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MULTILINEAR OPERATORS FACTORING THROUGH HILBERT SPACES

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Abstract. We characterize those bounded multilinear operators that factor through a Hilbert space in terms of its behavior in finite sequences. This extends a result, essentially due to Kwapień, from the linear to the multilinear setting. We prove that Hilbert–Schmidt and Lipschitz 2-summing multilinear operators naturally factor through a Hilbert space. We also prove that the class Γ of all multilinear operators that factor through a Hilbert space is a maximal multi-ideal; moreover, we give an explicit formulation of a finitely generated tensor norm γ which is in duality with Γ.

1. Introduction and preliminaries

The fact that a bounded linear operator between Banach spaces factors through a Hilbert space is a priori a fairly abstract property. It is possible, however, to describe it in terms of the behavior of the operator in a special type of finite sequences of the domain. Such a local expression of the property makes it possible to relate it with other fundamental notions of the geometry of Banach spaces. This is the case, for example, of Kwapień's characterization of the Banach spaces that are isomorphic to a Hilbert space as those having type 2 and cotype 2 (see [14]). (Regarding the factorization of linear operators through a Hilbert space,

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we refer the reader to the original papers [10] and [16], and to the corresponding chapters in [6], [20], and [23].)

The problem of factoring an operator through a Hilbert space has also been studied for mappings other than linear operators. This is the case of Lipschitz mappings between metric spaces and completely bounded operators between operator spaces developed in [4] and [21], respectively. This problem certainly makes sense for multilinear mappings. Compact (see [13]), nuclear (see [1]), p-summing (see [3], [7], [19]), integral (see [24]), and other classes of linear operators have been extended to the multilinear setting. Despite this, the case of factoring a multilinear operator through a Hilbert space, as far as we know, has not been studied or even defined. In this article, we provide a solution to this problem. We first briefly describe our results.

We say that $T : X_1 \times \cdots \times X_n \to Y$ *factors through a Hilbert space* if there exist a Hilbert space H, a subset M of H, a bounded multilinear operator $A : X_1 \times \cdots \times X_n \to H$, and a Lipschitz function $B : M \to Y$ such that $A(X_1 \times \cdots \times X_n) \subset M$ and

$$\begin{tikzcd}
X_1 \times \cdots \times X_n \arrow[rr, "T"] \arrow[dr, "A"'] & & Y \\
& M \arrow[ur, "B"'] \arrow[d, "i"'] & \\
& H &
\end{tikzcd} \tag{1.1}$$

commutes; that is, $T = B \circ A$. We define $\Gamma(T) = \inf \|A\| \operatorname{Lip}(B)$, where the infimum is taken over all possible factorizations as in (1.1).

Our main result, Theorem 3.3, says that if π denotes the projective tensor norm on $X_1 \otimes \cdots \otimes X_n$, then we have the following.

Kwapień-type characterization. The multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ factors through a Hilbert space if and only if there is a constant $C > 0$ such that

$$\sum_{i=1}^{m} \big\|T(x_i^1, \ldots, x_i^n) - T(z_i^1, \ldots, z_i^n)\big\|^2 \leq C^2 \sum_{i=1}^{m} \pi(s_i^1 \otimes \cdots \otimes s_i^n - t_i^1 \otimes \cdots \otimes t_i^n)^2$$

holds for all finite sequences $(x_i^1, \ldots, x_i^n)_{i=1}^m$, $(z_i^1, \ldots, z_i^n)_{i=1}^m$, $(s_i^1, \ldots, s_i^n)_{i=1}^m$, and $(t_i^1, \ldots, t_i^n)_{i=1}^m$ in $X_1 \times \cdots \times X_n$ with the property

$$\sum_{i=1}^{m} \big|\varphi(x_i^1, \ldots, x_i^n) - \varphi(z_i^1, \ldots, z_i^n)\big|^2 \leq \sum_{i=1}^{m} \big|\varphi(s_i^1, \ldots, s_i^n) - \varphi(t_i^1, \ldots, t_i^n)\big|^2$$

for all φ in $\mathcal{L}(X_1, \ldots, X_n)$. In this situation, $\Gamma(T)$ is the best constant C.

Note that in the case $n = 1$, (1.1) reduces to

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "A"'] & & Y \\
& H \arrow[ur, "B"'] &
\end{tikzcd}$$

where all the involved operators are linear and $M = A(X) = H$. In this case, we get an equivalent formulation of the well-known linear characterization (essentially due to Kwapień), namely, that a bounded linear operator $T : X \to Y$ factors through a Hilbert space if and only if there exists a constant $C > 0$ such that

$$\sum_{i=1}^{m} \left\|T(x_i)\right\|^2 \leq C^2 \sum_{i=1}^{m} \|a_i\|^2$$

holds for all finite sequences $(x_i)_{i=1}^m$ and $(a_i)_{i=1}^m$ in X with the property

$$\sum_{i=1}^{m} \left|x^*(x_i)\right|^2 \leq \sum_{i=1}^{m} \left|x^*(a_i)\right|^2 \quad \forall x^* \in X^*.$$

In this way, Theorem 3.3 extends Kwapień's formulation of linear operators factoring through a Hilbert space to the multilinear setting. (The interested reader is strongly encouraged to review Lindenstrauss and Pelczynski's early formulation of this property in [16, Proposition 5.2] and Kwapień's subsequent versions in [14, Proposition 3.1] and [15, Theorem 2′]; see also Pisier [20, Theorem 2.4] for an accessible proof and a good exposition of this class in the linear setting.)

In relation to other multilinear properties, we prove that every Hilbert–Schmidt multilinear operator (see [18, Definition 5.2]), as well as every Lipschitz 2-summing multilinear operator (see [3, Definition 3.1]), factors through a Hilbert space. We also prove that the class of multilinear operators which satisfy diagram (1.1) enjoys ideal properties. To explain this, let $\mathcal{L}$ denote the class of all bounded multilinear operators. If we denote by Γ the subclass of $\mathcal{L}$ that consists of all bounded multilinear operators that factor through a Hilbert space with the function $\Gamma(\cdot)$, then the pair $[\Gamma, \Gamma(\cdot)]$ is a maximal multi-ideal in the sense of Floret and Hunfeld [9]. This affirmation is a consequence of Proposition 4.1 and Theorem 3.2, which establish the multi-ideal nature of Γ and maximality, respectively.

In duality with the maximal multi-ideal nature of the class Γ, we exhibit a finitely generated tensor norm γ which satisfies that

$$(X_1 \otimes \cdots \otimes X_n \otimes Y, \gamma)^* = \Gamma(X_1, \ldots, X_n; Y^*)$$

and

$$(X_1 \otimes \cdots \otimes X_n \otimes Y^*, \gamma)^* \cap \mathcal{L}(X_1, \ldots, X_n, Y) = \Gamma(X_1, \ldots, X_n; Y)$$

hold isometrically. These results are presented as Theorem 4.4 and Corollary 4.5, respectively.

Following the ideas developed throughout the article, we also introduce the notion of polynomials that can be factored through a Hilbert space (see Definition 5.1), and we state a Kwapień-type characterization for polynomials (see Theorem 5.3). To obtain these results, we have applied the general approach introduced in [8]. This approach is, basically, to study a multilinear map T by means of its associated Σ-operator f_T (see Section 1.1). Posing the problem of factoring T through a Hilbert space in the context of Σ-operators allowed us to use the geometric richness of the tensor products of Banach spaces. Moreover, since bounded Σ-operators are Lipschitz mappings (see [8, Theorem 3.2]), this approach enables us to relate, naturally, multilinear operators that factor through a Hilbert space with the metric study carried out in [4] (see Section 2.1).

The material here is organized as follows. In Section 1.1, we fix some standard notation of Banach spaces and multilinear theories. We also recall from [8] the notion of a Σ-operator. In Section 2, we give the precise definition of a multilinear operator that factors through a Hilbert space, along with some examples. Section 3 is dedicated to proving the main result, Theorem 3.3. Section 4 is devoted to proving that the class Γ of all multilinear operators that factor through a Hilbert space is a maximal multi-ideal. The duality with the tensor norm is proved in Theorem 4.4 and Corollary 4.5. In Section 5, we study the polynomials that factor through a Hilbert space, proving a Kwapień-type characterization for them.

1.1. Notation and preliminaries. We use standard notation of the theory of Banach spaces. The letter $\mathbb{K}$ denotes the real or complex numbers. The unit ball of a the normed space X is denoted by B_X. We denote by $K_X : X \to X^{**}$ the canonical embedding.

Throughout this article, n denotes a positive integer and the capital letters $X_1, \dots, X_n, Y$ and Z denote Banach spaces over the same field. The symbol $\mathcal{L}(X_1, \dots, X_n, Y)$ denotes the Banach space of bounded multilinear operators $T : X_1 \times \cdots \times X_n \to Y$ with the usual norm $\|T\| = \sup\{\|T(x^1, \dots, x^n)\| \mid x^i \in B_{X_i}\}$. For simplicity of notation, we write $\mathcal{L}(X_1, \dots, X_n)$ in the case $Y = \mathbb{K}$.

The set of decomposable tensors of the algebraic tensor product $X_1 \otimes \cdots \otimes X_n$ is denoted by $\Sigma_{X_1,\dots,X_n}$. That is,

$$\Sigma_{X_1,\dots,X_n} := \{x^1 \otimes \cdots \otimes x^n \mid x^i \in X_i\}.$$

Let π be the projective tensor norm given by

$$\pi(u; X_1 \otimes \cdots \otimes X_n) = \inf\Big\{\sum_{i=1}^{m} \|x_i^1\| \dots \|x_i^n\| \;\Big|\; u = \sum_{i=1}^{m} x_i^1 \otimes \cdots \otimes x_i^n\Big\}.$$

The symbol $\Sigma^\pi_{X_1,\dots,X_n}$ denotes the resulting metric space obtained by restricting the norm π to $\Sigma_{X_1,\dots,X_n}$.

The universal property of the projective tensor product establishes that for every bounded multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ there exists a unique bounded linear operator $\widetilde{T} : X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n \to Y$ such that $T(x^1, \dots, x^n) = \widetilde{T}(x^1 \otimes \cdots \otimes x^n)$. In particular, the restriction of $\widetilde{T}$ to $\Sigma^\pi_{X_1 \dots X_n}$ is a Lipschitz function. In this situation, the linear map $\widetilde{T}$ is called the *linearization* of T and $f_T = \widetilde{T}|_{\Sigma^\pi_{X_1,\dots,X_n}} : \Sigma^\pi_{X_1,\dots,X_n} \to Y$ is called the Σ-*operator* associated to T. Moreover, we have $\|T\| = \mathrm{Lip}(f_T) = \|\widetilde{T}\|$ (for details on Σ-operators, the reader is referred to [8]).

A norm β on $X_1 \otimes \cdots \otimes X_n$ is said to be a *reasonable crossnorm* if

$$\varepsilon(u) \leq \beta(u) \leq \pi(u) \quad \forall u \in X_1 \otimes \cdots \otimes X_n,$$

where ε denotes the injective tensor norm defined by

$$\varepsilon(u; X_1 \otimes \cdots \otimes X_n) = \sup\big\{\big|x_1^* \otimes \cdots \otimes x_n^*(u)\big| \;\big|\; x_i^* \in B_{X_i^*}, 1 \leq i \leq n\big\}$$

for each u in $X_1 \otimes \cdots \otimes X_n$. (The interested reader may refer to [5], [22], and [9] for details on tensor norms.)

According to Theorem 2.1 of [8], we have that if β is a reasonable crossnorm on $X_1 \otimes \cdots \otimes X_n$, then the resulting metric space $\Sigma^{\beta}_{X_1,\dots,X_n}$ (obtained by restricting the norm β to $\Sigma_{X_1,\dots,X_n}$) is Lipschitz equivalent to $\Sigma^{\pi}_{X_1,\dots,X_n}$. Specifically, we have

$$\pi(p-q) \leq 2^{n-1}\beta(p-q) \quad \forall p,q \in \Sigma_{X_1,\dots,X_n} \tag{1.2}$$

for all reasonable crossnorms β on $X_1 \otimes \cdots \otimes X_n$.

2. Definition, examples, and the metric case

Definition 2.1. We say that the multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ *factors through a Hilbert space* if there exists a Hilbert space H, a subset M of H, a bounded multilinear operator $A : X_1 \times \cdots \times X_n \to H$ whose image is contained in M, and a Lipschitz function $B : M \to Y$ such that the diagram

$$\begin{tikzcd} X_1 \times \cdots \times X_n \arrow[rr, "T"] \arrow[dr, "A"'] & & Y \\ & M \arrow[ur, "B"'] \arrow[d] & \\ & H & \end{tikzcd} \tag{2.1}$$

commutes. We define $\Gamma(T)$ as $\inf \|A\| \operatorname{Lip}(B)$, where the infimum is taken over all possible factorizations as above.

In the previous definition it is enough to take M as $A(X_1 \times \cdots \times X_n)$ (or equivalently, its closure in H). The collection of multilinear operators $T : X_1 \times \cdots \times X_n \to Y$ which admit a factorization through a Hilbert space as in (2.1) is denoted by the symbol $\Gamma(X_1, \dots, X_n; Y)$. The symbol Γ denotes the class of all bounded multilinear operators that factor through a Hilbert space.

It is easy to see that the translation of the diagram (2.1) to the setting of Σ-operators acquires the form

$$\begin{tikzcd} \Sigma^{\pi}_{X_1,\dots,X_n} \arrow[rr, "f_T"] \arrow[dr, "f_A"'] & & Y \\ & f_A(\Sigma_{X_1,\dots,X_n}) \arrow[ur, "B"'] \arrow[d] & \\ & H & \end{tikzcd} \tag{2.2}$$

where f_T and f_A are the Σ-operators associated to the bounded multilinear operators T and A. In other words, $f_T = B f_A$.

In the rest of this article, $H_1, \dots, H_n$ and H denote Hilbert spaces, $H_1 \widehat{\otimes}_2 \cdots \widehat{\otimes}_2 H_n$ denotes its Hilbert tensor product, and $\|\cdot\|_2$ denotes its reasonable crossnorm (we refer the reader to [12, Section 2.6] for details of this construction).

Example 2.2 (The canonical multilinear map on Hilbert spaces). The canonical multilinear map

$$\otimes : H_1 \times \cdots \times H_n \to H_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi H_n,$$
$$(x^1, \ldots, x^n) \mapsto x^1 \otimes \cdots \otimes x^n$$

factors through the Hilbert space $H_1 \widehat{\otimes}_2 \cdots \widehat{\otimes}_2 H_n$. Moreover, (1.2) implies that

$$\Gamma(\otimes) \leq \| \otimes \| \operatorname{Lip}(\mathrm{Id} : \Sigma^{\|\cdot\|_2}_{H_1,\ldots,H_n} \to \Sigma^{\pi}_{H_1,\ldots,H_n}) \leq 2^{n-1}.$$

Note that the linearization $\tilde{\otimes}$ is the identity map on $H_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi H_n$. Hence, $\tilde{\otimes}$ does not factor through a Hilbert space in the linear sense since for $n > 1$, $H_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi H_n$ contains a subspace isometric to ℓ_1 (see [22, Example 2.10]).

As a consequence of the previous discussion, $T \in \Gamma(X_1, \ldots, X_n; Y)$ does not imply that its linearization $\widetilde{T} : X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n \to Y$ factors through a Hilbert space. However, the converse is naturally true.

Example 2.3 (Multilinear operators whose linearizations factor through a Hilbert space). Consider a bounded multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ whose linearization $\widetilde{T} : X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n \to Y$ factors through a Hilbert space. Note that a typical factorization $\widetilde{T} = BA$ implies that T factors as $T = B|_{f_A(\Sigma_{X_1,\ldots,X_n})}(A\otimes)$. Therefore, $T \in \Gamma(X_1, \ldots, X_n; Y)$ and $\Gamma(T) \leq \Gamma(\widetilde{T})$. In other words, the operator

$$\Gamma(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n; Y) \to \Gamma(X_1, \ldots, X_n; Y),$$
$$\widetilde{T} \mapsto T$$

is bounded and has norm at most 1.

Example 2.4 (When every factor of the domain is a Hilbert space). It is natural to expect that operators of the form $T : H_1 \times \cdots \times H_n \to Y$ factor through the Hilbert space $H_1 \widehat{\otimes}_2 \cdots \widehat{\otimes}_2 H_n$. Indeed, the identity

$$\Gamma(H_1, \ldots, H_n; Y) \to \mathcal{L}(H_1, \ldots, H_n; Y),$$
$$T \mapsto T$$

is a surjective isomorphism for every Y. To see this, let $T : H_1 \times \cdots \times H_n \to Y$ be a bounded multilinear operator. Recall that $f_T : \Sigma^{\pi}_{H_1,\ldots,H_n} \to Y$ is a Lipschitz function and $\otimes : H_1 \times \cdots \times H_n \to H_1 \widehat{\otimes}_2 \cdots \widehat{\otimes}_2 H_n$ is bounded. From (1.2) we have that $f_T : \Sigma^{\|\cdot\|_2}_{H_1,\ldots,H_n} \to Y$ is also Lipschitz. Hence, the factorization $T = f_T \otimes$ tells us that $T \in \Gamma(H_1, \ldots, H_n; Y)$ and

$$\Gamma(T) \leq \| \otimes \| \operatorname{Lip}(f_T : \Sigma^{\|\cdot\|_2}_{H_1,\ldots,H_n} \to Y) \leq 2^{n-1} \|T\|.$$

Furthermore, Proposition 4.1(iii) says that $\|T\| \leq \Gamma(T)$.

Example 2.5 (Hilbert–Schmidt multilinear operators). Following Matos [18], let $\mathcal{L}_{\mathrm{HS}}(H_1, \ldots, H_n; H)$ denote the Banach space of Hilbert–Schmidt multilinear operators $T : H_1 \times \cdots \times H_n \to H$ endowed with the norm

$$\|T\|_{\mathrm{HS}} = \Big(\sum_{\substack{j_i \in J_i \\ 1 \le i \le n}} \big\| T(e^1_{j_1}, \dots, e^n_{j_n}) \big\|^2 \Big)^{\frac{1}{2}},$$

where $(e^i_j)_{j \in J_i}$ is an orthonormal basis of H_i.

The previous example tells us that every Hilbert–Schmidt multilinear operator factors through a Hilbert space. Even more, we have the following.

Proposition 2.6. *We have* $\|\mathrm{Id} : \mathcal{L}_{\mathrm{HS}}(H_1, \dots, H_n; H) \to \Gamma(H_1, \dots, H_n; H)\| \le 1$.

Proof. Recall that the spaces $\mathcal{L}_{\mathrm{HS}}(H_1, \dots, H_n; H)$ and $\mathcal{L}_{\mathrm{HS}}(H_1 \widehat{\otimes}_2 \cdots \widehat{\otimes}_2 H_n; H)$ are isometrically isomorphic via the assignment $T \mapsto \widetilde{T}$ (see [18, Proposition 5.10]). Then every Hilbert–Schmidt multilinear operator $T : H_1 \times \cdots \times H_n \to H$ factors as $T = f_T \otimes$ through $H_1 \widehat{\otimes}_2 \cdots \widehat{\otimes}_2 H_n$. Moreover,

$$\big\| f_T(p) - f_T(q) \big\| \le \|p - q\|_2 \|T\|_{\mathrm{HS}} \quad \forall p, q \in \Sigma_{H_1, \dots, H_n}.$$

Hence $\Gamma(T) \le \| \otimes \| \, \mathrm{Lip}(f_T : \Sigma^{\|\cdot\|_2}_{H_1, \dots, H_n} \to H) \le \|T\|_{\mathrm{HS}}$. □

With this, we obtain that for every $2 \le p < \infty$, every fully absolutely p-summing operator $T \in \mathcal{L}^p_{\mathrm{fas}}(H_1, \dots, H_n; H)$ (for this notion, see [18, Definition 2.2]) factors through a Hilbert space. In [18, Proposition 5.5], the author proves that $\mathcal{L}^2_{\mathrm{fas}}(H_1, \dots, H_n; H)$ is isometrically isomorphic to $\mathcal{L}_{\mathrm{HS}}(H_1, \dots, H_n; H)$. Hence, the preceding proposition asserts that every absolutely 2-summing multilinear operator T between Hilbert spaces factors through a Hilbert space and $\Gamma(T) \le \|T\|_{\mathrm{fas},2}$. Even more, according to [18, Proposition 5.7], $\mathcal{L}^p_{\mathrm{fas}}(H_1, \dots, H_n; H)$ is isomorphic to $\mathcal{L}_{\mathrm{HS}}(H_1, \dots, H_n; H)$ for $2 \le p < \infty$. As a consequence, every fully absolutely p-summing multilinear operator T factors through a Hilbert space and $\Gamma(T) \le (b_p)^n \|T\|_{\mathrm{fas},p}$, where b_p is the greater constant from Khintchine's inequality for all $2 \le p < \infty$.

It is worth pointing out that the morphism in Proposition 2.6 is not surjective since $\otimes : H_1 \times \cdots \times H_n \to H_1 \widehat{\otimes}_2 \cdots \widehat{\otimes}_2 H_n$ is a bounded multilinear operator which is not Hilbert–Schmidt when each H_i is infinite-dimensional. The same observation is also valid for the case of fully absolutely summing multilinear operators that we have dealt with before.

Example 2.7 (Lipschitz 2-summing multilinear operators). In this example, we relate the notion of Lipschitz 2-summing multilinear operators developed in [3] with multilinear operators that factor through a Hilbert space.

One of the equivalences of the Lipschitz 2-summability of $T : X_1 \times \cdots \times X_n \to Y$ (see [3, Theorem 1.1]) establishes that T factors as

$$\begin{tikzcd}
X_1 \times \cdots \times X_n \arrow[r, "T"] \arrow[d, "i"'] & Y \\
i(\Sigma_{X_1, \dots, X_n}) \arrow[r, "j_2|_{i(\Sigma_{X_1, \dots, X_n})}"'] \arrow[d] & j_2 \circ i(\Sigma_{X_1, \dots, X_n}) \arrow[u, "u"'] \arrow[d] \\
C(B_{\mathcal{L}(X_1, \dots, X_n)^*}) \arrow[r, "j_2"'] & L_2(\mu)
\end{tikzcd}$$

where μ is a probability measure on $(B_{\mathcal{L}(X_1,\dots,X_n)^*}, w^*)$, $i : X_1 \times \cdots \times X_n \to C(B_{\mathcal{L}(X_1,\dots,X_n)^*})$ acts by evaluation, $j_2 : C(B_{\mathcal{L}(X_1,\dots,X_n)^*}) \to L_2(\mu)$ is the canonical inclusion, and u is a Lipschitz function such that $\pi_2^{\mathrm{Lip}}(T) = \mathrm{Lip}(u)$. Hence T is in $\Gamma(X_1,\dots,X_n;Y)$ and $\Gamma(T) \le \pi_2^{\mathrm{Lip}}(T)$.

Actually, we have proved that

$$\big\| \mathrm{Id} : \Pi_2^{\mathrm{Lip}}(X_1,\dots,X_n;Y) \to \Gamma(X_1,\dots,X_n;Y) \big\| \le 1$$

holds for all Banach spaces $X_1,\dots,X_n$ and Y.

2.1. Relation with the metric case. Now we turn our attention to Lipschitz mappings between metric spaces. Recall from [4] that a Lipschitz function between metric spaces $f : X \to Y$ factors through a subset of a Hilbert space if there exist a Hilbert space H and a subset Z of H (actually, we may take $Z = f(X)$) and two Lipschitz functions $A : X \to Z$, $B : Z \to Y$ such that $f = BA$. In this case $\gamma_2^{\mathrm{Lip}}(f) = \inf \mathrm{Lip}(A)\,\mathrm{Lip}(B)$, where the infimum is taken over all possible factorizations of f as before.

It is clear from (2.2) that if T is an element in $\Gamma(X_1,\dots,X_n;Y)$, then its associated Σ-operator $f_T : \Sigma^{\pi}_{X_1,\dots,X_n} \to Y$ is a Lipschitz function that can be factored through a subset of a Hilbert space in the sense of [4]. Moreover, we have

$$\gamma_2^{\mathrm{Lip}}(f_T) \le \Gamma(T).$$

In other words, every multilinear operator T in $\Gamma(X_1,\dots,X_n;Y)$ gives rise to a Lipschitz function f_T in $\Gamma_2^{\mathrm{Lip}}(\Sigma^{\pi}_{X_1,\dots,X_n};Y)$. That is, the operator

$$\begin{aligned} \Gamma(X_1,\dots,X_n;Y) &\to \Gamma_2^{\mathrm{Lip}}(\Sigma^{\pi}_{X_1,\dots,X_n};Y), \\ T &\mapsto f_T \end{aligned} \tag{2.3}$$

is bounded and has norm at most 1.

We do not know if the metric approach of [4] restricts well to the setting of multilinear operators we are proposing. Specifically, we have the following two questions.

Question 1. Is the map defined in (2.3) an isometry?

Question 2. We do not know if T factors through a Hilbert space whenever f_T does in the metric sense. That is, does f_T in $\Gamma_2^{\mathrm{Lip}}(X;Y)$ imply T in $\Gamma(X_1,\dots,X_n;Y)$?

3. Kwapień-type characterization

In this section, we characterize the multilinear operators that factor through a Hilbert space in terms of their behavior on some special finite sequences (see Theorem 3.3). This fact relies on the local character of the property of factoring through a Hilbert space, which is proved in Theorem 3.2. First, we need some facts and notation.

Sets of the form $f_A(\Sigma_{X_1,\dots,X_n}) = A(X_1 \times \cdots \times X_n)$, where $A : X_1 \times \cdots \times X_n \to Z$ is bounded, are fundamental for the proof of Theorem 3.2. We collect some relevant facts about these sets in the next lemma. We omit its proof since it can be done by standard arguments of the theory of Banach spaces.

Lemma 3.1. *Let $A : X_1 \times \cdots \times X_n \to Z$ be a bounded multilinear operator between Banach spaces. Then we have the following.*

(i) *The set $(f_A(\Sigma_{X_1,\dots,X_n}))^*$ defined by*

$$\{\psi : f_A(\Sigma_{X_1,\dots,X_n}) \to \mathbb{K} \mid \psi A \text{ is multilinear and } \psi \text{ is Lipschitz}\}$$

is a vector space endowed with the algebraic operations defined pointwise; moreover, it becomes a Banach space with the Lipschitz norm induced by Z.

(ii) *Let $B : f_A(\Sigma_{X_1,\dots,X_n}) \to Y$ be a Lipschitz function such that the composition $BA : X_1 \times \cdots \times X_n \to Y$ is multilinear. The function*

$$\begin{aligned} B^* : Y^* &\to \big(f_A(\Sigma_{X_1,\dots,X_n})\big)^*, \\ y^* &\mapsto y^* B \end{aligned} \tag{3.1}$$

is a well-defined bounded linear operator and $\|B^*\| \le \operatorname{Lip}(B)$. The linear operator B^* is called the adjoint *of B.*

Let E_i be a finite-dimensional subspace of X_i, for $1 \le i \le n$, and let L be a finite-codimensional subspace of Y. We define the multilinear map

$$\begin{aligned} I_{E_1,\dots,E_n} : E_1 \times \cdots \times E_n &\to X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n, \\ (x^1, \dots, x^n) &\mapsto x^1 \otimes \cdots \otimes x^n \end{aligned}$$

and denote by $Q_L : Y \to Y/L$ the natural quotient map.

Theorem 3.2. *The multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ admits a factorization through a Hilbert space if and only if*

$$s := \sup \Gamma(Q_L f_T I_{E_1,\dots,E_n}) < \infty,$$

where the supremum is taken over all finite-dimensional subspaces E_i of X_i and finite-codimensional subspaces L of Y. In this situation, $\Gamma(T) = s$.

Proof. Suppose that $T : X_1 \times \cdots \times X_n \to Y$ factors through a Hilbert space. Let E_i and L be as above. The factorization $T = BA$ implies that $Q_L f_T I_{E_1,\dots,E_n} = (Q_L B)(f_A I_{E_1,\dots,E_n})$ and that

$$\Gamma(Q_L f_T I_{E_1,\dots,E_n}) \le \Gamma(T).$$

Therefore, s must be finite.

For the converse, we have to translate a condition on finite-dimensional spaces to a global condition. To this end, we use the technique of ultraproducts. Basic facts about ultraproducts of Banach spaces can be found in [11].

Let us denote by $\mathcal{F}(X)$ the collection of all finite-dimensional subspaces of X and by $\mathcal{CF}(Y)$ the collection of all finite-codimensional subspaces of Y. Define $\mathcal{P} = \mathcal{F}(X_1) \times \cdots \times \mathcal{F}(X_n) \times \mathcal{CF}(Y)$. The relation $\le$ defined by $(E_1, \dots, E_n, L) \le (M_1, \dots, M_n, N)$ if $E_i \subset M_i$ and $N \subset L$ defines a partial order on $\mathcal{P}$. Let $\mathfrak{A}$ be an ultrafilter on $\mathcal{P}$ containing the sets

$$(E_1, \dots, E_n, L)^\# = \big\{(M_1, \dots, M_n, N) \mid (E_1, \dots, E_n, L) \le (M_1, \dots, M_n, N)\big\}.$$

For each $(E_1, \dots, E_n, L) \in \mathcal{P}$, there exists a factorization

$$\begin{tikzcd}
E_1 \times \cdots \times E_n \arrow[rr, "Q_L f_T I_{E_1,\dots,E_n}"] \arrow[dr, "A^{E_1,\dots,E_n L}"'] & & Y/L \\
& A^{E_1,\dots,E_n L}(E_1 \times \cdots \times E_n) \arrow[ur, "B^{E_1,\dots,E_n L}"'] \arrow[d] & \\
& H^{E_1,\dots,E_n L} &
\end{tikzcd}$$

with $\|A^{E_1,\dots,E_n,L}\| \leq 1$ and $\mathrm{Lip}(B^{E_1,\dots,E_n,L}) \leq s$. By the finite-dimensional hypothesis, we may assume that $H^{E_1,\dots,E_n,L} = \ell_2^{n(E_1,\dots,E_n,L)}$, where $n(E_1, \dots, E_n, L)$ is a positive integer.

For each $(E_1, \dots, E_n, L) \in \mathcal{P}$, define

$$\begin{aligned}
A_{E_1,\dots,E_n,L} : X_1 \times \cdots \times X_n &\to \ell_2^{n(E_1,\dots,E_n,L)}, \\
(x^1, \dots, x^n) &\mapsto \begin{cases} A^{E_1,\dots,E_n,L}(x^1, \dots, x^n) & \text{if } x^i \in E_i, 1 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

It is not difficult to see that

$$\begin{aligned}
A : X_1 \times \cdots \times X_n &\to (\ell_2^{n(E_1,\dots,E_n,L)})_{\mathfrak{A}}, \\
(x^1, \dots, x^n) &\mapsto \big(A_{E_1,\dots,E_n,L}(x^1, \dots, x^n)\big)_{\mathfrak{A}}
\end{aligned}$$

is a multilinear mapping. Moreover,

$$\begin{aligned}
\big\|A(x^1, \dots, x^n)\big\|_{\mathfrak{A}} &= \big\|\big(A_{E_1,\dots,E_n,L}(x^1, \dots, x^n)\big)_{\mathfrak{A}}\big\|_{\mathfrak{A}} \\
&= \lim_{\mathfrak{A}} \big\|A_{E_1,\dots,E_n,L}(x^1, \dots, x^n)\big\| \\
&\leq \|x^1\| \cdots \|x^n\|
\end{aligned}$$

implies that A is bounded and that $\|A\| \leq 1$.

We extend the operator $(B^{E_1,\dots,E_n,L})^* : (Y/L)^* \to (f_{A^{E_1,\dots,E_n,L}}(\Sigma_{E_1,\dots,E_n}))^*$ (see Lemma 3.1) as follows:

$$\begin{aligned}
\overline{(B^{E_1,\dots,E_n,L})^*} : Y^* &\to \big(f_{A^{E_1,\dots,E_n,L}}(\Sigma_{E_1,\dots,E_n})\big)^*, \\
y^* &\mapsto \begin{cases} (B^{E_1,\dots,E_n,L})^*(\zeta) & \text{if } y^* = Q_L^*(\zeta) \in Q_L^*((Y/L)^*), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Define

$$\begin{aligned}
B : A(X_1 \times \cdots \times X_n) &\to Y^{**}, \\
A(x^1, \dots, x^n) &\mapsto BA(x^1, \dots, x^n),
\end{aligned}$$

where

$$\begin{aligned}
BA(x^1, \dots, x^n) : Y^* &\to \mathbb{K}, \\
y^* &\mapsto \lim_{\mathfrak{A}} \big\langle \overline{(B^{E_1,\dots,E_n,L})^*}(y^*), A_{E_1,\dots,E_n,L}(x^1, \dots, x^n) \big\rangle.
\end{aligned}$$

The definitions of $\overline{(B^{E_1,\dots,E_n,L})^*}$ and $A_{E_1,\dots,E_n,L}$ imply that

$$\begin{aligned} &\left|\left\langle\overline{(B^{E_1,\dots,E_n,L})^*}(y^*), A_{E_1,\dots,E_n,L}(\mathbf{x})\right\rangle - \left\langle\overline{(B^{E_1,\dots,E_n,L})^*}(y^*), A_{E_1,\dots,E_n,L}(\mathbf{z})\right\rangle\right| \\ &\quad \leq s\|y^*\|\left\|A_{E_1,\dots,E_n,L}(\mathbf{x}) - A_{E_1,\dots,E_n,L}(\mathbf{z})\right\| \end{aligned} \tag{3.2}$$

holds for all y^* in Y^*, $\mathbf{x} = (x^1,\dots,x^n)$, $\mathbf{z} = (z^1,\dots,z^n)$ in $X_1 \times \cdots \times X_n$, and $(E_1,\dots,E_n,L)$ in $\mathcal{P}$. Inequality (3.2) has many implications. First, $z^1 \otimes \cdots \otimes z^n = 0$ implies that $BA(x^1,\dots,x^n)$ is well defined. Second, the equality $(A^{E_1,\dots,E_n,L}(x^1,\dots,x^n))_{\mathfrak{A}} = (A^{E_1,\dots,E_n,L}(z^1,\dots,z^n))_{\mathfrak{A}}$ asserts that B does not depend on representations since

$$\lim_{\mathfrak{A}}\left\|A^{E_1,\dots,E_n,L}(x^1,\dots,x^n) - A^{E_1,\dots,E_n,L}(z^1,\dots,z^n)\right\| = 0.$$

Third, the general case ensures that B is Lipschitz and that $\mathrm{Lip}(B) \leq s$.

To conclude, note that for every $(x^1,\dots,x^n) \in X_1 \times \cdots \times X_n$ and $y^* \in Y^*$ there exists $(E_1,\dots,E_n,L)$ in $\mathcal{P}$ such that $(x^1,\dots,x^n) \in E_1 \times \cdots \times E_n$ and $y^* \in Q_L^*((Y/L)^*)$. Then $(E_1,\dots,E_n,L)^\# \in \mathfrak{A}$ ensures that

$$\lim_{\mathfrak{A}}\left\langle\overline{(B^{E_1,\dots,E_n,L})^*}(y^*), A_{E_1,\dots,E_n,L}(x^1,\dots,x^n)\right\rangle = y^*\big(T(x^1,\dots,x^n)\big).$$

As a consequence, $BA(x^1,\dots,x^n) = K_Y T(x^1,\dots,x^n)$ for every $(x^1,\dots,x^n)$ in $X_1 \times \cdots \times X_n$. This means that

$$\begin{tikzcd} X_1 \times \cdots \times X_n \arrow[rr, "T"] \arrow[dr, "A"'] & & Y \\ & A(X_1 \times \cdots \times X_n) \arrow[ur, "K_Y^{-1}B"'] \arrow[d] & \\ & (\ell_2^{n(E_1,\dots,E_n,L)})_{\mathfrak{A}} & \end{tikzcd}$$

is commutative. Now, if we consider all the spaces $\ell_2^{n(E_1,\dots,E_n,L)}$ as abstract L_2-spaces, then the ultraproduct $(\ell_2^{n(E_1,\dots,E_n,L)})_{\mathfrak{A}}$ is an abstract L_2-space. Moreover, [17, Theorem 1.b.2] implies that this ultraproduct is (order) linearly isometric to $L_2(\mu)$ for some measure space (Ω,μ). This means that T factors through a Hilbert space and that$\Gamma(T) \leq s$. $\square$

Given finite sequences $(p_i)_{i=1}^m$, $(q_i)_{i=1}^m$, $(a_j)_{j=1}^l$, $(b_j)_{j=1}^l$ in $\Sigma_{X_1,\dots,X_n}$, we write $(p_i,q_i) \leq_\pi (a_j,b_j)$ if

$$\sum_{i=1}^m \left|f_\varphi(p_i) - f_\varphi(q_i)\right|^2 \leq \sum_{j=1}^l \left|f_\varphi(a_j) - f_\varphi(b_j)\right|^2 \quad \forall \varphi \in \mathcal{L}(X_1,\dots,X_n).$$

Note that it is enough, by adding zeros if necessary, to take $m = l$.

Theorem 3.3. *The multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ admits a factorization through a Hilbert space if and only if there exists a constant $C > 0$ such* *that*

$$\sum_{i=1}^{m}\left\|T(x_i^1,\ldots,x_i^n)-T(z_i^1,\ldots,z_i^n)\right\|^2 \le C^2\sum_{i=1}^{m}\pi(s_i^1\otimes\cdots\otimes s_i^n-t_i^1\otimes\cdots\otimes t_i^n)^2 \quad (3.3)$$

holds for all finite sequences $(x_i^1,\ldots,x_i^n)_{i=1}^m$, $(z_i^1,\ldots,z_i^n)_{i=1}^m$, $(s_i^1,\ldots,s_i^n)_{i=1}^m$, *and* $(t_i^1,\ldots,t_i^n)_{i=1}^m$ *such that*

$$(x_i^1\otimes\cdots\otimes x_i^n, z_i^1\otimes\cdots\otimes z_i^n) \le_\pi (s_i^1\otimes\cdots\otimes s_i^n, t_i^1\otimes\cdots\otimes t_i^n).$$

In this case, $\Gamma(T)$ *is the best possible constant* C *as above.*

Proof. First, let us suppose that $T : X_1\times\cdots\times X_n \to Y$ admits a factorization through a Hilbert space H, $T = BA$. If $(p_i,q_i)\le_\pi (a_i,b_i)$, then it is clear that $(f_A(p_i), f_A(q_i)) \le (f_A(a_i), f_A(b_i))$ in H. Given an orthonormal basis $(e_\alpha)_{\alpha\in I}$ of H, we have that $\|h\|^2=\sum_\alpha |\langle h,e_\alpha\rangle|^2$ holds for all $h\in H$. Then

$$\begin{aligned}\sum_{i=1}^{m}\sum_{\alpha\in F}\left|\left\langle f_A(p_i)-f_A(q_i),e_\alpha\right\rangle\right|^2 &= \sum_{\alpha\in F}\sum_{i=1}^{m}\left|\left\langle f_A(p_i)-f_A(q_i),e_\alpha\right\rangle\right|^2\\ &\le \sum_{\alpha\in F}\sum_{i=1}^{m}\left|\left\langle f_A(a_i)-f_A(b_i),e_\alpha\right\rangle\right|^2\end{aligned}$$

for all finite subsets F of I. Therefore,

$$\sum_{i=1}^{m}\left\|f_A(p_i)-f_A(q_i)\right\|^2 \le \sum_{i=1}^{m}\left\|f_A(a_i)-f_A(b_i)\right\|^2. \quad (3.4)$$

Finally, the combination of (3.4) and the Lipschitz conditions of B and f_A imply that

$$\sum_{i=1}^{m}\left\|f_T(p_i)-f_T(q_i)\right\|^2 \le \mathrm{Lip}(B)^2\|A\|^2\sum_{i=1}^{m}\beta(a_i-b_i)^2.$$

Consequently, (3.3) must be true and $\inf C\le \mathrm{Lip}(B)\|A\|$. Hence, $\inf C\le\Gamma(T)$.

Conversely, let us prove that whenever T satisfies (3.3), then T admits such a factorization. To this end, we will use Theorem 3.2. Let E_i be a finite-dimensional subspace of X_i. Let us denote by $\pi|$ the restriction of the norm $\pi(\cdot;X_1\otimes\cdots\otimes X_n)$ to $E_1\otimes\cdots\otimes E_n$. Set

$$K := \left\{\zeta\in(E_1\otimes\cdots\otimes E_n,\pi|)^* \;\middle|\; \|\zeta\|=1\right\}.$$

Since the spaces E_i are finite-dimensional, K is compact. Define S as the subset of $C(K)$ given by functions of the form

$$\phi(\zeta)=\sum_{i=1}^{m}\left|\zeta(p_i)-\zeta(q_i)\right|^2-\sum_{i=1}^{m}\left|\zeta(a_i)-\zeta(b_i)\right|^2,$$

where (a_i), (b_i), (p_i), and (q_i) are finite sequences in $\Sigma_{E_1,\ldots,E_n}$ such that

$$C^2\sum_{i=1}^{m}\pi\mid(a_i-b_i)^2<\sum_{i=1}^{m}\left\|f_T(p_i)-f_T(q_i)\right\|^2.$$

Every ϕ in S satisfies $\|\phi\| > 0$ since there exists ζ in K such that $\phi(\zeta) > 0$. Moreover, S is a convex cone disjoint of the negative open cone $C_- := \{\phi \mid \sup\phi < 0\}$. An application of the Hahn–Banach theorem ensures the existence of a measure μ on K which separates C_- and S. It is possible to adjust μ to be a positive measure such that

$$0 \leq \int_K \phi(\zeta)\, d\mu(\zeta) \quad \forall \phi \in S. \tag{3.5}$$

Since E_i is a finite-dimensional space,

$$D = \sup\Big\{ \Big(\int_K \big|\zeta(a) - \zeta(b)\big|^2 \, d\mu(\zeta) \Big)^{\frac{1}{2}} \;\Big|\; \pi \mid (a-b) \leq 1, a, b \in \Sigma_{E_1,\dots,E_n} \Big\} > 0.$$

Thus, we may adjust μ such that $D = C$.

For every $a, b, p, q \in \Sigma_{E_1,\dots,E_n}$ such that $C\pi|(a-b) \leq \|f_T(p) - f_T(q)\|$, (3.5) asserts that

$$\int_K \big|\zeta(a) - \zeta(b)\big|^2 \, d\mu(\zeta) \leq \int_K \big|\zeta(p) - \zeta(q)\big|^2 \, d\mu(\zeta). \tag{3.6}$$

In particular, (3.6) is also true for p and q in $\Sigma_{E_1,\dots,E_n}$ such that $C < \|f_T(p) - f_T(q)\|$ and a, b in $\Sigma_{E_1,\dots,E_n}$ with $\pi|(a-b) < 1$. As a consequence,

$$C \leq \Big(\int_K \big|\zeta(p) - \zeta(q)\big|^2 \, d\mu(\zeta) \Big)^{\frac{1}{2}}$$

for all p, q in $\Sigma_{E_1,\dots,E_n}$ with $C \leq \|f_T(p) - f_T(q)\|$. Take $c = \|f_T(p) - f_T(q)\|$ and $\varepsilon > 0$. The homogeneous property of f_T asserts that

$$C < (C+\varepsilon)\frac{c}{c} = \Big\| f_T\Big(\frac{C+\varepsilon}{c} p\Big) - f_T\Big(\frac{C+\varepsilon}{c} q\Big) \Big\|.$$

Hence,

$$\frac{C}{C+\varepsilon} \big\| f_T(p) - f_T(q) \big\| \leq \Big(\int_K \big|\zeta(p) - \zeta(q)\big|^2 \, d\mu \Big)^{\frac{1}{2}} \quad \forall \varepsilon > 0.$$

This way,

$$\big\| f_T(p) - f_T(q) \big\| \leq \Big(\int_K \big|\zeta(p) - \zeta(q)\big|^2 \, d\mu(\zeta) \Big)^{\frac{1}{2}} \quad \forall p, q \in \Sigma_{E_1,\dots,E_n}. \tag{3.7}$$

On the other hand, it is clear that

$$\Big(\int_K \big|\zeta(a) - \zeta(b)\big|^2 \, d\mu(\zeta) \Big)^{\frac{1}{2}} \leq C\pi \mid (a-b) \quad \forall a, b \in \Sigma_{E_1,\dots,E_n}. \tag{3.8}$$

Finally, we obtain a factorization

$$\begin{tikzcd} E_1 \times \cdots \times E_n \arrow[rr, "f_T I_{E_1,\ldots,E_n}"] \arrow[dr, "A"'] & & Y \\ & f_A(\Sigma_{E_1,\ldots,E_n}) \arrow[ur, "B"'] \arrow[d] & \\ & L_2(\mu) & \end{tikzcd}$$

where

$$A : E_1 \times \cdots \times E_n \to L_2(\mu),$$
$$(x^1,\ldots,x^n) \mapsto A(x^1,\ldots,x^n) : \zeta \mapsto \zeta(x^1 \otimes \cdots \otimes x^n)$$

and

$$B : f_A(\Sigma_{E_1,\ldots,E_n}) \to Y,$$
$$f_A(x^1 \otimes \cdots \otimes x^n) \mapsto T(x^1,\ldots,x^n).$$

The boundedness of A is deduced from (3.8); moreover, $\|A\| \leq C$. Inequality (3.7) asserts that B is a well-defined Lipschitz function and that $\mathrm{Lip}(B) \leq 1$.

Let L be a finite-codimensional subspace of Y, and consider the composition $Q_L f I_{E_1,\ldots,E_n} = (Q_L B)A$. Since $\|Q_L\| \leq 1$, we obtain that $Q_L f I_{E_1,\ldots,E_n}$ admits a factorization through a Hilbert space and $\Gamma(Q_L f_T I_{E_1,\ldots,E_n}) \leq C$. Theorem 3.2 implies that T belongs to $\Gamma(X_1,\ldots,X_n;Y)$ and $\Gamma(T) \leq \inf C$. $\square$

4. Ideal behavior and tensorial representation

The ideal features of Γ are contained in the next proposition. We omit its proof since it follows easily from the definition, using the characterization provided by Theorem 3.3.

Proposition 4.1. *Let $X_1,\ldots,X_n$ and Y be Banach spaces. Then we have the following.*

(i) *We have that Γ is a norm on $\Gamma(X_1,\ldots,X_n;Y)$.*

(ii) *Every rank 1 multilinear operator*

$$\varphi \cdot y : X_1 \times \cdots \times X_n \to Y,$$
$$(x^1,\ldots,x^n) \mapsto \varphi(x^1,\ldots,x^n)y$$

with $\varphi \in \mathcal{L}(X_1,\ldots,X_n)$ and $y \in Y$ is an element of $\Gamma(X_1,\ldots,X_n;Y)$ and $\Gamma(\varphi \cdot y) \leq \|\varphi\|\|y\|$.

(iii) *We have that $\|T\| \leq \Gamma(T)$ for all $T \in \Gamma(X_1,\ldots,X_n;Y)$.*

(iv) *Let m be a positive integer, and let $Z_1,\ldots,Z_m$, W be Banach spaces. Let $R : Z_1 \times \cdots \times Z_m \to X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n$ be a bounded multilinear operator such that $f_R(\Sigma_{Z_1,\ldots,Z_m}) \subset \Sigma_{X_1,\ldots,X_n}$, and let $S : Y \to W$ be a bounded linear operator. Then $S f_T R : Z_1 \times \cdots \times Z_m \to W$ is an element of $\Gamma(Z_1,\ldots,Z_m;W)$ whenever T is in $\Gamma(X_1,\ldots,X_n;Y)$ and $\Gamma(S f_T R) \leq \|R\|\Gamma(T)\|S\|$.*

A consequence of Corollary 4.5 is that every $\Gamma(X_1,\dots,X_n;Y)$ is a Banach space. This result, in addition to the preceding proposition and Theorem 3.2, tells us that the pair $[\Gamma,\Gamma(\cdot)]$ is a maximal multi-ideal in the sense of Floret and Hunfeld [9].

In [9] the authors prove that every maximal ideal is represented by a finitely generated tensor norm, extending, in this way, the representation theorem for maximal ideals (see [5, Section 17]). Consequently, Γ is represented by a finitely generated tensor norm γ with which it is in duality. Now, we give an explicit formulation of γ.

For a better understanding, it is convenient to have in mind that the mapping $x^1\otimes\cdots\otimes x^n\otimes y\mapsto(x^1\otimes\cdots\otimes x^n)\otimes y$ defines a linear isomorphism between $X_1\otimes\cdots\otimes X_n\otimes Y$ and $(X_1\otimes\cdots\otimes X_n)\otimes Y$. For example, under this identification if $p=x^1\otimes\cdots\otimes x^n$ and $q=z^1\otimes\cdots\otimes z^n$ are elements in $\Sigma_{X_1,\dots,X_n}$ and y is in Y, then

$$x^1\otimes\cdots\otimes x^n\otimes y-z^1\otimes\cdots\otimes z^n\otimes y=(p-q)\otimes y.$$

In order to define $\gamma(u)$, the Lipschitz condition of bounded Σ-operators leads us to consider representations of u of the form

$$\sum_{i=1}^{m}(p_i-q_i)\otimes y_i, \tag{4.1}$$

where p_i and q_i are elements in $\Sigma_{X_1,\dots,X_n}$ and y_i in Y. The first tensor norm which considers representations as in (4.1) was given by Angulo [2] in his doctoral dissertation. In that case, Angulo defined the tensor norm d_p which is in duality with the collection of Lipschitz p-summing multilinear operators defined in [3].

Before presenting the norm γ, we fix some notation. Given finite sequences $(a_j)_{j=1}^m$ and $(b_j)_{j=1}^m$ in $\Sigma_{X_1,\dots,X_n}$, we write

$$\big\|(a_j-b_j)\big\|_2^{\pi}:=\Big(\sum_{j=1}^{m}\pi(a_j-b_j)^2\Big)^{\frac{1}{2}}.$$

We also use the standard notation

$$\big\|(y_i)\big\|_2=\Big(\sum_{i=1}^{m}\|y_i\|^2\Big)^{\frac{1}{2}}$$

for a finite sequence $(y_i)_{i=1}^m$ in Y.

Definition 4.2. Let $X_1,\dots,X_n,Y$ be Banach spaces. For u in $X_1\otimes\cdots\otimes X_n\otimes Y$ define

$$\gamma(u)=\inf\big\|(a_i-b_i)\big\|_2^{\pi}\big\|(y_i)\big\|_2,$$

where the infimum is taken over all representations $u=\sum_{i=1}^m(p_i-q_i)\otimes y_i$ and $(p_i,q_i)\leq_\pi(a_i,b_i)$.

The next proposition is straightforward and only requires standard tensor product techniques. We omit the proof.

Proposition 4.3. *Let $X_1,\dots,X_n,Y$ be Banach spaces and let β be a reasonable crossnorm on the tensor product $X_1\otimes\cdots\otimes X_n$. Then we have the following.*

(i) *We have that* γ *is a norm on* $X_1 \otimes \cdots \otimes X_n \otimes Y$.
(ii) *We have* $\gamma((p-q)\otimes y) \leq \pi(p-q)\|y\|$ *for all* $p, q \in \Sigma_{X_1,\dots,X_n}$ *and* $y \in Y$.
(iii) *Let* $\varphi \in \mathcal{L}(X_1,\dots,X_n)$ *and* $y^* \in Y^*$. *The functional*

$$\varphi \otimes y^* : (X_1 \otimes \cdots \otimes X_n \otimes Y, \gamma) \to \mathbb{K},\qquad x^1 \otimes \cdots \otimes x^n \otimes y \mapsto f_\varphi(x^1 \otimes \cdots \otimes x^n) y^*(y)$$

is bounded and $\|\varphi \otimes y^*\| \leq \|\varphi\|\|y^*\|$.
(iv) *Let* $Z_1,\dots,Z_m, W$ *be Banach spaces. If* $R : Z_1 \times \cdots \times Z_m \to X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n$ *is a bounded multilinear operator such that* $f_R(\Sigma_{Z_1,\dots,Z_m}) \subset \Sigma_{X_1,\dots,X_n}$ *and* $S : W \to Y$ *is a bounded linear operator, then*

$$R \otimes S : (Z_1 \otimes \cdots \otimes Z_m \otimes W, \gamma) \to (X_1 \otimes \cdots \otimes X_n \otimes Y, \gamma),\qquad z_1 \otimes \cdots \otimes z_m \otimes w \mapsto f_R(z_1 \otimes \cdots \otimes z_m) \otimes S(w)$$

is bounded and $\|R \otimes S\| \leq \|R\|\|S\|$.
(v) *We have* $\gamma(u; X_1 \otimes \cdots \otimes X_n \otimes Y) = \inf \gamma(u; E_1 \otimes \cdots \otimes E_n \otimes F)$, *where the infimum is taken over all finite-dimensional subspaces* E_i *and* F *of* X_i *and* Y, *respectively, such that* $u \in E_1 \otimes \cdots \otimes E_n \otimes F$.

The preceding proposition tells us that γ is a finitely generated tensor norm in the sense of Floret and Hunfeld [9].

For a better understanding of the tensorial representation of the class Γ, we need the involved algebraic morphism. Every bounded multilinear operator $T : X_1 \times \cdots \times X_n \to Y^*$ gives rise to a bounded functional

$$\varphi_T : X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n \widehat{\otimes}_\pi Y \to \mathbb{K},\qquad x^1 \otimes \cdots \otimes x^n \otimes y \mapsto T(x^1,\dots,x^n)(y).$$

Conversely, every bounded functional φ on $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n \widehat{\otimes}_\pi Y$ defines a bounded multilinear operator

$$T_\varphi : X_1 \times \cdots \times X_n \to Y^*,\qquad x^1 \otimes \cdots \otimes x^n \mapsto T_\varphi(x^1 \otimes \cdots \otimes x^n) : y \mapsto \varphi(x^1 \otimes \cdots \otimes x^n \otimes y).$$

It is not difficult to prove that these assignments are linear isometries and inverse of each other. In other words, we have that

$$\Phi : (X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n \widehat{\otimes}_\pi Y)^* \to \mathcal{L}(X_1,\dots,X_n; Y^*),\qquad \varphi \mapsto T_\varphi \tag{4.2}$$

is an isometric linear isomorphism. The next theorem establishes that Φ in (4.2) also is an isometric linear isomorphism if we replace π by the norm γ and $\mathcal{L}(X_1,\dots,X_n;Y^*)$ by the normed space $\Gamma(X_1,\dots,X_n;Y^*)$.

Theorem 4.4. *Let* $X_1,\dots,X_n, Y$ *be Banach spaces. Then*

$$\Phi : (X_1 \otimes \cdots \otimes X_n \otimes Y, \gamma)^* \to \Gamma(X_1,\dots,X_n;Y^*)$$

is an isometric linear isomorphism.

Proof. We will use the linear isometry

$$(Y \oplus_2 \cdots \oplus_2 Y)^* = Y^* \oplus_2 \cdots \oplus_2 Y^*. \tag{4.3}$$

Suppose that T factors through a Hilbert space. The combination of (4.3) and Theorem 3.3, implies that for all $(y_i)_i$ and $(p_i, q_i) \leq_\pi (a_i, b_i)$,

$$\Big|\sum_{i=1}^{m} \big\langle f_T(p_i) - f_T(q_i), y_i \big\rangle\Big| \leq \Gamma(T)\big\|(a_i - b_i)\big\|_2^\pi \big\|(y_i)\big\|_2.$$

So, if $u = \sum_{i=1}^m (p_i - q_i) \otimes y_i$ is an element in $X_1 \otimes \cdots \otimes X_n \otimes Y$ and $(p_i, q_i) \leq_\pi (a_i, b_i)$, then

$$\big|\varphi_T(u)\big| \leq \Gamma(T)\big\|(a_j - b_j)\big\|_2^\pi \big\|(y_i)\big\|_2.$$

In other words, φ_T is bounded and $\|\varphi_T\| \leq \Gamma(T)$.

Conversely, suppose that $\varphi \in (X_1 \otimes \cdots \otimes X_n \otimes Y, \gamma)^*$. Let $(p_i, q_i) \leq_\pi (a_i, b_i)$ and $(y_i)_i$. Define $u = \sum_{i=1}^m (p_i - q_i) \otimes y_i$. Then

$$\Big|\sum_{i=1}^{m} \big\langle f_{T_\varphi}(p_i) - f_{T_\varphi}(q_i), y_i \big\rangle\Big| = \big|\varphi(u)\big| \leq \|\varphi\| \big\|(a_i - b_i)\big\|_2^\pi \big\|(y_i)\big\|_2.$$

After taking suprema over $\sum_{i=1}^m \|y_i\|^2 \leq 1$, (4.3) implies that

$$\Big(\sum_{i=1}^{m} \big\|f_{T_\varphi}(p_i) - f_{T_\varphi}(q_i)\big\|^2\Big)^{\frac{1}{2}} \leq \|\varphi\| \big\|(a_i - b_i)\big\|_2^\pi.$$

According to Theorem 3.3, $T_\varphi : X_1 \times \cdots \times X_n \to Y^*$ factors through a Hilbert space and $\Gamma(T_\varphi) \leq \|\varphi\|$. □

Corollary 4.5. *Let $X_1, \ldots, X_n$ and Y be Banach spaces. Then the spaces $(X_1 \otimes \cdots \otimes X_n \otimes Y^*, \gamma)^* \cap \mathcal{L}(X_1, \ldots, X_n, Y)$ and $\Gamma(X_1, \ldots, X_n; Y)$ are isometrically isomorphic.*

Proof. Let T be an operator in $\Gamma(X_1, \ldots, X_n; Y)$. Define

$$\begin{aligned} \zeta_T : X_1 \otimes \cdots \otimes X_n \otimes Y^* &\to \mathbb{K}, \\ x^1 \otimes \cdots \otimes x^n \otimes y^* &\mapsto y^*\big(T(x^1, \ldots, x^n)\big). \end{aligned}$$

The multilinear feature of T and the linearity of every y^* assert that ζ_T is well defined and linear. Let $u \in X_1 \otimes \cdots \otimes X_n \otimes Y^*$ and $\eta > 0$. The finitely generated property of γ (see Proposition 4.3) asserts that there exist $E_i \in \mathcal{F}(X_i)$ and $F \in \mathcal{F}(Y^*)$ such that $u \in E_1 \otimes \cdots \otimes E_n \otimes F$ and

$$\gamma(u; E_1, \ldots, E_n \otimes F) \leq (1 + \eta)\gamma(u; X_1 \otimes \cdots \otimes X_n \otimes Y^*).$$

The subspace F defines $L \in \mathcal{CF}(Y)$ such that $(Y/L)^* = F$ is isometrically isomorphic to F via Q_L^*. Then Theorem 4.4 implies that

$$\big(E_1 \otimes \cdots \otimes E_n \otimes F \otimes (Y/L)^*, \gamma\big)^* = \Gamma(E_1, \ldots, E_n; Y/L) \tag{4.4}$$

in which the equal sign stands for an isometric isomorphism. Note that in (4.4) we are identifying Y/L with its double topological dual. Algebraic manipulations lead us to the fact that, under (4.4), $Q_L f_T I_{E_1,\dots,E_n}$ is the multilinear operator that corresponds to the composition $\varphi_T \circ (I_{E_1,\dots,E_n} \otimes Q_L^*)$. Furthermore,

$$\begin{aligned}\big|\zeta_T(u)\big| &= \big|\varphi_T \circ (I_{E_1,\dots,E_n} \otimes Q_L^*)(u)\big| \\ &\le \big\|\varphi_T \circ (I_{E_1,\dots,E_n} \otimes Q_L^*) : \big(E_1,\dots,E_n \otimes (Y/L)^*\big),\gamma) \to \mathbb{K}\big\|\gamma(u) \\ &\le \Gamma(Q_L f_T I_{E_1,\dots,E_n})(1+\eta)\gamma(u; X_1 \otimes \cdots \otimes X_n \otimes Y^*) \\ &\le \Gamma(T)(1+\eta)\gamma(u; X_1 \otimes \cdots \otimes X_n \otimes Y^*).\end{aligned}$$

The election of η allows us to conclude that ζ_T is bounded and that $\|\zeta_T\| \le \Gamma(T)$.

For the converse, let $\varphi \in (X_1 \otimes \cdots \otimes X_n \otimes Y^*, \gamma)^* \cap \mathcal{L}(X_1,\dots,X_n,Y)$. We may assume that T_φ has range contained in Y. Reasoning as before (see (4.4)), we have that

$$\Gamma(Q_L f_{T_\varphi} I_{E_1,\dots,E_n}) = \big\|\varphi \circ (I_{E_1,\dots,E_n} \otimes Q_L^*)\big\| \le \|\varphi\|$$

holds for all $E_i \in \mathcal{F}(X_i)$ and $L \in \mathcal{CF}(Y)$. Hence, Theorem 3.2 asserts that $T \in \Gamma(X_1,\dots,X_n;Y)$ and that $\Gamma(T) \le \|\varphi\|$.

Finally, it is easy to check that the assignments $T \mapsto \zeta_T$ and $\varphi \mapsto T_\varphi$ are linear and inverse of each other. □

4.1. Preservation of the property of factoring through a Hilbert space.

Decreasing the degree by evaluations. For any bounded multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ and any x^n in X_n fixed, define

$$\begin{aligned}T^{x^n} : X_1 \times \cdots \times X_{n-1} &\to Y, \\ (x^1,\dots,x^{n-1}) &\mapsto T(x^1,\dots,x^n).\end{aligned}$$

Plainly, T^{x^n} is a bounded multilinear operator. Analogously, we can define a bounded multilinear operator $T^{x^{n-k+1},\dots,x^n} : X_1 \times \cdots \times X_{n-k} \to Y$ for $1 \le k < n$ once we fix x^j in X_j for $n-k+1 \le j \le n$.

Proposition 4.6. *Let T in $\Gamma(X_1,\dots,X_n;Y)$ and x^j in X_j for $n-k+1 \le j \le n$. Then $T^{x^{n-k+1},\dots,x^n}$ is an element of $\Gamma(X_1,\dots X_{n-k};Y)$ for all $1 \le k < n$ and $\Gamma(T^{x^{n-k+1},\dots,x^n}) \le \Gamma(T)\|x^{n-k+1}\|\dots\|x^n\|$.*

Proof. It suffices to apply the ideal property from Proposition 4.1(iv) to the map $R : X_1 \times \cdots \times X_{n-k} \to X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n$ defined by $R(x^1,\dots,x^{n-k}) = (x^1,\dots,x^n)$. □

The case $k = n-1$ produces a bounded linear operator $T_{n-(n-1)} : X_1 \to Y$ that factors through a Hilbert space in the linear sense. In this respect, we can say more. Let $p = (x^1,\dots,x^n)$ in $X_1 \times \cdots \times X_n$ be fixed, and let T in $\Gamma(X_1,\dots,X_n;Y)$. Denote by $T_i : X_i \to Y$ the linear map resulting by fixing all coordinates except the ith (see Proposition 4.6). Arguments analogous to those presented in the proof of Proposition 4.6 allow us to conclude that T_i is an element of $\Gamma(X_i;Y)$ and $\Gamma(T_i) \le \Gamma(T)\prod_{j\ne i}\|x^j\|$ for all $1 \le i \le n$. Hence

$$\Gamma(T_1)\cdots\Gamma(T_n) \le \Gamma(T)^n \pi(p)^{n-1}.$$

Increasing the degree by products. In the following proposition, we show how to construct multilinear operators that factor through a Hilbert space for given n linear operators with the same property.

Proposition 4.7. *Let n be a positive integer, and let $T_i : X_i \to Y_i$ in $\Gamma(X_i; Y_i)$ for $1 \leq i \leq n$. Then*

$$\otimes \circ (T_1, \ldots, T_n) : X_1 \times \cdots \times X_n \to Y_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi Y_n,$$
$$(x^1, \ldots, x^n) \mapsto T_1(x^1) \otimes \cdots \otimes T_n(x^n)$$

belongs to $\Gamma(X_1, \ldots, X_n; Y_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi Y_n)$ and

$$\Gamma\big(\otimes \circ (T_1, \ldots, T_n)\big) \leq 2^{n-1} \Gamma(T_1) \cdots \Gamma(T_n).$$

Proof. Let $T_i = B_i A_i$ be a factorization through the Hilbert space H_i for $1 \leq i \leq n$. Since A_i and B_i are bounded for all i, we have that

$$A_1 \otimes \cdots \otimes A_n : X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n \to H_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi H_n$$

and that

$$B_1 \otimes \cdots \otimes B_n : H_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi H_n \to Y_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi Y_n$$

are bounded. Applying the ideal property of Proposition 4.1(iv) to the operator $R : X_1 \times \cdots \times X_n \to H_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi H_n$, defined by $R(x^1, \ldots, x^n) = A_1(x^1) \otimes \cdots \otimes A_n(x^n)$, $T = \otimes : H_1 \times \cdots \times H_n \to H_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi H_n$, and $S = B_1 \otimes \cdots \otimes B_n$, we have that $(B_1 \otimes \cdots \otimes B_n) f_\otimes R : X_1 \times \cdots \times X_n \to Y_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi Y_n$ is in $\Gamma(X_1, \ldots, X_n; Y_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi Y_n)$ and $\Gamma((B_1 \otimes \ldots \otimes B_n) f_\otimes R) \leq 2^{n-1} \prod_{i=1}^n \|A_i\| \|B_i\|$. Hence $\Gamma((B_1 \otimes \cdots \otimes B_n) f_\otimes R) \leq 2^{n-1} \Gamma(T_1) \ldots \Gamma(T_n)$. We conclude the proof by noting that $\otimes \circ (T_1, \ldots, T_n) = (B_1 \otimes \cdots \otimes B_n) f_\otimes R$. $\square$

5. Polynomials factoring through a Hilbert space

Homogeneous polynomials that factorize through a Hilbert space can also be characterized in terms of their behavior in some special finite sequences of points. In this case we only state the main results. Their proofs are analogous to those of Theorems 3.2 and 3.3.

Recall that a mapping $P : X \to Y$ between Banach spaces is a *homogeneous polynomial of degree* n if there exists a multilinear mapping $T_P : X \times \cdots \times X \to Y$ such that $P(x) = T_P(x, \overset{n}{\ldots}, x)$.

Definition 5.1. A n-homogeneous polynomial $P : X \to Y$ factors through a Hilbert space if there exist a Hilbert space H, a bounded n-homogeneous polynomial $q : X \to H$, and a Lipschitz function $B : q(X) \to Y$such that $p = Bq$. We define $\Gamma(q) = \inf \|q\| \operatorname{Lip}(B)$.

It is clear that every T in $\Gamma(X \times \cdots \times X \to Y)$ defines an n-homogeneous polynomial $p : X \to Y$ that factors through a Hilbert space, and $\Gamma(p) \leq \Gamma(T)$. Also, a composition of the form SpR factors through a Hilbert space if p does and R and S are bounded linear operators; moreover, $\Gamma(RpS) \leq \|R\| \Gamma(p) \|S\|$.

Theorem 5.2. *The n-homogeneous polynomial $p : X \to Y$ admits a factorization through a Hilbert space if and only if*

$$s := \sup\{\Gamma(Q_L p I_E) \mid E \in \mathcal{F}(X), L \in \mathcal{CF}(Y)\} < \infty.$$

In this situation, $\Gamma(p) = s$.

If we denote by $\pi_{n,s}$ the symmetric projective tensor norm on the symmetric tensor product $\otimes^{n,s} X$ and $\otimes^n x := x \otimes \cdots \otimes x$, then we have the following.

Theorem 5.3. *The n-homogeneous polynomial $p : X \to Y$ admits a factorization through a Hilbert space if and only if there exists a constant $C > 0$ such that*

$$\sum_{i=1}^{m} \big\|p(x_i) - p(z_i)\big\|^2 \le C^2 \sum_{i=1}^{m} \pi_{n,s}(\otimes^n s_i - \otimes^n t_i)^2$$

for all finite sequences (x_i), (y_i), (s_i), and (t_i) in X such that

$$\sum_{i=1}^{m} \big|\varphi(x_i) - \varphi(z_i)\big|^2 \le \sum_{i=1}^{m} \big|\varphi(s_i) - \varphi(t_i)\big|^2$$

for all n-homogeneous polynomials $\varphi : X \to \mathbb{K}$. In this case, $\Gamma(p)$ is the best possible constant C as above.

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