BANACH JOURNAL of MATHEMATICAL ANALYSIS

Banach J. Math. Anal. 13 (2019), no. 1, 47–63
https://doi.org/10.1215/17358787-2018-0015
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

PARAMETRIC MARCINKIEWICZ INTEGRALS WITH ROUGH KERNELS ACTING ON WEAK MUSIELAK–ORLICZ HARDY SPACES

BO LI

Communicated by M. A. Ragusa

ABSTRACT. Let $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ satisfy that $\varphi(x, \cdot)$, for any given $x \in \mathbb{R}^n$, is an Orlicz function and that $\varphi(\cdot, t)$ is a Muckenhoupt A_∞ weight uniformly in $t \in (0, \infty)$. The weak Musielak–Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$ is defined to be the set of all tempered distributions such that their grand maximal functions belong to the weak Musielak–Orlicz space $WL^\varphi(\mathbb{R}^n)$. For parameter $\rho \in (0, \infty)$ and measurable function f on $\mathbb{R}^n$, the parametric Marcinkiewicz integral μ_Ω^ρ related to the Littlewood–Paley g-function is defined by setting, for all $x \in \mathbb{R}^n$,

$$\mu_\Omega^\rho(f)(x) := \Big(\int_0^\infty \Big| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)dy\Big|^2 \frac{dt}{t^{2\rho+1}}\Big)^{1/2},$$

where Ω is homogeneous of degree zero satisfying the cancellation condition.

In this article, we discuss the boundedness of the parametric Marcinkiewicz integral μ_Ω^ρ with rough kernel from weak Musielak–Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$ to weak Musielak–Orlicz space $WL^\varphi(\mathbb{R}^n)$. These results are new even for the classical weighted weak Hardy space of Quek and Yang, and probably new for the classical weak Hardy space of Fefferman and Soria.

1. Introduction

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ $(n \ge 2)$ equipped with normalized Lebesgue measure $d\sigma$. A function $\Omega(x)$ defined on $\mathbb{R}^n$ is said to be in $L^q(S^{n-1})$

Received Mar. 18, 2018; Accepted May 7, 2018.
First published online Oct. 30, 2018.
2010 *Mathematics Subject Classification.* Primary 42B25; Secondary 42B30, 46E30.
Keywords. Marcinkiewicz integral, weak Hardy space, Muckenhoupt weight, Musielak–Orlicz function.

with $q \geq 1$ if $\Omega(x)$ satisfies the following conditions:

$$\Omega(\lambda x) = \Omega(x) \quad \text{for any } x \in \mathbb{R}^n \text{ and } \lambda \in (0, \infty), \tag{1.1}$$

$$\int_{S^{n-1}} \Omega(x)\, d\sigma(x') = 0, \tag{1.2}$$

$$\int_{S^{n-1}} \left|\Omega(x)\right|^q d\sigma(x') < \infty, \tag{1.3}$$

where $x' := x/|x|$ for any $x \neq \mathbf{0}$. For parameter $\rho \in (0, \infty)$ and measurable function f on $\mathbb{R}^n$, the *parametric Marcinkiewicz integral* μ_Ω^ρ is defined by setting, for all $x \in \mathbb{R}^n$,

$$\mu_\Omega^\rho(f)(x) := \left(\int_0^\infty \left|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)dy\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2}.$$

The Marcinkiewicz integral μ_Ω^1 was introduced by Stein [23] in 1958. He showed that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ with $\alpha \in (0,1]$, then μ_Ω^1 is bounded on $L^p(\mathbb{R}^n)$ with $p \in (1,2]$ and bounded from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$. In 1960, Hörmander [7] proved that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ with $\alpha \in (0,1]$, then μ_Ω^ρ is bounded on $L^p(\mathbb{R}^n)$ provided that $p \in (1,\infty)$ and $\rho \in (0,\infty)$. Note that all the results mentioned above hold true depending on some smoothness condition of Ω. However, in 2009, Shi and Jiang [22] obtained the following celebrated result that μ_Ω^ρ is bounded on $L^p_\omega(\mathbb{R}^n)$ without any smoothness condition of Ω, where $\omega \in A_p$ and A_p denotes the Muckenhoupt weight class.

Theorem A ([22, Theorem 1.1])**.** *Let $\rho \in (0,\infty)$, let $p, q \in (1,\infty)$, let $q' := q/(q-1)$, and let $\Omega \in L^q(S^{n-1})$. If $\omega^{q'} \in A_p$, then μ_Ω^ρ is bounded on $L^p_\omega(\mathbb{R}^n)$.*

On the other hand, in the past four decades, there has been tremendous interest in developing the theory of Hardy spaces. Hardy space first appeared in complex analysis in the study of analytic functions on the unit disk, and its theory was one-dimensional. The higher dimensional Euclidean theory of Hardy spaces was developed by Fefferman and Stein [4] who proved a variety of characterizations for them. As everyone knows, many important operators are better behaved on Hardy space $H^p(\mathbb{R}^n)$ than on Lebesgue space $L^p(\mathbb{R}^n)$ in the range $p \in (0,1]$. For example, when $p \in (0,1]$, the Riesz transforms are bounded on Hardy space $H^p(\mathbb{R}^n)$, but not on the corresponding Lebesgue space $L^p(\mathbb{R}^n)$. Therefore, one can consider $H^p(\mathbb{R}^n)$ to be a very natural replacement for $L^p(\mathbb{R}^n)$ when $p \in (0,1]$. Moreover, when studying the endpoint estimate for various important operators, the weak Hardy space $WH^p(\mathbb{R}^n)$ naturally appears and proves to be a good substitute of Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0,1]$. For instance, if $\delta \in (0,1]$, T is a δ-Calderón–Zygmund operator and $T^*(1) = 0$, where T^* denotes the adjoint operator, of T, it is known that T is bounded on $H^p(\mathbb{R}^n)$ for any $p \in (n/(n+\delta), 1]$ (see [1]), but T may be not bounded on $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$; however, Liu [18] proved that T is bounded from $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$ to $WH^{\frac{n}{n+\delta}}(\mathbb{R}^n)$.

Recently, Ky [12] introduced a new Musielak–Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$, which unifies the classical Hardy space (see [4]), the weighted Hardy space (see [24]), the Orlicz Hardy space (see [8]–[11]), and the weighted Orlicz Hardy space.

Its spatial and time variables may not be separable. Later, Liang, Yang, and Jiang [16] further introduced a weak Musielak–Orlicz Hardy space $WH^{\varphi}(\mathbb{R}^n)$, which covers the weak Hardy space (see [5]), the weighted weak Hardy space (see [21]), the weak Orlicz Hardy space, and the weighted weak Orlicz Hardy space as special cases. Various equivalent characterizations of $WH^{\varphi}(\mathbb{R}^n)$ by means of maximal functions, atoms, molecules, and Littlewood–Paley functions, and the boundedness of Calderón–Zygmund operators in the critical case were obtained in [16]. Apart from interesting theoretical considerations, the motivation behind the study of Musielak–Orlicz-type space comes from applications to elasticity, fluid dynamics, image processing, nonlinear PDEs, and the calculus of variation (see, e.g., [2]; more Musielak–Orlicz-type spaces are referred to in [3], [6], [15], [19], [20]).

Motivated by the above facts, a natural and interesting question arises, namely, whether the parametric Marcinkiewicz integral μ_{Ω}^{ρ} is bounded from weak Musielak–Orlicz Hardy space $WH^{\varphi}(\mathbb{R}^n)$ to weak Musielak–Orlicz space $WL^{\varphi}(\mathbb{R}^n)$ under a weaker smoothness condition assumed on Ω. In this article, we give an affirmative answer to this problem. What is worth mentioning here is that our results are new even for classical weighted weak Hardy space and probably new for classical weak Hardy space.

This article is organized as follows. In Section 2, we recall some notions concerning Muckenhoupt weight, growth function, and weak Musielak–Orlicz Hardy space. Then we present the boundedness of μ_{Ω}^{ρ} from $WH^{\varphi}(\mathbb{R}^n)$ to $WL^{\varphi}(\mathbb{R}^n)$ (see Theorems 2.7, 2.8, and 2.10 and Corollary 2.9 below). In Section 3, with the help of some auxiliary lemmas and the atomic decomposition theory of $WH^{\varphi}(\mathbb{R}^n)$, we present the proofs of our main results.

Finally, we adopt the following notational conventions. Let $\mathbb{Z}_+ := \{1, 2, \ldots\}$ and $\mathbb{N} := \{0\} \cup \mathbb{Z}_+$. For any $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, let $|\beta| := \beta_1 + \cdots + \beta_n$ and $\partial^{\beta} := (\frac{\partial}{\partial x_1})^{\beta_1} \cdots (\frac{\partial}{\partial x_n})^{\beta_n}$. Throughout this article, the letter C will denote a *positive constant* that may vary from line to line but will remain independent of the main variables. The *symbol* $P \lesssim Q$ stands for the inequality $P \le CQ$. If $P \lesssim Q \lesssim P$, then we will write $P \sim Q$. For any sets $E, F \subset \mathbb{R}^n$, we use $E^{\complement}$ to denote the set $\mathbb{R}^n \setminus E$, $|E|$ its *n-dimensional Lebesgue measure*, χ_E its *characteristic function*, and $E + F$ the *algebraic sum* $\{x + y : x \in E, y \in F\}$. For any $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the unique integer such that $s - 1 < \lfloor s \rfloor \le s$. If there are no special instructions, any space $\mathcal{X}(\mathbb{R}^n)$ is denoted simply by $\mathcal{X}$. For instance, $L^2(\mathbb{R}^n)$ is simply denoted by L^2. For any set $E \subset \mathbb{R}^n$, $t \in [0, \infty)$, and measurable function $\varphi(\cdot, t)$, let $\varphi(E, t) := \int_E \varphi(x, t)\, dx$ and let $\{|f| > t\} := \{x \in \mathbb{R}^n : |f(x)| > t\}$. For any $x \in \mathbb{R}^n$, $r \in (0, \infty)$, and $\alpha \in (0, \infty)$, we use $B(x, r)$ to denote the ball $\{y \in \mathbb{R}^n : |y - x| < r\}$ and $\alpha B(x, r)$ to denote $B(x, \alpha r)$, as usual.

2. Notions and main results

In this section, we first recall the definition of the weak Musielak–Orlicz Hardy space WH^{φ}, and then we present the boundedness of the parametric Marcinkiewicz integral μ_{Ω}^{ρ} from weak Musielak–Orlicz Hardy space WH^{φ} to weak Musielak–Orlicz space WL^{φ}.

Recall that a nonnegative function φ on $\mathbb{R}^n \times [0,\infty)$ is called a *Musielak–Orlicz function* if, for any $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ is an Orlicz function on $[0,\infty)$ and, for any $t \in [0,\infty)$, $\varphi(\cdot,t)$ is measurable on $\mathbb{R}^n$. Here a function $\phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is nondecreasing, $\phi(0) = 0$, $\phi(t) > 0$ for any $t \in (0,\infty)$, and $\lim_{t\to\infty} \phi(t) = \infty$.

Given a Musielak–Orlicz function φ on $\mathbb{R}^n \times [0,\infty)$, φ is said to be of *uniformly lower-* (resp., *upper-*) *type* p with $p \in \mathbb{R}$ if there exists a positive constant $C := C_\varphi$ such that, for any $x \in \mathbb{R}^n$, $t \in [0,\infty)$, and $s \in (0,1]$ (resp., $s \in [1,\infty)$),

$$\varphi(x,st) \le Cs^p\varphi(x,t).$$

The *critical uniformly lower-type index* of φ is defined by

$$i(\varphi) := \sup\{p \in \mathbb{R} : \varphi \text{ is of uniformly lower-type } p\}. \tag{2.1}$$

Observe that $i(\varphi)$ may not be attainable, namely, φ may not be of uniformly lower-type $i(\varphi)$ (see [14, p. 415] for more details).

Definition 2.1.

(i) Let $q \in [1,\infty)$. A locally integrable function $\varphi(\cdot,t) : \mathbb{R}^n \to [0,\infty)$ is said to satisfy the *uniformly Muckenhoupt condition* $\mathbb{A}_q$, denoted by $\varphi \in \mathbb{A}_q$, if there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$ and $t \in (0,\infty)$, when $q = 1$,

$$\frac{1}{|B|}\int_B \varphi(x,t)\,dx\Big\{\operatorname*{ess\,sup}_{x\in B}\big[\varphi(x,t)\big]^{-1}\Big\} \le C$$

and, when $q \in (1,\infty)$,

$$\frac{1}{|B|}\int_B \varphi(x,t)\,dx\Big\{\frac{1}{|B|}\int_B \big[\varphi(x,t)\big]^{-\frac{1}{q-1}}\,dx\Big\}^{q-1} \le C.$$

(ii) Let $q \in (1,\infty]$. A locally integrable function $\varphi(\cdot,t) : \mathbb{R}^n \to [0,\infty)$ is said to satisfy the *uniformly reverse Hölder condition* $\mathbb{RH}_q$, denoted by $\varphi \in \mathbb{RH}_q$, if there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$ and $t \in (0,\infty)$, when $q \in (1,\infty)$,

$$\Big\{\frac{1}{|B|}\int_B \varphi(x,t)\,dx\Big\}^{-1}\Big\{\frac{1}{|B|}\int_B \big[\varphi(x,t)\big]^q\,dx\Big\}^{1/q} \le C$$

and, when $q = \infty$,

$$\Big\{\frac{1}{|B|}\int_B \varphi(x,t)\,dx\Big\}^{-1}\operatorname*{ess\,sup}_{x\in B}\varphi(x,t) \le C.$$

Define $\mathbb{A}_\infty := \bigcup_{q\in[1,\infty)} \mathbb{A}_q$. It is well known that if $\varphi \in \mathbb{A}_q$ with $q \in (1,\infty]$, then $\varphi^\varepsilon \in \mathbb{A}_{\varepsilon q+1-\varepsilon} \subset \mathbb{A}_q$ for any $\varepsilon \in (0,1]$ and $\varphi^\eta \in \mathbb{A}_q$ for some $\eta \in (1,\infty)$. Also, if $\varphi \in \mathbb{A}_q$ with $q \in (1,\infty)$, then $\varphi \in \mathbb{A}_r$ for any $r \in (q,\infty)$ and $\varphi \in \mathbb{A}_d$ for some $d \in (1,q)$. Thus, the *critical weight index* of $\varphi \in \mathbb{A}_\infty$ is defined as follows:

$$q(\varphi) := \inf\big\{q \in [1,\infty) : \varphi \in \mathbb{A}_q\big\}. \tag{2.2}$$

For the uniformly Muckenhoupt (resp., reverse Hölder) condition, we have the following property as the classical case.

Lemma 2.2 ([12, Lemma 4.5])**.** *Let $\varphi \in \mathbb{A}_q$ with $q \in [1, \infty)$. Then there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$, $\lambda \in (1, \infty)$, and $t \in (0, \infty)$,*

$$\varphi(\lambda B, t) \leq C\lambda^{nq}\varphi(B, t).$$

Lemma 2.3 ([13, Lemma 3.5])**.** *Let $r \in (1, \infty)$. Then $\varphi^r \in \mathbb{A}_\infty$ if and only if $\varphi \in \mathbb{RH}_r$.*

Definition 2.4 ([12, Definition 2.1]). A function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is called a *growth function* if the following conditions are satisfied:

(i) φ is a Musielak–Orlicz function;
(ii) $\varphi \in \mathbb{A}_\infty$;
(iii) φ is of uniformly lower-type p for some $p \in (0, 1]$ and of uniformly upper-type 1.

Throughout this paper, we always assume that φ is a growth function.

Recall that the *weak Musielak–Orlicz space* WL^φ is defined to be the space of all measurable functions f such that, for some $\lambda \in (0, \infty)$,

$$\sup_{t\in(0,\infty)} \varphi\Big(\big\{|f| > t\big\}, \frac{t}{\lambda}\Big) < \infty$$

equipped with the quasinorm

$$\|f\|_{WL^\varphi} := \inf\Big\{\lambda \in (0, \infty) : \sup_{t\in(0,\infty)} \varphi\Big(\big\{|f| > t\big\}, \frac{t}{\lambda}\Big) \leq 1\Big\}.$$

In what follows, we denote by $\mathcal{S}$ the *space of all Schwartz functions* and by $\mathcal{S}'$ its *dual space* (namely, the *space of all tempered distributions*). For any $m \in \mathbb{N}$, let $\mathcal{S}_m$ be the *space* of all $\psi \in \mathcal{S}$ satisfying $\|\psi\|_{\mathcal{S}_m} \leq 1$, where

$$\|\psi\|_{\mathcal{S}_m} := \sup_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\leq m+1}} \sup_{x\in\mathbb{R}^n} \big(1 + |x|\big)^{(m+2)(n+1)}\big|\partial^\alpha\psi(x)\big|.$$

Then, for any $m \in \mathbb{N}$ and $f \in \mathcal{S}'$, the *nontangential grand maximal function* f_m^* of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$f_m^*(x) := \sup_{\psi\in\mathcal{S}_m} \sup_{\substack{|y-x|<t \\ t\in(0,\infty)}} \big|f * \psi_t(y)\big|, \tag{2.3}$$

where, for any $t \in (0, \infty)$, $\psi_t(\cdot) := t^{-n}\psi(\frac{\cdot}{t})$. When

$$m = m(\varphi) := \Big\lfloor n\Big(\frac{q(\varphi)}{i(\varphi)} - 1\Big)\Big\rfloor, \tag{2.4}$$

we denote f_m^* simply by f^*, where $q(\varphi)$ and $i(\varphi)$ are as in (2.2) and (2.1), respectively.

Definition 2.5 ([16, Definition 2.3]). Let φ be a growth function as in Definition 2.4. The *weak Musielak–Orlicz Hardy space* WH^φ is defined as the space of all $f \in \mathcal{S}'$ such that $f^* \in WL^\varphi$ endowed with the quasinorm

$$\|f\|_{WH^\varphi} := \|f^*\|_{WL^\varphi}.$$

Remark 2.6. Let ω be a classic Muckenhoupt weight, and let ϕ be an Orlicz function.

(i) If $\varphi(x,t) := \omega(x)\phi(t)$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$, then WH^φ goes back to weighted weak Orlicz Hardy space WH^ϕ_ω, and particularly, when $\omega \equiv 1$, the corresponding unweighted space is also obtained.

(ii) If $\varphi(x,t) := \omega(x)t^p$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$ with $p \in (0,1]$, then WH^φ goes back to weighted weak Hardy space WH^p_ω, and particularly, when $\omega \equiv 1$, the corresponding unweighted space is also obtained.

Before stating our main results, we recall some notions about Ω. For any $q \in [1,\infty)$ and $\alpha \in (0,1]$, a function $\Omega \in L^q(S^{n-1})$ is said to satisfy the *$L^{q,\alpha}$-Dini condition* if

$$\int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\alpha}}\, d\delta < \infty,$$

where

$$\omega_q(\delta) := \sup_{\|\gamma\|<\delta} \Big(\int_{S^{n-1}} \big|\Omega(\gamma x') - \Omega(x')\big|^q \, d\sigma(x') \Big)^{1/q}$$

and γ denotes a rotation on S^{n-1} with $\|\gamma\| := \sup_{y' \in S^{n-1}} |\gamma y' - y'|$. For any $\alpha, \beta \in (0,1]$ with $\beta < \alpha$, it is trivial to see that if Ω satisfies the $L^{q,\alpha}$-Dini condition, then it also satisfies the $L^{q,\beta}$-Dini condition. We thus denote by $\mathrm{Din}^q_\alpha(S^{n-1})$ the class of all functions which satisfy the $L^{q,\beta}$-Dini conditions for all $\beta < \alpha$. For any $\alpha \in (0,1]$, we define

$$\mathrm{Din}^\infty_\alpha(S^{n-1}) := \bigcap_{q \geq 1} \mathrm{Din}^q_\alpha(S^{n-1}).$$

A routine computation gives rise to

$$\mathrm{Din}^r_\alpha(S^{n-1}) \subset \mathrm{Din}^q_\alpha(S^{n-1}) \quad \text{if } 1 \leq q < r \leq \infty,$$

and

$$\mathrm{Din}^q_\alpha(S^{n-1}) \subset \mathrm{Din}^q_\beta(S^{n-1}) \quad \text{if } 0 < \beta < \alpha \leq 1.$$

The main results of this article are as follows, the proofs of which are given in Section 3.

Theorem 2.7. *Let $\rho \in (0,\infty)$, let $\alpha \in (0,1]$, let $\beta := \min\{1/2, \alpha\}$, and let φ be a growth function as in Definition 2.4 with $p \in (n/(n+\beta), 1)$. Suppose that $\Omega \in L^r(S^{n-1}) \cap \mathrm{Din}^1_\alpha(S^{n-1})$ with $r \in (1,\infty]$. If q and φ satisfy one of the following conditions:*

(i) *$r \in (1, 1/p]$ and $\varphi^{r'} \in \mathbb{A}_{p\beta/[n(1-p)]}$,*

(ii) *$r \in (1/p, \infty]$ and $\varphi^{1/(1-p)} \in \mathbb{A}_{p\beta/[n(1-p)]}$,*

then μ^ρ_Ω is bounded from WH^φ to WL^φ.

Theorem 2.8. *Let $\rho \in (0,\infty)$, let $\alpha \in (0,1]$, let $\beta := \min\{1/2, \alpha\}$, and let φ be a growth function as in Definition 2.4 with $p \in (n/(n+\beta), 1]$. Suppose that $\Omega \in \mathrm{Din}^q_\alpha(S^{n-1})$ with $q \in (1,\infty)$. If $\varphi^{q'} \in \mathbb{A}_{(p+p\beta/n-1/q)q'}$, then μ^ρ_Ω is bounded from WH^φ to WL^φ.*

Corollary 2.9. *Let $\rho \in (0,\infty)$, let $\alpha \in (0,1]$, let $\beta := \min\{1/2, \alpha\}$, and let φ be a growth function as in Definition 2.4 with $p \in (n/(n+\beta), 1]$. Suppose that $\Omega \in \mathrm{Din}_\alpha^\infty(S^{n-1})$. If $\varphi \in \mathbb{A}_{p(1+\beta/n)}$, then μ_Ω^ρ is bounded from WH^φ to WL^φ.*

Theorem 2.10. *Let $\rho \in (0,\infty)$, let $\Omega \in L^q(S^{n-1})$ with $q \in (1,\infty]$, and let φ be a growth function as in Definition 2.4 with $p := 1$ and $\varphi^{q'} \in \mathbb{A}_1$. If there exists a positive constant C such that, for any $y, h \in \mathbb{R}^n$ and M, $t \in (0,\infty)$,*

$$\int_{|x|\geq M|y|} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| \varphi(x+h,t)\,dx \leq \frac{C}{M}\varphi(y+h,t), \tag{2.5}$$

then μ_Ω^ρ is bounded from WH^φ to WL^φ.

Remark 2.11.

(i) It is worth noting that Corollary 2.9 can be regarded as the limit case of Theorem 2.8 by letting $q \to \infty$.

(ii) Theorems 2.7 and 2.8 and Corollary 2.9 jointly answer the question: When $\Omega \in \mathrm{Din}_\alpha^q(S^{n-1})$ with $q = 1$, $q \in (1,\infty)$ or $q = \infty$, respectively, what kind of additional conditions on φ and Ω can we use to deduce the boundedness of μ_Ω^ρ from WH^φ to WL^φ?

(iii) Let ω be a classic Muckenhoupt weight, and let ϕ be an Orlicz function.
   (a) When $\varphi(x,t) := \omega(x)\phi(t)$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$, we have $WH^\varphi = WH_\omega^\phi$. In this case, Theorems 2.7, 2.8, and 2.10 and Corollary 2.9 hold true for weighted weak Orlicz Hardy space. Even when $\omega \equiv 1$, the corresponding unweighted results are also new.
   (b) When $\varphi(x,t) := \omega(x)t^p$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$, we have $WH^\varphi = WH_\omega^p$. In this case, Theorems 2.7, 2.8, and 2.10 and Corollary 2.9 are new for weighted weak Hardy space. Even when $\omega \equiv 1$, the corresponding unweighted results are probably new.

3. Proofs of main results

To show our main results, we need some auxiliary notions and lemmas.

Definition 3.1 ([12, Definition 2.4]). Let φ be a growth function as in Definition 2.4.

(i) A triplet (φ, q, s) is said to be *admissible* if $q \in (q(\varphi), \infty]$ and $s \in [m(\varphi), \infty) \cap \mathbb{N}$, where $q(\varphi)$ and $m(\varphi)$ are as in (2.2) and (2.4), respectively.

(ii) For an admissible triplet (φ, q, s), a measurable function a is called a (φ, q, s)-*atom* if there exists some ball $B \subset \mathbb{R}^n$ such that the following conditions are satisfied:
   (a) a is supported in B;
   (b) $\|a\|_{L_\varphi^q(B)} \leq \|\chi_B\|_{L^\varphi}^{-1}$, where

$$\|a\|_{L_\varphi^q(B)} := \begin{cases} \sup_{t\in(0,\infty)}\left[\frac{1}{\varphi(B,t)}\int_B |a(x)|^q \varphi(x,t)\,dx\right]^{1/q}, & q \in [1,\infty), \\ \|a\|_{L^\infty}, & q = \infty, \end{cases}$$

and

$$\|\chi_B\|_{L^\varphi} := \inf\{\lambda \in (0,\infty) : \varphi(B,\lambda^{-1}) \le 1\};$$

(c) $\int_{\mathbb{R}^n} a(x)x^\gamma\,dx = 0$ for any $\gamma \in \mathbb{N}^n$ with $|\gamma| \le s$.

Definition 3.2 ([16, Definition 3.2]). For an admissible triplet (φ,q,s) as in Definition 3.1, the *weak atomic Musielak–Orlicz Hardy space* $WH^{\varphi,q,s}_{\mathrm{at}}$ is defined as the space of all $f \in \mathcal{S}'$ satisfying that there exist a sequence of (φ,q,s)-atoms, $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$, associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$, and a positive constant C such that $\sum_{j\in\mathbb{Z}_+}\chi_{B_{i,j}}(x) \le C$ for any $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, and $f = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}_+}\lambda_{i,j}a_{i,j}$ in $\mathcal{S}'$, where $\lambda_{i,j} := \widetilde{C}2^i\|\chi_{B_{i,j}}\|_{L^\varphi}$ for any $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_+$, and $\widetilde{C}$ is a positive constant independent of f.

Moreover, define

$$\|f\|_{WH^{\varphi,q,s}_{\mathrm{at}}} := \inf\left\{\inf\left\{\lambda \in (0,\infty) : \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\frac{2^i}{\lambda}\right)\right\} \le 1\right\}\right\},$$

where the first infimum is taken over all decompositions of f, as above.

Lemma 3.3 ([16, Theorem 3.5]). *Let (φ,q,s) be an admissible triplet as in Definition 3.1. Then*

$$WH^\varphi = WH^{\varphi,q,s}_{\mathrm{at}}$$

with equivalent quasinorms.

Lemma 3.4. *For any $\alpha \in (0,1]$ and $q \in [1,\infty)$, suppose that Ω satisfies the $L^{q,\alpha}$-Dini condition. Let $\rho \in (0,\infty)$, let $\beta := \min\{1/2,\alpha\}$, and let b be a multiple of a (φ,∞,s)-atom associated with some ball $B(x_0,r) \subset \mathbb{R}^n$.*

(i) *If $q = 1$, then there exists a positive constant C independent of b such that, for any $R \in [2r,\infty)$,*

$$\int_{R\le|x-x_0|<2R}\left|\mu^\rho_\Omega(b)(x)\right|dx \le C\|b\|_{L^\infty}R^n\left(\frac{r}{R}\right)^{n+\beta}.$$

(ii) *If $q \in (1,\infty)$, then there exists a positive constant C independent of b such that, for any $R \in [2r,\infty)$ and $t \in (0,\infty)$,*

$$\int_{R\le|x-x_0|<2R}\left|\mu^\rho_\Omega(b)(x)\right|\varphi(x,t)\,dx \le C\|b\|_{L^\infty}\left[\varphi^{q'}\left(B(x_0,2R),t\right)\right]^{1/q'}R^{n/q}\left(\frac{r}{R}\right)^{n+\beta}.$$

Proof. The proof of this lemma, the details of which we omit, can be completed by the method analogous to that used in the proof of [17, Lemma 4.4]. □

Proof of Theorem 2.7. We need only consider the case $r \in (1,\infty)$, since the case $r = \infty$ can be derived from the case $r = 2$. Indeed, when $r = \infty$, a routine computation gives rise to $2 > 1/p$. If Theorem 2.7 holds true for $r = 2$, then by $\Omega \in L^\infty(S^{n-1}) \subset L^2(S^{n-1})$, $2 > 1/p$, and $\varphi^{1/(1-p)} \in \mathbb{A}_{p\beta/[n(1-p)]}$, we know that Theorem 2.7 holds true for $q = \infty$. We now turn to the proof of Theorem 2.7

under case $r \in (1,\infty)$. We claim that, in either Theorem 2.7(i) or Theorem 2.7(ii), there exists some $d \in (1, p\beta/[n(1-p)])$ such that

$$\varphi^{r'} \in \mathbb{A}_d \qquad \text{and} \qquad \varphi^{1/(1-p)} \in \mathbb{A}_d. \tag{3.1}$$

We only prove (3.1) under case (ii) since the proof under case (i) is similar. By $\varphi^{1/(1-p)} \in \mathbb{A}_{p\beta/n(1-p)}$, we see that there exists some $d \in (1, p\beta/[n(1-p)])$ such that $\varphi^{1/(1-p)} \in \mathbb{A}_d$. On the other hand, note that $r' < 1/(1-p)$. Then $\varphi^{r'} \in \mathbb{A}_d$, as claimed.

Let (φ, ∞, s) be an admissible triplet as in Definition 3.1. By Lemma 3.3, we know that, for any $f \in WH^\varphi = WH^{\varphi,\infty,s}_{\mathrm{at}}$, there exists a sequence of multiples of (φ,∞,s)-atoms, $\{b_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$, associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{Z}_+}$, such that

$$f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}_+} b_{i,j} \quad \text{in } \mathcal{S}',$$

$\sum_{j\in\mathbb{Z}_+} \chi_{B_{i,j}}(x) \lesssim 1$ for any $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, $\|b_{i,j}\|_{L^\infty} \lesssim 2^i$ for any $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_+$, and

$$\|f\|_{WH^\varphi} \sim \inf\left\{\lambda \in (0,\infty) : \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+} \varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right)\right\} \le 1\right\}.$$

Thus, our problem reduces to proving that, for any $\gamma, \lambda \in (0,\infty)$ and $f \in WH^\varphi$,

$$\varphi\left(\left\{\left|\mu^\rho_\Omega(f)\right| > \gamma\right\}, \frac{\gamma}{\lambda}\right) \lesssim \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+} \varphi\left(B_{i,j}, \frac{2^i}{\lambda}\right)\right\}.$$

To show this inequality, we may assume without loss of generality that there exists $i_0 \in \mathbb{Z}$ such that $\gamma = 2^{i_0}$. Let us write

$$f = \sum_{i=-\infty}^{i_0-1} \sum_{j\in\mathbb{Z}_+} b_{i,j} + \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{Z}_+} b_{i,j} =: F_1 + F_2.$$

We estimate F_1 first. From Theorem A with $\Omega \in L^r(S^{n-1})$ and $\varphi^{r'} \in \mathbb{A}_d$ (see (3.1)), Minkowski's inequality, $\sum_{j\in\mathbb{Z}_+} \chi_{B_{i,j}}(x) \lesssim 1$ for any $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, and the uniformly upper-type 1 property of φ, we deduce that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
&\varphi\left(\left\{\left|\mu^\rho_\Omega(F_1)\right| > 2^{i_0}\right\}, \frac{2^{i_0}}{\lambda}\right) \\
&\quad = \int_{\{|\mu^\rho_\Omega(F_1)|>2^{i_0}\}} \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx \\
&\quad \le 2^{-di_0} \int_{\mathbb{R}^n} \left|\mu^\rho_\Omega(F_1)(x)\right|^d \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx \\
&\quad \lesssim 2^{-di_0} \int_{\mathbb{R}^n} \left|F_1(x)\right|^d \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx \\
&\quad \lesssim 2^{-di_0} \left\{\sum_{i=-\infty}^{i_0-1} \left[\int_{\mathbb{R}^n} \left|\sum_{j\in\mathbb{Z}_+} b_{i,j}(x)\right|^d \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx\right]^{1/d}\right\}^d
\end{aligned}$$

$$
\begin{aligned}
&\lesssim 2^{-di_0}\Big\{\sum_{i=-\infty}^{i_0-1} 2^i\Big[\sum_{j\in\mathbb{Z}_+}\varphi\Big(B_{i,j},\frac{2^{i_0}}{\lambda}\Big)\Big]^{1/d}\Big\}^d\\
&\lesssim 2^{-di_0}\Big\{\sum_{i=-\infty}^{i_0-1} 2^i\Big[2^{i_0-i}\sum_{j\in\mathbb{Z}_+}\varphi\Big(B_{i,j},\frac{2^{i}}{\lambda}\Big)\Big]^{1/d}\Big\}^d\\
&\lesssim 2^{(1-d)i_0}\Big(\sum_{i=-\infty}^{i_0-1} 2^{(1-1/d)i}\Big)^d\sup_{i\in\mathbb{Z}}\Big\{\sum_{j\in\mathbb{Z}_+}\varphi\Big(B_{i,j},\frac{2^{i}}{\lambda}\Big)\Big\}\\
&\sim \sup_{i\in\mathbb{Z}}\Big\{\sum_{j\in\mathbb{Z}_+}\varphi\Big(B_{i,j},\frac{2^{i}}{\lambda}\Big)\Big\},
\end{aligned}
\tag{3.2}
$$

which is desired.

Next let us deal with F_2. Denote the center of $B_{i,j}$ by $x_{i,j}$ and the radius by $r_{i,j}$. Set

$$
A_{i_0} := \bigcup_{i=i_0}^{\infty}\bigcup_{j\in\mathbb{Z}_+}\widetilde{B_{i,j}},
$$

where $\widetilde{B_{i,j}} := B(x_{i,j}, 2(3/2)^{(i-i_0)/(n+\beta)}r_{i,j})$. To show that

$$
\varphi\Big(\big\{|\mu_\Omega^\rho(F_2)| > 2^{i_0}\big\},\frac{2^{i_0}}{\lambda}\Big) \lesssim \sup_{i\in\mathbb{Z}}\Big\{\sum_{j\in\mathbb{Z}_+}\varphi\Big(B_{i,j},\frac{2^{i}}{\lambda}\Big)\Big\},
$$

we cut $\{|\mu_\Omega^\rho(F_2)| > 2^{i_0}\}$ into A_{i_0} and $\{x\in(A_{i_0})^{\complement} : |\mu_\Omega^\rho(F_2)(x)| > 2^{i_0}\}$.

For A_{i_0}, from Lemma 2.2 with $\varphi\in\mathbb{A}_{p(1+\beta/n)}$ (since $\varphi^{1/(1-p)}\in\mathbb{A}_{p\beta/[n(1-p)]}$), and the uniformly lower-type p property of φ, it follows that, for any $\lambda\in(0,\infty)$,

$$
\begin{aligned}
\varphi\Big(A_{i_0},\frac{2^{i_0}}{\lambda}\Big) &\le \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+}\varphi\Big(\widetilde{B_{i,j}},\frac{2^{i_0}}{\lambda}\Big)\\
&\lesssim \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+}\Big(\frac{3}{2}\Big)^{(i-i_0)p}\varphi\Big(B_{i,j},\frac{2^{i_0}}{\lambda}\Big)\\
&\lesssim \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+}\Big(\frac{3}{4}\Big)^{(i-i_0)p}\varphi\Big(B_{i,j},\frac{2^{i}}{\lambda}\Big)\\
&\lesssim \sup_{i\in\mathbb{Z}}\Big\{\sum_{j\in\mathbb{Z}_+}\varphi\Big(B_{i,j},\frac{2^{i}}{\lambda}\Big)\Big\},
\end{aligned}
\tag{3.3}
$$

which is also desired.

It remains to estimate $(A_{i_0})^{\complement}$. Applying the inequality $\|\cdot\|_{\ell^1}\le\|\cdot\|_{\ell^p}$ with $p\in(0,1)$, we conclude that, for any $\lambda\in(0,\infty)$,

$$
\varphi\Big(\big\{x\in(A_{i_0})^{\complement} : |\mu_\Omega^\rho(F_2)(x)| > 2^{i_0}\big\},\frac{2^{i_0}}{\lambda}\Big)
$$

$$\leq 2^{-i_0 p} \int_{(A_{i_0})^\complement} \left|\mu_\Omega^\rho(F_2)(x)\right|^p \varphi\Big(x, \frac{2^{i_0}}{\lambda}\Big)\, dx$$
$$\leq 2^{-i_0 p} \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{Z}_+} \int_{(\widetilde{B_{i,j}})^\complement} \left|\mu_\Omega^\rho(b_{i,j})(x)\right|^p \varphi\Big(x, \frac{2^{i_0}}{\lambda}\Big)\, dx. \tag{3.4}$$

Below, we will give the estimate of integral

$$\mathrm{I} := \int_{(\widetilde{B_{i,j}})^\complement} \left|\mu_\Omega^\rho(b_{i,j})(x)\right|^p \varphi\Big(x, \frac{2^{i_0}}{\lambda}\Big)\, dx.$$

For any $k \in \mathbb{N}$, let

$$E_k := (2^{k+1}\widetilde{B_{i,j}}) \setminus (2^k \widetilde{B_{i,j}}).$$

It follows from Hölder's inequality that, for any $\lambda \in (0,\infty)$,

$$\mathrm{I} \leq \sum_{k=0}^{\infty} \Big[\int_{E_k} \left|\mu_\Omega^\rho(b_{i,j})(x)\right| dx\Big]^p \Big\{\int_{E_k} \Big[\varphi\Big(x, \frac{2^{i_0}}{\lambda}\Big)\Big]^{\frac{1}{1-p}} dx\Big\}^{1-p}.$$

On the one hand, by Lemma 2.3 with $\varphi^{1/(1-p)} \in \mathbb{A}_d \subset \mathbb{A}_\infty$ (see (3.1)), we have $\varphi \in \mathbb{RH}_{1/(1-p)}$. Thus, thanks to Lemma 2.2 with $\varphi^{1/(1-p)} \in \mathbb{A}_d$, and $\varphi \in \mathbb{RH}_{1/(1-p)}$, it follows that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}\Big\{\int_{E_k} \Big[\varphi\Big(x, \frac{2^{i_0}}{\lambda}\Big)\Big]^{\frac{1}{1-p}} dx\Big\}^{1-p} &\leq \Big[\varphi^{\frac{1}{1-p}}\Big(2^{k+1}\widetilde{B_{i,j}}, \frac{2^{i_0}}{\lambda}\Big)\Big]^{1-p} \\ &\lesssim \Big[\varphi^{\frac{1}{1-p}}\Big(B_{i,j}, \frac{2^{i_0}}{\lambda}\Big)\Big]^{1-p} \Big[2^k\Big(\frac{3}{2}\Big)^{\frac{i-i_0}{n+\beta}}\Big]^{nd(1-p)} \\ &\lesssim (r_{i,j})^{-np} \varphi\Big(B_{i,j}, \frac{2^{i_0}}{\lambda}\Big) \Big[2^k\Big(\frac{3}{2}\Big)^{\frac{i-i_0}{n+\beta}}\Big]^{nd(1-p)}.\end{aligned}$$

On the other hand, since $d < p\beta/[n(1-p)]$, we may choose an $\widetilde{\alpha} \in (0,\alpha)$ such that $d < p\widetilde{\beta}/[n(1-p)]$, where $\widetilde{\beta} := \min\{1/2, \widetilde{\alpha}\}$. By the assumption $\Omega \in \mathrm{Din}_\alpha^1(S^{n-1})$, we know that Ω satisfies the $L^{1,\widetilde{\alpha}}$-Dini condition. Then Lemma 3.4(i) yields that

$$\int_{E_k} \left|\mu_\Omega^\rho(b_{i,j})(x)\right| dx \lesssim 2^i (r_{i,j})^n \Big[2^k\Big(\frac{3}{2}\Big)^{\frac{i-i_0}{n+\beta}}\Big]^{-\widetilde{\beta}}.$$

The above three inequalities give us that, for any $\lambda \in (0,\infty)$,

$$\mathrm{I} \lesssim 2^{ip} \varphi\Big(B_{i,j}, \frac{2^{i_0}}{\lambda}\Big) \sum_{k=0}^{\infty} \Big[2^k\Big(\frac{3}{2}\Big)^{\frac{i-i_0}{n+\beta}}\Big]^{nd-ndp-p\widetilde{\beta}}.$$

Substituting this inequality into (3.4) and using the uniformly lower-type p property of φ, we obtain that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}&\varphi\Big(\Big\{x \in (A_{i_0})^\complement : \left|\mu_\Omega^\rho(F_2)(x)\right| > 2^{i_0}\Big\}, \frac{2^{i_0}}{\lambda}\Big) \\ &\quad\lesssim 2^{-i_0 p} \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{Z}_+} 2^{ip} \varphi\Big(B_{i,j}, \frac{2^{i_0}}{\lambda}\Big) \sum_{k=0}^{\infty} \Big[2^k\Big(\frac{3}{2}\Big)^{\frac{i-i_0}{n+\beta}}\Big]^{nd-ndp-p\widetilde{\beta}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sup_{i\in\mathbb{Z}}\Big\{\sum_{j\in\mathbb{Z}_+}\varphi\Big(B_{i,j},\frac{2^i}{\lambda}\Big)\Big\}\sum_{i=i_0}^{\infty}\sum_{k=0}^{\infty}\Big[2^k\Big(\frac{3}{2}\Big)^{\frac{i-i_0}{n+\beta}}\Big]^{nd-ndp-p\widetilde{\beta}}\\
&\sim \sup_{i\in\mathbb{Z}}\Big\{\sum_{j\in\mathbb{Z}_+}\varphi\Big(B_{i,j},\frac{2^i}{\lambda}\Big)\Big\}, \qquad (3.5)
\end{aligned}
$$

where the last $\sim$ is due to $d < p\widetilde{\beta}/[n(1-p)]$.

Finally, combining (3.2), (3.3), and (3.5), we obtain the desired inequality. This finishes the proof of Theorem 2.7. $\square$

Proof of Theorem 2.8. We need only consider the case $p < 1$. The proof of the case $p = 1$ is similar and easier. Once we prove Lemma 3.4(ii), the proof of this theorem is quite similar to that of Theorem 2.7, the major change being the substitution of

$$\mathrm{I} \le \sum_{k=0}^{\infty}\Big[\int_{E_k}\big|\mu_\Omega^\rho(b_{i,j})(x)\big|\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx\Big]^p\Big[\int_{E_k}\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx\Big]^{1-p}$$

for

$$\mathrm{I} \le \sum_{k=0}^{\infty}\Big[\int_{E_k}\big|\mu_\Omega^\rho(b_{i,j})(x)\big|\,dx\Big]^p\Big\{\int_{E_k}\Big[\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\Big]^{\frac{1}{1-p}}\,dx\Big\}^{1-p}.$$

But to limit the length of this article, we leave the details to the interested reader. $\square$

Proof of Corollary 2.9. By $\varphi \in \mathbb{A}_{p(1+\beta/n)}$, we see that there exists some $d \in (1,\infty)$ such that $\varphi^d \in \mathbb{A}_{p(1+\beta/n)}$. For any $q \in (1,\infty)$, by $p > n/(n+\beta)$, some tedious manipulation yields that $(p + p\beta/n - 1/q)q' > p(1+\beta/n)$ and hence $\varphi^d \in \mathbb{A}_{(p+p\beta/n-1/q)q'}$. Thus, we may choose $q := d/(d-1)$ such that

$$\varphi^{q'} = \varphi^d \in \mathbb{A}_{(p+\beta/n-1/q)q'}$$

and hence Corollary 2.9 follows from Theorem 2.8. $\square$

Proof of Theorem 2.10. Since the proof of Theorem 2.10 is similar to that of Theorem 2.7, we use the same notation as in the proof of Theorem 2.7. Rather than give a completed proof, we just point out the necessary modifications with respect to the estimate of $(A_{i_0})^\complement$. Reset

$$A_{i_0} := \bigcup_{i=i_0}^{\infty}\bigcup_{j\in\mathbb{Z}_+}\widetilde{B_{i,j}},$$

where $\widetilde{B_{i,j}} := B(x_{i,j}, 2(3/2)^{(i-i_0)/n}r_{i,j})$. For any $\lambda \in (0,\infty)$, we have

$$
\begin{aligned}
&\varphi\Big(\Big\{x\in(A_{i_0})^\complement : \big|\mu_\Omega^\rho(F_2)(x)\big| > 2^{i_0}\Big\},\frac{2^{i_0}}{\lambda}\Big)\\
&\quad\le 2^{-i_0}\int_{(A_{i_0})^\complement}\big|\mu_\Omega^\rho(F_2)(x)\big|\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\le 2^{-i_0}\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+}\int_{(\widetilde{B_{i,j}})^{\complement}}\left|\mu_\Omega^\rho(b_{i,j})(x)\right|\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx\\
&=: 2^{-i_0}\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+}\mathrm{I}.
\end{aligned}
$$

For any $\lambda\in(0,\infty)$, let us write

$$
\begin{aligned}
\mathrm{I}\le&\int_{(\widetilde{B_{i,j}})^{\complement}}\Big(\int_0^{|x-x_{i,j}|+r_{i,j}}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}b_{i,j}(y)dy\Big|^2\frac{dt}{t^{2\rho+1}}\Big)^{1/2}\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx\\
&+\int_{(\widetilde{B_{i,j}})^{\complement}}\Big(\int_{|x-x_{i,j}|+r_{i,j}}^{\infty}\cdots\Big)^{1/2}\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx=:\mathrm{I}_1+\mathrm{I}_2.
\end{aligned}
$$

Below, we will first estimate I_1 and then estimate I_2.

For I_1, noting that $x\in(\widetilde{B_{i,j}})^{\complement}$ and $y\in B_{i,j}$, we know that

$$
|x-y|\sim|x-x_{i,j}|\sim|x-x_{i,j}|+r_{i,j},
$$

which, together with the mean value theorem, implies that

$$
\Big|\frac{1}{|x-y|^{2\rho}}-\frac{1}{(|x-x_{i,j}|+r_{i,j})^{2\rho}}\Big|\lesssim\frac{r_{i,j}}{|x-y|^{2\rho+1}}.
$$

From Minkowski's inequality for integrals, the above inequality, and Hölder's inequality, it follows that, for any $\lambda\in(0,\infty)$,

$$
\begin{aligned}
\mathrm{I}_1\le&\int_{(\widetilde{B_{i,j}})^{\complement}}\Big[\int_{B_{i,j}}\Big|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}b_{i,j}(y)\Big|\Big(\int_{|x-y|}^{|x-x_{i,j}|+r_{i,j}}\frac{dt}{t^{2\rho+1}}\Big)^{1/2}dy\Big]\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx\\
\lesssim&\,2^i(r_{i,j})^{1/2}\int_{(\widetilde{B_{i,j}})^{\complement}}\Big(\int_{B_{i,j}}\frac{|\Omega(x-y)|}{|x-y|^{n+1/2}}\,dy\Big)\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx\\
\sim&\,2^i(r_{i,j})^{1/2}\sum_{k=0}^{\infty}\int_{B_{i,j}}\Big[\int_{E_k}\frac{|\Omega(x-y)|}{|x-y|^{n+1/2}}\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx\Big]\,dy\\
\lesssim&\,2^i(r_{i,j})^{1/2}\sum_{k=0}^{\infty}\int_{B_{i,j}}\Big(\int_{E_k}\frac{|\Omega(x-y)|^q}{|x-y|^{n+1/2}}\,dx\Big)^{1/q}\\
&\times\Big(\int_{E_k}\frac{1}{|x-y|^{n+1/2}}\Big[\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\Big]^{q'}\,dx\Big)^{1/q'}\,dy.
\end{aligned}
$$

On the one hand, $x\in E_k$ and $y\in B_{i,j}$ imply that $\theta r_{i,j}<|x-y|<5\theta r_{i,j}$, where $\theta:=2^k(3/2)^{(i-i_0)/n}$. Therefore, we have

$$
\begin{aligned}
\Big(\int_{E_k}\frac{|\Omega(x-y)|^q}{|x-y|^{n+1/2}}\,dx\Big)^{1/q}&\le\Big(\int_{\theta r_{i,j}<|z|<5\theta r_{i,j}}\frac{|\Omega(z)|^q}{|z|^{n+1/2}}\,dz\Big)^{1/q}\\
&=\Big(\int_{S^{n-1}}\int_{\theta r_{i,j}}^{5\theta r_{i,j}}\frac{|\Omega(z')|^q}{u^{n+1/2}}u^{n-1}\,du\,d\sigma(z')\Big)^{1/q}\\
&\sim(\theta r_{i,j})^{-1/2q}.
\end{aligned}
$$

On the other hand, according to Lemmas 2.2 and 2.3 with $\varphi^{q'} \in \mathbb{A}_1$, it follows that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
\Big(\int_{E_k} & \frac{1}{|x-y|^{n+1/2}}\Big[\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\Big]^{q'}\,dx\Big)^{1/q'} \\
&\sim (\theta r_{i,j})^{-n/q'-1/2q'}\Big\{\int_{E_k}\Big[\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\Big]^{q'}\,dx\Big\}^{1/q'} \\
&\lesssim (\theta r_{i,j})^{-n/q'-1/2q'}\Big[\varphi^{q'}\Big(4\theta B_{i,j},\frac{2^{i_0}}{\lambda}\Big)\Big]^{1/q'} \\
&\lesssim (\theta r_{i,j})^{-n/q'-1/2q'}\theta^{n/q'}\Big[\varphi^{q'}\Big(B_{i,j},\frac{2^{i_0}}{\lambda}\Big)\Big]^{1/q'} \\
&\lesssim (\theta r_{i,j})^{-n/q'-1/2q'}\theta^{n/q'}(r_{i,j})^{-n/q}\varphi\Big(B_{i,j},\frac{2^{i_0}}{\lambda}\Big).
\end{aligned}$$

If we plug the above two inequalities back into I_1, we obtain that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
\mathrm{I}_1 &\lesssim 2^i (r_{i,j})^{1/2}\sum_{k=0}^{\infty}\int_{B_{i,j}}(\theta r_{i,j})^{-1/2q}(\theta r_{i,j})^{-n/q'-1/2q'}\theta^{n/q'}(r_{i,j})^{-n/q}\varphi\Big(B_{i,j},\frac{2^{i_0}}{\lambda}\Big)\,dy \\
&\sim 2^i\sum_{k=0}^{\infty}\theta^{-1/2}\varphi\Big(B_{i,j},\frac{2^{i_0}}{\lambda}\Big) \sim 2^i\Big(\frac{2}{3}\Big)^{\frac{i-i_0}{2n}}\varphi\Big(B_{i,j},\frac{2^{i_0}}{\lambda}\Big).
\end{aligned}$$

For I_2, it is apparent from $t > |x-x_{i,j}|+r_{i,j}$ that $B_{i,j} \subset \{y \in \mathbb{R}^n : |x-y| \le t\}$. From this, the vanishing moments of $b_{i,j}$, and Minkowski's inequality for integrals, it follows that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
\mathrm{I}_2 &= \int_{(\widetilde{B_{i,j}})^{\complement}}\Big(\int_{|x-x_{i,j}|+r_{i,j}}^{\infty}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}b_{i,j}(y)dy\Big|^2\frac{dt}{t^{2\rho+1}}\Big)^{1/2}\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx \\
&\le \int_{(\widetilde{B_{i,j}})^{\complement}}\Big[\int_{B_{i,j}}\Big|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x-x_{i,j})}{|x-x_{i,j}|^{n-\rho}}\Big||b_{i,j}(y)|\Big(\int_{|x-x_{i,j}|}^{\infty}\frac{dt}{t^{2\rho+1}}\Big)^{1/2}dy\Big] \\
&\quad\times\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx \\
&= C\int_{(\widetilde{B_{i,j}})^{\complement}}\Big(\int_{B_{i,j}}\Big|\frac{\Omega(x-y)}{|x-y|^{n-\rho}|x-x_{i,j}|^{\rho}}-\frac{\Omega(x-x_{i,j})}{|x-x_{i,j}|^{n}}\Big||b_{i,j}(y)|\,dy\Big) \\
&\quad\times\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx \\
&\le C\int_{(\widetilde{B_{i,j}})^{\complement}}\Big(\int_{B_{i,j}}\Big|\frac{\Omega(x-y)}{|x-y|^{n-\rho}|x-x_{i,j}|^{\rho}}-\frac{\Omega(x-y)}{|x-y|^{n}}\Big||b_{i,j}(y)|\,dy\Big) \\
&\quad\times\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx \\
&\quad+C\int_{(\widetilde{B_{i,j}})^{\complement}}\Big(\int_{B_{i,j}}\Big|\frac{\Omega(x-y)}{|x-y|^{n}}-\frac{\Omega(x-x_{i,j})}{|x-x_{i,j}|^{n}}\Big||b_{i,j}(y)|\,dy\Big)\varphi\Big(x,\frac{2^{i_0}}{\lambda}\Big)\,dx \\
&=: C(\mathrm{I}_{21}+\mathrm{I}_{22}).
\end{aligned}$$

On the one hand, the mean value theorem yields that, for any $x \in (\widetilde{B_{i,j}})^{\complement}$ and $y \in B_{i,j}$,

$$\left|\frac{1}{|x-x_{i,j}|^\rho}-\frac{1}{|x-y|^\rho}\right| \sim \frac{|y-x_{i,j}|}{|x-y|^{\rho+1}} \lesssim \frac{|y-x_{i,j}|^{1/2}}{|x-y|^{\rho+1/2}} \lesssim \frac{(r_{i,j})^{1/2}}{|x-y|^{\rho+1/2}},$$

which, together with the same argument as that used in I_1, implies that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
\mathrm{I}_{21} &\lesssim 2^i \int_{(\widetilde{B_{i,j}})^{\complement}} \Big(\int_{B_{i,j}} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} \left|\frac{1}{|x-x_{i,j}|^\rho}-\frac{1}{|x-y|^\rho}\right| dy\Big) \varphi\Big(x, \frac{2^{i_0}}{\lambda}\Big)\, dx \\
&\lesssim 2^i (r_{i,j})^{1/2} \int_{(\widetilde{B_{i,j}})^{\complement}} \Big(\int_{B_{i,j}} \frac{|\Omega(x-y)|}{|x-y|^{n+1/2}}\, dy\Big) \varphi\Big(x, \frac{2^{i_0}}{\lambda}\Big)\, dx \\
&\lesssim 2^i \Big(\frac{2}{3}\Big)^{\frac{i-i_0}{2n}} \varphi\Big(B_{i,j}, \frac{2^{i_0}}{\lambda}\Big).
\end{aligned}$$

On the other hand, the condition (2.5) gives that, for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
\mathrm{I}_{22} &\lesssim 2^i \int_{|y-x_{i,j}|<r_{i,j}} \int_{|x-x_{i,j}|>(3/2)^{(i-i_0)/n} r_{i,j}} \left|\frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x-x_{i,j})}{|x-x_{i,j}|^n}\right| \\
&\quad \times \varphi\Big(x, \frac{2^{i_0}}{\lambda}\Big)\, dx\, dy \\
&\sim 2^i \int_{|y|<r_{i,j}} \int_{|x|>(3/2)^{(i-i_0)/n} r_{i,j}} \left|\frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n}\right| \varphi\Big(x+x_{i,j}, \frac{2^{i_0}}{\lambda}\Big)\, dx\, dy \\
&\lesssim 2^i \int_{|y|<r_{i,j}} \Big(\frac{2}{3}\Big)^{\frac{i-i_0}{n}} \varphi\Big(y+x_{i,j}, \frac{2^{i_0}}{\lambda}\Big)\, dy \lesssim 2^i \Big(\frac{2}{3}\Big)^{\frac{i-i_0}{2n}} \varphi\Big(B_{i,j}, \frac{2^{i_0}}{\lambda}\Big).
\end{aligned}$$

Collecting the estimates of I_1, I_{21}, and I_{22}, we obtain that, for any $\lambda \in (0,\infty)$,

$$\mathrm{I} \lesssim \mathrm{I}_1 + \mathrm{I}_{21} + \mathrm{I}_{22} \lesssim 2^i \Big(\frac{2}{3}\Big)^{\frac{i-i_0}{2n}} \varphi\Big(B_{i,j}, \frac{2^{i_0}}{\lambda}\Big)$$

and hence

$$\begin{aligned}
\varphi\Big(\big\{x \in (A_{i_0})^{\complement} : \big|\mu_\Omega^\rho(F_2)(x)\big| > 2^{i_0}\big\}, \frac{2^{i_0}}{\lambda}\Big) &\le 2^{-i_0} \sum_{i=i_0}^\infty \sum_{j\in\mathbb{Z}_+} \mathrm{I} \\
&\lesssim 2^{-i_0} \sum_{i=i_0}^\infty \sum_{j\in\mathbb{Z}_+} 2^i \Big(\frac{2}{3}\Big)^{\frac{i-i_0}{2n}} \varphi\Big(B_{i,j}, \frac{2^{i_0}}{\lambda}\Big) \\
&\lesssim \sup_{i\in\mathbb{Z}} \Big\{\sum_{j\in\mathbb{Z}_+} \varphi\Big(B_{i,j}, \frac{2^i}{\lambda}\Big)\Big\},
\end{aligned}$$

where the last $\lesssim$ is due to the uniformly lower-type 1 property of φ. The proof is completed. $\square$

Remark 3.5. We should point out that if φ is a growth function of uniformly lower-type 1 and of uniformly upper-type 1, then $WH^\varphi = WH^1_{\varphi(\cdot,1)}$ and $WL^\varphi =$

$WL^1_{\varphi(\cdot,1)}$. In fact, there exists a positive constant C such that, for any $x \in \mathbb{R}^n$ and $t \in (0, \infty)$,

$$C^{-1}t\varphi(x,1) = C^{-1}t\varphi(x,t/t) \le \varphi(x,t) \le Ct\varphi(x,1),$$

which implies that

$$\sup_{t\in(0,\infty)} \varphi\big(\{|f| > t\}, t\big) \sim \sup_{t\in(0,\infty)} \varphi\big(\{|f| > t\}, 1\big)t.$$

Hence, we have $WL^{\varphi} = WL^1_{\varphi(\cdot,1)}$. Analogously, $WH^{\varphi} = WH^1_{\varphi(\cdot,1)}$.

Acknowledgments. The author is greatly indebted to Professor Baode Li for many useful discussions and guidance over the past years. The author is also grateful to the anonymous referees for useful comments and suggestions.

The author's works was partially supported by the National Natural Science Foundation of China grants 11671039 and 11771043.

References

1. J. Álvarez and M. Milman, *H^p continuity properties of Calderón-Zygmund-type operators*, J. Math. Anal. Appl. **118** (1986), no. 1, 63–79. Zbl 0596.42006. MR0849442. DOI 10.1016/0022-247X(86)90290-8. 48
2. L. Diening, P. Hästö, and S. Roudenko, *Function spaces of variable smoothness and integrability*, J. Funct. Anal. **256** (2009), no. 6, 1731–1768. Zbl 1179.46028. MR2498558. DOI 10.1016/j.jfa.2009.01.017. 49
3. X. T. Duong and T. D. Tran, *Musielak-Orlicz Hardy spaces associated to operators satisfying Davies-Gaffney estimates and bounded holomorphic functional calculus*, J. Math. Soc. Japan **68** (2016), no. 1, 1–30. Zbl 1344.42017. MR3454550. DOI 10.2969/jmsj/06810001. 49
4. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3–4, 137–193. Zbl 0257.46078. MR0447953. DOI 10.1007/BF02392215. 48
5. R. Fefferman and F. Soria, *The space weak H^1*, Studia Math. **85** (1986), no. 1, 1–16. Zbl 0626.42013. MR0879411. DOI 10.4064/sm-85-1-1-16. 49
6. K.-P. Ho, *Intrinsic atomic and molecular decompositions of Hardy-Musielak-Orlicz spaces*, Banach J. Math. Anal. **10** (2016), no. 3, 565–591. Zbl 1346.42024. MR3528348. DOI 10.1215/17358787-3607354. 49
7. L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140. Zbl 0093.11402. MR0121655. DOI 10.1007/BF02547187. 48
8. R. Jiang and D. Yang, *New Orlicz-Hardy spaces associated with divergence form elliptic operators*, J. Funct. Anal. **258** (2010), no. 4, 1167–1224. Zbl 1205.46014. MR2565837. DOI 10.1016/j.jfa.2009.10.018. 48
9. R. Jiang and D. Yang, *Orlicz-Hardy spaces associated with operators satisfying Davies-Gaffney estimates*, Commun. Contemp. Math. **13** (2011), no. 2, 331–373. Zbl 1221.42042. MR2794490. DOI 10.1142/S0219199711004221. 48
10. R. Jiang and D. Yang, *Predual spaces of Banach completions of Orlicz-Hardy spaces associated with operators*, J. Fourier Anal. Appl. **17** (2011), no. 1, 1–35. Zbl 1213.42079. MR2765590. DOI 10.1007/s00041-010-9123-8. 48
11. R. Jiang, D. Yang, and Y. Zhou, *Orlicz-Hardy spaces associated with operators*, Sci. China Ser. A **52** (2009), no. 5, 1042–1080. Zbl 1177.42018. MR2505009. DOI 10.1007/s11425-008-0136-6. 48
12. L. D. Ky, *New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators*, Integral Equations Operator Theory **78** (2014), no. 1, 115–150. Zbl 1284.42073. MR3147406. DOI 10.1007/s00020-013-2111-z. 48, 51, 53

13. Bo Li, M. Liao, and B. Li, *Boundedness of Marcinkiewicz integrals with rough kernels on Musielak-Orlicz Hardy spaces*, J. Inequal. Appl. **2017**, no. 228. Zbl 1375.42017. MR3703531. DOI 10.1186/s13660-017-1501-1. 51
14. Y. Liang, J. Huang, and D. Yang, *New real-variable characterizations of Musielak-Orlicz Hardy spaces*, J. Math. Anal. Appl. **395** (2012), no. 1, 413–428. Zbl 1256.42035. MR2943633. DOI 10.1016/j.jmaa.2012.05.049. 50
15. Y. Liang, E. Nakai, D. Yang, and J. Zhang, *Boundedness of intrinsic Littlewood-Paley functions on Musielak-Orlicz Morrey and Campanato spaces*, Banach J. Math. Anal. **8** (2014), no. 1, 221–268. Zbl 1280.42016. MR3161693. 49
16. Y. Liang, D. Yang, and R. Jiang, *Weak Musielak-Orlicz Hardy spaces and applications*, Math. Nachr. **289** (2016), no. 5–6, 634–677. Zbl 1338.42023. MR3486149. DOI 10.1002/mana.201500152. 49, 51, 54
17. C.-C. Lin and Y.-C. Lin, *H^p_ω-L^p_ω boundedness of Marcinkiewicz integral*, Integral Equations Operator Theory **58** (2007), no. 1, 87–98. Zbl 1136.42015. MR2312447. DOI 10.1007/s00020-006-1475-8. 54
18. H. Liu, "The weak H^p spaces on homogeneous groups" in *Harmonic Analysis (Tianjin, 1988)*, Lecture Notes in Math. **1494**, Springer, Berlin, 1991, 113–118. Zbl 0801.42012. MR1187071. DOI 10.1007/BFb0087762. 48
19. F.-Y. Maeda, Y. Mizuta, T. Ohno, and T. Shimomura, *Duality of non-homogeneous central Herz-Morrey-Musielak-Orlicz spaces*, Potential Anal. **47** (2017), no. 4, 447–460. Zbl 1387.46029. MR3717349. DOI 10.1007/s11118-017-9621-2. 49
20. Y. Mizuta, E. Nakai, T. Ohno, and T. Shimomura, *Maximal functions, Riesz potentials and Sobolev embeddings on Musielak-Orlicz-Morrey spaces of variable exponent in $\mathbb{R}^n$*, Rev. Mat. Complut. **25** (2012), no. 2, 413–434. Zbl 1273.31005. MR2931419. DOI 10.1007/s13163-011-0074-7. 49
21. T. Quek and D. Yang, *Calderón-Zygmund-type operators on weighted weak Hardy spaces over $\mathbb{R}^n$*, Acta Math. Sin. (Engl. Ser.) **16** (2000), no. 1, 141–160. Zbl 0956.42011. MR1760530. DOI 10.1007/s101149900022. 49
22. X. Shi and Y. Jiang, *Weighted boundedness of parametric Marcinkiewicz integral and higher order commutator*, Anal. Theory Appl. **25** (2009), no. 1, 25–39. Zbl 1199.42076. MR2506774. DOI 10.1007/s10496-009-0025-z. 48
23. E. M. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466. Zbl 0105.05104. MR0112932. DOI 10.2307/1993226. 48
24. J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math. **1381**, Springer, Berlin, 1989. Zbl 0676.42021. MR1011673. DOI 10.1007/BFb0091154. 48

Center for Applied Mathematics, Tianjin University, Tianjin 300072, People's Republic of China.
E-mail address: bli.math@outlook.com