CONTINUOUS GENERALIZATION OF CLARKSON–MCCARTHY INEQUALITIES

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ABSTRACT. Let G be a compact Abelian group, let μ be the corresponding Haar measure, and let $\hat{G}$ be the Pontryagin dual of G. Furthermore, let $\mathcal{C}_p$ denote the Schatten class of operators on some separable infinite-dimensional Hilbert space, and let $L^p(G; \mathcal{C}_p)$ denote the corresponding Bochner space. If $G \ni \theta \mapsto A_\theta$ is the mapping belonging to $L^p(G; \mathcal{C}_p)$, then

$$\sum_{k \in \hat{G}} \left\| \int_G \overline{k(\theta)} A_\theta \, d\theta \right\|_p^p \leq \int_G \|A_\theta\|_p^p \, d\theta, \quad p \geq 2,$$

$$\sum_{k \in \hat{G}} \left\| \int_G \overline{k(\theta)} A_\theta \, d\theta \right\|_p^p \leq \left(\int_G \|A_\theta\|_p^q \, d\theta \right)^{p/q}, \quad p \geq 2,$$

$$\sum_{k \in \hat{G}} \left\| \int_G \overline{k(\theta)} A_\theta \, d\theta \right\|_p^q \leq \left(\int_G \|A_\theta\|_p^p \, d\theta \right)^{q/p}, \quad p \leq 2.$$

If G is a finite group, then the previous equations comprise several generalizations of Clarkson–McCarthy inequalities obtained earlier (e.g., $G = \mathbf{Z}_n$ or $G = \mathbf{Z}_2^n$), as well as the original inequalities, for $G = \mathbf{Z}_2$. We also obtain other related inequalities.

1. Introduction

While investigating uniformly convex spaces, Clarkson [5] proved the following inequalities for L^p-norms:

$$2\big(\|f\|_p^p + \|g\|_p^p\big) \le \|f + g\|_p^p + \|f - g\|_p^p \le 2^{p-1}\big(\|f\|_p^p + \|g\|_p^p\big), \quad p \ge 2,$$
$$2\big(\|f\|_p^p + \|g\|_p^p\big) \ge \|f + g\|_p^p + \|f - g\|_p^p \ge 2^{p-1}\big(\|f\|_p^p + \|g\|_p^p\big), \quad p \le 2, \tag{1.1}$$

$$\|f + g\|_p^p + \|f - g\|_p^p \le 2\big(\|f\|_p^q + \|g\|_p^q\big)^{p/q}, \quad p \ge 2, q = p/(p-1), \tag{1.2}$$

$$\|f + g\|_p^q + \|f - g\|_p^q \le 2^{q-1}\big(\|f\|_p^p + \|g\|_p^p\big)^{q/p}, \quad p \le 2, q = p/(p-1). \tag{1.3}$$

McCarthy [19] later generalized these inequalities to Schatten classes of operators. He replaced measurable functions f and g by compact operators A and B, and the L^p-norm by the C_p-norm defined as

$$\|A\|_p = \big(\mathrm{tr}\big(|A|^p\big)\big)^{1/p}.$$

The inequalities he obtained were exactly (1.1), (1.2), and (1.3). In an operator framework, these are usually referred to as *Clarkson–McCarthy inequalities*. In what follows, we will use the abbreviation CMC.

There are many generalizations of CMC inequalities. Among others, Bhatia and Kittaneh [3, Theorem 2] proved the following inequalities for n-tuples of operators:

$$n \sum_{j=0}^{n-1} \|A_j\|_p^p \le \sum_{k=0}^{n-1} \Big\|\sum_{j=0}^{n-1} \omega_j^k A_j\Big\|_p^p \le n^{p-1} \sum_{j=0}^{n-1} \|A_j\|_p^p, \tag{1.4}$$

for $p \ge 2$, and the corresponding reversed inequalities for $p \le 2$, where $\omega_j = e^{2\pi i j/n}$ is the jth degree of the nth root of unity. They also proved the stronger inequality

$$n^{-p/2}\Big\|\!\!\sum_{k=0}^{n-1}\Big|\sum_{j=0}^{n-1} \omega_j^k A_j\Big|^p\Big\|\! \le \Big\|\Big(\sum_{j=0}^{n-1} |A_j|^2\Big)^{p/2}\Big\| \le \frac{1}{n}\Big\|\!\!\sum_{k=0}^{n-1}\Big|\sum_{j=0}^{n-1} \omega_j^k A_j\Big|^p\Big\| \tag{1.5}$$

for all unitarily invariant norms $\|\!|\cdot|\!\|$ and the same complex numbers ω_j.

After this work there were several further generalizations. Hirzallah and Kittaneh [12] replaced $t \mapsto t^{p/2}$ by an arbitrary convex (concave) function f and obtained

$$\Big\|\!\!\sum_{k=0}^{n-1} f\Big(\frac{1}{n}\Big|\sum_{j=0}^{n-1} \omega_j^k A_j\Big|^2\Big)\Big\| \le \Big\|f\Big(\Big(\sum_{j=0}^{n-1} |A_j|^2\Big)^{1/2}\Big)\Big\|$$

$$\le \frac{1}{n}\Big\|\!\!\sum_{k=0}^{n-1} f\Big(\Big|\sum_{j=0}^{n-1} \omega_j^k A_j\Big|^2\Big)\Big\|, \tag{1.6}$$

for any convex $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$ and any unitarily invariant norm.

The aim of this article is to generalize the preceding three inequalities to the framework of compact Abelian groups as stated in the Abstract. The most technical part of the article is Theorem 3.1, which establishes the Parseval identity for operator-valued abstract Fourier series. Otherwise, we mainly follow the argument from [12, Section 3].

It is worth mentioning that the other approach from [3] can be further extended, as was done in [15] and [9], to obtain more general results. It seems that the results in the present article do not imply the results of [15] and [9].

This article is organized as follows. In Section 2, we quote known results concerning abstract harmonic analysis on compact Abelian groups, unitarily invariant norms, and Bochner integrals. We also derive some minor auxiliary statements. Section 3 is devoted to the main results. In Section 4, we obtain a number of corollaries by varying the group G. For instance, choosing $G = \mathbf{Z}_n$ we obtain (1.4), (1.5) and (1.6), choosing $G = \mathbf{Z}_2$ we obtain classic CMC inequalities, choosing $G = \mathbf{Z}_2^n$ we obtain a generalization of some results from [14] and [11], whereas for other choices of G we get completely new results. Finally, in the last section we list problems that naturally arise from this work.

2. Preliminaries

Compact Abelian groups. Let us recall some basic facts regarding abstract harmonic analysis on compact Abelian groups. (For more details, the reader is referred to [6] or [8].)

For any locally compact Abelian topological group there is a left (and also right) invariant regular Borel measure μ which is unique up to multiplication by a positive scalar. This measure is known as the *Haar measure*. If, moreover, G is compact, then μ is finite and usually normalized such that $\mu(G) = 1$.

Haar measures exist for non-Abelian locally compact groups as well. In this case, it is only left invariant. However, the further theory cannot be applied to non-Abelian groups. In what follows, G will always be Abelian.

A character on G is a continuous homomorphism $k : G \to \mathbf{T} = \{z \in \mathbf{C} \mid |z| = 1\}$. It is well known that the set of all characters on G equipped with open-compact topology, denoted by $\hat{G}$, is also a topological group. The group $\hat{G}$ is called the *Pontryagin dual* of G. The topology on $\hat{G}$ is discrete if and only if G is compact.

Throughout this article, G will always denote the *compact* Abelian group. The elements of G will be denoted by small Greek letters θ, φ, and so on. Since its Pontryagin dual $\hat{G}$ is a discrete group, its elements will be denoted by k, j, m, n, and so on, and integration with the corresponding Haar measure will be denoted by the $\sum$ symbol.

For a function $f \in L^1(G)$, the abstract Fourier coefficient of f is given by

$$\hat{f} : \hat{G} \to \mathbf{C}, \qquad \hat{f}(k) := \int_G \overline{k(\theta)} f(\theta) \, \mathrm{d}\mu(\theta). \tag{2.1}$$

The abstract Fourier series of f is

$$\sum_{k \in \hat{G}} \hat{f}(k) k(\theta), \tag{2.2}$$

whereas for finite $\Delta \subseteq \hat{G}$, the sum

$$S_\Delta(f) = \sum_{k \in \Delta} \hat{f}(k) k(\theta)$$

is called the *partial sum* of the expansion (2.2).

We will use the notation $\Delta \to \hat{G}$ for the usual summation over arbitrary families. Namely, $S_\Delta(f) \to f$ means, for instance, that for each $\varepsilon > 0$ there is a finite $\Delta_\varepsilon \subseteq \hat{G}$ such that $d(S_\Delta(f), f) < \varepsilon$ for all $\Delta \supseteq \Delta_\varepsilon$. Any convergence that appears in this note is a convergence within a certain metric space. Hence, we will always be able to choose a sequence $\Delta_n \subseteq \Delta_{n+1}$ such that $S_{\Delta_n} f \to f$.

We will use the following facts.

Proposition 2.1. *Let $f \in L^2(G)$. Then*

(1) *For k_1, $k_2 \in \hat{G}$, we have $\int_G k_1(\theta)\overline{k_2(\theta)} \, d\mu(\theta) = 0$ if $k_1 \neq k_2$. If $k_1 = k_2$, then this integral is equal to 1.*

(2) *The Fourier series converges to f in L^2-norm as $\Delta \to \hat{G}$; that is, for any $\varepsilon > 0$, there is a finite $\Delta_\varepsilon \subseteq \hat{G}$ such that $\|S_\Delta(f) - f\|_{L^2(G)} < \varepsilon$ whenever $\Delta \supseteq \Delta_\varepsilon$.*

(3) *Parseval's identity $\|\hat{f}\|_{L^2(\hat{G})} = \|f\|_{L^2(G)}$ holds, that is,*

$$\sum_{k \in \hat{G}} |\hat{f}(k)|^2 = \sum_{k \in \hat{G}} \left| \int_G \overline{k(\theta)} f(\theta) \, d\mu(\theta) \right|^2 = \int_G |f(\theta)|^2 \, d\theta. \tag{2.3}$$

(Note that $L^2(G) \subseteq L^1(G)$, due to $\mu(G) < +\infty$.)

Proof. Part (1) is proved in [8, Proposition 4.4]. Parts (2) and (3) are proved in [8, Corollary 4.7] (see also [6, Sections 2.7.2, 2.7.3, 2.9.1]). □

Unitarily invariant norms. Let $B(H)$ be the space of all bounded linear operators on a separable complex Hilbert space H. The absolute value of an operator $A \in B(H)$ is defined by $|A| = (A^*A)^{1/2}$, and the singular values of A, denoted by $s_j(A)$, are defined as eigenvalues of $|A|$ arranged in nonincreasing order counting multiplicity, that is, $s_j(A) = \lambda_j(|A|)$.

A *unitarily invariant norm*, denoted by $\|\!|\cdot|\!\|$, is a norm defined on a norm ideal $J_{\|\!|\cdot|\!\|}$ in $B(H)$ satisfying the property that $\|\!|UAV|\!\| = \|\!|A|\!\|$ for all operators $A \in J_{\|\!|\cdot|\!\|}$ and all unitary operators U, $V \in B(H)$. We also assume that $\|\!|\cdot|\!\|$ is normalized, that is, $\|\!|A|\!\| = \|A\|$ for all rank 1 operators ($\|\cdot\|$ stands for *usual operator norm*). We will abbreviate $J_{\|\!|\cdot|\!\|}$ to J when there is no risk of ambiguity. Each unitarily invariant norm $\|\!|\cdot|\!\|$ is a symmetric gauge function of the singular values, and $J_{\|\!|\cdot|\!\|}$ is a Banach space contained in the ideal of compact operators. The only exception are norms equivalent to the usual operator norm which are defined on the whole $B(H)$.

Among all unitarily invariant norms, the most examined are Schatten norms

$$\|A\|_p = \left(\operatorname{tr} |A|^p \right)^{1/p} = \left(\sum_{j=1}^{+\infty} s_j^p(A) \right)^{1/p}, \tag{2.4}$$

where tr is the usual trace functional and $1 \leq p < +\infty$. The corresponding ideals will be denoted by $\mathcal{C}_p$. We retain the definition of $\mathcal{C}_p$ for $0 < p < 1$, though, for $p < 1$, $\|\cdot\|_p$ defined by (2.4) is not a norm but a quasinorm. Some results in this article are valid for $p < 1$, for example, Theorem 3.4 and its corollaries.

The other examples are Ky Fan norms

$$\|A\|_{(n)} = \sum_{j=1}^{n} s_j(A).$$

The importance of the latter is contained in part (2) of the following proposition where other basic properties of unitarily invariant norms are listed.

Proposition 2.2. *We have the following.*

(1) *For any unitarily invariant norm $\|\cdot\|$ we have $\|A\| \leq \|A\| \leq \|A\|_1$, where $\|\cdot\|$ is the usual operator norm and $\|\cdot\|_1$ is the Schatten 1-norm.*

(2) *The inequality $\|A\| \leq \|B\|$ holds for all unitarily invariant norms if and only if $\|A\|_{(n)} \leq \|B\|_{(n)}$ for all n. This is known as the* Ky Fan dominance *property. The inequality $\|A\| \leq \|B\|$ for all unitarily invariant norms should be understood as follows: If $B \in J_{\|\cdot\|}$, then $A \in J_{\|\cdot\|}$ and the inequality holds.*

(3) *The Ky Fan norms can be computed as*

$$\|A\|_{(n)} = \sum_{j=1}^{n} s_j(A) = \max \sum_{j=1}^{n} |\langle A\varphi_j, \psi_j \rangle|,$$

where $\max$ is taken over all orthonormal systems φ_j and all orthonormal systems ψ_j. The maximum is attained if $|A|\varphi_j = s_j(A)\varphi_j$ and $\psi_j = U^*\varphi_j$, where $A = U|A|$ is the polar decomposition of A.

(4) *If $0 \leq A \leq B$, then $\sum_{j=1}^{n} s_j(A) \leq \sum_{j=1}^{n} s_j(B)$ for all n, and therefore $\|A\| \leq \|B\|$ for all unitarily invariant norms.*

Proof. The proofs of parts (1), (2), and (3) can be found in [10, Chapter III, Section 3], [10, Chapter III, Section 4], and [10, Chapter III, Section 3], respectively. Finally, (4) is an immediate consequence of (3). □

Proposition 2.3. *Let A_n be an increasing sequence of positive operators from $J_{\|\cdot\|}$, and let $\|A_n\| \leq C$. Then A_n weakly converges to some $A \in J_{\|\cdot\|}$ and $\|A_n\| \to \|A\|$.*

Proof. For each ξ we have $\langle A_n\xi, \xi \rangle \leq C\|\xi\|^2$. Therefore, the sequence $\langle A_n\xi, \xi \rangle$ is increasing and bounded and hence convergent. By the polarization identity, the sequence $\langle A_n\xi, \eta \rangle$ is convergent for each ξ, η. Thus, A_n weakly converges to some A.

By Proposition 2.2(4), the sequence $\|A_n\|$ is nondecreasing. Hence, the limit can be replaced by a supremum. By the same argument, $\sup_n \|A_n\| \leq \|A\|$. The opposite inequality follows from the lower semicontinuity of $\|\cdot\|$, that is, $\|A\| \leq \liminf \|A_n\|$ (see [20, Theorem 2.7]; see also [10, Theorem III.5.1.] and [20, Theorem 2.16]). □

We will deal with convex functions $\varphi : [0, +\infty) \to [0, +\infty)$, that is, those functions which satisfy $\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$ for $t \in [0, 1]$. Note that these functions must be nondecreasing. Although such a function is never operator monotone (i.e., $A \leq B$ does not imply $\varphi(A) \leq \varphi(B)$) and not necessarily

operator convex (i.e., $\varphi(\lambda A + (1 - \lambda)B) \leq \lambda \varphi(A) + (1 - \lambda)\varphi(B)$ need not be true in general), many scalar-valued inequalities can be extended to unitarily invariant norms.

Lemma 2.4. *Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a convex function with $\varphi(0) = 0$.*

(1) *If $\|\!|A|\!\| \leq \|\!|B|\!\|$ in any unitarily invariant norm, then also $\|\!|\varphi(A)|\!\| \leq \|\!|\varphi(B)|\!\|$ in any unitarily invariant norm. In particular, the conclusion follows for positive A and B such that $A \leq B$.*

(2) *If A, B are any two positive operators and $0 \leq \lambda \leq 1$, then*

$$\left\|\!\left|\varphi\big(\lambda A + (1 - \lambda)B\big)\right|\!\right\| \leq \left\|\!\left|\lambda\varphi(A) + (1 - \lambda)\varphi(B)\right|\!\right\|$$

in any unitarily invariant norm.

(3) *If A_n is a sequence of positive operators, then*

$$\left\|\!\left|\sum_{n=1}^{+\infty} \varphi(A_n)\right|\!\right\| \leq \left\|\!\left|\varphi\Big(\sum_{n=1}^{+\infty} A_n\Big)\right|\!\right\|$$

in any unitarily invariant norm. The preceding inequality is reversed if $\varphi : [0, +\infty) \to [0, +\infty)$ is a concave function with $\varphi(0) = 0$, $\varphi(+\infty) = +\infty$.

Proof. (1) Let $s_j(A)$ and $s_j(B)$ be the singular values of A and B, respectively. We have from Proposition 2.2(2) for all Ky Fan norms $\|A\|_{(n)} \leq \|B\|_{(n)}$, that is,

$$\sum_{j=1}^{n} s_j(A) \leq \sum_{j=1}^{n} s_j(B).$$

Since φ is convex and nondecreasing, by the elementary Karamata inequality (see [13, p. 148]) we have

$$\sum_{j=1}^{n} \varphi\big(s_j(A)\big) \leq \sum_{j=1}^{n} \varphi\big(s_j(B)\big), \qquad \text{that is,} \qquad \left\|\varphi(A)\right\|_{(n)} \leq \left\|\varphi(B)\right\|_{(n)}.$$

Now the result follows from the Ky Fan dominance property (see Proposition 2.2(2)).

(2) This is [2, Theorem 2.3].

(3) For finite sums and convex φ, this was proved in [16, Corollary 3.6] (see also [21, Theorem 4.4], [1, Section 6.1]). Let us prove this for infinite sums. For any $n \in \mathbf{N}$ we have $A_1 + \cdots + A_n \leq \sum_{n=1}^{+\infty} A_n$, which, by Lemma 2.4(1) implies that

$$\left\|\!\left|\varphi(A_1 + \cdots + A_n)\right|\!\right\| \leq \left\|\!\left|\varphi\Big(\sum_{n=1}^{+\infty} A_n\Big)\right|\!\right\|.$$

Hence, by the finite case,

$$\left\|\!\left|\sum_{k=1}^{n} \varphi(A_k)\right|\!\right\| \leq \left\|\!\left|\varphi\Big(\sum_{k=1}^{n} A_k\Big)\right|\!\right\| \leq \left\|\!\left|\varphi\Big(\sum_{n=1}^{+\infty} A_n\Big)\right|\!\right\|.$$

The result for convex φ follows by taking a limit $n \to +\infty$ according to Proposition 2.3.

Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a concave function with $\varphi(0) = 0$, $\varphi(+\infty) = +\infty$. Then it has the inverse function $\varphi^{-1} : [0, +\infty) \to [0, +\infty)$ which is convex, with $\varphi^{-1}(0) = 0$. Let $B_n = \varphi(A_n)$. By the result for convex functions, we have

$$\left\|\varphi^{-1}\Big(\sum_{n=1}^{+\infty} B_n\Big)\right\| \leq \left\|\sum_{k=1}^{n} \varphi^{-1}(B_n)\right\|, \qquad \text{that is,}$$

$$\left\|\varphi^{-1}\Big(\sum_{n=1}^{+\infty} \varphi(A_n)\Big)\right\| \leq \left\|\sum_{k=1}^{n} A_n\right\|.$$

Apply Lemma 2.4(1) to the previous inequality and φ to obtain the conclusion. $\qquad \square$

Bochner spaces. Let (Ω, μ) be a measurable space, and let X be a Banach space. The Bochner space $L^p(\Omega; X)$ is defined as the set of strongly measurable functions $f : \Omega \to X$ such that

$$\|f\|_{L^p(\Omega;X)} := \Big(\int_\Omega \|f(t)\|_X^p \, \mathrm{d}\mu(t)\Big)^{1/p} < +\infty,$$

after identification of μ-almost everywhere equal functions. Here, strong measurability is equivalent to weak measurability (i.e., the measurability of scalar functions $t \mapsto \Lambda(f(t))$ for all $\Lambda \in X^*$) and separability of the image of f. The Bochner integral is linear and additive with respect to disjoint unions. Also,

$$T \int_\Omega f(t) \, \mathrm{d}\mu(t) = \int_\Omega T f(t) \, \mathrm{d}\mu(t) \tag{2.5}$$

holds for all $f \in L^p(\Omega; X)$ and all bounded linear $T : X \to Y$.

Jensen's inequality for unitarily invariant norms, Proposition 2.4(2), can be extended to Bochner integrals using the same argument as in [2, Theorem 2.3].

Proposition 2.5. *Let $\|\cdot\|$ be some unitarily invariant norm, let J be the corresponding ideal, and let $L^1(\Omega; J)$ be the Bochner space, where $(\Omega; \mu)$ is a measurable space such that $\mu(\Omega) = 1$. For all $A : \Omega \to J$, $A \in L^1(\Omega; J)$ such that $A(t) \geq 0$ for almost all $t \in \Omega$ (i.e., $A(t)$ is a positive operator), and all convex functions $\varphi : [0, +\infty) \to [0, +\infty)$, the following inequality holds:*

$$\left\|\varphi\Big(\int_\Omega A(t) \, \mathrm{d}\mu(t)\Big)\right\| \leq \left\|\int_\Omega \varphi\big(A(t)\big) \, \mathrm{d}\mu(t)\right\|. \tag{2.6}$$

If $\varphi : [0, +\infty) \to [0, +\infty)$ is concave with $\varphi(0) = 0$, $\varphi(+\infty) = +\infty$, then the inequality (2.6) is reversed.

Proof. Denote $X = \int_\Omega A(t) \, \mathrm{d}\mu(t)$, and let s_n be the eigenvalues of X arranged in nonincreasing order counting possible multiplicities, and let ξ_n be the corresponding unit eigenvectors. Then $\varphi(s_k)$ are eigenvalues of $\varphi(X)$ with respect to the same eigenvectors and

$$\varphi(s_k) = \varphi\big(\langle X\xi_k, \xi_k\rangle\big)$$
$$= \varphi\Big(\int_\Omega \langle A(t)\xi_k, \xi_k\rangle \, \mathrm{d}\mu(t)\Big) \leq \int_\Omega \varphi\big(\langle A(t)\xi_k, \xi_k\rangle\big) \, \mathrm{d}\mu(t)$$

by the scalar version of Jensen's inequality. The convexity of φ also implies that $\varphi(\langle A\xi, \xi\rangle) \leq \langle \varphi(A)\xi, \xi\rangle$ for any positive A and any unit vector ξ. Indeed, if E_A denotes the spectral measure for A, then $\langle E_A\xi, \xi\rangle$ is a probabilistic (scalar) measure, and hence

$$\varphi(\langle A\xi, \xi\rangle) = \varphi\left(\int_0^{+\infty} \lambda \, d(\langle E_A(\lambda)\xi, \xi\rangle)\right)$$
$$\leq \int_0^{+\infty} \varphi(\lambda) \, d(\langle E_A(\lambda)\xi, \xi\rangle) = \langle \varphi(A)\xi, \xi\rangle.$$

Therefore,

$$\sum_{k=1}^n \varphi(s_k) \leq \sum_{k=1}^n \int_\Omega \langle \varphi(A(t))\xi_k, \xi_k\rangle \, d\mu(t) = \sum_{k=1}^n \left\langle \int_\Omega \varphi(A(t)) \, d\mu(t)\xi_k, \xi_k \right\rangle.$$

By Proposition 2.2(3), we obtain

$$\sum_{k=1}^n \varphi(s_k) \leq \sum_{k=1}^n s_k\left(\int_\Omega \varphi(A(t)) \, d\mu(t)\right),$$

that is,

$$\left\|\varphi\left(\int_\Omega A(t) \, d\mu(t)\right)\right\|_{(n)} \leq \left\|\int_\Omega \varphi(A(t)) \, d\mu(t)\right\|_{(n)},$$

for all n. The result follows from the Ky Fan dominance property (Proposition 2.2(2)).

If φ is concave, then φ^{-1} is convex, and applying the previously proved inequality to φ^{-1} and $\varphi(A(t))$ we obtain

$$\left\|\varphi^{-1}\left(\int_\Omega \varphi(A(t)) \, d\mu(t)\right)\right\| \leq \left\|\int_\Omega \varphi^{-1}(\varphi(A(t))) \, d\mu(t)\right\| = \left\|\int_\Omega A(t) \, d\mu(t)\right\|.$$

Apply Proposition 2.4(1) to obtain the conclusion. $\qquad\square$

3. Main results

First, we establish Parseval's identity for functions in Bochner spaces, which is the key technical tool in this article.

Theorem 3.1. *Let G be a compact Abelian group, let $\hat{G}$ be its Pontryagin dual, and let $\|\cdot\|$ be a unitarily invariant norm on an ideal J. For $A = \{A_\theta\} \in L^2(G; J)$ and $k \in \hat{G}$, the operators*

$$B_k = \int_G \overline{k(\theta)}A_\theta \, d\mu(\theta) \tag{3.1}$$

are well defined, and also

$$\sum_{k \in \hat{G}} |B_k|^2 = \int_G |A_\theta|^2 \, d\mu(\theta), \tag{3.2}$$

where the series on the left-hand side converges strongly.

Proof. Since $\mu(G) < +\infty$, we have $L^2(G; J) \subseteq L^1(G; J)$. Therefore, as $|k(\theta)| = 1$ we have

$$\int_G \left\| \overline{k(\theta)} A_\theta \right\| \, \mathrm{d}\mu(\theta) = \int_G \|A_\theta\| \, \mathrm{d}\mu(\theta) < +\infty,$$

and the B_k's are well defined.

Next, for $\xi \in H$ and $X_\theta = A_\theta - \sum_{k \in \Delta} B_k k(\theta)$ we have

$$\left\langle \int_G X_\theta^* X_\theta \, \mathrm{d}\mu(\theta) \xi, \xi \right\rangle = \int_G \langle X_\theta \xi, X_\theta \xi \rangle \, \mathrm{d}\mu(\theta) \geq 0,$$

that is, $\int_G X_\theta^* X_\theta \, \mathrm{d}\mu(\theta) \geq 0$. (Here we apply (2.5) to $T : J \to \mathbf{C}$, $T(X) = \langle X\xi, \xi \rangle$.) Hence

$$\int_G \left(A_\theta - \sum_{k \in \Delta} B_k k(\theta) \right)^* \left(A_\theta - \sum_{k \in \Delta} B_k k(\theta) \right) \mathrm{d}\mu(\theta) \geq 0.$$

By expanding the left-hand side (which is correct, since Δ is finite) and taking into account Proposition 2.1(1) as well as (3.1), we obtain

$$\sum_{k \in \Delta} B_k^* B_k \leq \int_G A_\theta^* A_\theta \, \mathrm{d}\mu(\theta). \tag{3.3}$$

(Alternatively, we can invoke the Bessel inequality for Hilbert C^*-modules to get (3.3).) In particular,

$$\sup_{\substack{\Delta \subseteq \hat{G} \\ \Delta \text{ finite}}} \sum_{k \in \Delta} \langle B_k^* B_k \xi, \xi \rangle < +\infty. \tag{3.4}$$

We establish the first conclusion, that the series in (3.2) converges weakly, and even more strongly, due to its monotonicity.

Next, let us compute the difference between $\int |A_\theta|^2$ and the partial sum of $\sum_{k \in \hat{G}} |B_k|^2$. For finite $\Delta \subseteq \hat{G}$, let $(S_\Delta A)_\theta = \sum_{k \in \Delta} B_k k(\theta)$ be the partial sum of the abstract Fourier series of A_θ. Then we have

$$\sum_{k \in \Delta} B_k^* B_k = \sum_{k \in \Delta} \int_G k(\theta) A_\theta^* B_k \, \mathrm{d}\mu(\theta) = \int_G A_\theta^* (S_\Delta A)_\theta \, \mathrm{d}\mu(\theta), \tag{3.5}$$

once again invoking (2.5), and hence

$$\int_G |A_\theta|^2 \, \mathrm{d}\mu(\theta) - \sum_{k \in \Delta} |B_k|^2 = \int_G A_\theta^* \big(A_\theta - (S_\Delta A)_\theta \big) \, \mathrm{d}\mu(\theta).$$

Choose unit vectors $\xi, \eta \in H$ to get

$$\left\langle \left(\int_G |A_\theta|^2 \, \mathrm{d}\mu(\theta) - \sum_{k \in \Delta} |B_k|^2 \right) \xi, \eta \right\rangle = \int_G \big\langle \big(A_\theta - (S_\Delta A)_\theta \big) \xi, A_\theta \eta \big\rangle \, \mathrm{d}\mu(\theta). \tag{3.6}$$

We will prove that the right-hand side of (3.6) tends to zero for a suitable sequence of finite $\Delta_n \subseteq \hat{G}$.

Set $f_{A,\xi,\eta}(\theta) = \langle A(\theta)\xi, \eta \rangle$. Then we have

$$\langle (S_\Delta A)_\theta \xi, \eta \rangle = \sum_{k \in \Delta} \langle B_k \xi, \eta \rangle k(\theta)$$

$$= \sum_{k \in \Delta} \left(\int_G \overline{k(\varphi)} f_{A,\xi,\eta}(\varphi) \, d\mu(\varphi) \right)$$

$$= \sum_{k \in \Delta} \hat{f}_{A,\xi,\eta}(k) k(\theta).$$

As $|\langle A_\theta \xi, \eta \rangle| \le \|A_\theta\| \, \|\xi\| \, \|\eta\| \le \|A_\theta\| \, \|\xi\| \, \|\eta\|$, the function $f_{A,\xi,\eta}$ belongs to $L^2(G)$. Therefore, by Proposition 2.1((2)), the functions $\theta \mapsto f_{A,\xi,\eta}^\Delta(\theta) = \sum_{k \in \Delta} \hat{f}_{A,\xi,\eta}(k) \times k(\theta)$ converge to $f_{A,\xi,\eta}$ in L^2-norm when Δ increases. Moreover, we can choose a sequence of finite sets $\Delta_n \subseteq \Delta_{n+1}$ such that

$$\langle (S_{\Delta_n} A)_\theta \xi, \eta \rangle \to \langle A_\theta \xi, \eta \rangle \tag{3.7}$$

for almost all $\theta \in G$. To show that we can pass the limit on the right-hand side of (3.6) to the integrand, we show that the family of functions $\theta \mapsto \langle (A_\theta - (S_\Delta A)_\theta)\xi, A_\theta \eta \rangle$ is uniformly integrable.

Let $\varepsilon > 0$ be arbitrary. By (3.4) there is a finite set $\Delta_\varepsilon \subseteq \hat{G}$ such that

$$\sum_{k \notin \Delta_\varepsilon} \langle B_k^* B_k \xi, \xi \rangle < \frac{\varepsilon^2}{4M}, \tag{3.8}$$

where

$$M = \int_G \|A_\theta\|^2 \, d\mu(\theta) \le \int_G \|A_\theta\|^2 \, d\mu(\theta) < +\infty.$$

Let p be the cardinality of Δ_ε. The function $\theta \mapsto \|A_\theta\|^2 + pM^{1/2}\|A_\theta\|$ is integrable. Therefore, there is $\delta > 0$ such that

$$\int_E \left(\|A_\theta\|^2 + pM^{1/2}\|A_\theta\| \right) d\mu(\theta) < \frac{\varepsilon}{2}, \tag{3.9}$$

for all $E \subseteq G$ such that $\mu(E) < \delta$.

For Δ_ε and E_δ for which (3.9) holds, we have that for each Δ

$$\left| \int_{E_\delta} \langle (A_\theta - (S_\Delta A)_\theta)\xi, A_\theta \eta \rangle \, d\mu(\theta) \right|$$

$$\le \int_E \left| \left\langle \left(A_\theta - \sum_{k \in \Delta} B_k k(\theta) \right)\xi, A_\theta \eta \right\rangle \right| d\mu(\theta)$$

$$\le \int_{E_\delta} \left| \left\langle \left(A_\theta - \sum_{k \in \Delta \cap \Delta_0} B_k k(\theta) \right)\xi, A_\theta \eta \right\rangle \right| d\mu(\theta)$$

$$+ \int_{E_\delta} \left| \left\langle \sum_{k \in \Delta \backslash \Delta_0} B_k k(\theta)\xi, A_\theta \eta \right\rangle \right| d\mu(\theta)$$

$$= S_1 + S_2.$$

Since
$$\|B_k\| \le \int_G \|A_\theta \overline{k(\theta)}\|\, \mathrm{d}\mu(\theta) \le \Big(\int_G \|A_\theta\|^2\, \mathrm{d}\mu(\theta)\Big)^{1/2} = M^{1/2},$$
we can estimate the first summand S_1 as
$$S_1 \le \int_{E_\delta} \Big(\big|\langle A_\theta\xi, A_\theta\eta\rangle\big| + \sum_{k\in\Delta\cap\Delta_\varepsilon} \big|\langle B_k k(\theta)\xi, A_\theta\eta\rangle\big| \Big)\, \mathrm{d}\mu(\theta)$$
$$\le \int_{E_\delta} \Big(\|A_\theta\xi\|\|A_\theta\eta\| + \sum_{k\in\Delta_\varepsilon} \|B_k\xi\|\|A_\theta\eta\| \Big)\, \mathrm{d}\mu(\theta)$$
$$\le \int_{E_\delta} \big(\|A_\theta\|^2 + pM^{1/2}\|A_\theta\| \big)\, \mathrm{d}\mu(\theta) < \frac{\varepsilon}{2}$$
for unit vectors $\xi,\ \eta \in H$ by (3.9)

Let us estimate the second summand S_2. We have
$$S_2 \le \int_{E_\delta} \Big| \Big\langle \sum_{k\in\Delta\setminus\Delta_\varepsilon} B_k k(\theta)\xi, A_\theta\eta \Big\rangle \Big|\, \mathrm{d}\mu(\theta)$$
$$\le \int_G \Big\| \sum_{k\in\Delta\setminus\Delta_\varepsilon} B_k k(\theta)\xi \Big\| \|A_\theta\|\, \mathrm{d}\mu(\theta)$$
$$\le \Big(\int_G \Big\| \sum_{k\in\Delta\setminus\Delta_\varepsilon} B_k k(\theta)\xi \Big\|^2\, \mathrm{d}\mu(\theta) \Big)^{1/2} \Big(\int_G \|A_\theta\|^2\, \mathrm{d}\mu(\theta) \Big)^{1/2}$$
$$\le M^{1/2} \Big(\int_G \Big\langle \sum_{k\in\Delta\setminus\Delta_\varepsilon} B_k k(\theta)\xi, \sum_{j\in\Delta\setminus\Delta_\varepsilon} B_j j(\theta)\xi \Big\rangle\, \mathrm{d}\mu(\theta) \Big)^{1/2}$$
$$= M^{1/2} \Big(\int_G \sum_{k,j\in\Delta\setminus\Delta_\varepsilon} \langle B_j^* B_k\xi, \xi\rangle k(\theta)\overline{j(\theta)}\, \mathrm{d}\mu(\theta) \Big)^{1/2}$$
$$\le M^{1/2} \Big(\sum_{k\notin\Delta_\varepsilon} \langle B_k^* B_k\xi, \xi\rangle \Big)^{1/2} < M^{1/2} \Big(\frac{\varepsilon^2}{4M} \Big)^{1/2} = \frac{\varepsilon}{2},$$
where we use Proposition 2.1(1) and (3.8).

Thus
$$\Big| \int_E \big\langle (A_\theta - (S_\Delta A)_\theta)\xi, A_\theta\eta \big\rangle\, \mathrm{d}\mu(\theta) \Big| < \varepsilon,$$
whenever $\mu(E) < \delta$, where δ does not depend on Δ. This ensures that $\theta \mapsto \langle (A_\theta - (S_\Delta A)_\theta)\xi, A_\theta\eta\rangle$ is uniformly integrable, and by Vitali's convergence theorem, we can pass the limit as $\Delta \to \hat{G}$ to the integrand on the right-hand side of (3.6). By (3.7) we obtain (3.2), where the series converges in the weak operator topology. The entries of the sum are positive, hence partial sums are increasing. Therefore, the convergence is, moreover, strong. □

Remark 3.2. We have $L^2(G; J) \subseteq L^2(G; B(H))$. A careful reading of the proof shows that we use only the following properties of $B(H)$: (i) it is a C^* algebra, (ii) it is closed under weak and strong limits, and (iii) it has a unit. Therefore, equality (3.2) holds for $A_\theta \in L^2(G; \mathcal{A})$, where $\mathcal{A}$ is an arbitrary W^*-algebra.

Also, $L^2(G; \mathcal{A})$ can be regarded as a Hilbert W^*-module with right multiplication $A_\theta \cdot X = A_\theta X$ and the $\mathcal{A}$-valued inner product $\langle A_\theta, B_\theta \rangle = \int_G A_\theta^* B_\theta \, d\mu(\theta)$. The idea of proving (3.2) by showing that $\{k(\theta) \cdot I \mid k \in \hat{G}\}^\perp = \{0\}$, where I is the unit of $\mathcal{A}$, is misleading. Namely, there are examples of subspaces of some Hilbert modules with trivial orthogonal complement and that are not dense.

Using the operator-valued Parseval's identity, we are able to derive continuous counterparts of CMC inequalities.

Theorem 3.3. *Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a convex function such that $\varphi(0) = 0$, and let A_θ, $\theta \in G$ be such that the functions $\theta \mapsto A_\theta$ and $\theta \mapsto \varphi(|A_\theta|^2)$ belong to $L^1(G; J)$, $J = J_{\|\cdot\|}$ arbitrary. Then it holds that*

$$\left\| \sum_{k \in \hat{G}} \varphi\left(\left| \int_G \overline{k(\theta)} A_\theta \, d\theta \right|^2 \right) \right\| \leq \left\| \int_G \varphi(|A_\theta|^2) \, d\theta \right\|. \qquad (3.10)$$

If $\varphi : [0, +\infty) \to [0, +\infty)$ is a concave function, $\varphi(0) = 0$, $\varphi(+\infty) = +\infty$, then the inequality is reversed.

Proof. By (3.2) and (2.6), we have

$$\left\| \varphi\left(\sum_{k \in \hat{G}} \left| \int_G \overline{k(\theta)} A_\theta \, d\theta \right|^2 \right) \right\| = \left\| \varphi\left(\int_G |A_\theta|^2 \, d\theta \right) \right\| \leq \left\| \int_G \varphi(|A_\theta|^2) \, d\theta \right\|. \qquad (3.11)$$

Also by (3.2), the operator $\varphi(\sum_{k \in \hat{G}} |\int_G A_\theta \, d\theta|^2)$ belongs to J. By Proposition 2.4(3), we have

$$\left\| \sum_{k \in \hat{G}} \varphi\left(\left| \int_G \overline{k(\theta)} A_\theta \, d\theta \right|^2 \right) \right\| \leq \left\| \varphi\left(\sum_{k \in \hat{G}} \left| \int_G \overline{k(\theta)} A_\theta \, d\theta \right|^2 \right) \right\|. \qquad (3.12)$$

The conclusion (3.10) follows from (3.11) and (3.12). If φ is concave, note that the inequality in (3.11) is reversed due to Proposition 2.5, and in (3.12) by Proposition 2.4(3). $\qquad\square$

Theorem 3.4. *Let $1 \leq p < +\infty$, and let the function $\theta \mapsto A_\theta$ belong to $L^p(G; \mathcal{C}_p)$. Then for $p \geq 2$ we have*

$$\sum_{k \in \hat{G}} \left\| \int_G \overline{k(\theta)} A_\theta \, d\theta \right\|_p^p \leq \int_G \|A_\theta\|_p^p \, d\theta, \qquad (3.13)$$

whereas for $0 < p \leq 2$ the inequality is reversed.

Proof. Put $\varphi(t) = t^{p/2}$, $p \geq 2$, which is a convex function, for $p \geq 2$ and $\|A\| = \|A\|_1 = \mathrm{tr}(|A|)$ in (3.10). We obtain

$$\mathrm{tr}\left(\sum_{k \in \hat{G}} \left| \int_G \overline{k(\theta)} A_\theta \, d\theta \right|^p \right) \leq \mathrm{tr}\left(\int_G |A_\theta|^p \, d\theta \right). \qquad (3.14)$$

The trace tr is a bounded (with respect to $\| \cdot \|_1$) linear functional. Hence, by (2.5) the right-hand side of (3.14) is equal to

$$\int_G \mathrm{tr}\, |A_\theta|^p \, d\theta = \int_G \|A_\theta\|_p^p \, d\theta.$$

The left-hand side of (3.14) can be replaced by the sum over a finite set $\Delta \subseteq \hat{G}$. Then by linearity of tr we get

$$\sum_{k \in \Delta} \left\| \int_G \overline{k(\theta)} A_\theta \, d\theta \right\|_p^p \leq \int_G \|A_\theta\|_p^p \, d\theta,$$

which leads to (3.13) by taking a supremum over all finite $\Delta \subseteq \hat{G}$.

For $0 < p \leq 2$, the function $\varphi(t) = t^{p/2}$ is concave, $\varphi(0) = 0$ and $\varphi(+\infty) = +\infty$, so by the same argument we obtain the reversed inequality. $\square$

Formula (3.13) is a generalization of the right inequality in (1.4), which we will prove in the next section. Concerning the left inequality in (1.4), it follows from the right inequality by the substitution $B_n = \sum_{j=0}^{n-1} \omega_j^k A_j$. This is possible due to the fact that $\mathbf{Z}_n$ is self-dual in the sense of Pontryagin. Nothing similar can be said for the general compact Abelian group G. It need not be isomorphic to its dual group $\hat{G}$ in general.

Nevertheless, a partial substitution for the left inequality in (1.4) might be the following.

Theorem 3.5. *Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a convex function $\varphi(0) = 0$, and let α_k, $k \in \hat{G}$ be a family of positive reals such that*

$$\sum_{k \in \hat{G}} \alpha_k = 1. \tag{3.15}$$

If $A_\theta \in L^1(G; \mathcal{C}_p)$ for some $p \geq 2$, then

$$\left\| \int_G |A_\theta| \, d\theta \right\|_p^p \leq \sum_{k \in \hat{G}} \alpha_k^{1-p/2} \left\| \int_G \overline{k(\theta)} A_\theta \, d\theta \right\|_p^p, \tag{3.16}$$

provided that the term on the right-hand side is finite.

Proof. For $\Delta \subseteq \hat{G}$, let $\nu(\Delta) = \sum_{k \in \Delta} \alpha_k$. Then ν is a measure with $\nu(\hat{G}) = 1$. By (2.6), we have

$$\left\| \varphi \Big(\sum_{k \in \hat{G}} |B_k|^2 \Big) \right\| = \left\| \varphi \Big(\sum_{k \in \hat{G}} \alpha_k \frac{1}{\alpha_k} |B_k|^2 \Big) \right\| \leq \left\| \sum_{k \in \hat{G}} \alpha_k \varphi \Big(\frac{1}{\alpha_k} |B_k|^2 \Big) \right\|,$$

where $B_k = \int_G \overline{k(\theta)} A_\theta \, d\theta$ for any unitarily invariant norm. It follows by (3.2) that

$$\left\| \varphi \Big(\int_G |A_\theta|^2 \, d\theta \Big) \right\| = \left\| \varphi \Big(\sum_{k \in \hat{G}} |B_k|^2 \Big) \right\| \leq \left\| \sum_{k \in \hat{G}} \alpha_k \varphi \Big(\frac{1}{\alpha_k} |B_k|^2 \Big) \right\|.$$

Once again, take $\varphi(t) = t^{p/2}$, $p \geq 2$, and $\|A\| = \|A\|_1 = \mathrm{tr}(|A|)$. We obtain

$$\mathrm{tr}\Big(\int_G |A_\theta|^2 \, d\theta \Big)^{p/2} \leq \mathrm{tr}\Big(\sum_{k \in \hat{G}} \alpha_k \Big(\frac{1}{\alpha_k} |B_k|^2 \Big)^{p/2} \Big) = \sum_{k \in \hat{G}} \alpha_k^{1-p/2} \|B_k\|_p^p. \tag{3.17}$$

Indeed, the right-hand sides of (3.16) and (3.17) coincide and hence they are finite. This ensures that the series in the middle term in (3.17) converges in $\|\cdot\|_1$-norm. Therefore the trace, as a bounded functional, commutes with the summation.

However, as $f(t) = t^2$ is convex, by (2.6), we have

$$\left\| \left(\int_G |A_\theta| \, \mathrm{d}\theta \right)^2 \right\|_1 = \left\| f\left(\int_G |A_\theta| \, \mathrm{d}\theta \right) \right\|_1 \le \left\| \int_G f(|A_\theta|) \, \mathrm{d}\theta \right\| = \left\| \int_G |A_\theta|^2 \, \mathrm{d}\theta \right\|_1.$$

Now, as $\varphi(t) = t^{p/2}$ is convex, it follows from Lemma 2.4(1) that

$$\mathrm{tr}\left(\int_G |A_\theta| \, \mathrm{d}\theta \right)^p = \left\| \left(\int_G |A_\theta| \, \mathrm{d}\theta \right)^p \right\|_1 = \left\| \varphi\left(\left(\int_G |A_\theta| \, \mathrm{d}\theta \right)^2 \right) \right\|_1$$

$$\le \left\| \varphi\left(\int_G |A_\theta|^2 \, \mathrm{d}\theta \right) \right\|_1 = \left\| \left(\int_G |A_\theta|^2 \, \mathrm{d}\theta \right)^{p/2} \right\|_1$$

$$= \mathrm{tr}\left(\int_G |A_\theta|^2 \, \mathrm{d}\theta \right)^{p/2}. \tag{3.18}$$

From (3.17) and (3.18), we get (3.16) $\qquad\qquad\qquad\qquad\qquad$ □

In the next two theorems, we prove the counterparts of Clarkson inequalities (1.2) and (1.3) using complex interpolation, which is a standard procedure. These results hold only for $p \ge 1$.

Theorem 3.6. *For all $1 \le p \le 2$ and $A_\theta \in L^p(G; \mathcal{C}_p)$, there holds*

$$\sum_{k \in \hat{G}} \left\| \int_G \overline{k(\theta)} A_\theta \, \mathrm{d}\mu(\theta) \right\|_p^q \le \left(\int_G \|A_\theta\|_p^p \right)^{q/p}, \tag{3.19}$$

where q is conjugate to p, that is, $q = p/(p-1)$.

Proof. The proof can be obtained using complex interpolation as it was done in [7] and later repeated in [3]. Therefore, we only give an outline.

First, prove the inequality

$$\left| \mathrm{tr} \sum_{k \in \Delta} Y_k B_k \right| \le \left(\sum_{k \in \Delta} \|Y_k\|_q^p \right)^{1/p} \left(\int_G \|A_\theta\|_p^p \, \mathrm{d}\mu(\theta) \right)^{1/p}, \tag{3.20}$$

where $Y_k \in \mathcal{C}_q$, $B_k = \int_G \overline{k(\theta)} A_\theta \, \mathrm{d}\mu(\theta)$, and $\Delta \subseteq \hat{G}$ is finite. (We choose a finite subset of $\hat{G}$ to avoid complications with convergence until the end of the proof.)

Indeed, consider the function $f(z)$ defined for $1/2 \le \Re z \le 1$ by

$$f(z) = \mathrm{tr} \sum_{k \in \Delta} Y_k(z) B_k(z),$$

where

$$Y_k(z) = \|Y_k\|_q^{(p+q)z-q} V_k |Y_k|^{q-qz}, \qquad Y_k = V_k |Y_k|,$$

$$B_k(z) = \int_G \overline{k(\theta)} A_\theta(z) \, \mathrm{d}\mu(\theta), \qquad A_\theta(z) = |A_\theta|^{pz} W_\theta, \qquad A_\theta = |A_\theta| W_\theta.$$

Then estimate

$$\left| \mathrm{tr}\left(Y_k(1+it) A_\theta(1+it) \right) \right| \le \|Y_k\|_q^p \|A_\theta\|_p^p,$$

and hence

$$\left| f(1+it) \right| \le \left(\sum_{k \in \Delta} \|Y_k\|_q^p \right) \left(\int_G \|A_\theta\|_p^p \, \mathrm{d}\mu(\theta) \right). \tag{3.21}$$

Also,

$$\big|f(1/2 + it)\big| \leq \sum_{k \in \Delta} \big\|Y_k(1/2 + it)\big\|_2 \big\|B_k(1/2 + it)\big\|_2$$

$$\leq \Big(\sum_{k \in \Delta} \big\|Y_k(1/2 + it)\big\|_2^2\Big)^{1/2} \Big(\sum_{k \in \Delta} \big\|B_k(1/2 + it)\big\|_2^2\Big)^{1/2}.$$

However, by (3.13) with $p = 2$ we obtain

$$\sum_{k \in \Delta} \big\|B_k(1/2 + it)\big\|_2^2 \leq \int_G \big\|A_\theta(1/2 + it)\big\|_2^2 \, \mathrm{d}\mu(\theta) = \int_G \|A_\theta\|_p^p \, \mathrm{d}\mu(\theta),$$

which yields (note that $\|Y_k(1/2 + it)\|_2 = \|Y_k\|_q^{p/2}$) that

$$\big|f(1/2 + it)\big| \leq \Big(\sum_{k \in \Delta} \|Y_k\|_q^p\Big)^{1/2} \Big(\int_G \|A_\theta\|_p^p \, \mathrm{d}\mu(\theta)\Big)^{1/2}. \tag{3.22}$$

From (3.21) and (3.22) we get (3.20) using the three-lines theorem (see [10, Chapter III, Section 13] or [20, Theorem 2.9]), since the left-hand side of (3.20) is equal to $f(1/p)$. Once (3.20) is proved, set $Y_k = \|B_k\|_p^{q-p}|B_k|^{p-1}U_k^*$, where $B_k = U_k|B_k|$, and the conclusion follows by passing to the limit $\Delta \to \hat{G}$. $\qquad\square$

Theorem 3.7. *For all $p \geq 2$ and $A_\theta \in L^q(G; \mathcal{C}_p)$, there holds*

$$\sum_{k \in \hat{G}} \Big\|\int_G \overline{k(\theta)} A_\theta \, \mathrm{d}\mu(\theta)\Big\|_p^p \leq \Big(\int_G \|A_\theta\|_p^q \, \mathrm{d}\mu(\theta)\Big)^{p/q}, \tag{3.23}$$

where q is conjugate to p, that is, $q = p/(p-1)$.

Proof. The proof is very similar to that of Theorem 3.6. Therefore, we only highlight the differences. First, we prove the inequality

$$\Big|\mathrm{tr}\sum_{k \in \Delta} Y_k B_k\Big| \leq \Big(\sum_{k \in \Delta} \|Y_k\|_q^q\Big)^{1/q} \Big(\int_G \|A_\theta\|_p^q \, \mathrm{d}\mu(\theta)\Big)^{1/q}, \tag{3.24}$$

where $Y_k \in \mathcal{C}_q$, $B_k = \int_G \overline{k(\theta)} A_\theta \, \mathrm{d}\mu(\theta)$, and $\Delta \subseteq \hat{G}$ is finite.

Consider the function $f(z)$ (slightly different from that of Theorem 3.6) defined in the same strip $1/2 \leq \Re z \leq 1$ by

$$f(z) = \mathrm{tr}\sum_{k \in \Delta} Y_k(z) B_k(z),$$

where

$$Y_k(z) = V_k|Y_k|^{qz}, \qquad Y_k = V_k|Y_k|,$$

$$B_k(z) = \int_G \overline{k(\theta)} A_\theta(z) \, \mathrm{d}\mu(\theta),$$

$$A_\theta(z) = \|A_\theta\|_p^{(p+q)z-p}|A_\theta|^{p-pz}W_\theta, \qquad A_\theta = |A_\theta|W_\theta.$$

Then use the same estimates for $z = 1 + it$ and $z = 1/2 + it$, and finally use the three-lines theorem for $z = 1/q \in (1/2, 1)$. Once (3.24) is proved, set $Y_k =$

$|B_k|^{p-1}U_k^*$, where $B_k = U_k|B_k|$, and the conclusion follows by passing to the limit $\Delta \to \hat{G}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Corollaries

Varying the group G, from Theorems 3.3, 3.4, 3.5, 3.6, and 3.7, we obtain different earlier published results, as well as some new ones. For $G = \mathbf{Z}_n$ we get results from [3, Theorems 2, 4] and [12, Corollary 3.2 and following Remark].

Corollary 4.1. *Let* $n \in \mathbf{N}$, *let* $\omega_j = e^{2\pi i j/n}$, *and let* $A_j \in \mathcal{C}_p$, $j = 1, 2, \ldots, n$. *Then*

$$n \sum_{j=0}^{n-1} \|A_j\|_p^p \le \sum_{k=0}^{n-1}\Big\|\sum_{j=0}^{n-1} \omega_j^k A_j\Big\|_p^p \le n^{p-1} \sum_{j=0}^{n-1} \|A_j\|_p^p, \qquad (4.1)$$

for $p \ge 2$. *For* $0 < p \le 2$ *the inequalities are reversed. We have*

$$\sum_{k=0}^{n-1}\Big\|\sum_{j=0}^{n-1} \omega_j^k A_j\Big\|_p^q \le n\Big(\sum_{j=0}^{n-1} \|A_j\|_p^p\Big)^{q/p}, \qquad (4.2)$$

for $1 \le p \le 2$, $q = p/(p-1)$, *and*

$$\sum_{k=0}^{n-1}\Big\|\sum_{j=0}^{n-1} \omega_j^k A_j\Big\|_p^p \le n\Big(\sum_{j=0}^{n-1} \|A_j\|_p^q\Big)^{p/q}, \qquad (4.3)$$

for $p \ge 2$ *and* $q = p/(p-1)$.

Proof. Consider $G = \mathbf{Z}^n$. Its Haar measure is the counting measure divided by n, and the Pontryagin dual is also $\mathbf{Z}_n$. Indeed, since $\mathbf{Z}_n = \{1, a, \ldots, a^{n-1}\}$ for some generator a, any homomorphism $k : \mathbf{Z}_n \to \mathbf{T}$ is determined by $k(a)$. From $a^n = 1$ we deduce that $k(a)^n = 1$. Hence $k_j(a) = \omega_j$ for some $j = 0, 1, \ldots, n-1$. Then $k_j(a^l) = \omega_j^l$. Hence (3.13), (3.19), and (3.23) are reduced to

$$\sum_{j=0}^{n-1}\Big\|\frac{1}{n}\sum_{l=0}^{n-1}(\overline{\omega}_j)^l A_l\Big\|_p^p \le \frac{1}{n}\sum_{j=0}^{n-1}\|A_j\|_p^p,$$

$$\sum_{j=0}^{n-1}\Big\|\frac{1}{n}\sum_{l=0}^{n-1}(\overline{\omega}_j)^l A_l\Big\|_p^q \le \Big(\frac{1}{n}\sum_{j=0}^{n-1}\|A_j\|_p^p\Big)^{q/p},$$

$$\sum_{j=0}^{n-1}\Big\|\frac{1}{n}\sum_{l=0}^{n-1}(\overline{\omega}_j)^l A_l\Big\|_p^p \le \Big(\frac{1}{n}\sum_{j=0}^{n-1}\|A_j\|_p^q\Big)^{p/q}.$$

These inequalities are equivalent to the right inequalities in (4.1), (4.2), and (4.3). Indeed, complex conjugation is the automorphism of the group $\{1, \omega_j, \omega_j^2, \ldots, \omega_j^{n-1}\}$ and it only affects the corresponding sums by permutation of A_j.

The left inequality in (4.1) can be obtained either from the right inequality by substitutions $B_k = \sum_{j=0}^{n-1} \omega_j^k A_j$ or by Theorem 3.5, choosing $\alpha_k = 1/k$. $\qquad\square$

Remark 4.2. Inequalities (4.1) and (4.2) were proved in [3] and [12], as well as the inequality (1.6) which is the consequence of (3.10).

Remark 4.3. Note that all constants that appear in any CMC inequality become 1 if we normalize the Haar measure.

Remark 4.4. As a special case of the preceding Corollary, if $G = \mathbf{Z}_2$, then we obtain the original CMC inequalities (1.1), (1.2), and (1.3).

For $G = \mathbf{Z}_2^n$, we get the following result concerning Littlewood matrices. They are defined inductively as

$$L_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad L_{n+1} = \begin{bmatrix} L_n & L_n \\ L_n & -L_n \end{bmatrix}. \tag{4.4}$$

Corollary 4.5. *Let* $n \in \mathbf{N}$, *let* $A_j \in \mathcal{C}_p$ *for* $1 \le j \le 2^n$, *and let* ε_{ij} *be the entries of the Littlewood matrix* L_n. *Then*

$$\sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} A_j \right\|_p^p \le 2^{n(p-1)} \sum_{j=1}^{2^n} \|A_j\|_p^p, \quad p \ge 2, \tag{4.5}$$

$$\sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} A_j \right\|_p^q \le 2^n \left(\sum_{j=1}^{2^n} \|A_j\|_p^p \right)^{q/p}, \quad 1 \le p \le 2, q = \frac{p}{p-1}, \tag{4.6}$$

$$\sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} A_j \right\|_p^p \le 2^n \left(\sum_{j=1}^{2^n} \|A_j\|_p^q \right)^{p/q}, \quad p \ge 2, q = \frac{p}{p-1}. \tag{4.7}$$

Proof. Consider the group $\mathbf{Z}_2^n$. It has n generators, say, $a_1, a_2, \ldots, a_n$, all of them of order 2. Therefore,

$$\mathbf{Z}_2^n = \left\{ a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n} \mid (m_1, \ldots, m_n) \in \{0,1\}^n \right\}.$$

Any character $k : \mathbf{Z}_2^n \to \mathbf{T}$ is determined by $k(a_j)$, $1 \le j \le n$. It has to be $k(a_j) = \pm 1$, since $k(a_j)^2 = k(a_j^2) = k(1) = 1$. Therefore, there are 2^n distinct characters on $\mathbf{Z}_2^n$.

Let us show that the rows of the Littlewood matrix are exactly the images $k(a)$, $a \in \mathbf{Z}_2^n$, in the lexicographic order, namely, in the order

$$a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n} \le a_1^{m_1'} a_2^{m_2'} \ldots a_n^{m_n'} \qquad \text{iff}$$

$$m_1 < m_1' \qquad \text{or} \qquad m_1 = m_1' \wedge m_2 < m_2', \ldots.$$

For $n = 1$, $\mathbf{Z}_2^1 = \{a_1^0, a_1^1\}$ there are exactly two characters $k_1 \equiv 1$ and $k_2(a_1^0) = 1$, $k_2(a_1^1) = -1$. This corresponds to the rows of L_1 (see (4.4)).

Let the statement be true for some $n \in \mathbf{N}$. Then the first 2^n elements of $\mathbf{Z}_2^{n+1}$ are $a_1^0 a_2^{m_2} \ldots a_{n+1}^{m_{n+1}}$, whereas the other 2^n elements are $a_1^1 a_2^{m_2} \ldots a_{n+1}^{m_{n+1}}$. Divide characters on $\mathbf{Z}_2^{n+1}$ into two slots. Let the first consist of those k for which $k(a_1) = 1$, and let the second consist of those k for which $k(a_1) = -1$.

For $k \in \hat{\mathbf{Z}}_2^{n+1}$ in the first slot, there is a unique $k' \in \hat{\mathbf{Z}}_2^n$ such that

$$k(a_1^0 a_2^{m_2} \ldots a_{n+1}^{m_{n+1}}) = k(a_1^1 a_2^{m_2} \ldots a_{n+1}^{m_{n+1}}) = k'(a_2^{m_2} \ldots a_{n+1}^{m_{n+1}}).$$

If the ith row of L_n corresponds to k', then two copies of this row correspond to k, and these two copies make exactly the ith row of L_{n+1}.

For $k \in \hat{\mathbf{Z}}_2^{n+1}$ in the second slot, there is a unique $k' \in \hat{\mathbf{Z}}_2^n$ such that

$$k(a_1^0 a_2^{m_2} \ldots a_{n+1}^{m_{n+1}}) = k'(a_2^{m_2} \ldots a_{n+1}^{m_{n+1}}),$$
$$k(a_1^1 a_2^{m_2} \ldots a_{n+1}^{m_{n+1}}) = -k'(a_2^{m_2} \ldots a_{n+1}^{m_{n+1}}).$$

If the ith row of L_n corresponds to k', then two copies of this row with the second copy multiplied by -1 correspond to k, and these two copies make exactly the $(2^n + i)$th row of L_{n+1}.

Thus it is proved that the ε_{ij}'s are values of the ith character in $\mathbf{Z}_2^n$, that is, $\varepsilon_{ij} = k_i(b_j)$, where $\mathbf{Z}_2^n = \{b_1, \ldots, b_{2^n}\}$, $\hat{\mathbf{Z}}_2^n = \{k_1, \ldots, k_{2^n}\}$. Hence

$$\sum_{j=1}^{2^n} \varepsilon_{ij} A_j = \sum_{j=1}^{2^n} k_i(b_j) A_j = 2^n \int_{\mathbf{Z}^n} k_i(b) A_b \, d\mu(b),$$

since the Haar measure on Z_2^n is the counting measure, divided by 2^n. Therefore, (4.5) becomes

$$\sum_{k \in \hat{\mathbf{Z}}_2^n} \left\| 2^n \int_{\mathbf{Z}_2^n} k_i(b) A_b \, d\mu(b) \right\|_p^p \leq 2^{n(p-1)} 2^n \int_{\mathbf{Z}_2^n} \|A_b\|_p^p \, d\mu(b),$$

which is equivalent to (3.13). Similarly, (4.6) becomes

$$\sum_{i=1}^{2^n} \left\| 2^n \int_{\mathbf{Z}_2^n} k_i(b) A_b \, d\mu(b) \right\|_p^q \leq 2^n \left(2^n \int_{\mathbf{Z}_2^n} \|A_b\|_p^p \, d\mu(b) \right)^{q/p},$$

which is equivalent to (3.19). Finally, (4.7) becomes

$$\sum_{i=1}^{2^n} \left\| 2^n \int_{\mathbf{Z}_2^n} k_i(b) A_b \, d\mu(b) \right\|_p^p \leq 2^n \left(2^n \int_{\mathbf{Z}_2^n} \|A_b\|_q^q \, d\mu(b) \right)^{q/p},$$

which is equivalent to (3.23)   □

Remark 4.6. A related result was given in [14, p. 164] (see also [18, Theorem 3.3]). It was proved that

$$\left(\sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^v \right)^{1/v} \leq 2^{nc(u,v;p)} \left(\sum_{j=1}^{2^n} \|f_j\|_p^u \right)^{1/u} \tag{4.8}$$

for various choices of u, v, and p, and for $f_i \in L^p$—the standard Lebesgue space. It was later generalized in [11, Theorem 2.4] and [18, Corollary 4.2] for f_i from the Lebesgue–Bochner space.

Corollary 4.4 is an expansion of (4.8) to $\mathcal{C}_p$ spaces for some choices of u, v. Namely, for $u = v = p \geq 2$, for $u = p \leq 2$, $v = q = p/(p-1)$ and for $v = p \geq 2$, $u = q = p/(p-1)$. The constants in these cases match. We conjecture that the same can be done for other choices of u, v, p (see Problem 5.1.)

For $G = \mathbf{T}$, we get the following.

Corollary 4.7. *For all $A_\theta \in L^p((0, 2\pi); \mathcal{C}_p)$ we have*

$$\sum_{k=-\infty}^{+\infty} \left\| \int_0^{2\pi} e^{-ik\theta} A_\theta \, \mathrm{d}\theta \right\|_p^p \leq (2\pi)^{p-1} \int_0^{2\pi} \|A_\theta\|_p^p \, \mathrm{d}\theta$$

$$\leq \sum_{k=-\infty}^{+\infty} \alpha_k^{1-p/2} \left\| \int_0^{2\pi} e^{-ik\theta} A_\theta \, \mathrm{d}\theta \right\|_p^p,$$

for any sequence $\alpha_k > 0$, $k \in \mathbf{Z}$, such that $\sum_{k=-\infty}^{+\infty} \alpha_k = 1$. We have

$$\sum_{k=-\infty}^{+\infty} \left\| \int_0^{2\pi} e^{-ik\theta} A_\theta \, \mathrm{d}\theta \right\|_p^p \leq 2\pi \left(\int_0^{2\pi} \|A_\theta\|_p^q \, \mathrm{d}\theta \right)^{p/q},$$

for $p \geq 2$ and $q = p/(p-1)$, whereas for $1 \leq p \leq 2$ and $q = p/(p-1)$, we have

$$\sum_{k=-\infty}^{+\infty} \left\| \int_0^{2\pi} e^{-ik\theta} A_\theta \, \mathrm{d}\theta \right\|_p^q \leq 2\pi \left(\int_0^{2\pi} \|A_\theta\|_p^p \, \mathrm{d}\theta \right)^{q/p}.$$

Proof. Consider $G = \mathbf{T} = \{z \in \mathbf{C} \mid |z| = 1\}$. Its Pontryagin dual is $\hat{G} \cong \mathbf{Z}$, and the corresponding Haar measure is $\mathrm{d}\theta/2\pi$, that is, the usual Lebesgue measure normalized by the factor 2π. Characters on G are mappings $\theta \mapsto e^{k\theta}$ for all $k \in \mathbf{Z}$. Hence, the result immediately follows from Theorems 3.4, 3.5, 3.6, and 3.7. □

Remark 4.8. The preceding corollary also estimates the Fourier coefficients for functions in Bochner spaces $L^p(\mathbf{T}; \mathcal{C}_p)$. More precisely, let $l^r(\mathcal{C}_p)$ denote the space of all sequences $B_k \in \mathcal{C}_p$, $k \in \mathbf{Z}$ such that $\sum_{k \in \mathbf{Z}} \|B_k\|_p^r < +\infty$, and let $\mathcal{F}$ stand for the mapping which A_θ maps to $B_k = (1/2\pi) \int_0^{2\pi} e^{-ik\theta} A_\theta \, \mathrm{d}\theta$. Then Corollary 4.7 establishes norm estimates

$$\|\mathcal{F}\|_{L^p((0,2\pi);\mathcal{C}_p) \to l^p(\mathcal{C}_p)}, \|\mathcal{F}\|_{L^q((0,2\pi);\mathcal{C}_p) \to l^p(\mathcal{C}_p)} \leq 1, \quad p \geq 2, q = p/(p-1),$$

$$\|\mathcal{F}\|_{L^p((0,2\pi);\mathcal{C}_p) \to l^q(\mathcal{C}_p)}, \quad p \leq 2, q = p/(p-2).$$

5. Problems

We list some questions that naturally arise from the results of this article.

Problem 5.1. Do Boas–Koskela type inequalities (see [4] and [17]), that is,

$$\left(\sum_{k \in \hat{G}} \left\| \int_G \overline{k(\theta)} A_\theta \, \mathrm{d}\mu(\theta) \right\|_p^r \right)^{1/r} \leq \left(\int_G \|A_\theta\|_p^s \, \mathrm{d}\mu(\theta) \right)^{1/s}$$

for $s \leq p \leq r$ and $r/(r-1) \leq s \leq r$, hold? If the answer is positive, it is likely that the Kato inequality (4.8) might be extended to $\mathcal{C}_p$ classes for all choices of u, v, and p.

Problem 5.2. Can we prove that the inequalities in Theorems 3.4, 3.6, and 3.7 are sharp? Recall that the inequalities (4.8) are sharp for f_i in Lebesgue spaces and Lebesgue–Bochner spaces. This suggests, since the constants match, that (3.13), (3.19), and (3.23) are also sharp.

Problem 5.3. Is it possible to get convergence in $\|\cdot\|$-norm in Theorem 3.1? We do have strong convergence. If this convergence is in the norm of $J = B(H)$, then this would lead to the conclusion that functions $k(\theta)I$ make a basis for Hilbert module $L^2(G; B(H))$. See Remark 3.2.

Problem 5.4. What can be done if G is not assumed to be compact? Some classical results on Fourier transforms may be useful. Namely, for any $p > 2$, there is $f \in L^p(\mathbf{R}^n)$ such that its Fourier transform is not a function, but a tempered distribution. For $1 \leq p \leq 2$, however, the Fourier transform is a bounded operator from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, where $q = p/(p-1)$ with norm equal to 1. For any locally compact group, this is known as the Hausdorff–Young inequality (see [8, Proposition 4.28]), and is usually proved by Riesz–Thorin theorem. This would suggest that (in the case where G need not be compact) only inequality (3.19) might be generalized.

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