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BOUNDEDNESS OF CESÀRO AND RIESZ MEANS IN VARIABLE DYADIC HARDY SPACES

KRISTÓF SZARVAS and FERENC WEISZ*

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Abstract. We consider two types of maximal operators. We prove that, under some conditions, each maximal operator is bounded from the classical dyadic martingale Hardy space H_p to the classical Lebesgue space L_p and from the variable dyadic martingale Hardy space $H_{p(\cdot)}$ to the variable Lebesgue space $L_{p(\cdot)}$. Using this, we can prove the boundedness of the Cesàro and Riesz maximal operator from $H_{p(\cdot)}$ to $L_{p(\cdot)}$ and from the variable Hardy–Lorentz space $H_{p(\cdot),q}$ to the variable Lorentz space $L_{p(\cdot),q}$. As a consequence, we can prove theorems about almost everywhere and norm convergence.

1. Introduction and preliminaries

Variable Lebesgue spaces, a new field of mathematics, is currently a topic of intensive study. Instead of the classical L_p-norm, the variable $L_{p(\cdot)}$-norm is defined by

$$\|f\|_{p(\cdot)} := \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^d} \Big|\frac{f(x)}{\lambda}\Big|^{p(x)}\, dx \le 1\Big\},$$

where $0 < p(x) < \infty$ for all $x \in \mathbb{R}^d$. The variable $L_{p(\cdot)}$ spaces contain all measurable functions f, for which $\|f\|_{p(\cdot)} < \infty$. When the exponent function $p(\cdot)$ is constant, we get back the classical Lebesgue spaces. Variable Lebesgue spaces

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*Corresponding author.

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share many common properties with classical Lebesgue spaces (see, e.g., Kováčik and Rákosník [26], Cruz-Uribe and Fiorenza [6], Diening, Hästö, and Růžička [11], Cruz-Uribe, Fiorenza, and Neugebauer [8], Cruz-Uribe, Fiorenza, Martell, and Pérez [7]). The classical Hardy–Littlewood maximal operator is bounded on the variable $L_{p(\cdot)}$ spaces if the exponent function $p(\cdot)$ is log-Hölder continuous and $1 < p_- \le p_+ < \infty$, where p_- denotes the infimum and p_+ the supremum of $p(\cdot)$ (see, e.g., [8], [10], [30]). This field could experience rapid development with the help of log-Hölder continuity. The variable Hardy spaces $H_{p(\cdot)}(\mathbb{R}^d)$ were defined by Nakai and Sawano [29] and Cruz-Uribe and Yang [9].

The martingale Hardy spaces have given new direction and impetus to the development of harmonic analysis. These spaces are defined on a complete probability space (see, e.g., [1]) which has no natural metric structure. Therefore, it is necessary to redefine log-Hölder continuity. In this paper, instead of the log-Hölder conditions, we suppose (2.1), which in some sense is a generalization of log-Hölder continuity. The condition (2.1) was introduced very recently in [24] (see also [23]). We investigate the martingale Hardy space $H_{p(\cdot)}$ defined by the variable $L_{p(\cdot)}$-norm of the maximal function. Other martingale Hardy spaces can also be defined (see, e.g., Weisz [35], [37] for the classical case or Jiao, Zhou, Hao, and Chen [22], [23] or Jiao, Zhou, Weisz, and Wu [24] for the variable case). Jiao, Zuo, Zhou, and Wu [25] recently introduced and investigated the variable Hardy–Lorentz space $H_{p(\cdot),q}$ for the first time in the case where $0 < q < \infty$, while Yan, Yang, Yuan, and Zhou [38] did the same in the setting $q = \infty$.

In the classical case, Herz [21] and Weisz [35] gave one of the most powerful techniques in the theory of martingale Hardy spaces, the so-called *atomic decomposition*. Some boundedness results, duality theorems, martingale inequalities, and interpolation results can be proved with the help of atomic decomposition (see, e.g., [2], [3], [4]). These results were generalized in [22], [23], [31], and [24] for variable Hardy spaces $H_{p(\cdot)}$ and variable Hardy–Lorentz spaces $H_{p(\cdot),q}$.

The atomic decomposition and martingale inequalities can be applied in Fourier analysis. In the classical case, Schipp, Wade, and Simon [32] and Weisz [37] studied the boundedness of the maximal Fejér operator on the classical L_p space and on the dyadic martingale Hardy space H_p. The second author proved in [37] that the maximal Fejér operator is bounded from the space $H_{p,q}$ to the space $L_{p,q}$ in the cases where $1/2 < p < \infty$ and $0 < q \le \infty$. Similar results were obtained in numerous other papers (see, e.g., Gát [13]–[15] and Goginava [17]–[19]). This boundedness result was generalized for variable Hardy–Lorentz spaces in [24]. There, the authors showed that if the exponent function $p(\cdot)$ satisfies condition (2.1) and $1/2 < p_- < \infty$, then the maximal Fejér operator is bounded from $H_{p(\cdot)}$ to $L_{p(\cdot)}$ and from $H_{p(\cdot),q}$ to $L_{p(\cdot),q}$ $(0 < q \le \infty)$. They used the Marcinkiewicz–Zygmund density theorem (see, e.g., [27, Theorem 1]) to prove a number of convergence theorems in [24].

The so-called *Cesàro means* and *Riesz means* are generalizations of the Fejér means (see Section 3 for the definition). In [36] and [37], Weisz considered the maximal operators of these means and proved that the Cesàro and Riesz maximal operators are bounded from $H_{p,q}$ to $L_{p,q}$ $(0 < \alpha \le 1 \le \gamma,\ 1/(\alpha+1) < p < \infty,\ 0 < q \le \infty)$. In this paper, we will generalize this result for Hardy spaces with variable exponents and show that if the exponent function $p(\cdot)$ satisfies condition (2.1) and $1/(\alpha+1) < p_- < \infty$, then the Cesàro and Riesz maximal operators are bounded from $H_{p(\cdot)}$ to $L_{p(\cdot)}$ and from $H_{p(\cdot),q}$ to $L_{p(\cdot),q}$ $(0 < q \leq \infty)$. We will also prove almost everywhere and norm convergence theorems.

Throughout this article, we will denote the set of natural numbers by $\mathbb{N}$ and the set of integer numbers by $\mathbb{Z}$. The symbol $\alpha \sim \beta$ means that there exist constants $A, B > 0$ such that $A\beta \leq \alpha \leq B\beta$. We denote by C a positive constant which can vary from line to line, and we denote by $C_{p(\cdot)}$ a constant depending only on $p(\cdot)$.

2. Background

2.1. Variable Lebesgue spaces. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A function $p(\cdot)$ belongs to $\mathcal{P}(\Omega)$ if $p(\cdot) : \Omega \to (0,\infty)$, $p(\cdot)$ is measurable, and $0 < p_- \leq p_+ < \infty$, where $p_- := p_-(\Omega) := \operatorname{ess\,inf}\{p(x) : x \in \Omega\}$ and $p_+ := p_+(\Omega) := \operatorname{ess\,sup}\{p(x) : x \in \Omega\}$. Then we say that $p(\cdot)$ is an *exponent function*. For a measurable set $A \subset \Omega$, we will use the notation

$$p_-(A) := \operatorname{ess\,inf}\big\{p(x) : x \in A\big\} \qquad \text{and} \qquad p_+(A) := \operatorname{ess\,sup}\big\{p(x) : x \in A\big\}.$$

Let us define the modular

$$\varrho_{p(\cdot)}(f) := \int_\Omega \big|f(x)\big|^{p(x)}\, dx.$$

We can define the space $L_{p(\cdot)}$ with the help of this modular. A measurable function f belongs to the space $L_{p(\cdot)}$ if there exists $\lambda > 0$ such that $\varrho_{p(\cdot)}(f/\lambda) < \infty$. This modular generates the quasinorm

$$\|f\|_{p(\cdot)} := \inf\Big\{\lambda > 0 : \varrho_{p(\cdot)}\Big(\frac{f}{\lambda}\Big) \leq 1\Big\}.$$

If the space $L_{p(\cdot)}$ is equipped with this quasinorm, then we get a quasi-Banach space. In the case where $p(\cdot) = p$ is a constant, we get back the usual L_p spaces (for details, see the monographs [6] and [11]). Let us denote $\underline{p} := \min\{p_-, 1\}$. The function $p'(\cdot)$ is the *conjugate exponent function* of $p(\cdot)$ if $1/p(x) + 1/p'(x) = 1$ $(x \in \Omega)$. The well-known *Hölder's inequality* can be generalized for variable Lebesgue spaces (see [6, p. 27] or [11, p. 74]).

The next formula is a very useful statement. The proof can be found in [11, p. 77].

Lemma 2.1 (Norm conjugate formula)**.** *Let $p(\cdot) \in \mathcal{P}(\Omega)$, let $p_- \geq 1$, and let $p'(\cdot)$ be the conjugate exponent function of $p(\cdot)$. Then for all measurable functions f*

$$\frac{1}{2}\|f\|_{p(\cdot)} \leq \sup_{\|g\|_{p'(\cdot)} \leq 1} \int_\Omega |fg|\, d\lambda \leq 2\|f\|_{p(\cdot)}.$$

The following lemma can be found in [9, Lemma 2.3].

Lemma 2.2. *Let $p(\cdot) \in \mathcal{P}(\Omega)$. Then for all $s > 0$ and $f \in L_{p(\cdot)}(\Omega)$,*

$$\big\||f|^s\big\|_{p(\cdot)} = \|f\|_{sp(\cdot)}^s.$$

2.2. Variable Lorentz spaces. Let $p(\cdot) \in \mathcal{P}(\Omega)$ and $0 < q \le \infty$. Then the *variable Lorentz spaces* $L_{p(\cdot),q}$ contain those measurable functions f for which

$$\|f\|_{L_{p(\cdot),q}} := \begin{cases} \left(\int_0^\infty t^q \|\chi_{\{|f|>t\}}\|_{p(\cdot)}^q \frac{dt}{t}\right)^{1/q} & \text{if } q < \infty, \\ \sup_{t>0} t\|\chi_{\{|f|>t\}}\|_{p(\cdot)} & \text{if } q = \infty \end{cases}$$

is finite. These spaces are *quasi-Banach spaces*. The space $\mathcal{L}_{p(\cdot),\infty}$ $(p(\cdot) \in \mathcal{P}(\Omega))$ is defined as the set of all measurable functions f for which

$$\lim_{n\to\infty} \|f\chi_{A_n}\|_{L_{p(\cdot),\infty}} = 0$$

for every sequence $(A_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \mathbb{P}(A_n) = 0$. Then $\mathcal{L}_{p(\cdot),\infty}$ is a closed subspace of $L_{p(\cdot),\infty}$. Moreover, for every $p(\cdot) \in \mathcal{P}(\Omega)$, $L_{p(\cdot)} \subset \mathcal{L}_{p(\cdot),\infty} \subset L_{p(\cdot),\infty}$ (see [24]). Note that if $0 < q < \infty$, then for every $f \in L_{p(\cdot),q}$

$$\lim_{n\to\infty} \|f\chi_{A_n}\|_{L_{p(\cdot),q}} = 0$$

for every sequence $(A_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \mathbb{P}(A_n) = 0$.

2.3. Variable martingale Hardy spaces. Let $\Omega := [0, 1)$, $\mathbb{P}$ be the Lebesgue measure. The interval $I_{k,n} := [k2^{-n}, (k+1)2^{-n})$ $(n \in \mathbb{N}, k = 0, \ldots, 2^n - 1)$ is said to be a *dyadic interval*. For $x \in [0, 1)$ and $n \in \mathbb{N}$, let us denote by $I_n(x)$ the dyadic interval of length 2^{-n} which contains x. Let $\mathcal{F}_n$ $(n \in \mathbb{N})$ be the σ-algebra generated by $\{I_n(x) : x \in [0, 1)\}$. In this case, $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is *regular*; that is, there exists a positive constant R (which is independent of n) such that for all $A \in \mathcal{F}_n$ there exists $B \in \mathcal{F}_{n-1}$: $A \subset B$ and $\mathbb{P}(B) \le R \cdot \mathbb{P}(A)$. Moreover, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and for every $n \in \mathbb{N}$, $\mathcal{F}_n \subset \mathcal{F}_{n+1}$.

The expectation and conditional expectation operators relative to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. An integrable sequence $f = (f_n)_{n\in\mathbb{N}}$ is said to be a *martingale* if f_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$ and $\mathbb{E}_n f_m = f_n$ in case $n \le m$. For $n \in \mathbb{N}$, the *martingale difference* is defined by $d_n f := f_n - f_{n-1}$, where $f = (f_n)_{n\in\mathbb{N}}$ is a martingale and $f_0 := f_{-1} := 0$. Thus $d_0 = 0$.

The *maximal function* is defined by

$$M(f) := \sup_{n\in\mathbb{N}} |f_n|,$$

where $f = (f_n)_{n\in\mathbb{N}}$ is a martingale. Now we can define the *variable martingale Hardy spaces* by

$$H_{p(\cdot)} := \left\{f = (f_n)_{n\in\mathbb{N}} : \|f\|_{H_{p(\cdot)}} := \big\|M(f)\big\|_{p(\cdot)} < \infty\right\}.$$

The *variable martingale Hardy–Lorentz spaces* can be defined similarly:

$$H_{p(\cdot),q} := \left\{f = (f_n)_{n\in\mathbb{N}} : \|f\|_{H_{p(\cdot),q}} := \big\|M(f)\big\|_{L_{p(\cdot),q}} < \infty\right\}.$$

The space $\mathcal{H}_{p(\cdot),\infty}$ is the space of all martingales such that $M(f) \in \mathcal{L}_{p(\cdot),\infty}$. More details about martingale Hardy spaces in the classical case can be found in [35] and [37]; see [22], [23], and [24] for the variable case.

2.4. Atomic decomposition. First of all, we need the definition of *atoms*. For $p(\cdot) \in \mathcal{P}(\Omega)$, a measurable function a is called a $p(\cdot)$-*atom* if there exists a stopping time τ such that

(1) $\mathbb{E}_n(a) = 0$ for all $n \le \tau$,
(2) $\|M(a)\|_\infty \le \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1}$.

The set of dyadic intervals of length 2^{-n} is denoted by $A(\mathcal{F}_n)$. For all $f \in L_1$, the conditional expectation operator relative to $\mathcal{F}_n$ can be written as

$$\mathbb{E}_n(f) = \sum_{A \in A(\mathcal{F}_n)} \Big(\frac{1}{\mathbb{P}(A)} \int_A f \, d\mathbb{P} \Big) \chi_A \quad (n \in \mathbb{N}).$$

Now we can define the most important condition of the exponent function $p(\cdot)$. We will suppose that there exists a constant $K_{p(\cdot)} \ge 1$ such that

$$\mathbb{P}(A)^{p_-(A)-p_+(A)} \le K_{p(\cdot)} \quad \Big(A \in \bigcup_n A(\mathcal{F}_n) \Big). \tag{2.1}$$

If $\Omega = [0,1)$ and the exponent function $p(\cdot)$ satisfies the so-called *locally log-Hölder condition*, then $p(\cdot)$ satisfies (2.1) (see [6]). The spaces $H_{p(\cdot)}$ and $H_{p(\cdot),q}$ have the following atomic decomposition (see, e.g., [24]). The classical case can be found in [35] and [37].

Theorem 2.3. *Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (2.1), and let $0 < q \le \infty$. Then the martingale $f = (f_n)_{n\in\mathbb{N}} \in H_{p(\cdot)}$ or $f = (f_n)_{n\in\mathbb{N}} \in H_{p(\cdot),q}$, respectively, if and only if there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $p(\cdot)$-atoms and a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of real numbers such that for every $n \in \mathbb{N}$,*

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n a^k \quad \textit{almost everywhere}, \tag{2.2}$$

where $\mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}$. Moreover,

$$\|f\|_{H_{p(\cdot)}} \sim \inf \Big\| \Big(\sum_{k\in\mathbb{Z}} \Big(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}} \Big)^t \Big)^{1/t} \Big\|_{p(\cdot)},$$

$$\|f\|_{H_{p(\cdot),q}} \sim \inf \Big(\sum_{k\in\mathbb{Z}} 2^{kq} \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^q \Big)^{1/q},$$

respectively, where $0 < t \le \underline{p}$ is fixed and the infimum is taken over all decompositions of the form (2.2).

The next results will be applied many times in this paper. The proofs can be found in [22], [23], or [24].

Lemma 2.4. *Let $p(\cdot) \in \mathcal{P}(\Omega)$, let $1 \le p_- \le p_+ < \infty$, and suppose that $p(\cdot)$ satisfies (2.1). If $f \in L_{p(\cdot)}$ and $\|f\|_{p(\cdot)} \le 1/2$, then for any atom $A \in \cup_n A(\mathcal{F}_n)$,*

$$\Big(\frac{1}{\mathbb{P}(A)} \int_A |f(x)| \, d\mathbb{P} \Big)^{p(x)} \le \frac{K}{\mathbb{P}(A)} \int_A \big(|f(x)|^{p(x)} + 1 \big) \, d\mathbb{P} \quad (x \in A).$$

Theorem 2.5. *Let $0 < q \le \infty$, let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (2.1), and let $1 < p_- \le p_+ < \infty$. Then the space $H_{p(\cdot)}$ is equivalent to the space $L_{p(\cdot)}$, and the space $H_{p(\cdot),q}$ is equivalent to the space $L_{p(\cdot),q}$.*

3. Walsh system and the Cesàro and Riesz means

The *Rademacher system* is defined by

$$r(x) := \begin{cases} 1, & \text{if } x \in [0,1/2), \\ -1, & \text{if } x \in [1/2,1) \end{cases} \qquad \text{and} \qquad r_n(x) := r(2^n x) \quad \big(x \in [0,1), n \in \mathbb{N}\big).$$

The *Walsh system* is the product system generated by the Rademacher system. For $n \in \mathbb{N}$, let

$$w_n := \prod_{k=0}^{\infty} r_k^{n_k}, \quad \text{where } n = \sum_{k=0}^{\infty} n_k 2^k \quad (0 \le n_k < 2).$$

The *Walsh–Dirichlet kernels*

$$D_n := \sum_{k=0}^{n-1} w_k$$

satisfy

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in [0,2^{-n}), \\ 0 & \text{if } x \in [2^{-n},1) \end{cases} \quad (n \in \mathbb{N}). \tag{3.1}$$

For $f \in L_1$, the *nth Walsh–Fourier coefficient of f* is

$$\widehat{f}(n) := \mathbb{E}(f w_n) \quad (n \in \mathbb{N}).$$

The definition can be extended to martingales as well (see, e.g., [37]). Let $s_n f$ be the *nth partial sum of the Walsh–Fourier series* of a martingale f, that is,

$$s_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k \quad (n \in \mathbb{N}).$$

It can be seen that $s_{2^n} f = f_n$ $(n \in \mathbb{N})$, therefore $\lim_{n\to\infty} s_{2^n} f = f$ in the L_p-norm, if $f \in L_p$ $(1 \le p < \infty)$. Schipp, Wade, and Simon [32] extended this result for $s_n f$: if $f \in L_p$ $(1 < p < \infty)$, then $\lim_{n\to\infty} s_n f = f$ in the L_p-norm. Jiao, Zhou, Weisz, and Wu [24] generalized this result for $L_{p(\cdot)}$: if $p(\cdot) \in \mathcal{P}(\Omega)$ satisfies (2.1), and $1 < p_- \le p_+ < \infty$, then for all $f \in L_{p(\cdot)}$, $\lim_{n\to\infty} s_n f = f$ in the $L_{p(\cdot)}$-norm.

Unfortunately, these results are not true if $p \le 1$ or if $p_- \le 1$ (see, e.g., [5], [20]). However, for $p \le 1$, or $p_- \le 1$, we can prove convergence results with the help of summability means.

For $k \in \mathbb{N}$ and $\alpha \in \mathbb{R} \setminus \mathbb{Z} \cup \mathbb{N}$ (i.e., $\mathbb{R} \ni \alpha \ne -1, -2, \ldots$), let us denote

$$A_k^\alpha := \binom{k+\alpha}{k} = \frac{(\alpha+k)(\alpha+k-1)\ldots(\alpha+1)}{k!}.$$

For $\alpha > 0$, the *Cesàro means* of a martingale f is defined by

$$\sigma_n^\alpha f := \frac{1}{A_{n-1}^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} s_k f = \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-k-1}^{\alpha} \widehat{f}(k) w_k \quad (1 \le n \in \mathbb{N}).$$

For $0 < \alpha \le 1 \le \gamma$, the *Riesz means* of a martingale f is defined by

$$\sigma_n^{\alpha,\gamma} f := \frac{1}{n^{\alpha\gamma}} \sum_{k=0}^{n-1} (n^\gamma - k^\gamma)^\alpha \widehat{f}(k) w_k \quad (1 \le n \in \mathbb{N}).$$

If $f \in L_1$, then these means can be written as

$$\sigma_n^\alpha f(x) = \int_0^1 f(t) K_n^\alpha (x \dot{+} t)\, dt \quad \big(x \in [0,1)\big),$$

$$\sigma_n^{\alpha,\gamma} f(x) = \int_0^1 f(t) K_n^{\alpha,\gamma} (x \dot{+} t)\, dt \quad \big(x \in [0,1)\big),$$

where "$\dot{+}$" is the dyadic addition (see [37], [32]), and the kernels are defined by

$$K_n^\alpha := \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-k-1}^\alpha w_k \quad (1 \le n \in \mathbb{N})$$

and

$$K_n^{\alpha,\gamma} := \frac{1}{n^{\alpha\gamma}} \sum_{k=0}^{n-1} (n^\gamma - k^\gamma)^\alpha w_k \quad (1 \le n \in \mathbb{N}).$$

The previous kernel functions can be estimated pointwise. The proof of the following theorem can be found in Weisz [37, Theorem 3.3].

Theorem 3.1. *Let* $0 < \alpha \le 1 \le \gamma$. *If*

$$n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_v},$$

where $v \in \mathbb{N}$, $n_1 > n_2 > \cdots > n_v$, $n_k \in \mathbb{N}$ $(k = 1, \dots, v)$, *then*

$$\big|K_n^\alpha(x)\big|, \big|K_n^{\alpha,\gamma}(x)\big| \le C n^{-\alpha} \sum_{k=1}^{v} \sum_{j=0}^{n_k-1} \sum_{i=j}^{n_k-1} 2^{i(\alpha-1)} 2^j D_{2^i}(x \dot{+} 2^{-j-1})$$
$$+ C n^{-\alpha} \sum_{k=1}^{v} 2^{n_k \alpha} D_{2^{n_k}}(x).$$

The *Cesàro* and *Riesz maximal operators* are defined, respectively, by

$$\sigma_*^\alpha f := \sup_{n \in \mathbb{N}} |\sigma_n^\alpha f| \qquad \text{and} \qquad \sigma_*^{\alpha,\gamma} f := \sup_{n \in \mathbb{N}} |\sigma_n^{\alpha,\gamma} f|.$$

4. Boundedness of maximal operators

Motivated by Theorem 3.1, we will consider two maximal operators: $U^{(\alpha)}$ and $V^{(\alpha)}$. It will be shown that, under some conditions, each maximal operator is bounded from H_p to L_p and bounded on $L_{p(\cdot)}$ with $p_- > 1$. These maximal operators differ from the maximal operators U and V in [24].

4.1. Boundedness of the maximal operator $U^{(\alpha)}$. Let $\alpha \in (0,1]$ and $t,r>0$. Then the maximal operator $U^{(\alpha)}$ is defined by

$$U^{(\alpha)}f(x) := \sup_{x\in I} \sum_{m=1}^{n} \sum_{j=0}^{m-1} 2^{(j-n)t} \sum_{i=j}^{m-1} 2^{\alpha(i-n)t} 2^{(n-i)r/(r-t)} \times \frac{1}{\mathbb{P}(I \dotplus [2^{-j-1}, 2^{-j-1} \dotplus 2^{-i}))} \Big| \int_{I \dotplus [2^{-j-1}, 2^{-j-1} \dotplus 2^{-i})} f_n \Big|,$$

where I is a dyadic interval of length 2^{-n} and $f=(f_n)$ is a martingale. If $I_{k,n} := [k2^{-n},(k+1)2^{-n})$ ($n\in\mathbb{N}$, $k=0,\dots,2^n-1$), then

$$U^{(\alpha)}f(x) = \sup_{n\in\mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}} \sum_{m=1}^{n} \sum_{j=0}^{m-1} 2^{(j-n)t} \sum_{i=j}^{m-1} 2^{\alpha(i-n)t} 2^{(n-i)r/(r-t)} \times \frac{1}{\mathbb{P}(I_{k,n} \dotplus [2^{-j-1}, 2^{-j-1} \dotplus 2^{-i}))} \Big| \int_{I_{k,n} \dotplus [2^{-j-1}, 2^{-j-1} \dotplus 2^{-i})} f_n \Big|.$$

A measurable function a is called a *simple p-atom* if there exist $j\in\mathbb{N}$ and $I\in A(\mathcal{F}_j)$ such that

(1) the support of a is contained in I,
(2) $\|M(a)\|_\infty \le |I|^{-1/p}$,
(3) $\mathbb{E}_j(a)=0$.

The following theorem can be found in Weisz [37, Theorem 1.34].

Theorem 4.1. *Let $0<p<\infty$, and suppose that the σ-sublinear operator $T : L_\infty \to L_\infty$ is bounded from L_∞ to L_∞. If*

$$\|Ta\chi_{I^c}\|_p \le C_p$$

for all simple p-atoms a, where I is the support associated with a and I^c denotes the complement of I, then for all $f\in H_p$,

$$\|Tf\|_p \le C\|f\|_{H_p}.$$

We use Theorem 4.1 to help us prove that the operator $U^{(\alpha)}$ is bounded from H_p to L_p.

Theorem 4.2. *Let $\alpha\in(0,1]$, $0<p\le\infty$, and $t,r>0$ be such that $\alpha t < r/(r-t) < (1+\alpha)t$. Then for all $f\in H_p$,*

$$\|U^{(\alpha)}f\|_p \le C_p\|f\|_{H_p}.$$

Proof. Observe that

$$\|U^{(\alpha)}f\|_\infty \le \sup_{n\in\mathbb{N}} 2^{n(r/(r-t)-(1+\alpha)t)} \sum_{m=1}^{n} \sum_{j=0}^{m-1} 2^{jt} \sum_{i=j}^{m-1} 2^{i(\alpha t - r/(r-t))} \|f\|_\infty \le C\|f\|_\infty.$$

Let a be a simple p-atom with support $J=[0,2^{-K})$. If $i\le K$, then

$$\int_{I\dotplus[2^{-j-1},2^{-j-1}\dotplus 2^{-i})} a = 0;$$

that is, we can suppose that $i > K$, and thus $n \geq m > K$. If $x \notin [0, 2^{-K})$, $x \in I$, and $j \geq K$, then $I \dotplus [2^{-j-1}, 2^{-j-1} \dotplus 2^{-i}) \cap [0, 2^{-K}) = \emptyset$. Therefore, we can suppose that $j < K$. Similarly, if $x \in [2^{-j-1} + 2^{-K}, 2^{-j})$, then the set $I \dotplus [2^{-j-1}, 2^{-j-1} \dotplus 2^{-i}) \cap [0, 2^{-K}) = \emptyset$, so we may assume that $x \in [2^{-j-1}, 2^{-j-1} + 2^{-K})$. Using this and the assumption that a is a simple p-atom, we have

$$\begin{aligned}
&\left|U^{(\alpha)}a(x)\right| \\
&\quad \leq \sup_{n>K} \chi_I(x) \sum_{m=K}^{n} \sum_{j=0}^{K-1} 2^{(j-n)t} \sum_{i=K}^{m-1} 2^{\alpha(i-n)t} 2^{(n-i)r/(r-t)} \\
&\qquad \times \frac{1}{\mathbb{P}(I \dotplus [2^{-j-1}, 2^{-j-1} \dotplus 2^{-i}))} \left| \int_{I \dotplus [2^{-j-1}, 2^{-j-1} \dotplus 2^{-i})} a \right| \chi_{j,K}(x) \\
&\quad \leq 2^{K/p} \sup_{n>K} \chi_I(x) 2^{n(r/(r-t)-(1+\alpha)t)} \sum_{m=K}^{n} \sum_{j=0}^{K-1} 2^{jt} \sum_{i=K}^{m-1} 2^{i(\alpha t - r/(r-t))} \chi_{j,K}(x) \\
&\quad =: A(x),
\end{aligned}$$

where $\chi_{j,K} := \chi_{[2^{-j-1}, 2^{-j-1}+2^{-K})}$. We can estimate $A(x)$ by

$$\begin{aligned}
A(x) &\leq 2^{K/p} \sup_{n>K} (n-K) 2^{(n-K)(r/(r-t)-(1+\alpha)t)} 2^{-Kt} \sum_{j=0}^{K-1} 2^{jt} \chi_{j,K}(x) \\
&\leq C_{\alpha,r,t} 2^{K/p} 2^{-Kt} \sum_{j=0}^{K-1} 2^{jt} \chi_{j,K}(x).
\end{aligned}$$

We have used the fact that for all $\gamma > 0$ and $x > 0$, the function $x \mapsto x 2^{-\gamma x}$ is bounded, where $\gamma := (1+\alpha)t - r/(r-t) > 0$, $x := n - K > 0$. We obtain that

$$\int_{J^c} A(x)^p \, dx \leq C_p 2^K 2^{-Ktp} \sum_{j=0}^{K-1} 2^{jtp} 2^{-K} \leq C_p,$$

and therefore

$$\int_{J^c} \left|U^{(\alpha)}a(x)\right|^p dx \leq C_p.$$

The theorem follows from Theorem 4.1. □

Theorem 4.3. *Let $\alpha \in (0,1]$ and $t, r > 0$ be such that $\alpha t < r/(r-t) < (1+\alpha)t$, $p(\cdot) \in \mathcal{P}(\Omega)$, $1 < p_- \leq p_+ < \infty$, and suppose that $p(\cdot)$ satisfies (2.1). Then for all $f \in L_{p(\cdot)}$,*

$$\|U^{(\alpha)} f\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{p(\cdot)}.$$

Proof. We assume that $\|f\|_{p(\cdot)} \leq 1/2$. Since for arbitrary fixed $x \in \Omega$, the function $t \mapsto t^{p(x)/p_-}$ is convex and the intervals $I_{k,n}$ $(k = 0 \dots 2^n - 1)$ are disjoint we get that

$$\begin{aligned}\int_\Omega \big|U^{(\alpha)}f(x)\big|^{p(x)}\,d\mathbb{P} \\ &\le C\int_\Omega \Big(\sup_{n\in\mathbb{N}}\sum_{k=0}^{2^n-1}\chi_{I_{k,n}}\Big(\sum_{m=1}^{n}\sum_{j=0}^{m-1}2^{(j-n)t}\sum_{i=j}^{m-1}2^{\alpha(i-n)t}2^{(n-i)r/(r-t)} \\ &\quad\times\frac{1}{\mathbb{P}(I_{k,n}\dot{+}[2^{-j-1},2^{-j-1}\dot{+}2^{-i}))}\int_{I_{k,n}\dot{+}[2^{-j-1},2^{-j-1}\dot{+}2^{-i})}|f|\Big)^{\frac{p(x)}{p_-}}\Big)^{p_-}\,d\mathbb{P} \\ &\le C\int_\Omega \Big(\sup_{n\in\mathbb{N}}\sum_{k=0}^{2^n-1}\chi_{I_{k,n}}\sum_{m=1}^{n}\sum_{j=0}^{m-1}2^{(j-n)t}\sum_{i=j}^{m-1}2^{\alpha(i-n)t}2^{(n-i)r/(r-t)} \\ &\quad\times\frac{1}{\mathbb{P}(I_{k,n}\dot{+}[2^{-j-1},2^{-j-1}\dot{+}2^{-i}))}\int_{I_{k,n}\dot{+}[2^{-j-1},2^{-j-1}\dot{+}2^{-i})}\big(|f|^{\frac{p(x)}{p_-}}+1\big)\Big)^{p_-}\,d\mathbb{P} \\ &\le C\big\|U^{(\alpha)}\big(|f|^{\frac{p(x)}{p_-}}+1\big)\big\|_{p_-}^{p_-}\le C_p,\end{aligned}$$

where we have used Lemma 2.4 and Theorem 4.2. □

4.2. Boundedness of the maximal operator $V^{(\alpha)}$. For $\alpha\in(0,1]$, $t,r>0$, the maximal operator $V^{(\alpha)}$ is defined by

$$V^{(\alpha)}f(x):=\sup_{x\in I}\sum_{m=0}^{n-1}2^{(n-m)(r/(r-t)-(1+\alpha)t)}\frac{1}{\mathbb{P}(I\dot{+}[0,2^{-m}))}\Big|\int_{I\dot{+}[0,2^{-m})}f_n\Big|,$$

where I is a dyadic interval of length 2^{-n} and $f=(f_n)$ is a martingale. Of course,

$$\begin{aligned}V^{(\alpha)}f(x)=\sup_{n\in\mathbb{N}}\sum_{k=0}^{2^n-1}\chi_{I_{k,n}}\sum_{m=0}^{n-1}2^{(n-m)(r/(r-t)-(1+\alpha)t)} \\ \times\frac{1}{\mathbb{P}(I_{k,n}\dot{+}[0,2^{-m}))}\Big|\int_{I_{k,n}\dot{+}[0,2^{-m})}f_n\Big|.\end{aligned}$$

Theorem 4.4. *Let $\alpha\in(0,1]$, $0<p\le\infty$, and $t,r>0$ be such that $\alpha t< r/(r-t)<(1+\alpha)t$. Then for all $f\in H_p$,*

$$\|V^{(\alpha)}f\|_p\le C_p\|f\|_{H_p}.$$

Proof. Since

$$\|V^{(\alpha)}f\|_\infty\le\sup_{n\in\mathbb{N}}2^{n(r/(r-t)-(1+\alpha)t)}\sum_{m=0}^{n-1}2^{m((1+\alpha)t-r/(r-t))}\|f\|_\infty\le C\|f\|_\infty,$$

$V^{(\alpha)}$ is bounded on L_∞.

Let a be a simple p-atom with support J, where $J=[0,2^{-K})$. If $m\le K$, then $\int_{I\dot{+}[0,2^{-m})}a=0$. If $m>K$, $x\notin J$, but $x\in I$, where I is a dyadic interval of length 2^{-n}, then $I\dot{+}[0,2^{-m})\cap J=\emptyset$. This means that $V^{(\alpha)}a(x)=0$ $(x\notin J)$. Hence

$$\|V^{(\alpha)}f\|_p\le C_p\|f\|_{H_p}$$

for all $f\in H_p$. □

Theorem 4.5. *Let $\alpha \in (0,1]$ and $t, r > 0$ be such that $\alpha t < r/(r-t) < (1+\alpha)t$, $p(\cdot) \in \mathcal{P}(\Omega)$, $1 < p_- \leq p_+ < \infty$, and suppose that $p(\cdot)$ satisfies (2.1). Then for all $f \in L_{p(\cdot)}$,*

$$\|V^{(\alpha)} f\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{p(\cdot)}.$$

Proof. The proof is similar to the proof of Theorem 4.3. □

5. Boundedness of the Cesàro and Riesz means in $H_{p(\cdot)}$

In this section we will use the boundedness results of $U^{(\alpha)}$ and $V^{(\alpha)}$ above. The following theorem can be found in [24].

Theorem 5.1. *Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (2.1), and let $0 < t < \underline{p}$. If the σ-sublinear operator $T : L_\infty \to L_\infty$ is bounded and*

$$\left\| \sum_k 2^{kt} \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}^t T(a^k)^t \chi_{\{\tau_k = \infty\}} \right\|_{\frac{p(\cdot)}{t}} \leq C \left\| \sum_k 2^{kt} \chi_{\{\tau_k < \infty\}} \right\|_{\frac{p(\cdot)}{t}}, \tag{5.1}$$

where τ_k is the stopping time associated with the $p(\cdot)$-atom a^k, then

$$\|Tf\|_{p(\cdot)} \leq C \|f\|_{H_{p(\cdot)}}$$

for all $f \in H_{p(\cdot)}$.

Now we will prove that the Cesàro and Riesz maximal operators satisfy condition (5.1).

Theorem 5.2. *Let $0 < \alpha \leq 1 \leq \gamma$, let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (2.1), and suppose that $1/(\alpha+1) < t < \underline{p}$. Then*

$$\left\| \sum_k 2^{kt} \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}^t \sigma_*^\alpha(a^k)^t \chi_{\{\tau_k = \infty\}} \right\|_{\frac{p(\cdot)}{t}} \leq C \left\| \sum_k 2^{kt} \chi_{\{\tau_k < \infty\}} \right\|_{\frac{p(\cdot)}{t}},$$

where τ_k is the stopping time associated with the $p(\cdot)$-atom a^k. The same holds for $\sigma_*^{\alpha,\gamma}$.

Proof. We will use the estimation for the kernels K_n^α and $K_n^{\alpha,\gamma}$ given in Theorem 3.1; therefore, it is sufficient to prove the theorem only for the Cesàro means. The sets $\{\tau = j\}$ are disjoint, and there exist disjoint dyadic intervals $I_j^i \in \mathcal{F}_j$ such that

$$\{\tau = j\} = \bigcup_i I_j^i.$$

Hence

$$\{\tau < \infty\} = \bigcup_{j \in \mathbb{N}} \bigcup_i I_j^i,$$

where the dyadic intervals I_j^i are disjoint. Since a is an atom, $\int_{I_j^i} a \, d\mathbb{P} = 0$. For simplicity, instead of I_j^i (resp., $a\chi_{I_j^i}$), we write I_l (resp., b^l). Then

$$a = \sum_{j \in \mathbb{N}} \sum_i a\chi_{I_{j,i}} = \sum_l b^l.$$

Suppose that $\mathbb{P}(I_l) = |I_l| := 2^{-K_l}$ with a suitable $K_l \in \mathbb{N}$. The sets I_l are disjoint and $\int_{I_l} b^l \, d\mathbb{P} = 0$.

Assume that $x \in \{\tau = \infty\}$. If $n < 2^{K_l}$, then $\widehat{b^l}(n) = 0$; therefore, $\sigma_n^\alpha b^l = 0$. Thus we can suppose that $n \geq 2^{K_l}$. If $j \geq K_l$ and $x \notin I_l$, then $x \dotplus 2^{-j-1} \notin I_l$. Thus for $x \notin I_l$, $t \in I_l$ and $i \geq j \geq K_l$,

$$b^l(t) D_{2^i}(x \dotplus t) = b^l(t) D_{2^i}(x \dotplus t \dotplus 2^{-j-1}) = 0.$$

Since a is a $p(\cdot)$-atom, we have that $\|b^l\|_\infty \leq \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1}$. By this and Theorem 3.1, we get for $x \notin I_l$ that

$$\begin{aligned}
\left|\sigma_n^\alpha b^l(x)\right| &\leq n^{-\alpha} \sum_{k=1}^{v} \sum_{j=0}^{n_k-1} \sum_{i=j}^{n_k-1} 2^{i(\alpha-1)} 2^j \int_0^1 \left|b^l(t)\right| D_{2^i}(x \dotplus t \dotplus 2^{-j-1}) \, dt \\
&\qquad + C n^{-\alpha} \sum_{k=1}^{v} 2^{n_k \alpha} \int_0^1 \left|b^l(t)\right| D_{2^{n_k}}(x \dotplus t) \, dt \\
&\leq C \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} n^{-\alpha} \sum_{\substack{k=1 \\ n_k \geq K_l}}^{v} \sum_{j=0}^{K_l-1} \sum_{i=j}^{K_l-1} 2^{i(\alpha-1)} 2^j \int_{I_l} D_{2^i}(x \dotplus t \dotplus 2^{-j-1}) \, dt \\
&\qquad + C \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} n^{-\alpha} \\
&\qquad \times \sum_{\substack{k=1 \\ n_k \geq K_l}}^{v} \sum_{j=0}^{K_l-1} \sum_{i=K_l}^{n_k-1} 2^{i(\alpha-1)} 2^j \int_{I_l} D_{2^i}(x \dotplus t \dotplus 2^{-j-1}) \, dt \\
&\qquad + C \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} n^{-\alpha} \\
&\qquad \times \sum_{\substack{k=1 \\ n_k < K_l}}^{v} \sum_{j=0}^{n_k-1} \sum_{i=j}^{n_k-1} 2^{i(\alpha-1)} 2^j \int_{I_l} D_{2^i}(x \dotplus t \dotplus 2^{-j-1}) \, dt \\
&\qquad + C \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} n^{-\alpha} \sum_{\substack{k=1 \\ n_k < K_l}}^{v} 2^{n_k \alpha} \int_{I_l} D_{2^{n_k}}(x \dotplus t) \, dt \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

By (3.1), we can see (cf. [37]) that for $x \notin I_l$, if $j \leq i \leq K_l - 1$, then

$$\int_{I_l} D_{2^i}(x \dotplus t \dotplus 2^{-j-1}) \, dt = 2^{i-K_l} \chi_{I_l \dotplus [2^{-j-1}, 2^{-j-1} \dotplus 2^{-i})}(x),$$

if $i \geq K_l$, then

$$\int_{I_l} D_{2^i}(x \dotplus t \dotplus 2^{-j-1}) \, dt = \chi_{I_l \dotplus [2^{-j-1}, 2^{-j-1} \dotplus 2^{-i})}(x) = \chi_{I_l \dotplus 2^{-j-1}}(x),$$

and for all $i \in \mathbb{N}$,

$$\int_{I_l} D_{2^i}(x \dotplus t) \, dt = 2^{i-K_l} \chi_{I_l \dotplus [2^{-K_l}, 2^{-i})}(x).$$

Since $n \geq 2^{n_1}$ and $n_1 \geq K_l$, we have that

$$\begin{aligned} \mathrm{I} &\leq C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} 2^{-n_1\alpha}(n_1-K_l+1) \\ &\quad\times \sum_{j=0}^{K_l-1} 2^j \sum_{i=j}^{K_l-1} 2^{i(\alpha-1)} 2^{i-K_l} \chi_{I_l \dot{+} [2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i})}(x) \\ &\leq C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} 2^{-K_l\alpha} \sum_{j=0}^{K_l-1} 2^j \sum_{i=j}^{K_l-1} 2^{i(\alpha-1)} 2^{i-K_l} \chi_{I_l \dot{+} [2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i})}(x) \\ &= C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} 2^{-(1+\alpha)K_l} \sum_{j=0}^{K_l-1} 2^j \sum_{i=j}^{K_l-1} 2^{i\alpha} \chi_{I_l \dot{+} [2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i})}(x), \end{aligned} \tag{5.2, 5.3}$$

where we have used the fact that the function $x \mapsto x2^{-x}$ is bounded in case $x \geq 1$.

To estimate II observe that $n^{-\alpha} = n^{-\alpha/3} n^{-2\alpha/3}$ and $n^{-2\alpha/3} \leq 2^{-2\alpha/3i}$. Therefore,

$$\begin{aligned} \mathrm{II} &\leq C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} n^{-\alpha/3}(n_1-K_l+1) \sum_{j=0}^{K_l-1} 2^j \chi_{I_l \dot{+} 2^{-j-1}}(x) \sum_{i=K_l}^{\infty} 2^{i(\alpha/3-1)} \\ &\leq C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} 2^{-K_l\alpha/3} \sum_{j=0}^{K_l-1} 2^j \chi_{I_l \dot{+} 2^{-j-1}}(x) 2^{K_l(\alpha/3-1)} \\ &= C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} 2^{-K_l} \sum_{j=0}^{K_l-1} 2^j \chi_{I_l \dot{+} 2^{-j-1}}(x). \end{aligned} \tag{5.4}$$

Furthermore,

$$\begin{aligned} \mathrm{III} &\leq C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} 2^{-\alpha K_l} \\ &\quad\times \sum_{\substack{k=1 \\ n_k<K_l}}^{v} \sum_{j=0}^{n_k-1} 2^j \sum_{i=j}^{n_k-1} 2^{i(\alpha-1)} 2^{i-K_l} \chi_{I_l \dot{+} [2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i})}(x) \\ &= C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} 2^{-(1+\alpha)K_l} \\ &\quad\times \sum_{m=1}^{K_l-1} \sum_{j=0}^{m-1} 2^j \sum_{i=j}^{m-1} 2^{i\alpha} \chi_{I_l \dot{+} [2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i})}(x) \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \mathrm{IV} &\leq C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} 2^{-\alpha K_l} \sum_{m=0}^{K_l-1} 2^{m\alpha} 2^{m-K_l} \chi_{I_l \dot{+} [2^{-K_l}, 2^{-m})}(x) \\ &\leq C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1} 2^{-(1+\alpha)K_l} \sum_{m=0}^{K_l-1} 2^{(1+\alpha)m} \chi_{I_l \dot{+} [0, 2^{-m})}(x). \end{aligned} \tag{5.6}$$

Note that the estimations (5.3), (5.4), (5.5), and (5.6) are independent of n. From this it follows that

$$\begin{aligned}\sigma_* a(x) &\le \sum_l \sigma_*^\alpha b^l(x)\\ &\le C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1}\sum_l 2^{-(1+\alpha)K_l}\sum_{j=0}^{K_l-1}2^j\sum_{i=j}^{K_l-1}2^{i\alpha}\chi_{I_l\dot{+}[2^{-j-1},2^{-j-1}\dot{+}2^{-i})}(x)\\ &\quad + C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1}\sum_l 2^{-K_l}\sum_{j=0}^{K_l-1}2^j\chi_{I_l\dot{+}2^{-j-1}}(x)\\ &\quad + C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1}\\ &\quad \times\sum_l 2^{-(1+\alpha)K_l}\sum_{m=1}^{K_l-1}\sum_{j=0}^{m-1}2^j\sum_{i=j}^{m-1}2^{i\alpha}\chi_{I_l\dot{+}[2^{-j-1},2^{-j-1}\dot{+}2^{-i})}(x)\\ &\quad + C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1}\sum_l 2^{-(1+\alpha)K_l}\sum_{m=0}^{K_l-1}2^{(1+\alpha)m}\chi_{I_l\dot{+}[0,2^{-m})}(x)\\ &=: C\|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1}\big(A(x)+B(x)+C(x)+D(x)\big).\end{aligned}\tag{5.7}$$

For the atom a^k, we denote l, K_l, A, B, C, and D above by l_k, K_{l_k}, A_k, B_k, C_k, and D_k, respectively. That is, we have

$$\begin{aligned}&\Big\|\sum_k 2^{kt}\Big\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^t\sigma_*^\alpha(a^k)^t\chi_{\{\tau_k=\infty\}}\Big\|_{\frac{p(\cdot)}{t}}\\ &\le C\Big(\Big\|\sum_k 2^{kt}A_k^t\Big\|_{\frac{p(\cdot)}{t}}+\Big\|\sum_k 2^{kt}B_k^t\Big\|_{\frac{p(\cdot)}{t}}+\Big\|\sum_k 2^{kt}C_k^t\Big\|_{\frac{p(\cdot)}{t}}+\Big\|\sum_k 2^{kt}D_k^t\Big\|_{\frac{p(\cdot)}{t}}\Big)\\ &=: Z_1+Z_2+Z_3+Z_4.\end{aligned}\tag{5.8}$$

Estimation of Z_1 *and* Z_3: Because of $(1+\alpha)t>1$ and $\lim_{r\to\infty} r/(r-t)=1$, we can choose $\max\{1,p_+\}<r<\infty$ satisfying $r/(r-t)<(1+\alpha)t$. Note that $\alpha t<r/(r-t)$, because $\alpha t<1<r/(r-t)$, and thus the inequality

$$\alpha t<\frac{r}{r-t}<(1+\alpha)t$$

holds. Using Lemma 2.1, there exists a function $g\in L_{(p(\cdot)/t)'}$ with $\|g\|_{(p(\cdot)/t)'}\le 1$ such that

$$\begin{aligned}Z_1+Z_3 &\le C\int_\Omega\sum_k 2^{kt}\sum_l\sum_{m=1}^{K_{l_k}}\sum_{j=0}^{m-1}\sum_{i=j}^{m-1}2^{-(1+\alpha)K_{l_k}t}2^{jt}2^{i\alpha t}\chi_{I_{l_k}\dot{+}[2^{-j-1},2^{-j-1}\dot{+}2^{-i})}|g|\,d\mathbb{P}\\ &\le C\sum_k 2^{kt}\sum_l\sum_{m=1}^{K_{l_k}}\sum_{j=0}^{m-1}\sum_{i=j}^{m-1}2^{-(1+\alpha)K_{l_k}t}2^{jt}2^{i\alpha t}\|\chi_{I_{l_k}\dot{+}[2^{-j-1},2^{-j-1}\dot{+}2^{-i})}\|_{\frac{r}{t}}\\ &\quad\times\|\chi_{I_{l_k}\dot{+}[2^{-j-1},2^{-j-1}\dot{+}2^{-i})}g\|_{(\frac{r}{t})'}\end{aligned}$$

$$= C \int_\Omega \sum_k 2^{kt} \sum_l \chi_{I_{l_k}} \sum_{m=1}^{K_{l_k}} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(i\alpha+j-(1+\alpha)K_{k_l})t(1/(r/t)+1/(r/t)')} 2^{K_{l_k}-i}$$
$$\times \Big(\frac{1}{\mathbb{P}(I_{l_k} \dot+ [2^{-j-1}, 2^{-j-1} \dot+ 2^{-i}))} \int_{I_{l_k} \dot+ [2^{-j-1}, 2^{-j-1} \dot+ 2^{-i})} |g|^{(\frac{r}{t})'} \Big)^{1/(r/t)'} \, d\mathbb{P}.$$

By Hölder's inequality,

$$Z_1 + Z_3 \le C \int_\Omega \sum_k 2^{kt} \sum_l \chi_{I_{l_k}} \Big(\sum_{m=1}^{K_{l_k}} \sum_{j=0}^{m-1} 2^{(j-K_{l_k})t} \sum_{i=j}^{m-1} 2^{\alpha(i-K_{l_k})t} 2^{(K_{l_k}-i)(r/t)'}$$
$$\times \frac{1}{\mathbb{P}(I_{l_k} \dot+ [2^{-j-1}, 2^{-j-1} \dot+ 2^{-i}))} \int_{I_{l_k} \dot+ [2^{-j-1}, 2^{-j-1} \dot+ 2^{-i})} |g|^{(\frac{r}{t})'} \Big)^{1/(r/t)'} \, d\mathbb{P}$$
$$\le C \int_\Omega \sum_k 2^{kt} \sum_l \chi_{I_{l_k}} \big[U^{(\alpha)} \big(|g|^{(r/t)'} \big) \big]^{1/(r/t)'} \, d\mathbb{P}.$$

Since $(r/t)' < (p(\cdot)/t)'$, by Hölder's inequality, Theorem 4.3, and Lemma 2.2, we get that

$$Z_1 + Z_3 \le C \Big\| \sum_k 2^{kt} \sum_l \chi_{I_{l_k}} \Big\|_{\frac{p(\cdot)}{t}} \big\| \big[U^{(\alpha)} \big(|g|^{(r/t)'} \big) \big]^{1/(r/t)'} \big\|_{(\frac{p(\cdot)}{t})'}$$
$$\le C \Big\| \sum_k 2^{kt} \chi_{\{\tau_k < \infty\}} \Big\|_{\frac{p(\cdot)}{t}}.$$

Estimation of Z_2: Again by Lemma 2.1, there exists a function $g \in L_{(p(\cdot)/t)'}$ with $\|g\|_{(p(\cdot)/t)'} \le 1$ such that

$$Z_2 \le C \int_\Omega \sum_k 2^{kt} \sum_l \sum_{j=0}^{K_{k_l}-1} 2^{(j-K_{k_l})t} \chi_{I_{k_l} \dot+ 2^{-j-1}} |g| \, d\mathbb{P}.$$

This is exactly the same as Z_1 in [24]. We obtain that

$$Z_2 \le C \Big\| \sum_k 2^{kt} \sum_l \chi_{I_{k_l}} \Big\|_{\frac{p(\cdot)}{t}} = C \Big\| \sum_k 2^{kt} \chi_{\{\tau_k < \infty\}} \Big\|_{\frac{p(\cdot)}{t}}.$$

Estimation of Z_4: There exists a function $g \in L_{(p(\cdot)/t)'}$ with $\|g\|_{(p(\cdot)/t)'} \le 1$ such that

$$Z_4 \le C \int_\Omega \sum_k 2^{kt} \sum_l \sum_{m=0}^{K_{k_l}-1} 2^{-(1+\alpha)K_{k_l}t} 2^{(1+\alpha)mt} \chi_{I_{k_l} \dot+ [0, 2^{-m})} |g| \, d\mathbb{P}$$
$$\le C \sum_k 2^{kt} \sum_l \sum_{m=0}^{K_{k_l}-1} 2^{-(1+\alpha)K_{k_l}t} 2^{(1+\alpha)mt} \|\chi_{I_{k_l} \dot+ [0,2^{-m})}\|_{\frac{r}{t}}$$
$$\times \|\chi_{I_{k_l} \dot+ [0,2^{-m})} g\|_{(\frac{r}{t})'}$$
$$= C \int_\Omega \sum_k 2^{kt} \sum_l \chi_{I_{k_l}} \sum_{m=0}^{K_{k_l}-1} 2^{(-(1+\alpha)K_{k_l}t+(1+\alpha)mt)(1/(r/t)+1/(r/t)')} 2^{K_{k_l}-m}$$

$$\times \Big(\frac{1}{\mathbb{P}(I_{k_l} \dot{+} [0,2^{-m}))} \int_{I_{k_l}\dot{+}[0,2^{-m})} |g|^{(\frac{r}{t})'}\Big)^{1/(r/t)'} d\mathbb{P}$$

$$\le C \int_\Omega \sum_k 2^{kt} \sum_l \chi_{I_{k_l}} \Big(\sum_{m=0}^{K_{k_l}-1} 2^{(K_{k_l}-m)(r/(r-t)-(1+\alpha)t)}$$

$$\times \frac{1}{\mathbb{P}(I_{k_l} \dot{+} [0,2^{-m}))} \int_{I_{k_l}\dot{+}[0,2^{-m})} |g|^{(\frac{r}{t})'}\Big)^{1/(r/t)'} d\mathbb{P}$$

$$\le C \int_\Omega \sum_k 2^{kt} \sum_l \chi_{I_{k_l}} \big[V^{(\alpha)}\big(|g|^{(r/t)'}\big)\big]^{1/(r/t)'} d\mathbb{P}.$$

By Hölder's inequality, Theorem 4.5, and Lemma 2.2, we get that

$$Z_4 \le C \Big\| \sum_k 2^{kt} \sum_l \chi_{I_{k_l}} \Big\|_{\frac{p(\cdot)}{t}} \Big\| \big[V^{(\alpha)}\big(|g|^{(r/t)'}\big)\big]^{1/(r/t)'} \Big\|_{(\frac{p(\cdot)}{t})'}$$

$$\le C \Big\| \sum_k 2^{kt} \chi_{\{\tau_k<\infty\}} \Big\|_{\frac{p(\cdot)}{t}}.$$

We get (see (5.8)) that

$$\Big\| \sum_k \mu_k^t \sigma_*^\alpha (a^k)^t \chi_{\{\tau_k=\infty\}} \Big\|_{\frac{p(\cdot)}{t}} \le C \Big\| \sum_k 2^{kt} \chi_{\{\tau_k<\infty\}} \Big\|_{\frac{p(\cdot)}{t}},$$

which finishes the proof. □

By Theorems 5.1 and 5.2, we immediately get the following corollary.

Corollary 5.3. *Let $0 < \alpha \le 1 \le \gamma$, $p(\cdot) \in \mathcal{P}(\Omega)$, $1/(\alpha+1) < p_- < \infty$, and suppose that $p(\cdot)$ satisfies (2.1). Then for all $f \in H_{p(\cdot)}$,*

$$\|\sigma_*^\alpha f\|_{p(\cdot)} \le C \|f\|_{H_{p(\cdot)}}.$$

The same holds for $\sigma_*^{\alpha,\gamma}$.

This theorem was proved for the maximal Fejér operator ($\alpha = 1$) in [24]. Fujii [12] proved the theorem for $p = 1$. For the Cesàro and Riesz maximal operators and for constant p, the theorem is due to the second author [37]. If the exponent function is constant and $p \le 1/(\alpha+1)$, then the Cesàro and Riesz maximal operators are not bounded from H_p to L_p (see Simon and Weisz [34], Simon [33], and Gát and Goginava [16]).

The restriction of a martingale f to the dyadic interval I of length 2^{-k} is defined by $f\chi_I := (E_n f \chi_I : n \ge k)$. This result implies the next corollary. The proof is similar to that given in [24].

Corollary 5.4. *Let $0 < \alpha \le 1 \le \gamma$, $p(\cdot) \in \mathcal{P}(\Omega)$, $1/(\alpha+1) < p_- < \infty$, and suppose that $p(\cdot)$ satisfies (2.1) and $f \in H_{p(\cdot)}$.*

(1) *Then* $\sigma_n^\alpha f$ *and* $\sigma_n^{\alpha,\gamma} f$ *converge almost everywhere on* $[0,1)$ *as well as in the* $L_{p(\cdot)}$*-norm.*

(2) *If in addition* $f\chi_I \in L_1$*, where* I *is a dyadic interval, then*

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x) \quad \text{for almost every } x \in I \text{ and in the } L_{p(\cdot)}\text{-norm.}$$

The same holds for $\sigma_n^{\alpha,\gamma}$.

If $p_- \geq 1$ and $f \in H_{p(\cdot)}$, then $f \in L_1$. The next corollary follows from this.

Corollary 5.5. *Let* $0 < \alpha \leq 1 \leq \gamma$*,* $p(\cdot) \in \mathcal{P}(\Omega)$*,* $1 \leq p_- < \infty$*, and suppose that* $p(\cdot)$ *satisfies (2.1). Then for all* $f \in H_{p(\cdot)}$*,*

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x) \quad \text{for almost every } x \in [0,1) \text{ as well as in the } L_{p(\cdot)}\text{-norm.}$$

The same holds for $\sigma_n^{\alpha,\gamma}$.

6. Boundedness of the Cesàro and Riesz means in $H_{p(\cdot),q}$

The following theorem was proved in [24, Theorem 7.28].

Theorem 6.1. *Let* $p(\cdot) \in \mathcal{P}(\Omega)$ *satisfy (2.1), and let* $0 < q \leq \infty$*. If the* σ*-sublinear operator* $T : L_\infty \to L_\infty$ *is bounded and*

$$\left\| |Ta|^\beta \chi_{\{\tau=\infty\}} \right\|_{p(\cdot)} \leq C \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{1-\beta}$$

for some $0 < \beta < 1$ *and all* $p(\cdot)$*-atoms* a*, where* τ *is the stopping time associated with* a*, then for all* $f \in H_{p(\cdot),q}$*,*

$$\|Tf\|_{L_{p(\cdot),q}} \leq C \|f\|_{H_{p(\cdot),q}}.$$

If $q = \infty$*, then the theorem holds in case* $f \in \mathcal{H}_{p(\cdot),\infty}$ *instead of* $H_{p(\cdot),\infty}$.

Theorem 6.2. *Let* $0 < \alpha \leq 1 \leq \gamma$*,* $p(\cdot) \in \mathcal{P}(\Omega)$*,* $1/(\alpha+1) < p_- < \infty$*, and suppose that* $p(\cdot)$ *satisfies (2.1). Then*

$$\left\| |\sigma_*^\alpha a|^\beta \chi_{\{\tau=\infty\}} \right\|_{p(\cdot)} \leq C \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{1-\beta} \tag{6.1}$$

for some $0 < \beta < 1$ *and all* $p(\cdot)$*-atoms* a*, where* τ *is the stopping time associated with* a*. The same holds for* $\sigma_*^{\alpha,\gamma}$.

Proof. Let $t := \beta\varepsilon$, where $0 < \beta < 1$, $1/(\alpha+1) < \varepsilon < \underline{p}$ such that $\beta\varepsilon > 1/(\alpha+1)$. Observe that (6.1) is equivalent with

$$\left\| |\sigma_*^\alpha a|^{\beta\varepsilon} \chi_{\{\tau=\infty\}} \right\|_{\frac{p(\cdot)}{\varepsilon}} \leq C \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-\beta\varepsilon} \cdot \|\chi_{\{\tau<\infty\}}\|_{\frac{p(\cdot)}{\varepsilon}}. \tag{6.2}$$

From the proof of Theorem 5.2 (see (5.7)), we get

$$\begin{aligned}\left\| |\sigma_*^\alpha a|^{\beta\varepsilon} \chi_{\{\tau=\infty\}} \right\|_{\frac{p(\cdot)}{\varepsilon}} &\leq C \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-\beta\varepsilon} \big(\|A^{\beta\varepsilon} \chi_{\{\tau=\infty\}}\|_{\frac{p(\cdot)}{\varepsilon}} + \|B^{\beta\varepsilon} \chi_{\{\tau=\infty\}}\|_{\frac{p(\cdot)}{\varepsilon}} \\ &\qquad + \|C^{\beta\varepsilon} \chi_{\{\tau=\infty\}}\|_{\frac{p(\cdot)}{\varepsilon}} + \|D^{\beta\varepsilon} \chi_{\{\tau=\infty\}}\|_{\frac{p(\cdot)}{\varepsilon}} \big) \\ &=: C \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-\beta\varepsilon} (X_1 + X_2 + X_3 + X_4).\end{aligned}$$

We will show that each term satisfies

$$X_i \leq C \|\chi_{\{\tau<\infty\}}\|_{\frac{p(\cdot)}{\varepsilon}} \quad (i = 1,2,3,4).$$

Estimation of X_1 and X_3: Choose $\max\{1, \beta p_+\} < r < \infty$ large enough to satisfy that $r/(r - \beta\varepsilon) < (1 + \alpha)\beta\varepsilon$. Using Lemma 2.1, there exists a function $g \in L_{(p(\cdot)/\varepsilon)'}$ with $\|g\|_{(p(\cdot)/\varepsilon)'} \le 1$ such that

$$\begin{aligned}
X_1 + X_3 &\le C \int_\Omega \sum_l \sum_{m=1}^{K_l} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{-(1+\alpha)K_l\beta\varepsilon} 2^{j\beta\varepsilon} 2^{i\alpha\beta\varepsilon} \chi_{I_l \dot{+} [2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i})} |g| \, d\mathbb{P} \\
&\le C \int_\Omega \sum_l \sum_{m=1}^{K_l} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{-(1+\alpha)K_l\beta\varepsilon} 2^{j\beta\varepsilon} 2^{i\alpha\beta\varepsilon} \|\chi_{I_l \dot{+} [2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i})}\|_{\frac{r}{\beta\varepsilon}} \\
&\quad \times \|\chi_{I_l \dot{+} [2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i})} g\|_{(\frac{r}{\beta\varepsilon})'} \\
&\le C \int_\Omega \sum_l \chi_{I_l} \sum_{m=1}^{K_l} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(i\alpha + j - (1+\alpha)K_l)\beta\varepsilon(1/(\frac{r}{\beta\varepsilon}) + 1/(\frac{r}{\beta\varepsilon})')} \, 2^{K_l - i} \\
&\quad \times \Big(\frac{1}{\mathbb{P}(I_l \dot{+} [2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i}))} \int_{I_l \dot{+} [2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i})} |g|^{(\frac{r}{\beta\varepsilon})'} \Big)^{1/(\frac{r}{\beta\varepsilon})'} d\mathbb{P}.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
&\le C \int_\Omega \sum_l \chi_{I_l} \Big(\sum_{m=1}^{K_l} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(i\alpha + j - (1+\alpha)K_l)\beta\varepsilon} \Big)^{1/(\frac{r}{\beta\varepsilon})} \\
&\quad \times \Big(\sum_{m=1}^{K_l} \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(i\alpha + j - (1+\alpha)K_l)\beta\varepsilon} 2^{(K_l - i)(\frac{r}{\beta\varepsilon})'} \\
&\quad \times \frac{1}{\mathbb{P}(I_l \dot{+} [2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i}))} \int_{I_l \dot{+} [2^{-j-1}, 2^{-j-1} \dot{+} 2^{-i})} |g|^{(\frac{r}{\beta\varepsilon})'} \Big)^{1/(\frac{r}{\beta\varepsilon})'} d\mathbb{P} \\
&\le C \int_\Omega \sum_l \chi_{I_l} \big[U^{(\alpha)} \big(|g|^{(\frac{r}{\beta\varepsilon})'} \big) \big]^{1/(\frac{r}{\beta\varepsilon})'} d\mathbb{P}.
\end{aligned}$$

Note that since $r > \beta p_+$ and $\varepsilon < p_-$, we get that $(r/(\beta\varepsilon))' < (p(\cdot)/\varepsilon)'$ and $((p(\cdot)/\varepsilon)')_+ < \infty$. By Hölder's inequality, Theorem 4.3, and Lemma 2.2, we get that

$$\begin{aligned}
X_1 + X_2 &\le C \Big\| \sum_l \chi_{I_l} \Big\|_{\frac{p(\cdot)}{\varepsilon}} \Big\| \big[U^{(\alpha)} \big(|g|^{(\frac{r}{\beta\varepsilon})'} \big) \big]^{1/(\frac{r}{\beta\varepsilon})'} \Big\|_{(\frac{p(\cdot)}{\varepsilon})'} \\
&\le C \Big\| \sum_l \chi_{I_l} \Big\|_{\frac{p(\cdot)}{\varepsilon}} \|g\|_{(\frac{p(\cdot)}{\varepsilon})'} \\
&\le C \|\chi_{\{\tau < \infty\}}\|_{\frac{p(\cdot)}{\varepsilon}}.
\end{aligned}$$

Estimation of X_2: The estimation

$$X_2 \le C \|\chi_{\{\tau < \infty\}}\|_{\frac{p(\cdot)}{\varepsilon}}$$

can be found in [24].

Estimation of X_4: There exists a function $g \in L_{(p(\cdot)/\varepsilon)'}$ with $\|g\|_{(p(\cdot)/\varepsilon)'} \le 1$ such that

$$\begin{aligned}
X_4 &\le C \int_\Omega \sum_l \sum_{m=0}^{K_l-1} 2^{-(1+\alpha)K_l\beta\varepsilon} 2^{(1+\alpha)m\beta\varepsilon} \chi_{I_l \dot{+} [0,2^{-m})} |g| \, d\mathbb{P} \\
&\le C \sum_l \sum_{m=0}^{K_l-1} 2^{-(1+\alpha)K_l\beta\varepsilon} 2^{(1+\alpha)m\beta\varepsilon} \|\chi_{I_l \dot{+} [0,2^{-m})}\|_{\frac{r}{\beta\varepsilon}} \\
&\quad \times \|\chi_{I_l \dot{+} [0,2^{-m})} g\|_{(\frac{r}{\beta\varepsilon})'} \\
&= C \sum_l \sum_{m=0}^{K_l-1} 2^{(-(1+\alpha)K_l\beta\varepsilon + (1+\alpha)m\beta\varepsilon)(1/(\frac{r}{\beta\varepsilon}) + 1/(\frac{r}{\beta\varepsilon})')} 2^{K_l - m} \\
&\quad \times \int_\Omega \chi_{I_l} \Big(\frac{1}{\mathbb{P}(I_l \dot{+} [0,2^{-m}))} \int_{I_l \dot{+} [0,2^{-m})} |g|^{(\frac{r}{\beta\varepsilon})'} \Big)^{1/(\frac{r}{\beta\varepsilon})'} d\mathbb{P} \\
&\le C \int_\Omega \sum_l \chi_{I_l} \big[V^{(\alpha)}\big(|g|^{(\frac{r}{\beta\varepsilon})'}\big)\big]^{1/(\frac{r}{\beta\varepsilon})'} d\mathbb{P}.
\end{aligned}$$

Hence

$$X_4 \le C \Big\| \sum_l \chi_{I_l} \Big\|_{\frac{p(\cdot)}{\varepsilon}} \Big\| \big[V^{(\alpha)}\big(|g|^{(\frac{r}{\beta\varepsilon})'}\big)\big]^{1/(\frac{r}{\beta\varepsilon})'} \Big\|_{(\frac{p(\cdot)}{\varepsilon})'} \le C \|\chi_{\{\tau<\infty\}}\|_{\frac{p(\cdot)}{\varepsilon}}.$$

Combining the estimations, we have proved (6.2) as well as the theorem. □

By Theorems 6.1 and 6.2, we immediately get the following corollary.

Corollary 6.3. *Let* $0 < \alpha \le 1 \le \gamma$, $p(\cdot) \in \mathcal{P}(\Omega)$, $1/(\alpha+1) < p_- < \infty$, *and suppose that* $p(\cdot)$ *satisfies (2.1). Then for all* $0 < q \le \infty$,

$$\|\sigma_*^\alpha f\|_{L_{p(\cdot),q}} \le C \|f\|_{H_{p(\cdot),q}}$$

for all $f \in H_{p(\cdot),q}$. *The same holds for* $\sigma_*^{\alpha,\gamma}$.

The following corollaries can be proved similarly to Corollaries 5.4 and 5.5.

Corollary 6.4. *Let* $0 < \alpha \le 1 \le \gamma$, $0 < q \le \infty$, $p(\cdot) \in \mathcal{P}(\Omega)$, $1/(\alpha+1) < p_- < \infty$, *and suppose that* $p(\cdot)$ *satisfies (2.1) and* $f \in H_{p(\cdot),q}$.

(1) *Then* $\sigma_n^\alpha f$ *and* $\sigma_n^{\alpha,\gamma} f$ *converge almost everywhere on* $[0,1)$ *as well as in the* $L_{p(\cdot),q}$*-norm.*

(2) *If in addition* $f\chi_I \in L_1$, *where* I *is a dyadic interval, then*

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x) \quad \textit{for almost every } x \in I.$$

(3) *If* $1 \le p_- < \infty$, *then*

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x) \quad \textit{for almost every } x \in [0,1).$$

The same holds for $\sigma_n^{\alpha,\gamma}$.

Since $L_1 \subset \mathcal{H}_{1,\infty}$, almost everywhere convergence also holds for $f \in L_1$, which was similarly shown in Weisz [37].

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Department of Numerical Analysis, Eötvös Loránd University, Pázmány Péter sétány 1/c., H-1117 Budapest, Hungary.

E-mail address: kristof@inf.elte.hu; weisz@inf.elte.hu