can obtain the estimate

$$\big\|\mathbf{1}_n - W(t)\big\| \leq 2\sin\Big(\frac{t\|K\|}{2}\Big) \leq 2\sin\Big(\frac{\|K\|}{2}\Big) \leq 2\sin\big(\arcsin(\varepsilon/4)\big) \leq \frac{\varepsilon}{2} \tag{3.40}$$

for each $t \in [0,1]$.

Let us set $\Phi_t = \mathrm{Ad}[W(t)]$; we then have $\Phi_0 = \mathrm{id}_{M_n}$, and by (3.37) we have

$$\Phi_1(Y_j) = \Psi^{-1}(Y_j) = X_j \tag{3.41}$$

for each $1 \leq j \leq m$. Furthermore, as a consequence of (3.38) and (3.40), we have

$$\begin{aligned}\big\|\Phi_t(Y_j) - Y_j\big\| &= \big\|W(t)Y_jW(t)^* - Y_j\big\|\\ &= \big\|W(t)Y_j - Y_jW(t)\big\|\\ &\leq \big\|W(t)Y_j - Y_j\big\| + \big\|Y_j - Y_jW(t)\big\|\\ &\leq 2\big\|\mathbf{1}_n - W(t)\big\| \leq \varepsilon \end{aligned}\tag{3.42}$$

for each $0 \leq t \leq 1$. If we set $\Psi_t = \Phi_{1-t}$, then each path $\{\Psi_t(Y_j)\}_{t\in[0,1]}$ is differentiable with respect to t and satisfies the relation

$$\begin{cases} \Psi_0(Y_j) = X_j, \\ \Psi_1(Y_j) = Y_j, \end{cases} \tag{3.43}$$

together with the constraint

$$\left\| \Psi_t(Y_j) - Y_j \right\| \le \varepsilon \tag{3.44}$$

for each $1 \le j \le m$ and each $t \in [0,1]$. By (3.43) and (3.44), we have

$$\begin{cases} \breve{\Psi}_0(Y_1,\dots,Y_m) = (\Psi_0(Y_1),\dots,\Psi_0(Y_m)) = (X_1,\dots,X_m), \\ \breve{\Psi}_1(Y_1,\dots,Y_m) = (\Psi_1(Y_1),\dots,\Psi_1(Y_m)) = (Y_1,\dots,Y_m), \end{cases}$$

and

$$\eth\big(\breve{\Psi}_t(Y_1,\dots,Y_m),(Y_1,\dots,Y_m)\big) = \max_{1\le j\le m} \left\| \Psi_t(Y_j) - Y_j \right\| \le \varepsilon$$

for each $0 \le t \le 1$. This completes the proof. □

Theorem 3.1. *Given any $\varepsilon \ge 0$ and r nonconstant polynomials $p_1(x_1,\dots,x_m)$, $\dots, p_r(x_1,\dots,x_m)$ of m complex variables, with coefficients over $\mathbb{C}$, and with finite zero set $\mathbf{Z}(p_1,\dots,p_r) \subset [-1,1]^m$, there is $\delta > 0$ such that, for any integer $n \ge 1$ and any 2 m-tuples $(X_1,\dots,X_m)$, $(Y_1,\dots,Y_m)$ in $\mathbb{I}_n^m$ which satisfy the relations*

$$\begin{cases} p_j(X_1,\dots,X_m) = p_j(Y_1,\dots,Y_m) = \mathbf{0}_n, \quad 1 \le j \le r, \\ \eth((X_1,\dots,X_m),(Y_1,\dots,Y_m)) \le \delta, \end{cases}$$

there is a path $\varphi = (\varphi_1,\dots,\varphi_m) \in C([0,1],\mathbb{I}_n^m)$ that satisfies the relations

$$\begin{cases} \varphi(0) = (X_1,\dots,X_m), \\ \varphi(1) = (Y_1,\dots,Y_m), \end{cases} \tag{3.45}$$

together with the constraints

$$\begin{cases} p_j(\varphi(t)) = \mathbf{0}_n, \quad 1 \le j \le r, \\ \eth(\varphi(t),(Y_1,\dots,Y_m)) \le \varepsilon, \end{cases} \tag{3.46}$$

for each $0 \le t \le 1$.

Proof. Let r polynomials $p_1(x_1,\dots,x_m),\dots,p_r(x_1,\dots,x_m)$ of m complex variables, with coefficients over $\mathbb{C}$ be given, as in the statement of this theorem. Let us set $L = |\mathbf{Z}(p_1,\dots,p_r)| < \infty$. Then $\mathbf{Z}(p_1,\dots,p_r)$ can be listed in the form

$$\mathbf{Z}(p_1,\dots,p_r) = \big\{(x_{j,1},\dots,x_{j,m}) \bigm| 1 \le j \le L\big\}. \tag{3.47}$$

For each $1 \le k \le m$, let us set $Z_k = \{x_{j,k} \mid 1 \le j \le L\}$, and let us write $\check{Z}_k$ to denote the set consisting of all distinct numbers in Z_k counted without multiplicity. We then have $\check{Z}_k \subseteq Z_k$ and, for each $x \in Z_k$, there is $y \in \check{Z}_k$ such that $x = y$ for each $1 \le k \le m$. Let us set

$$\hat{p}_k(x_k) = \prod_{y\in\check{Z}_k} (x_k - y), \tag{3.48}$$

for each $1 \leq k \leq m$. We have that each $\hat{p}_k(x_k)$ is a polynomial over $\mathbb{C}$ such that $1 \leq \deg(\hat{p}_k) = |\check{Z}_k| \leq |Z_k|$, and that $\hat{p}_k(x) = 0$ for every $x \in Z_k$ and each $1 \leq k \leq m$.

Given any two $(X_1, \ldots, X_m), (Y_1, \ldots, Y_m)$ in $\mathbb{I}_n^m$ as in the statement of this theorem, as a direct application of multivariate functional calculus for commuting matrices, we have, for each $1 \leq j \leq r$ and each $1 \leq k \leq n$,

$$\begin{cases} p_j(\Lambda^{(k)}(X_1, \ldots, X_m)) = p_j(\Lambda^{(k)}(X_1), \ldots, \Lambda^{(k)}(X_m)) = 0, \\ p_j(\Lambda^{(k)}(Y_1, \ldots, Y_m)) = p_j(\Lambda^{(k)}(Y_1), \ldots, \Lambda^{(k)}(Y_m)) = 0. \end{cases} \tag{3.49}$$

By (3.49), $\Lambda(X_1, \ldots, X_m), \Lambda(Y_1, \ldots, Y_m) \subseteq \mathbf{Z}(p_1, \ldots, p_r)$, and this implies that

$$\hat{p}_j(X_j) = \hat{p}_j\big(\Lambda(X_j)\big) = \mathbf{0}_n = \hat{p}_j\big(\Lambda(Y_j)\big) = \hat{p}_j(Y_j) \tag{3.50}$$

for each $1 \leq j \leq m$.

Given $\varepsilon > 0$, by Lemma 3.8 applied to $\hat{p}_1, \ldots, \hat{p}_m$ and also to any 2 m-tuples $(X_1, \ldots, X_m), (Y_1, \ldots, Y_m) \in \mathbb{I}_n^m$ as in the statement of this theorem, there are $\delta > 0$ and a family of $*$-homomorphisms $\Psi_t : M_n \to M_n$ such that

$$\begin{cases} \breve{\Psi}_0(Y_1, \ldots, Y_m) = (X_1, \ldots, X_m), \\ \breve{\Psi}_1(Y_1, \ldots, Y_m) = (Y_1, \ldots, Y_m), \end{cases} \tag{3.51}$$

and

$$\eth\big(\breve{\Psi}_t(Y_1, \ldots, Y_m), (Y_1, \ldots, Y_m)\big) \leq \varepsilon \tag{3.52}$$

for each $0 \leq t \leq 1$, whenever $\eth((X_1, \ldots, X_m), (Y_1, \ldots, Y_m)) \leq \delta$. Let us set $\varphi(t) = (\varphi_1(t), \ldots, \varphi_m(t))$, with $\varphi_j(t) = \Psi_t(Y_j)$ for $1 \leq j \leq m$ and $0 \leq t \leq 1$; then by (3.51) and (3.52), we have

$$\begin{cases} \varphi(0) = (X_1, \ldots, X_m), \\ \varphi(1) = (Y_1, \ldots, Y_m), \\ \eth(\varphi(t), (Y_1, \ldots, Y_m)) \leq \varepsilon. \end{cases} \tag{3.53}$$

Furthermore, for each $1 \leq i, j \leq m$, $1 \leq k \leq r$, and each $0 \leq t \leq 1$, we get

$$\varphi_j(t)^* = \big(\Psi_t(Y_j)\big)^* = \Psi_t(Y_j^*) = \Psi_t(Y_j) = \varphi_j(t), \tag{3.54}$$

$$\begin{aligned} \varphi_j(t)\varphi_i(t) - \varphi_i(t)\varphi_j(t) &= \Psi_t(Y_j)\Psi_t(Y_i) - \Psi_t(Y_i)\Psi_t(Y_j) \\ &= \Psi_t(Y_jY_i - Y_iY_j) = \Psi_t(\mathbf{0}_n) = \mathbf{0}_n, \end{aligned} \tag{3.55}$$

$$\begin{aligned} p_k\big(\varphi(t)\big) &= p_k\big(\Psi_t(Y_1), \ldots, \Psi_t(Y_m)\big) \\ &= \Psi_t\big(p_k(Y_1, \ldots, Y_m)\big) = \Psi_t(\mathbf{0}_n) = \mathbf{0}_n. \end{aligned} \tag{3.56}$$

By the definition of φ, and by (3.54) and (3.55), we get $\varphi \in C^1([0,1], \mathbb{I}_n^m)$. And by (3.53) and (3.56), it holds that the path $\varphi \in C^1([0,1], \mathbb{I}_n^m)$ satisfies the conditions (3.45) and (3.46). This completes the proof. $\square$

Definition 3.2. We say that a matrix set $\mathbb{S}_n^m \subseteq M_n^m$ is uniformly piecewise differentiably path-connected with respect to the metric $\eth$ if, given $\varepsilon > 0$, there is $\delta > 0$ such that, for any $\mathbf{X} \in \mathbb{S}_n^m$ and any $\mathbf{Y} \in \mathbb{S}_n^m \cap B_\eth(\mathbf{X}, \delta)$, there is a piecewise C^1-path $\gamma \in C([0,1], \mathbb{S}_n^m)$ such that $\gamma(0) = \mathbf{X}$, $\gamma(1) = \mathbf{Y}$ and $\gamma(t) \in B_\eth(\mathbf{X}, \varepsilon)$ for each $0 \leq t \leq 1$.

Corollary 3.2. *Let there be given r nonconstant polynomials $p_1(x_1,\dots,x_m),\dots,p_r(x_1,\dots,x_m)$ of m complex variables, with coefficients over $\mathbb{C}$, and with finite zero set $\mathbf{Z}(p_1,\dots,p_r)\subset\mathbb{R}^m$. The algebraic matrix m-cube $\mathbb{ZI}_n^m(p_1,\dots,p_r)$ is uniformly piecewise differentiably path-connected with respect to the metric $\eth$.*

Proof. Let $\varepsilon>0$. Let us consider any r nonconstant polynomials $p_1(x_1,\dots,x_m),\dots,p_r(x_1,\dots,x_m)$ of m complex variables with coefficients over $\mathbb{C}$ and with finite zero set $\mathbf{Z}(p_1,\dots,p_r)\subset\mathbb{R}^m$. By Theorem 3.1, there is $\delta>0$ such that, for any $\mathbf{X}\in\mathbb{ZI}_n^m(p_1,\dots,p_r)$ and any $\mathbf{Y}\in\mathbb{ZI}_n^m(p_1,\dots,p_r)\cap B_\eth(\mathbf{X},\delta)$, there is a piecewise C^1-path $\gamma\in C([0,1],\mathbb{ZI}_n^m(p_1,\dots,p_r))$ such that $\gamma(0)=\mathbf{X}$, $\gamma(1)=\mathbf{Y}$ and $\gamma(t)\in B_\eth(\mathbf{X},\varepsilon)$ for each $0\le t\le 1$. This completes the proof. □

Theorem 3.2. *Let there be given r nonconstant polynomials $p_1(z_1,\dots,z_m),\dots,p_r(z_1,\dots,z_m)$ of m complex variables with coefficients over $\mathbb{C}$ and with finite zero set $\mathbf{Z}(p_1,\dots,p_r)\subset\mathbb{C}^m$. The algebraic matrix m-disk $\mathbb{ZD}_n^m(p_1,\dots,p_r)$ is uniformly piecewise differentiably path-connected with respect to the metric $\eth$.*

Proof. Let $\varepsilon>0$. Let us consider any r nonconstant polynomials $p_1(x_1,\dots,x_m),\dots,p_r(x_1,\dots,x_m)$ of m complex variables, with coefficients over $\mathbb{C}$ and with finite zero set $\mathbf{Z}(p_1,\dots,p_r)\subset\mathbb{C}^m$. By elementary theory of complex valued functions of several complex variables, we have that there are $2r$ polynomials $\mathrm{Re}(p_1),\dots,\mathrm{Re}(p_r)$, $\mathrm{Im}(p_1),\dots,\mathrm{Im}(p_r)$ in $2m$ real variables $\mathrm{Re}(z_1),\dots,\mathrm{Re}(z_m)$, $\mathrm{Im}(z_1),\dots,\mathrm{Im}(z_m)$ with coefficients over $\mathbb{R}$ and also such that $\mathbf{z}=(z_1,\dots,z_m)\in\mathbf{Z}(p_1,\dots,p_r)\subset\mathbb{C}^m$ if and only if it holds that

$$\big(\mathrm{Re}(\mathbf{z}),\mathrm{Im}(\mathbf{z})\big)\in\mathbf{Z}\big(\mathrm{Re}(p_1),\mathrm{Im}(p_1),\dots,\mathrm{Re}(p_r),\mathrm{Im}(p_r)\big)\subset\mathbb{R}^{2m},$$

where $\mathrm{Re}(\mathbf{z})=(\mathrm{Re}(z_1),\dots,\mathrm{Re}(z_m))$ and $\mathrm{Im}(\mathbf{z})=(\mathrm{Im}(z_1),\dots,\mathrm{Im}(z_m))$.

Let us consider the maps $\imath:\mathbb{C}^m\to\mathbb{R}^{2m}$, $\mathbf{z}\mapsto(\mathrm{Re}(\mathbf{z}),\mathrm{Im}(\mathbf{z}))$, and $\kappa:\mathbb{R}^{2m}\to\mathbb{C}^m$, $\mathbf{x}\mapsto(x_1+\mathbf{i}x_{m+1},\dots,x_m+\mathbf{i}x_{2m})$. By the arguments in the preceding paragraph, we have that there is a one-to-one correspondence between $\mathbf{Z}(p_1,\dots,p_r)$ and $\mathbf{Z}(\mathrm{Re}(p_1),\mathrm{Im}(p_1),\dots,\mathrm{Re}(p_r),\mathrm{Im}(p_r))$ induced by $\imath\circ\kappa$ and $\kappa\circ\imath$. This in turn implies that there is a one-to-one correspondence between $\mathbb{ZD}_n^m(p_1,\dots,p_r)$ and $\mathbb{ZI}_n^{2m}(\mathrm{Re}(p_1),\ \mathrm{Im}(p_1),\dots,\mathrm{Re}(p_r),\ \mathrm{Im}(p_r))$, induced by the maps $\hat\pi\circ\upsilon$ and $\upsilon\circ\hat\pi$ defined in Section 2.

By the definition of $\hat\pi$ and υ and by the one-to-one correspondence between $\mathbb{ZD}_n^m(p_1,\dots,p_r)$ and $\mathbb{ZI}_n^{2m}(\mathrm{Re}(p_1),\mathrm{Im}(p_1),\dots,\mathrm{Re}(p_r),\mathrm{Im}(p_r))$ described in the previous paragraph, we have the following. Given $\mathbf{X},\mathbf{Y}\in\mathbb{ZD}_n^m(p_1,\dots,p_r)$, on one hand, we have $\hat\pi(\mathbf{X}),\hat\pi(\mathbf{Y})\in\mathbb{ZI}_n^{2m}(\mathrm{Re}(p_1),\mathrm{Im}(p_1),\dots,\mathrm{Re}(p_r),\mathrm{Im}(p_r))$ and $\eth(\hat\pi(\mathbf{X}),\hat\pi(\mathbf{Y}))\le\eth(\mathbf{X},\mathbf{Y})$. On the other hand, for any $\mathbf{S},\mathbf{T}\in\mathbb{ZI}_n^{2m}(\mathrm{Re}(p_1),\mathrm{Im}(p_1),\dots,\mathrm{Re}(p_r),\mathrm{Im}(p_r))$, $\upsilon(\mathbf{S}),\upsilon(\mathbf{T})\in\mathbb{ZD}_n^m(p_1,\dots,p_r)$ and $\eth(\upsilon(\mathbf{X}),\upsilon(\mathbf{X}))\le 2\eth(\mathbf{S},\mathbf{T})$.

By Theorem 3.1 and by the arguments above, there is $\delta>0$ such that, for any $\mathbf{X}\in\mathbb{ZD}_n^m(p_1,\dots,p_r)$ and any $\mathbf{Y}\in\mathbb{ZD}_n^m(p_1,\dots,p_r)\cap B_\eth(\mathbf{X},\delta)$, there is a piecewise C^1-path $\gamma_H\in C([0,1],\mathbb{ZI}_n^{2m}(\mathrm{Re}(p_1),\mathrm{Im}(p_1),\dots,\mathrm{Re}(p_r),\mathrm{Im}(p_r)))$ such that $\gamma_H(0)=\hat\pi(\mathbf{X})$, $\gamma_H(1)=\hat\pi(\mathbf{Y})$ and $\gamma_H(t)\in B_\eth(\hat\pi(\mathbf{X}),\varepsilon/2)$ for each $0\le t\le 1$. This in turn implies that the path $\gamma=\upsilon\circ\gamma_H\in C([0,1],\mathbb{ZD}_n^m(p_1,\dots,p_r))$, which is clearly piecewise C^1, satisfies the conditions $\gamma(0)=\mathbf{X}$, $\gamma(1)=\mathbf{Y}$ and $\gamma(t)\in B_\eth(\hat\pi(\mathbf{X}),\varepsilon)$ for each $0\le t\le 1$. This completes the proof. □

3.4. Uniform path connectivity of nearly algebraic normal contractions. In order to extend the applicability of the results presented in Section 3.3, in this section we solve some connectivity problems on what we call *nearly algebraic matrix sets*.

Given $\epsilon > 0$, any $n \in \mathbb{Z}^+$, and any r nonconstant polynomials $p_1(x_1, \ldots, x_m), \ldots, p_r(x_1, \ldots, x_m)$ of m complex variables with coefficients over $\mathbb{R}$, we write $\mathbb{ZI}^m_{n,\epsilon}(p_1, \ldots, p_r)$ to denote the subset of $\mathbb{I}^m(n)$ determined by the expression

$$\begin{aligned}&\mathbb{ZI}^m_{n,\epsilon}(p_1, \ldots, p_r)\\ &\quad = \left\{(X_1, \ldots, X_m) \in \mathbb{I}^m_n \mid \left\|p_j(X_1, \ldots, X_m)\right\| \leq \epsilon, 1 \leq j \leq r\right\}.\end{aligned} \tag{3.57}$$

The Hermitian nearly algebraic matrix set $\mathbb{ZI}^m_{n,\epsilon}(p_1, \ldots, p_r)$ is called an *ϵ-nearly algebraic matrix m-cube* in this document.

Given $\epsilon > 0$, any $n \in \mathbb{Z}^+$, and any r nonconstant polynomials $p_1(z_1, \ldots, z_m), \ldots, p_r(z_1, \ldots, z_m)$ of m complex variables with coefficients over $\mathbb{C}$, we write $\mathbb{ZD}^m_{n,\epsilon}(p_1, \ldots, p_r)$ to denote the subset of $\mathbb{D}^m_n$ determined by the expression

$$\begin{aligned}&\mathbb{ZD}^m_{n,\epsilon}(p_1, \ldots, p_r)\\ &\quad = \left\{(X_1, \ldots, X_m) \in \mathbb{D}^m_n \mid \left\|p_j(X_1, \ldots, X_m)\right\| \leq \epsilon, 1 \leq j \leq r\right\}.\end{aligned} \tag{3.58}$$

The normal nearly algebraic matrix set $\mathbb{ZD}^m_{n,\epsilon}(p_1, \ldots, p_r)$ is called an *ϵ-nearly algebraic matrix m-disk* in this document.

Given $\delta > 0$ and any collection of simultaneously commuting normal matrix contractions $X_1, \ldots, X_m \in M_n$, we prove that there are a collection of simultaneously commuting normal matrix contractions $\tilde{X}_1, \ldots, \tilde{X}_m \in M_n$, together with m-polynomials $p_1(x_1), \ldots, p_r(x_m)$ with coefficients over $\mathbb{C}$, such that $\|X_j - \tilde{X}_j\| \leq \delta$, $X_j \hat{X}_k = \hat{X}_k X_j$, and $p_j(\tilde{X}_j) = \mathbf{0}_n$, $1 \leq j, k \leq m$. Moreover, we have that, for each $1 \leq j \leq m$, $\deg(p_j)$ does not depend on n. The collection $\tilde{X}_1, \ldots, \tilde{X}_m$ is called *δ-clustered pseudospectral approximants* (CPA_δ) of $X_1, \ldots, X_m$.

In terms of m-tuples in $\mathbb{D}^m_n$, we write $\mathbf{CPA}_\delta(X_1, \ldots, X_m) = (\tilde{X}_1, \ldots, \tilde{X}_m)$ to indicate that the components $\tilde{X}_1, \ldots, \tilde{X}_m$ of the m-tuple $(\tilde{X}_1, \ldots, \tilde{X}_m) \in \mathbb{ZD}^m_n(p_1, \ldots, p_j) \cap B_{\eth}((X_1, \ldots, X_m), \delta)$ are δ-clustered pseudospectral approximants of $X_1, \ldots, X_m$. Using this notation, it is enough to prove the following lemma, in order to solve the matrix approximation problem stated above.

Lemma 3.9. *Given any $(X_1, \ldots, X_m) \in \mathbb{D}^m_n$, the problem $\mathbf{CPA}_\delta(X_1, \ldots, X_m) = (\tilde{X}_1, \ldots, \tilde{X}_m)$ is solvable for any $\delta > 0$.*

Proof. Let $\delta > 0$. Because of the one-to-one correspondence between $\mathbb{D}^m_n$ and $\mathbb{I}^{2m}_n$, induced by $\hat{\pi} \circ \upsilon$ and $\upsilon \circ \hat{\pi}$, together with the constraint $\eth(\mathbf{X}, \mathbf{X}') \leq 2\eth(\hat{\pi}(\mathbf{X}), \hat{\pi}(\mathbf{X}'))$ that is satisfied for any $\mathbf{X}, \mathbf{X}' \in \mathbb{D}^m_n$, we have that it is enough to solve the problem in $\mathbf{CPA}_\delta(X_1, \ldots, X_m) = (\tilde{X}_1, \ldots, \tilde{X}_m)$ for any $(X_1, \ldots, X_m) \in \mathbb{I}^m_n$.

Let $\mathbf{X} = (X_1, \ldots, X_m) \in \mathbb{I}^m_n$ be given. Let us assume for simplicity that $1/\delta = N_\delta \in \mathbb{Z}^+$. Let us consider the grid $X_\delta := \{x_k = -1 + k\delta \mid 0 \leq k \leq N_\delta - 1\} \subset [-1, 1]$. It can be seen that the set X_δ is δ-dense in $[-1, 1]$. Moreover, if $\chi_{[a,b)}$ denotes the characteristic function of the interval $[a, b)$, then it is clear that the simple function $\hat{p}(x) = \sum_{j=0}^{N_\delta-2}(-1+(j+1/2)\delta)\chi_{[x_j,x_{j+1})}$ is a δ-approximation of the identity function id on $L^\infty([-1,1])$.

By Borel normal (matrix) functional calculus, we have that, if for each $1 \leq j \leq m$ we set $\tilde{X}_j = \hat{p}(X_j)$, then $\|\tilde{X}_j - X_j\| = \|\hat{p}(X_j) - \mathrm{id}(X_j)\| \leq \|\hat{p} - \mathrm{id}\|_{L^\infty([-1,1])} \leq \delta$. By Borel functional calculus, we also have $[\tilde{X}_j, X_k] = [\hat{p}(X_j), X_k] = \mathbf{0}_n$ for each $1 \leq j, k \leq m$.

Now let us set $p(x) = \prod_{j=0}^{N_\delta-2}(x-(-1+(j+1/2)\delta))$, and let us write $p_j(x_j)$ to denote the minimal polynomial of $\tilde{X}_j$ for each $1 \leq j \leq m$. By the definition of each $\tilde{X}_j$, we have that $\deg(p_j) \leq \deg(p) < N_\delta$ does not depend on n in general, and in particular when $N_\delta \geq 1$. It can be seen that the m-tuple $(\tilde{X}_1, \ldots, \tilde{X}_m) \in \mathbb{ZI}_n^m(p_1, \ldots, p_m)$ solves the approximation problem $\mathbf{CPA}_\delta(X_1, \ldots, X_m) = (\tilde{X}_1, \ldots, \tilde{X}_m)$. This completes the proof. $\square$

Remark 3.3. Given $\delta > 0$ and any $(X_1, \ldots, X_m) \in \mathbb{D}_n^m$, from the proof of Lemma 3.9 we have that one can find two disjoint finite grids $X_\delta, X'_\delta \subset [-1,1]$ such that, by applying some retraction $\rho : [-1,1]\backslash X_\delta \to X'_\delta$ (if necessary) to each component $\tilde{X}_j$ of the solution of the problem $\mathbf{CPA}_\delta(X_1, \ldots, X_m) = (\tilde{X}_1, \ldots, \tilde{X}_m)$ determined by Lemma 3.9, one can obtain a *preconditioned* m-tuple $(\hat{X}_1, \ldots, \hat{X}_m) \in \mathbb{ZD}_{n,\epsilon}(p_1, \ldots, p_r)$, where $\hat{X}_j = \rho(\tilde{X}_j)$, for each $1 \leq j \leq m$, and $p_1, \ldots, p_r$ are determined by ρ, X_δ, and X'_δ.

Theorem 3.3. *Given any $\varepsilon \geq 0$ and r nonconstant polynomials $p_1(x_1, \ldots, x_m)$, $\ldots, p_r(x_1, \ldots, x_m)$ of m complex variables, with coefficients over $\mathbb{C}$, and with finite zero set $\mathbf{Z}(p_1, \ldots, p_r) \subset [-1,1]^m$, there are $\delta, \delta' > 0$ such that for any integer $n \geq 1$ and any 2 m-tuples $(X_1, \ldots, X_m)$, $(Y_1, \ldots, Y_m)$ in $\mathbb{I}^m(n)$ which satisfy the relations*

$$\begin{cases} \|p_j(X_1, \ldots, X_m)\| \leq \delta', \\ \|p_j(Y_1, \ldots, Y_m)\| \leq \delta', \\ \eth((X_1, \ldots, X_m), (Y_1, \ldots, Y_m)) \leq \delta. \end{cases}$$

For each $1 \leq j \leq r$ there is a path $\varphi \in C^1([0,1], \mathbb{I}_n^m)$ that satisfies the relations

$$\begin{cases} \varphi(0) = (X_1, \ldots, X_m), \\ \varphi(1) = (Y_1, \ldots, Y_m), \end{cases} \tag{3.59}$$

together with the constraints

$$\begin{cases} \|p_j(\varphi(t))\| < \delta', \quad 1 \leq j \leq r, \\ \eth(\varphi(t), (Y_1, \ldots, Y_m)) \leq \varepsilon \end{cases} \tag{3.60}$$

for each $0 \leq t \leq 1$.

Proof. Let $\varepsilon > 0$ be given, and without loss of generality let us assume that $\varepsilon < 4\sin(1/8) < 1/2$. Given any polynomials $p_1, \ldots, p_r$ as in the statement of this theorem, let us assume that $|\mathbf{Z}(p_1, \ldots, p_r)| \geq 2$, as the following argument can be easily modified when $|\mathbf{Z}(p_1, \ldots, p_r)| = 1$. Since $\mathbf{Z}(p_1, \ldots, p_r)$ is finite, if we set

$$\delta_1 = \frac{1}{3} \min_{\mathbf{x},\mathbf{y}\in\mathbf{Z}(p_1,\ldots,p_r)} \{\|\mathbf{x}-\mathbf{y}\|_2 \mid \mathbf{x}\neq\mathbf{y}\} > 0, \tag{3.61}$$

then for any $(x_1,\ldots,x_m),(x'_1,\ldots,x'_m) \in \mathbf{Z}(p_1,\ldots,p_r)$ we then have that $B((x_1,\ldots,x_m),\delta_1/2)\cap B((x'_1,\ldots,x'_m),\delta_1/2)=\emptyset$.

By continuity of each $p_j(x_1,\ldots,x_m)$, we have that, for any $\delta'>0$ chosen so that $\delta'\leq\min\{\delta_1,\varepsilon/2\}$, there is $\delta_2>0$ such that, for each $(x_1,\ldots,x_m)\in\mathbf{Z}(p_1,\ldots,p_r)$ and any $(y_1,\ldots,y_m)\in B((x_1,\ldots,x_m),\delta_2/2)$, we have that the inequality $|p_j(y_1,\ldots,y_m)|<\delta'$ is satisfied for each $1\leq j\leq r$.

Let us set $\delta_3=\min\{\delta_1,\delta_2\}$. Given $\nu>0$, let us write $Z_\nu(p_1,\ldots,p_r)$ to denote the set determined by the following expression.

$$Z_\nu(p_1,\ldots,p_r) = \bigcup_{\mathbf{x}\in\mathbf{Z}(p_1,\ldots,p_r)} B(\mathbf{x},\nu). \tag{3.62}$$

Since $|\mathbf{Z}(p_1,\ldots,p_r)|\geq 2$, given $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ in the approximate zero set $Z_{\delta_3/2}(p_1,\ldots,p_r)$, such that $\tilde{\mathbf{x}}\in B(\mathbf{x},\delta_3/2)$ and $\tilde{\mathbf{y}}\in B(\mathbf{y},\delta_3/2)$ for some $\mathbf{x},\mathbf{y}\in\mathbf{Z}(p_1,\ldots,p_r)$, by (3.61) and (3.62) we have that $\|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}\|_2>\delta_2$. This implies that for any $\tilde{\mathbf{x}},\tilde{\mathbf{x}}'\in Z_{\delta_3/2}(p_1,\ldots,p_3)$ such that $\|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'\|<\delta_3$, there is $\mathbf{x}\in\mathbf{Z}(p_1,\ldots,p_r)$ such that $\tilde{\mathbf{x}},\tilde{\mathbf{x}}'\in B(\mathbf{x},\delta_3/2)$. This implies that, for any $\mathbf{x},\mathbf{y}\in\mathbb{R}^m$ that satisfy the inequality $\|\mathbf{x}-\mathbf{x}\|_2<\delta_3$, together with the constraints $|p_j(\mathbf{x})|<\delta'$ and $|p_j(\mathbf{y})|<\delta'$ for each $1\leq j\leq r$, we have that $\mathbf{x},\mathbf{y}\in Z_{\delta_3/2}(p_1,\ldots,p_r)$, otherwise we get a contradiction.

As a consequence of the preceding arguments, we then have that for any two $\mathbf{X}=(X_1,\ldots,X_m)$ and $\mathbf{X}'=(X'_1,\ldots,X'_m)$ in $\mathbb{I}_n^m$ that satisfy the inequality $\eth(\mathbf{X},\mathbf{X}')<\delta_3$, together with the constraints $\|p_j(X_j,\ldots,X_m)\|<\delta'$ and $\|p_j(X'_j,\ldots,X'_m)\|<\delta'$ for each $1\leq j\leq r$, we have that $\Lambda(X_1,\ldots,X_m)$, $\Lambda(X'_1,\ldots,X'_m)\in Z_{\delta_3/2}(p_1,\ldots,p_r)$.

Given any two m-tuples $\mathbf{H}=(H_1,\ldots,H_m)$, $\mathbf{H}'=(H'_1,\ldots,H'_m)$ in $\mathbb{I}_n^m$ such that $\Lambda(H_1,\ldots,H_m)$, $\Lambda(H'_1,\ldots,H'_m)\in Z_{\delta_3/2}(p_1,\ldots,p_r)$, let us consider a basis in which $\Lambda(H'_j)=\mathrm{diag}[\Lambda^{(1)}(H_j),\ldots,\Lambda^{(n)}(H_j)]=H'_j$ for each $1\leq j\leq m$. By Lemma 3.7, we have that there is a $*$-homomorphism $\Phi: M_n\to M_n$ that satisfies the conditions $\Lambda(\Phi(H_j))=\Phi(H_j)$ and $\Phi(H_j)\Lambda(H'_k)=\Lambda(H'_k)\Phi(H_j)$, together with the constraints

$$\begin{aligned}\big\|\Lambda\big(\Phi(H_j)\big)-\Lambda(H'_j)\big\| &\leq \big\|\Phi(H_j)-H_j\big\|+\|H_j-H'_j\| \\ &\leq (K_m+1)\eth\big((H_1,\ldots,H_m),(H'_1,\ldots,H'_m)\big)\end{aligned} \tag{3.63}$$

for each $1\leq j,k\leq m$. By (3.63), we have

$$\begin{aligned}\big\|\Lambda^{(k)}\big(\breve{\Phi}(\mathbf{H})\big)-\Lambda^{(k)}(\mathbf{H}')\big\|_2 &\leq \sqrt{m}\max_{1\leq j\leq m}\big|\Lambda^{(k)}\big(\Phi(H_j)\big)-\Lambda^{(k)}(H'_j)\big| \\ &\leq \sqrt{m}\max_{1\leq j\leq m}\max_{1\leq k\leq n}\big|\Lambda^{(k)}\big(\Phi(H_j)\big)-\Lambda^{(k)}(H'_j)\big| \\ &\leq \sqrt{m}\max_{1\leq j\leq m}\big\|\Lambda\big(\Phi(H_j)\big)-\Lambda(H'_j)\big\| \\ &\leq \sqrt{m}(K_m+1)\eth(\mathbf{H},\mathbf{H}')\end{aligned} \tag{3.64}$$

for each $1\leq k\leq n$. Let us set $\delta_4=\delta_3/(\sqrt{m}(K_m+1))$. By (3.64) and by the previous arguments, we have that if, in addition,

$$\eth\big((H_1,\dots,H_m),(H_1',\dots,H_m')\big) < \delta_4, \tag{3.65}$$

then there is $\hat{\mathbf{h}}_k = (\hat{h}_{k,1},\dots,\hat{h}_{k,m}) \in \mathbf{Z}(p_1,\dots,p_r)$ such that

$$\begin{cases} \|\hat{\mathbf{h}}_k - \Lambda^{(k)}(\breve{\Phi}(\mathbf{H}))\|_2 < \delta_4/2, \\ \|\hat{\mathbf{h}}_k - \Lambda^{(k)}(\mathbf{H}')\|_2 < \delta_4/2, \end{cases} \tag{3.66}$$

for each $1 \leq k \leq n$. Let us set

$$\hat{H}_j' = \mathrm{diag}[\hat{h}_{1,j},\dots,\hat{h}_{n,j}] \tag{3.67}$$

for each $1 \leq j \leq m$. We have that the m-tuples $\hat{\mathbf{H}}' = (\hat{H}_1',\dots,\hat{H}_m')$, $\hat{\mathbf{H}} = (\hat{H}_1,\dots,\hat{H}_m)$ in $\mathbb{I}_n^m$, with $\hat{H}_j = \Phi^{-1}(\hat{H}_j')$ for $1 \leq j \leq m$, satisfy the relations,

$$\begin{cases} [\hat{H}_j,\hat{H}_k] = [\hat{H}_j',\hat{H}_k'] = \mathbf{0}_n, \\ \Lambda(\Phi(\hat{H}_j)) = \Lambda(\hat{H}_j'), \\ p_k(\hat{H}_1,\dots,\hat{H}_m) = p_k(\hat{H}_1',\dots,\hat{H}_m') = \mathbf{0}_n, \\ \eth(\mathbf{H},\hat{\mathbf{H}}) < \delta_3/2, \\ \eth(\mathbf{H}',\hat{\mathbf{H}}') < \delta_3/2, \end{cases} \tag{3.68}$$

for each $1 \leq j \leq m$ and each $1 \leq k \leq r$. It can be seen that $\hat{\mathbf{H}}' = \breve{\Phi}(\hat{\mathbf{H}})$, and by (3.68), we have

$$\begin{aligned} \eth\big(\hat{\mathbf{H}},\Phi(\hat{\mathbf{H}})\big) &\leq \eth(\hat{\mathbf{H}},\mathbf{H}) + \eth(\mathbf{H},\mathbf{H}') + \eth(\mathbf{H}',\hat{\mathbf{H}}') \\ &< \delta_3/2 + \delta_4 + \delta_3/2 \leq 2\delta_3. \end{aligned} \tag{3.69}$$

Let us consider $\mathbf{X} = (X_1,\dots,X_m)$ and $\mathbf{Y} = (Y_1,\dots,Y_m)$ in $\mathbb{I}_n^m$ that satisfy the inequality $\eth(\mathbf{X},\mathbf{Y}) < \delta_4$ together with the constraints $\|p_j(X_1,\dots,X_m)\| < \delta'$ and $\|p_j(Y_1,\dots,Y_m)\| < \delta'$, for each $1 \leq j \leq r$. By the preceding arguments, together with (3.68) and (3.69), we have that there exist $\hat{\mathbf{X}} = (\hat{X}_1,\dots,\hat{X}_m)$ and $\hat{\mathbf{Y}} = (\hat{Y}_1,\dots,\hat{Y}_m)$ in $\mathbb{Z}\mathbb{I}_n^m(p_1,\dots,p_r)$ such that $\eth(\mathbf{X},\hat{\mathbf{X}}) < \delta_3/2$, $\eth(\mathbf{Y},\hat{\mathbf{Y}}) < \delta_3/2$ and $\eth(\hat{\mathbf{X}},\hat{\mathbf{Y}}) < 2\delta_3$.

By Theorem 3.1, by continuity of $p_1,\dots,p_r$, and by (3.36), we have that there are a number $\delta > 0$ that can be chosen so that $\delta < \min\{\delta',\varepsilon/2\}$ and a piecewise C^1-path $\phi \in C([0,1],\mathbb{Z}\mathbb{I}_n^m(p_1,\dots,p_r))$ such that, if $\eth((X_1,\dots,X_m),(Y_1,\dots,Y_m)) \leq \delta_3 < \delta$, then $\phi(0) = \hat{\mathbf{X}}$, $\phi(1) = \hat{\mathbf{Y}}$ and $\eth(\phi(t),\hat{\mathbf{Y}}) < \varepsilon/2$ for each $t \in [0,1]$.

As a consequence of Lemma 3.1, we have that there are piecewise C^1-paths $\phi_x,\phi_y \in C([0,1],\mathbb{I}_n^m)$ such that $\phi_x(0) = \mathbf{X}$, $\phi_x(1) = \hat{\mathbf{X}}$, $\phi_y(0) = \hat{\mathbf{Y}}$, $\phi_y(1) = \mathbf{Y}$, and in addition $\max\{\eth(\phi_x(t),\hat{\mathbf{X}}),\eth(\phi_y(t),\mathbf{Y})\} \leq \delta_3/2 < \delta/2$. By definition of δ' and by continuity of each p_j for $1 \leq j \leq r$, we then have that $\max\{\|p_j(\phi_x(t))\|, \|p_j(\phi_y(t))\|\} \leq \delta'$ for each $t \in [0,1]$.

Let us set $\varphi = (\phi_x \circledast \phi) \circledast \phi_y$. We have that the path $\varphi \in C([0,1],\mathbb{I}_n^m)$ is piecewise C^1. By the preceding arguments, we have that $\|p_j(\varphi(t))\| < \delta'$ for each $1 \leq j \leq r$ and each $t \in [0,1]$. Moreover, we have that the path φ satisfies the estimates,

$$\eth\big(\varphi(t),\mathbf{Y}\big) < \delta_3/2 + \max\{\varepsilon/2,\delta_4\} + \delta_3/2 \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \tag{3.70}$$

for each $0 \leq t \leq 1$. This completes the proof. $\square$

Corollary 3.3. *Let there be given r nonconstant polynomials $p_1(x_1,\ldots,x_m),\ldots,$ $p_r(x_1,\ldots,x_m)$ of m complex variables, with coefficients over $\mathbb{C}$, and with finite zero set $\mathbf{Z}(p_1,\ldots,p_r)\subset\mathbb{R}^m$. There is $\epsilon>0$ such that, for each $0<\epsilon'\leq\epsilon$, the ϵ'-nearly algebraic matrix m-cube $\mathbb{ZI}^m_{n,\epsilon'}(p_1,\ldots,p_r)$ is uniformly piecewise differentiably path connected with respect to the metric $\eth$.*

Proof. Let $\varepsilon>0$. Let us consider any r nonconstant polynomials $p_1(x_1,\ldots,x_m)$, $\ldots,p_r(x_1,\ldots,x_m)$ of m complex variables, with coefficients over $\mathbb{C}$, and with finite zero set $\mathbf{Z}(p_1,\ldots,p_r)\subset\mathbb{R}^m$. By Theorem 3.3, we then have that there are $\epsilon,\delta>0$ such that, for any $\epsilon'\leq\epsilon$, any $\mathbf{X}\in\mathbb{ZI}^m_{n,\epsilon'}(p_1,\ldots,p_r)$, and any $\mathbf{Y}\in\mathbb{ZI}^m_{n,\epsilon'}(p_1,\ldots,p_r)\cap B_\eth(\mathbf{X},\delta)$, there is a piecewise C^1-path $\gamma\in C([0,1],\mathbb{ZI}^m_{n,\epsilon'}(p_1,\ldots,p_r))$ such that $\gamma(0)=\mathbf{X}$, $\gamma(1)=\mathbf{Y}$ and $\gamma(t)\in B_\eth(\mathbf{X},\varepsilon)$ for each $0\leq t\leq 1$. $\square$

Theorem 3.4. *Let there be given r nonconstant polynomials $p_1(x_1,\ldots,x_m),\ldots,$ $p_r(x_1,\ldots,x_m)$ of m complex variables, with coefficients over $\mathbb{C}$, and with finite zero set $\mathbf{Z}(p_1,\ldots,p_r)\subset\mathbb{C}^m$. There is $\epsilon>0$ such that, for each $0<\epsilon'\leq\epsilon$, the ϵ'-nearly algebraic matrix m-disk $\mathbb{ZD}^m_{n,\epsilon'}(p_1,\ldots,p_r)$ is uniformly piecewise differentiably path connected with respect to the metric $\eth$.*

Proof. Let $\varepsilon>0$. Let us consider any r nonconstant polynomials $p_1(x_1,\ldots,x_m)$, $\ldots,p_r(x_1,\ldots,x_m)$ of m complex variables, with coefficients over $\mathbb{C}$, and with finite zero set $\mathbf{Z}(p_1,\ldots,p_r)\subset\mathbb{C}^m$. By elementary theory of complex valued functions of several complex variables, we have that there are $2r$ polynomials $\mathrm{Re}(p_1),\ldots,\mathrm{Re}(p_r),\mathrm{Im}(p_1),\ldots,\mathrm{Im}(p_r)$ in $2m$ real variables $\mathrm{Re}(z_1),\ldots,\mathrm{Re}(z_m)$, $\mathrm{Im}(z_1),\ldots,\mathrm{Im}(z_m)$, with coefficients over $\mathbb{R}$ such that $\mathbf{z}=(z_1,\ldots,z_m)\in\mathbf{Z}(p_1,\ldots,p_r)\subset\mathbb{C}^m$ if and only if $(\mathrm{Re}(\mathbf{z}),\mathrm{Im}(\mathbf{z}))\in\mathbf{Z}(\mathrm{Re}(p_1),\mathrm{Im}(p_1),\ldots,\mathrm{Re}(p_r),$ $\mathrm{Im}(p_r))\subset\mathbb{R}^{2m}$, where $\mathrm{Re}(\mathbf{z})=(\mathrm{Re}(z_1),\ldots,\mathrm{Re}(z_m))$ and $\mathrm{Im}(\mathbf{z})=(\mathrm{Im}(z_1),\ldots,$ $\mathrm{Im}(z_m))$.

Let us consider the maps $\imath:\mathbb{C}^m\to\mathbb{R}^{2m},\mathbf{z}\mapsto(\mathrm{Re}(\mathbf{z}),\mathrm{Im}(\mathbf{z}))$ and $\kappa:$ $\mathbb{R}^{2m}\to\mathbb{C}^m,\mathbf{x}\mapsto(x_1+\mathbf{i}x_{m+1},\ldots,x_m+\mathbf{i}x_{2m})$. By the preceding arguments, we have that there is a one-to-one correspondence between $\mathbf{Z}(p_1,\ldots,p_r)$ and $\mathbf{Z}(\mathrm{Re}(p_1),\mathrm{Im}(p_1),\ldots,\mathrm{Re}(p_r),\mathrm{Im}(p_r))$ induced by $\imath\circ\kappa$ and $\kappa\circ\imath$. This in turn implies that there is a one-to-one correspondence between $\mathbb{ZD}^m_n(p_1,\ldots,p_r)$ and $\mathbb{ZI}^m_n(\mathrm{Re}(p_1),\mathrm{Im}(p_1),\ldots,\mathrm{Re}(p_r),\mathrm{Im}(p_r))$, induced by the maps $\hat{\pi}\circ\upsilon$ and $\upsilon\circ\hat{\pi}$ defined in Section 2.

By the definition of $\hat{\pi}$ and υ and by the one-to-one correspondence between $\mathbb{ZD}^m_n(p_1,\ldots,p_r)$ and $\mathbb{ZI}^{2m}_n(\mathrm{Re}(p_1),\mathrm{Im}(p_1),\ldots,\mathrm{Re}(p_r),\mathrm{Im}(p_r))$ described in the previous paragraph, we have the following. Given any $\mathbf{X},\mathbf{Y}\in\mathbb{ZD}^m_n(p_1,\ldots,p_r)$, on one hand, we have that $\hat{\pi}(\mathbf{X}),\hat{\pi}(\mathbf{Y})\in\mathbb{ZI}^{2m}_n(\mathrm{Re}(p_1),\mathrm{Im}(p_1),\ldots,\mathrm{Re}(p_r),\mathrm{Im}(p_r))$ and $\eth(\hat{\pi}(\mathbf{X}),\hat{\pi}(\mathbf{Y}))\leq\eth(\mathbf{X},\mathbf{Y})$. On the other hand, for any $\mathbf{S},\mathbf{T}\in\mathbb{ZI}^{2m}_n(\mathrm{Re}(p_1),$ $\mathrm{Im}(p_1),\ldots,\mathrm{Re}(p_r),\mathrm{Im}(p_r)),\upsilon(\mathbf{S}),\upsilon(\mathbf{T})\in\mathbb{ZD}^m_n(p_1,\ldots,p_r)$ and $\eth(\upsilon(\mathbf{S}),\upsilon(\mathbf{T}))\leq$ $2\eth(\mathbf{S},\mathbf{T})$.

One one hand, given $\nu>0$ and the continuity of $p_1,\ldots,p_r$, we have that for any $\mathbf{X}=(X_1,\ldots,X_m)\in\mathbb{ZD}^m_{n,\nu}(p_1,\ldots,p_r),\max\{\|\mathrm{Re}(p_j(\hat{\pi}(\mathbf{X})))\|,\|\mathrm{Im}(p_j(\hat{\pi}(\mathbf{X})))\|\}\leq$

$(1/2)(\|p_j(X_1,\ldots,X_m)\| + \|(p_j(X_1,\ldots,X_m))^*\|) \leq \|p_j(X_1,\ldots,X_m))\| \leq \nu$, $1 \leq j \leq r$. On the other hand, we have that for any $\mathbf{H} = (H_1,\ldots,H_{2m}) \in \mathbb{ZI}^{2m}_{n,\nu/2}(\mathrm{Re}(p_1) , \mathrm{Im}(p_1), \ldots, \mathrm{Re}(p_r), \mathrm{Im}(p_r))$, $\|p_j(\upsilon(\mathbf{H}))\| \leq \|\mathrm{Re}(p_j(\mathbf{H}))\| + \|\mathrm{Im}(p_j(\mathbf{H}))\| \leq (2\nu/2) = \nu$, $1 \leq j \leq r$.

By Theorem 3.3 and by the arguments above, we have that there are $\delta, \epsilon > 0$ such that, for any $\epsilon' \leq \epsilon$, any $\mathbf{X} \in \mathbb{ZD}^m_{n,\epsilon'}(p_1,\ldots,p_r)$, and any $\mathbf{Y} \in \mathbb{ZD}^m_{n,\epsilon'}(p_1,\ldots,p_r) \cap B_{\mathfrak{d}}(\mathbf{X},\delta)$, there is a piecewise C^1-path $\gamma_H \in C([0,1], \mathbb{ZI}^{2m}_{n,\epsilon'/2}(\mathrm{Re}(p_1), \mathrm{Im}(p_1), \ldots, \mathrm{Re}(p_r), \mathrm{Im}(p_r)))$ such that $\gamma_H(0) = \hat{\pi}(\mathbf{X})$, $\gamma_H(1) = \hat{\pi}(\mathbf{Y})$, and $\gamma_H(t) \in B_{\mathfrak{d}}(\hat{\pi}(\mathbf{X}), \varepsilon/2)$ for each $0 \leq t \leq 1$. This in turn implies that the path $\gamma = \upsilon \circ \gamma_H \in C([0,1], \mathbb{ZD}^m_{n,\epsilon'}(p_1,\ldots,p_r))$, which is clearly piecewise C^1, satisfies the conditions $\gamma(0) = \mathbf{X}$, $\gamma(1) = \mathbf{Y}$, and $\gamma(t) \in B_{\mathfrak{d}}(\hat{\pi}(\mathbf{X}), \varepsilon)$ for each $0 \leq t \leq 1$. This completes the proof. □

4. Hints and future directions

In future work, we will study the potential extension of our techniques to almost-normal matrices, in particular, to the computation of normal approximants, normal dilations, and normal compressions, for nearly normal matrices. We will explore the computability of projective refinements and approximate joint diagonalizers, together with their impact on the ε-δ relations, studied in Sections 3.3 and 3.4.

We will study the applications of Lemma 3.3 and Theorem 3.2, in model order reduction for discrete-time control systems, more specifically, in approximate numerical solution of structured matrix equations of the form

$$\begin{cases} (QX - XQ)P = 0 \\ Q^4 = Q^2 \\ Q^2 = ZQ = (Q^2)^* \\ Q^2 P = PQ^2 = P, \end{cases}$$

where $X, Q, P, Z \in M_{2n}$, and where P and Z are given and satisfy the relations

$$\begin{cases} P = P^2 = P^* \\ Z = Z^* \\ Z^2 = \mathbf{1}_n, \end{cases}$$

while Q is to be determined, and X is to be completed, as X is partially known and has the structure

$$X = \begin{pmatrix} A & \times \\ \times & * \end{pmatrix},$$

for some given $A \in M_n$, where $*$ and $\times$ denote the unknown matrix blocks to be determined.

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