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BIMONOTONE MAPS ON SEMIPRIME BANACH ALGEBRAS

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To my beloved teacher Jesús Escoriza, in memoriam

Communicated by C. Badea

ABSTRACT. In this article, we investigate the properties of the sharp partial order in unital rings, and we study additive maps preserving the minus partial order in both directions in the setting of unital semiprime Banach algebras with essential socle.

1. Introduction and background

We start by recalling some of the *partial orders* defined historically first over rings of matrices, and extended later to more general settings. Then we will focus on the results known about maps preserving these orders.

Let A be a unital Banach algebra, and let $A^\bullet$ be its set of idempotent elements. An element $a \in A$ is *regular* if there is $b \in A$ such that $aba = a$. If such an element b exists, then it is called an *inner inverse* of a. Similarly, if there is $b \in A$ for which $b = bab$ holds, then b is said to be an *outer inverse* of a. If b is both an inner and outer inverse of a, then b is a *generalized inverse* of a. Note that if b is an inner inverse of a, then $ab, ba \in A^\bullet$ and $b' = bab$ is a generalized inverse of a. We write $A^\wedge$ for the set of regular elements in A. If a has a generalized inverse commuting with a, then it is unique and it is called the *group inverse* of a. In this case a is said to be *group invertible*, and its group inverse is denoted by $a^\sharp$. The set of all group invertible elements of A is denoted by $A^\sharp$. Since the late 1970s, many

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authors have made an effort to understand partial orders over abstract structures like semigroups, rings of matrices, or algebras (see, e.g., [6], [10], [14], [16] and the references therein).

Let $\mathrm{M}_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices. The *minus partial order* on $\mathrm{M}_n(\mathbb{C})$ was introduced by Hartwig and Styan [11], as follows:

$$A \leq^- B \quad \text{if } A^+A = A^+B \text{ and } AA^+ = BA^+,$$

where A^+ denotes a generalized inverse of A. The minus partial order is also known as the *rank subtractivity order* because (see [10, Theorem 2]) for $A, B \in \mathrm{M}_n(\mathbb{C})$,

$$A \leq^- B \qquad \text{if and only if} \qquad \operatorname{rank}(B - A) = \operatorname{rank}(B) - \operatorname{rank}(A).$$

Mitra [15] used the group inverse of a matrix to define the *sharp partial order* on group invertible matrices as

$$A \leq_\sharp B \quad \text{if } A^\sharp A = A^\sharp B \text{ and } AA^\sharp = BA^\sharp,$$

and provided many equivalent formulations to this and other partial orders.

Let X and Y be complex Banach spaces, and let $\mathrm{B}(X, Y)$ be the set of all bounded linear operators from X to Y. For short, write $\mathrm{B}(X) = \mathrm{B}(X, X)$. By $\mathrm{N}(T)$ and $\mathrm{R}(T)$ we denote the null space and the range of an operator $T \in B(X, Y)$, respectively. Rakić and Djordjević [20] extended the definition of minus partial order to the class of bounded linear operators between Banach spaces. If $T, S \in \mathrm{B}(X, Y)$ are regular operators, then T *is below* S *under the minus partial order*, denoted by $T \leq^- S$, if there exists an inner inverse T^- of T such that $TT^- = ST^-$ and $T^-T = T^-S$ (see [20, Definition 1.6]). They showed that the relation $\leq^-$ is a partial order on the class of regular operators on Banach spaces (see [20, Theorem 3.3]).

Let H be an infinite-dimensional complex Hilbert space, and let $\mathrm{B}(H)$ be the C*-algebra of all bounded linear operators on H. Taking into account that an operator in $\mathrm{B}(H)$ is regular if and only if it has closed range (see [17]), Šemrl [22] extended the minus partial order from $\mathrm{M}_n(\mathbb{C})$ to $\mathrm{B}(H)$, finding an appropriate equivalent definition of the minus partial order on $\mathrm{M}_n(\mathbb{C})$ that does not involve inner inverses: $A \preceq B$ if there exist idempotent operators $P, Q \in \mathrm{B}(H)$ such that

$$\mathrm{R}(P) = \overline{\mathrm{R}(A)}, \qquad \mathrm{N}(Q) = \mathrm{N}(A), \qquad PA = PB, \qquad \text{and} \qquad AQ = BQ.$$

Šemrl [22, Corollary 3] also proved that the relation $\preceq$ is a partial order in $\mathrm{B}(H)$ extending the minus partial order on matrices.

Recall that for every subset M of A, the *right annihilator* of M is

$$\operatorname{ann}_r(M) = \{x \in A \colon mx = 0 \text{ for all } m \in M\},$$

and the *left annihilator* of M is given by

$$\operatorname{ann}_l(M) = \{x \in A \colon xm = 0 \text{ for all } m \in M\}.$$

For an element $a \in A$, we write $\operatorname{ann}_r(a) = \operatorname{ann}_r(\{a\})$ and $\operatorname{ann}_l(a) = \operatorname{ann}_l(\{a\})$.

Djordjević, Rakić, and Marovt [4], with the help of annihilators, provided the following algebraic version of Šemrl's minus partial order. Let A be a unital ring. For $a, b \in A$, $a \leq^- b$ if there exist $p, q \in A^\bullet$ such that $\operatorname{ann}_l(a) = \operatorname{ann}_l(p)$,

$\mathrm{ann}_r(a) = \mathrm{ann}_r(q)$, $pa = pb$, and $aq = bq$. The authors showed that this relation on $\mathrm{B}(H)$ is equivalent to that given by Šemrl (see [4, Theorem 1.4]). Moreover, it was also proved in [4] that, in the setting of Rickart rings, the relation $\leq^-$ is a partial order. (Recall that a unital ring A is a *Rickart ring* if the left and right annihilators of any element are generated by idempotent elements.)

Following Šemrl's approach, Efimov [7] generalized the sharp relation to the set of those bounded operators on a Banach space X for which the closure of the image and kernel are topologically complementary subspaces. Given $I_X = \{T \in \mathrm{B}(X)\colon \overline{\mathrm{R}(T)} \oplus \mathrm{N}(T) = X\}$, $T \in I_X$ if and only if there exists an idempotent $P \in B(X)$ such that $\mathrm{R}(P) = \overline{\mathrm{R}(T)}$ and $\mathrm{N}(P) = \mathrm{N}(T)$. This idempotent is unique and satisfies $T = PT = TP$. Later, Rakić [19] extended the sharp partial order to more general rings, and to elements that do not necessarily have group inverses, by using the notion of annihilators instead of dealing with ranges or kernels. For a unital ring A, Rakić considers the set

$$\mathcal{I}_A = \big\{a \in A\colon\ \mathrm{ann}_l(a) = \mathrm{ann}_l(p_a) \text{ and } \mathrm{ann}_r(a) = \mathrm{ann}_r(p_a),\ \text{for some } p_a \in A^\bullet\big\}.$$

If $a \in \mathcal{I}_A$, then the idempotent element p_a such that $\mathrm{ann}_l(a) = \mathrm{ann}_l(p_a)$ and $\mathrm{ann}_r(a) = \mathrm{ann}_r(p_a)$ is unique (see [19, Lemma 2.1]). Given $a, b \in A$, a is below b under the sharp partial order if $a \in \mathcal{I}_A$ and $a = p_a b = b p_a$. It is shown that this relation defines a partial order on $\mathcal{I}_A$ (see [19, Theorem 3.1]).

During the last three decades, several results involving linear preservers of order relations have been published. In 1993, Ovchinnikov [18] showed that every bijective map ϕ defined on the set $\mathrm{B}(H)^\bullet$ of idempotent operators on a complex Hilbert space, satisfying that $\phi(P) \leq \phi(Q)$ if and only if $P \leq Q$, can be expressed either as $\phi(P) = APA^{-1}$ for every $P \in \mathrm{B}(H)^\bullet$ or as $\phi(P) = AP^*A^{-1}$ for every $P \in \mathrm{B}(H)^\bullet$, where A is a linear or conjugate-linear bijection on H. Later, many results concerning order-preserving maps in matrix algebras appeared in the literature (the reader is referred to [9], [13], [21]).

Šemrl [22] studied (not necessarily linear) bijective maps preserving the minus partial order on $\mathrm{B}(H)$ for an infinite-dimensional complex Hilbert space H. It is shown in [22, Theorem 8] that if $\phi : \mathrm{B}(H) \to \mathrm{B}(H)$ is a bijective map preserving the minus partial order in both directions, then there exist bounded bijective maps, both linear or both conjugate-linear, $T, S : H \to H$ such that either

$$\phi(A) = TAS \quad \text{for every } A \in \mathrm{B}(H)$$

or

$$\phi(A) = TA^*S \quad \text{for every } A \in \mathrm{B}(H).$$

Efimov and Guterman [8] studied the structure of additive bijective maps on $\mathrm{B}(H)$ preserving the sharp partial order in both directions. They showed (see [8, Theorem 3.7]) that if $\dim(H) \geq 3$ and $T : \mathrm{B}(H) \to \mathrm{B}(H)$ is an additive bijective map preserving the sharp partial order in both directions, then there exist $\alpha \in \mathbb{C} \setminus \{0\}$ and a linear or semilinear invertible bounded operator $S : H \to H$ such that either $T(A) = \alpha SAS^{-1}$ for all $A \in \mathrm{B}(H)$ or $T(A) = \alpha SA^*S^{-1}$ for all $A \in \mathrm{B}(H)$.

Recently, linear maps preserving the minus and sharp partial orders were considered in [2] and [3], respectively, in the more general setting of unital semisimple Banach algebras having essential socle, or for unital real rank zero C^*-algebras. When A and B are unital semisimple Banach algebras with essential socle, it is proved in [2, Theorem 3.2] that every bijective linear mapping $T : A \to B$ such that $T(A^\wedge) = B^\wedge$, and $a \leq^- b$ if and only if $T(a) \leq^- T(b)$, for every $a, b \in A^\wedge$, is a Jordan isomorphism multiplied by an invertible element. The condition $T(A^\wedge) = B^\wedge$ can be removed either when $B = \mathrm{B}(X)$ for a complex Banach space X (see [2, Theorem 3.4]) or when B is a prime C^*-algebra (see [2, Theorem 3.6]).

Theorem 2.7 in [3] states that a bijective linear map preserving the sharp order on group invertible elements from a unital semisimple Banach algebra with essential socle into a Banach algebra is a Jordan isomorphism multiplied by a invertible central element. Inspired by the works of Rakić [19] and Efimov and Guterman [8], we extend the study of sharp partial order and its preservers to the more general setting of unital rings.

This article is organized as follows. Section 2 is devoted to the study of the sharp partial relation $\preceq_\sharp$ in unital rings. We obtain some algebraic properties of this relation and show in Proposition 2.4 that this relation defines a partial order in every unital ring. In Section 3, we link Jordan isomorphisms and additive surjective maps preserving this relation in both directions. We prove in Theorem 3.2 that every Jordan isomorphism $T : A \to A$ in a unital semiprime ring satisfies that

$$a \preceq_\sharp b \qquad \text{if and only if} \qquad T(a) \preceq_\sharp T(b).$$

Reciprocally, the main result of this paper states that an additive surjective map $T : A \to A$ satisfying the above condition is a scalar multiple of either a linear or conjugate-linear Jordan isomorphism whenever A is a unital semiprime Banach algebra with essential socle with $\mathrm{Z}(A) = \mathbb{C}1$ (see Theorem 3.7).

2. Sharp order

Let A be a unital ring. We define the binary relations $\leq_\sharp$ and $\preceq_\sharp$ on A as follows:

- $a \leq_\sharp b$ if either $a = b$ or $a \in A^\sharp$, $a^\sharp a = a^\sharp b$, and $aa^\sharp = ba^\sharp$,
- $a \preceq_\sharp b$ if either $a = b$ or there exists $p \in A^\bullet$ such that $\mathrm{ann}_l(a) = \mathrm{ann}_l(p)$, $\mathrm{ann}_r(a) = \mathrm{ann}_r(p)$, $ap = bp$, and $pa = pb$.

It turns out that if $a \in A^\sharp$, then the elements over a are the same under both relations. This is shown in the following two lemmas.

Lemma 2.1. *Let A be a unital ring, and let $a \in A^\sharp$. For any $b \in A$, $a \leq_\sharp b$ if and only if $a \preceq_\sharp b$.*

Proof. (Necessity.) By definition, $a^\sharp a = a^\sharp b$ and $aa^\sharp = ba^\sharp$. Take $p = a^\sharp a$. It is clear that $p \in A^\bullet$ with $\mathrm{ann}_l(a) = \mathrm{ann}_l(p)$ and $\mathrm{ann}_r(a) = \mathrm{ann}_r(p)$. Also, $a = ap = pa$ and $bp = ba^\sharp a = aa^\sharp a = a = ap$. Similarly, $pb = a = pa$. Consequently, $a \preceq_\sharp b$.

(Sufficiency.) As $a \in A^\sharp$, $1-p \in \mathrm{ann}_l(p) = \mathrm{ann}_l(a)$, and $1-p \in \mathrm{ann}_r(p) = \mathrm{ann}_r(a)$, it follows that $a = pa = ap$ and $a^\sharp = pa^\sharp = a^\sharp p$. Then $a = pa = pb$, $a = ap = bp$, and $a^\sharp a = a^\sharp(pb) = (a^\sharp p)b = a^\sharp b$. Analogously, $aa^\sharp = ba^\sharp$. □

Lemma 2.2. *Let $a \in A^\wedge$. If $a \preceq_\sharp b$ for some $b \in A \setminus \{a\}$, then $a \in A^\sharp$ and $a \leq_\sharp b$.*

Proof. By definition, there exits $p \in A^\bullet$ such that $\mathrm{ann}_l(a) = \mathrm{ann}_l(p)$, $\mathrm{ann}_r(a) = \mathrm{ann}_r(p)$, $a = pa = pb$, and $a = ap = bp$. We first prove that a has group inverse. Since a is regular, there exists $x \in A$ such that $axa = a$ and $xax = x$. Then $(1-ax)a = 0$ and $a(1-xa) = 0$. As $\mathrm{ann}_l(a) = \mathrm{ann}_l(p)$ and $\mathrm{ann}_r(a) = \mathrm{ann}_r(p)$, we derive that $(1-ax)p = 0$ and $p(1-xa) = 0$, or equivalently, $p = axp$ and $p = pxa$.

Let us see that $a^\sharp = pxp$. As

$$a(pxp)a = (ap)x(pa) = axa = a$$

and

$$(pxp)a(pxp) = px(pa)pxp = pxapxp = px(ap)xp = pxaxp = pxp,$$

it follows that pxp is a generalized inverse of a. Moreover, since $a(pxp) = (ap)xp = axp = p$ and $(pxp)a = px(pa) = pxa = p$, it follows that $a(pxp) = (pxp)a$ and hence that pxp is the group inverse of a. Finally, $a \leq_\sharp b$ in light of Lemma 2.1. □

The next lemma shows the connection between the annihilator of two elements related under $\preceq_\sharp$. This will be used in Proposition 2.4 in order to establish that this binary relation is a partial order on every unital ring.

Lemma 2.3. *Let $a, b \in A$. If $a \preceq_\sharp b$, then*

(i) $a^2 = ab = ba$,
(ii) $\mathrm{ann}_l(b) \subseteq \mathrm{ann}_l(a)$ *and* $\mathrm{ann}_r(b) \subseteq \mathrm{ann}_r(a)$.

Proof. If $a = b$, then the result follows trivially. Otherwise, there exists $p \in A^\bullet$ such that $a = ap = pa$ and $a = bp = pb$.

(i) As $a = bp$ and $a = pa$, it follows that $a^2 = bpa = ba$. Similarly, it can be proved that $a^2 = ab$.
(ii) If $cb = 0$ for some c, then $0 = cbp = ca$. Analogously, the equality $bc = 0$ implies that $ca = 0$. □

Proposition 2.4. *The binary relation $\preceq_\sharp$ is a partial ordering.*

Proof. By definition, the relation is trivially reflexive. Let $a, b \in A$, and assume that $a \preceq_\sharp b$ and $b \preceq_\sharp a$. Given $p, q \in A^\bullet$ such that $a = ap = pa = bp = pb$ and $b = bq = qb = aq = qa$, it follows that

$$b = qa = q(bp) = (qb)p = bp = a.$$

This proves that $\preceq_\sharp$ is antisymmetric. For the transitivity, assume that $a \preceq_\sharp b$ and $b \preceq_\sharp c$ for some $a, b, c \in A$. Then there exists some $p, q \in A^\bullet$ such that

$$\begin{aligned} \mathrm{ann}_l(p) &= \mathrm{ann}_l(a), \qquad & \mathrm{ann}_r(p) &= \mathrm{ann}_r(a), \\ \mathrm{ann}_l(q) &= \mathrm{ann}_l(b), \qquad & \mathrm{ann}_r(q) &= \mathrm{ann}_r(b), \end{aligned}$$

$$a = ap = pa = bp = pb,$$
$$b = bq = qb = cq = qc.$$

By Lemma 2.3, $a^2 = ab = ba$, $b^2 = bc = cb$, and $a(c-b) = (c-b)a = 0$. Therefore, $ac = ab = ba = ca = a^2$. But then $a(c-a) = 0$, which implies that $p(c-a) = 0$ (since $\mathrm{ann}_l(a) = \mathrm{ann}_l(p)$). Similarly, $(c-a)p = 0$. This yields $pc = pa$ and $cp = ap$. Consequently, $a \preceq_\sharp c$, as desired. □

For $a \in A$, if there exists $p \in A^\bullet$ such that

$$\mathrm{ann}_l(a) = \mathrm{ann}_l(p), \qquad \mathrm{ann}_r(a) = \mathrm{ann}_r(p),$$

then this p is unique (see [19, Lemma 2.1]). We will denote it by $\pi_A(a)$, or simply $\pi(a)$, when there is no chance for confusion.

Following the notation proposed by Rakić [19], set

$$\mathcal{I}_A = \big\{a \in A\colon \text{there exists } \pi(a)\big\}.$$

With this notation, we can restate the definition of $\preceq_\sharp$ as follows:

$$a \preceq_\sharp b \qquad \text{if and only if} \qquad \begin{cases} a = b, \quad \text{or} \\ a \in \mathcal{I}_A \qquad \text{and} \qquad a = \pi(a)b = b\pi(a). \end{cases}$$

Lemma 2.5. *Let $a, b \in A$ with $b \in \mathcal{I}_A$. The following conditions are equivalent:*

(1) *$a \preceq_\sharp b$,*
(2) *there exists $p \in A^\bullet$ such that $a = bp = pb$,*
(3) *there exists $q \in A^\bullet$ such that $q \le \pi(b)$ and $a = bq = qb$.*

Proof. (1) ⇒ (2). If $a = b$, then it suffices to take $p = 1$. Otherwise, $a \in \mathcal{I}_A$, and we choose $p = \pi(a)$.

(2) ⇒ (3). Let $q = \pi(b)p$. As $b \in \mathcal{I}_A$ and $pb = bp$, it follows that $\pi(b)pb = pb$ and $bp\pi(b) = bp$. Hence $\pi(b)p - p \in \mathrm{ann}_l(b)$ and $p\pi(b) - p \in \mathrm{ann}_r(b)$. Therefore, $\pi(b)p\pi(b) = p\pi(b)$ and $\pi(b)p\pi(b) = \pi(b)p$. In particular, this proves that $q \in A^\bullet$ and $q = q\pi(b) = \pi(b)q$, that is, $q \le \pi(b)$. Finally, $a = bp = b\pi(b)p = bq$ and, similarly, $a = pb = p\pi(b)b = qb$.

(3) ⇒ (1). If $a = b$, then we are done. Otherwise, we only need to show that $q = \pi(a)$. The equalities $a = bq = qb$ imply that $\mathrm{ann}_l(q) \subseteq \mathrm{ann}_l(a)$ and $\mathrm{ann}_r(q) \subseteq \mathrm{ann}_r(a)$. Moreover, if $xa = 0$, then $xqb = 0$, and thus $xq\pi(b) = 0$. But $q \le \pi(b)$ ensures that $q = q\pi(b)$, and thus $xq = xq\pi(b) = 0$. This shows that $\mathrm{ann}_l(a) \subseteq \mathrm{ann}_l(q)$, and hence $\mathrm{ann}_l(q) = \mathrm{ann}_l(a)$. In the same way, we obtain $\mathrm{ann}_r(q) = \mathrm{ann}_r(a)$. By [19, Lemma 2.1], $q = \pi(a)$. □

Given $a, b \in A$, let us write $a \perp b$ whenever $ab = ba = 0$. Denote

$$\{a\}^\perp = \{x \in A\colon a \perp x\}.$$

Lemma 2.6. *Let $a, b \in \mathcal{I}_A$. The following are equivalent:*

(1) *$a \perp b$,*
(2) *$a + b \in \mathcal{I}_A$ with $\pi(a+b) = \pi(a) + \pi(b)$.*

Proof. Observe that $a \perp b \Leftrightarrow \pi(a) \perp b \Leftrightarrow a \perp \pi(b) \Leftrightarrow \pi(a) \perp \pi(b)$. Clearly, $\operatorname{ann}_l(a) \cap \operatorname{ann}_l(b) \subseteq \operatorname{ann}_l(a+b)$ and $\operatorname{ann}_r(a) \cap \operatorname{ann}_r(b) \subseteq \operatorname{ann}_r(a+b)$, for every $a, b \in A$.

(1) $\Rightarrow$ (2). Let $a, b \in \mathcal{I}_A$ with $a \perp b$, and take $x \in \operatorname{ann}_l(a+b)$. Then $xa = -xb$. By multiplying on the right by $\pi(a)$ (resp., by $\pi(b)$), we get $xa = xa\pi(a) = -xb\pi(a) = 0$ (resp., $xb = 0$). Therefore, $\operatorname{ann}_l(a+b) \subseteq \operatorname{ann}_l(a) \cap \operatorname{ann}_l(b)$. Similarly, we get that $\operatorname{ann}_r(a+b) \subseteq \operatorname{ann}_r(a) \cap \operatorname{ann}_r(b)$. This proves that $\operatorname{ann}_l(a+b) = \operatorname{ann}_l(a) \cap \operatorname{ann}_l(b)$ and $\operatorname{ann}_r(a+b) = \operatorname{ann}_r(a) \cap \operatorname{ann}_r(b)$. From this it is clear that $a + b \in \mathcal{I}_A$ with $\pi(a+b) = \pi(a) + \pi(b)$.

(2) $\Rightarrow$ (1). Let $a, b \in \mathcal{I}_A$ such that $a+b \in \mathcal{I}_A$ with $\pi(a+b) = \pi(a)+\pi(b)$. Since $\pi(a) + \pi(b)$ is an idempotent,

$$\pi(a)\pi(b) + \pi(b)\pi(a) = 0. \tag{2.1}$$

Moreover, from $a + b = (a + b)\pi(a + b) = (a + b)(\pi(a) + \pi(b))$ and $a + b = \pi(a + b)(a + b) = (\pi(a) + \pi(b))(a + b)$, we get

$$\pi(a)b + \pi(b)a = 0 \tag{2.2}$$

and

$$b\pi(a) + a\pi(b) = 0. \tag{2.3}$$

Multiplying (2.2) by $\pi(b)$ on the right and (2.3) by $\pi(b)$ on the left, respectively, we derive

$$\pi(a)b + \pi(b)a\pi(b) = 0, \qquad b\pi(a) + \pi(b)a\pi(b) = 0.$$

In particular,

$$\pi(a)b = b\pi(a). \tag{2.4}$$

Multiplying (2.1) by b on the right, we have

$$\pi(a)b + \pi(b)\pi(a)b = 0,$$

and taking into account (2.4), we conclude that $0 = b\pi(a) + \pi(b)b\pi(a) = 2b\pi(a)$. Thus, $\pi(a)b = b\pi(a) = 0$ or, equivalently, $ab = ba = 0$. Hence, $a \perp b$, as claimed. $\square$

It is clear that if $p \in A^\bullet$ and $a \preceq_\sharp p$, then $a \in A^\bullet$ and $a = ap = pa$. Moreover, as a direct consequence of Lemma 2.5, it follows that if $b \in \mathcal{I}_A$ and $a \preceq_\sharp b$, then $a \in \mathcal{I}_A$ and $\pi(a) \leq \pi(b)$. Observe also that for every element $a \in \mathcal{I}_A$, and $x \perp a$,

$$a \preceq_\sharp a + x.$$

The next proposition characterizes the set of maximal elements of the sharp partial order.

Proposition 2.7. *Let A be a unital ring. Then*

$$\operatorname{Maximals}_{\preceq_\sharp}(\mathcal{I}_A) = \big\{a \in A\colon\ \operatorname{ann}_l(a) = \operatorname{ann}_r(a) = \{0\}\big\}.$$

Proof. Let $a \in \text{Maximals}_{\preceq_\sharp}(\mathcal{I}_A)$. From $a \in \mathcal{I}_A$, it follows that $a \preceq_\sharp a + (1 - \pi(a))$. The maximality of a forces $a = a + (1 - \pi(a))$, that is, $1 = \pi(a)$. This proves one inclusion.

Now assume that $a \in A$ is such that $\text{ann}_l(a) = \text{ann}_r(a) = \{0\}$. Clearly, $a \in \mathcal{I}_A$ and $\pi(a) = 1$. If $a \preceq_\sharp b$ for some $b \in \mathcal{I}_A$, then $a = b\pi(a) = \pi(a)b$, but we know that $\pi(a) = 1$, and consequently $a = b$. □

Note that for $a \in \mathcal{I}_A$, $\text{ann}_l(a) = \text{ann}_r(a) = \{0\}$ if and only if $\pi(a) = 1$. In what follows, let us denote by $\mathcal{I}_A^0$ the set of elements in $\mathcal{I}_A$ that are not maximal for the sharp partial relation, that is,

$$\mathcal{I}_A^0 = \mathcal{I}_A \setminus \text{Maximals}_{\preceq_\sharp}(\mathcal{I}_A) = \big\{a \in \mathcal{I}_A \colon \pi(a) \neq 1\big\}.$$

3. Jordan isomorphisms and the sharp order

Let A be a ring. We denote by $\circ$ the usual Jordan product $a \circ b = \frac{1}{2}(ab + ba)$, and we denote by $\{a, b, c\} = \frac{1}{2}(abc + cba)$ the Jordan triple product in A. It is well known that

$$\{a, b, c\} = a \circ (c \circ b) + c \circ (a \circ b) - (a \circ c) \circ b, \tag{3.1}$$

$$\{a, b \circ d, c\} = \{a \circ d, b, c\} + \{a, b, c \circ d\} - \{a, b, c\} \circ d. \tag{3.2}$$

An additive map $T : A \to A$ is a *Jordan homomorphism* if $T(a^2) = T(a)^2$, for all $a \in A$, or equivalently, if $T(a \circ b) = T(a) \circ T(b)$, for every $a, b \in A$. It is well known that if $T : A \to A$ is a Jordan homomorphism, then T is a *Jordan triple homomorphism*, that is

$$T\big(\{a, b, c\}\big) = \big\{T(a), T(b), T(c)\big\}, \quad \text{for all } a, b, c \in A.$$

An additive map $T : A \to A$ is *monotone (with respect to the sharp partial order)* if $a \preceq_\sharp b$ implies $T(a) \preceq_\sharp T(b)$. The map T is *bimonotone (with respect to the sharp partial order)* whenever $a \preceq_\sharp b$ if and only if $T(a) \preceq_\sharp T(b)$. We begin this section by noting that every Jordan isomorphism in a semiprime ring is bimonotone. We will use the following property.

Proposition 3.1. *Let A be a unital semiprime ring, and let $T : A \to A$ be a Jordan isomorphism. Then*

$$T(\mathcal{I}_A) \subseteq \mathcal{I}_A.$$

Moreover, for every $a \in \mathcal{I}_A$, $T(\pi(a)) = \pi(T(a))$.

Proof. Note that as T is a Jordan isomorphism, $T(A^\bullet) \subseteq A^\bullet$.

Let $a \in \mathcal{I}_A$. Then $a = a\pi(a) = \pi(a)a$. Hence

$$2T(a) = T(a)T\big(\pi(a)\big) + T\big(\pi(a)\big)T(a),$$

which implies that $T(a)T(\pi(a)) = T(\pi(a))T(a)T(\pi(a)) = T(\pi(a))T(a)$, and thus $T(a) = T(a)T(\pi(a)) = T(\pi(a))T(a)$. In particular,

$$\text{ann}_l\Big(T\big(\pi(a)\big)\Big) \subseteq \text{ann}_l\big(T(a)\big) \qquad \text{and} \qquad \text{ann}_r\Big(T\big(\pi(a)\big)\Big) \subseteq \text{ann}_r\big(T(a)\big).$$

For the other inclusion, assume that $bT(a) = 0$ for some $b \in A$. As T is surjective, there exists $x \in A$ with $T(x) = b$. For every $y \in A$, we have that

$T(a)T(y)T(x)T(a) = 0$. Let $z \in A$ be such that $T(z) = T(y)T(x)$. Then $0 = T(a)T(z)T(a) = T(aza)$, which forces $aza = 0$. It follows that $\pi(a)za = az\pi(a) = \pi(a)z\pi(a) = 0$. Hence

$$0 = T\big(\pi(a)z\pi(a)\big) = T\big(\pi(a)\big)T(z)T\big(\pi(a)\big) = T\big(\pi(a)\big)T(y)T(x)T\big(\pi(a)\big),$$

and this holds for all $y \in A$. As T is surjective and A is semiprime, [1, Lemma 1.1] ensures that $T(x)T(\pi(a))T(y)T(\pi(a)) = 0$ for all $y \in A$ and, in particular, that $0 = T(x)T(\pi(a))T(\pi(a)) = T(x)T(\pi(a)) = bT(\pi(a))$. Therefore, $\mathrm{ann}_l(T(a)) \subseteq \mathrm{ann}_l(T(\pi(a)))$. The inclusion $\mathrm{ann}_r(T(a)) \subseteq \mathrm{ann}_r(T(\pi(a)))$ is shown in a similar way. This proves that $T(a) \in \mathcal{I}_A$ with $\pi(T(a)) = T(\pi(a))$. □

Theorem 3.2. *Let A be a unital semiprime ring, and let $T : A \to A$ be a Jordan isomorphism. Then*

$$a \preceq_\sharp b \qquad \text{if and only if} \qquad T(a) \preceq_\sharp T(b).$$

Proof. It suffices to show that $a \preceq_\sharp b$ implies $T(a) \preceq_\sharp T(b)$, whenever $a \neq b$. If $a \preceq_\sharp b$ with $a \neq b$, then $a \in \mathcal{I}_A$, and $a = b\pi(a) = \pi(a)b$. By the above proposition, $T(a) \in \mathcal{I}_A$, and $T(\pi(a)) = \pi(T(a))$. Moreover, since T is a Jordan homomorphism, from $a = b\pi(a) = \pi(a)b$, we deduce that

$$2T(a) = T(b)T\big(\pi(a)\big) + T\big(\pi(a)\big)T(b) = T(b)\pi\big(T(a)\big) + \pi\big(T(a)\big)T(b).$$

It follows that

$$2T(a) = T(b)\pi\big(T(a)\big) + \pi\big(T(a)\big)T(b)\pi\big(T(a)\big)$$

and

$$2T(a) = \pi\big(T(a)\big)T(b) + \pi\big(T(a)\big)T(b)\pi\big(T(a)\big).$$

In particular, $T(b)\pi(T(a)) = \pi(T(a))T(b)$, which allows us to conclude that

$$T(a) = T(b)\pi\big(T(a)\big) = \pi\big(T(a)\big)T(b).$$

Therefore, $T(a) \preceq_\sharp T(b)$, as claimed. □

Next we deal with the natural converse question: When are bimonotone maps (multiples) of Jordan isomorphisms? The following result gathers the properties of these mappings needed later to prove the main result of this section.

Proposition 3.3. *Let A be unital ring, and let $T : A \to A$ be a bimonotone additive surjective map. Then*

(i) *T is bijective,*
(ii) *$T(\mathcal{I}_A^0)) = \mathcal{I}_A^0$,*
(iii) *for $a \in \mathcal{I}_A^0$, $a \perp b$ if and only if $T(a) \perp T(b)$,*
(iv) *$\pi(a) = \pi(b)$ if and only if $\pi(T(a)) = \pi(T(b))$, for every $a, b \in \mathcal{I}_A^0$.*

Proof. We prove each item in the same order as in the statement.

(i) Let $a \in A$ with $T(a) = 0$. Then $T(a) \preceq_\sharp T(b)$ for every $b \in A$ and, by hypothesis, $a \preceq_\sharp b$ for all $b \in A$. This shows that $a = 0$, and hence T is injective.

(ii) Since T is bijective and preserves the sharp partial order in both directions, it is enough to prove that $T(\mathcal{I}_A^0) \subseteq \mathcal{I}_A^0$. To this end, take $a \in \mathcal{I}_A^0$. It is clear that $a \preceq_\sharp a + (1 - \pi(a))$ and, by assumption, that $T(a) \preceq_\sharp T(a) + T(1 - \pi(a))$. Since T is injective and $\pi(a) \neq 1$, this shows that $T(a) \in \mathcal{I}_A$ and

$$T(a)T\big(1 - \pi(a)\big) = 0 \qquad \text{and} \qquad T\big(1 - \pi(a)\big)T(a) = 0.$$

In particular, $\mathrm{ann}_l(T(a)) \neq \{0\}$ and $\mathrm{ann}_r(T(a)) \neq \{0\}$, which ensures that $T(a) \in \mathcal{I}_A^0$.

(iii) Keeping in mind the hypothesis and assertion (ii) just proved, it suffices to show that $T(a) \perp T(b)$ for $a \in \mathcal{I}_A^0$, and $b \perp a$. Indeed, $a \preceq_\sharp (a+b)$, and thus $T(a) \preceq_\sharp T(a+b) = T(a) + T(b)$. Hence, $T(b) \perp T(a)$.

(iv) Note that for every $a, b \in \mathcal{I}_A^0$, $\pi(a) = \pi(b)$ if and only if $\{a\}^\perp = \{b\}^\perp$. Then, from (ii) and (iii), we get

$$\begin{aligned}\pi(a) = \pi(b) \Leftrightarrow \{a\}^\perp = \{b\}^\perp &\Leftrightarrow \big\{T(a)\big\}^\perp = \big\{T(b)\big\}^\perp \\ &\Leftrightarrow \pi\big(T(a)\big) = \pi\big(T(b)\big).\end{aligned}$$ □

Let A be a unital semiprime Banach algebra. The *socle* of A, $\mathrm{Soc}(A)$, is the sum of all minimal left ideals of A, which coincides with the sum of all minimal right ideals of A. Recall that every minimal left ideal of A is of the form Ae for some minimal idempotent e, that is, $e^2 = e \neq 0$ with $eAe = \mathbb{C}e$. In what follows, we denote by $\mathrm{Min}(A)$ the set of minimal idempotents of A. If A has no minimal one-sided ideals, then $\mathrm{Soc}(A) = \{0\}$.

A nonzero element $u \in A$ is said to be of *rank-one* if u belongs to some minimal left ideal of A, that is, if $u = ue$ for some minimal idempotent e of A. It is known that u has rank 1 if and only if $uau = \mathbb{C}u \neq 0$. For every rank 1 element u in A there exists $\tau(u) \in \mathbb{C}$ such that $u^2 = \tau(u)u$. Moreover, $\tau(u) = 0$ or $\tau(u)$ is the only nonzero point of the spectrum of u. Thus, if $\tau(u) \neq 0$, then $\tau(u)^{-1}u$ is a minimal idempotent and $u = \tau(u)(\tau(u)^{-1}u)$.

Let us denote by $\mathrm{F}_1(A)$ the set of rank 1 elements of A,

$$\mathrm{F}_1^1(A) = \big\{u \in \mathrm{F}_1(A) \colon \tau(u) \neq 0\big\}$$

and

$$\mathrm{F}_1^0(A) = \big\{u \in \mathrm{F}_1(A) \colon \tau(u) = 0\big\}.$$

It is well known that every element of the socle is a finite sum of rank 1 elements and that $\mathrm{Soc}(A)$ consists of regular elements.

Given $u \in \mathrm{F}_1^0(A)$, let $x \in A$ and $\lambda \in \mathbb{C}$ be such that $uxu = u$ and $x - \lambda 1$ is invertible. Therefore, $e_1 = ux$ and $e_2 = u(x - \lambda)$ are minimal idempotents satisfying $u = \lambda^{-1}(e_1 - e_2)$. In particular, it follows from this fact that every element of the socle of a semisimple Banach algebra is a linear combination of minimal idempotents.

Recall that a nonzero ideal I of A is called *essential* if it has nonzero intersection with every nonzero ideal of A. For a semisimple Banach algebra A, this is equivalent to the condition that $aI = 0$, for $a \in A$, implies $a = 0$. Note that from

[5, Theorem 4.1], every unital semiprime Banach algebra with essential socle is semisimple.

Remark 3.4. Let A be a unital semiprime Banach algebra with nonzero socle.

(1) For every $u \in \mathrm{F}_1(A)$, set

$$L_u := \{ux : x \in A\} \qquad \text{and} \qquad R_u := \{xu : x \in A\}.$$

It was proved in [2, Lemma 2.18] that these are the maximal linear subspaces of $\mathrm{Soc}(A)$ consisting of elements with rank at most 1. In fact, for every $u, v \in \mathrm{F}_1(A)$, if $u+v \in \mathrm{F}_1(A)$, then $v \in R_u$ (consequently $R_u = R_v$) or $v \in L_u$ (which yields $L_u = L_v$).

(2) Assume that A has essential socle, and hence it is semisimple. For every $a \in A\backslash\{0\}$ there exists $w \in \mathrm{F}_1(A)$ such that $aw \neq 0$. Moreover, $w \in \mathrm{F}_1(A)$ can be chosen so that $\tau(aw) = \tau(wa) = 1$. In particular, if a is a nonzero idempotent element, $x = awa$ is a minimal idempotent such that $x \preceq_\sharp a$ (cf. [2, Proposition 2.15]).

(3) If $a \in A$ satisfies $xax = 0$ for all $x \in \mathrm{Soc}(A)$, then $xa = ax = 0$ for every $x \in \mathrm{Soc}(A)$. Indeed, if $au_0 \neq 0$ for some rank 1 element u of A, then, as A is semisimple, there exists $b \in A$ such that $\sigma(au_0b) \neq \{0\}$, which contradicts the fact that $(aub)^2 = 0$. Thus, if $\mathrm{Soc}(A)$ is essential, from $xax = 0$ for all $x \in \mathrm{Soc}(A)$, then it follows that $a = 0$.

In Propositions 3.5 and 3.6, we study the behavior of a bimonotone additive surjective map with respect to the elements of the socle of a unital semiprime Banach algebra.

Proposition 3.5. *Let A be a unital semiprime Banach algebra with essential socle. Let $T : A \to A$ be a bimonotone additive surjective map. Then*

(1) $\pi(T(p)) \in \mathrm{Min}(A)$ *for every* $p \in \mathrm{Min}(A)$,
(2) $T(\mathrm{F}_1^1(A)) = \mathrm{F}_1^1(A)$,
(3) $T(\mathrm{F}_1^0(A)) = \mathrm{F}_1^0(A)$,
(4) $T(\mathrm{Soc}(A)) = \mathrm{Soc}(A)$,
(5) $T(\mathbb{C}1) \subset \mathrm{Soc}(A)'$, *where* $\mathrm{Soc}(A)' = \{a \in A \colon xa = ax,\ \textit{for all}\ x \in \mathrm{Soc}(A)\}$.

Proof. Note that by Lemma 2.3 and the assumptions on T, for every $p \in A^\bullet$,

$$T(p)^2 = T(p)T(1) = T(1)T(p).$$

In particular, $T(p) \perp (T(1) - T(p))$.

(1) Take $p \in \mathrm{Min}(A)$. As $p \in \mathcal{I}_A^0$, Proposition 3.3(ii) ensures that $T(p) \in \mathcal{I}_A^0$. By Remark 3.4(2), there is $T(x) \in \mathrm{Min}(A)$ such that $T(x) \preceq_\sharp \pi(T(p))$. Taking into account Lemma 2.3 and the fact that $T(1-p) = T(1) - T(p) \perp \pi(T(p))$, we get $T(1-p) \perp T(x)$. From Proposition 3.3, $x \perp (1-p)$, that is, $x = xp = px$. If $p \in \mathrm{Min}(A)$, then it follows that $x \in \mathrm{F}_1(A)$ and $x = px = pxp = \tau(px)p = \tau(x)p$, with $\tau(x) \neq 0$ since $x \in \mathcal{I}_A^0$. Then $\pi(x) = p = \pi(p)$, which implies in view of Proposition 3.3(iv) that $\pi(T(x)) = \pi(T(p))$. Hence $\pi(T(p)) = T(x) \in \mathrm{Min}(A)$.

(2) Clearly, it is enough to prove that $T(\mathrm{F}_1^1(A)) \subseteq \mathrm{F}_1^1(A)$. To this aim, take $p \in \mathrm{Min}(A)$, $\lambda \in \mathbb{C} \setminus \{0\}$, and let us show that $T(\lambda p) \in \mathrm{F}_1^1(A)$. As we have just

proved, $\pi(T(p)) \in \mathrm{Min}(A)$, which implies that

$$T(p) = \pi\big(T(p)\big)T(p) = T(p)\pi\big(T(p)\big)$$
$$= \pi\big(T(p)\big)T(p)\pi\big(T(p)\big) = \tau\big(T(p)\big)\pi\big(T(p)\big),$$

and thus $T(p) \in \mathrm{F}_1^1(A)$. Moreover, as $p \preceq_\sharp 1$, it is clear that $\lambda p \preceq_\sharp \lambda 1$ and then $T(\lambda p) \preceq_\sharp T(\lambda 1)$. Keeping in mind that $\pi(T(\lambda p)) = \pi(T(p)) \in \mathrm{Min}(B)$ and

$$T(\lambda p) = \pi\big(T(p)\big)T(\lambda 1) = T(\lambda 1)\pi\big(T(p)\big),$$

we conclude that $T(\lambda p) \in \mathrm{F}_1^1(A)$.

(3) As above, we only need to show that the inclusion $T(\mathrm{F}_1^0(A)) \subseteq \mathrm{F}_1^0(A)$ holds. Let $u \in \mathrm{F}_1^0(A)$. Take $x \in A$ and $\lambda \in \mathbb{Q}$ such that $uxu = u$ and $x - \lambda 1$ is invertible. Therefore, $e_1 = ux$ and $e_2 = u(x - \lambda)$ are minimal idempotents satisfying $e_1e_2 = e_2$, $e_2e_1 = e_1$, and $u = \frac{1}{\lambda}(e_1 - e_2)$.

Note that e_1, e_2 and $\frac{1}{2}(e_1 + e_2)$ are minimal idempotents in A. Therefore, assertion (2) just proved ensures that $T(e_1), T(e_2)$ and $T(\frac{1}{2}(e_1 + e_2))$ lie in $\mathrm{F}_1^1(A)$. From Remark 3.4(1), $T(e_1) \in L_{T(e_2)}$ or $T(e_1) \in R_{T(e_2)}$. In the first case, $T(u) = \frac{1}{\lambda}(T(e_1) - T(e_2)) \in L_{T(e_2)}$; in the second case, $T(u) \in R_{T(e_2)}$. In particular, $T(u) \in \mathrm{F}_1(B)$.

As e_1, e_2, and $\frac{1}{2}(e_1 + e_2)$ are idempotents and T is additive, we know that

$$T(e_1)^2 = T(e_1)T(1) = T(1)T(e_1), \qquad T(e_2)^2 = T(e_2)T(1) = T(1)T(e_2),$$

and

$$\frac{1}{4}\big(T(e_1) + T(e_2)\big)^2 = \frac{1}{2}\big(T(e_1) + T(e_2)\big)T(1) = \frac{1}{2}T(1)\big(T(e_1) + T(e_2)\big).$$

By merging these identities, it follows that

$$T(e_1)^2 + T(e_2)^2 = T(e_1)T(e_2) + T(e_2)T(e_1).$$

Hence

$$\lambda^2 T(u)^2 = \big(T(e_1) - T(e_2)\big)^2$$
$$= T(e_1)^2 + T(e_2)^2 - T(e_1)T(e_2) - T(e_2)T(e_1) = 0.$$

We have proved that $T(u) \in \mathrm{F}_1^0(A)$.

(4) This can be deduced directly from the preceding assertions by just taking into account that every element of the socle of a semisimple Banach algebra is the sum of rank 1 elements.

(5) Pick $\lambda \in \mathbb{C} \setminus \{0\}$ and $e \in \mathrm{Min}(A)$. We know that $\pi(T(\lambda e)) = \pi(T(e)) \in \mathrm{Min}(A)$ and

$$T(\lambda e)) = \pi\big(T(e)\big)T(\lambda 1) = T(\lambda 1)\pi\big(T(e)\big).$$

Therefore

$$T(\lambda e) = \pi\big(T(e)\big)T(\lambda 1)\pi\big(T(e)\big) = \sigma(\lambda, e)\pi\big(T(e)\big)$$
$$= \tau\big(T(e)\big)^{-1}\alpha(\lambda, e)T(e),$$

where $\alpha(\lambda, e) = \tau(\pi(T(e))T(\lambda 1))$. This implies that $\{T(e)\}' = \{T(\lambda e)\}'$, and hence $T(1) \in \{T(\lambda e)\}'$, for every $\lambda \in \mathbb{C} \setminus \{0\}$ and $e \in \mathrm{Min}(A)$.

Now, given $\mu \in \mathbb{C} \setminus \{0\}$, because $e \perp \mu(1-e)$, Proposition 3.3(iv) guarantees that $T(e) \perp T(\mu(1-e))$, that is,

$$T(\mu 1)T(e) = T(\mu e)T(e) = \tau\big(T(e)\big)T(\mu e)\pi\big(T(e)\big) = \tau\big(T(e)\big)T(\mu e),$$

and similarly, $T(e)T(\mu 1) = \pi(T(e))T(\mu e)$. From this fact, we conclude that

$$T(\mu 1) \in \big\{T(e)\big\}' = \big\{T(\lambda e)\big\}',$$

for every $\lambda, \mu \in \mathbb{C}$ and $e \in \mathrm{Min}(A)$. This proves that $T(\mathbb{C}1)$ is included in $\mathrm{Soc}(A)'$, as claimed. □

Proposition 3.6. *Let A be a unital semiprime Banach algebra with essential socle. Assume that $\mathrm{Z}(A) = \mathbb{C}1$. Let $T : A \to A$ be a bimonotone additive surjective map. Then there exists a ring automorphism $\rho : \mathbb{C} \to \mathbb{C}$ such that*

$$T(\lambda x) = \rho(\lambda)T(x), \quad \textit{for all } x \in \mathrm{Soc}(A).$$

Proof. As we have shown in Proposition 3.5(5), $T(\mathbb{C}1) \subset \mathrm{Soc}(A)'$. With $\mathrm{Soc}(A)$ being essential, this implies that $T(\mathbb{C}1) \subset \mathrm{Z}(A)$. Indeed, given $x, y \in A$, if $xu = ux$ for every $u \in \mathrm{Soc}(A)$, then $(xy - yx)u = x(yu) - y(xu) = yux - yux = 0$ for every $u \in \mathrm{Soc}(A)$, which implies that $xy - yx = 0$, and hence that $xy = yx$. As by hypothesis $\mathrm{Z}(A) = \mathbb{C}1$, it follows that for every $\lambda \in \mathbb{C}$ there exists $\sigma(\lambda) \in \mathbb{C}$ such that $T(\lambda 1) = \sigma(\lambda)1$.

For every $e \in \mathrm{Min}(A)$ and $\lambda \in \mathbb{C}$, we know that $\pi(T(e)) \in \mathrm{Min}(A)$ and that

$$\begin{aligned} T(\lambda e) &= T(\lambda 1)\pi\big(T(e)\big) = \sigma(\lambda)\pi\big(T(e)\big), \\ T(e)T(\lambda e) &= T(1)T(\lambda e) = \sigma(1)T(\lambda e). \end{aligned}$$

From these identities, we deduce that

$$\sigma(1)T(\lambda e) = T(e)T(\lambda e) = \sigma(\lambda)T(e).$$

Therefore,

$$T(\lambda e) = \frac{\sigma(\lambda)}{\sigma(1)}T(e) \tag{3.3}$$

for every $e \in \mathrm{Min}(A)$ and $\lambda \in \mathbb{C}$.

Given $u \in \mathrm{F}_1^0(A)$, as we know, there exist $\beta \in \mathbb{Q}$ and $e_1, e_2 \in \mathrm{Min}(A)$ such that $u = \beta(e_1 - e_2)$. Since T is additive ($\mathbb{Q}$-linear), from (3.3) it follows that

$$\begin{aligned} T(\lambda u) &= T\big(\lambda\beta(e_1 - e_2)\big) \\ &= \beta T(\lambda e_1) - \beta T(\lambda e_2) \\ &= \beta\frac{\sigma(\lambda)}{\sigma(1)}T(e_1) - \beta\frac{\sigma(\lambda)}{\sigma(1)}T(e_2) = \frac{\sigma(\lambda)}{\sigma(1)}T(u). \end{aligned}$$

Now, take $u \in \mathrm{F}_1(A)$, with $\tau(u) \neq 1$ and $\tau(u) \neq 0$. Keeping in mind that $\frac{u}{\tau(u)} \in \mathrm{Min}(A)$ and (3.3), we deduce that

$$T(u) = T\Big(\tau(u)\frac{u}{\tau(u)}\Big) = \frac{\sigma(\tau(u))}{\sigma(1)}T\Big(\frac{u}{\tau(u)}\Big)$$

and

$$\begin{aligned}T(\lambda u) &= T\Big(\lambda\tau(u)\frac{u}{\tau(u)}\Big)\\ &= \frac{\sigma(\lambda\tau(u))}{\sigma(1)}T\Big(\frac{u}{\tau(u)}\Big)\\ &= \frac{\sigma(\lambda\tau(u))}{\sigma(\tau(u))}T(u).\end{aligned}$$

For the sake of simplicity, we write $\mu(\lambda) = \frac{\sigma(\lambda\tau(u))}{\sigma(\tau(u))}$. As $\tau(u) \neq 0$, $u \in A^\sharp$ with $u^\sharp = \tau(u)^{-2}u$. Let $v = uu^\sharp$ and $x \in A$ be such that $w = (1 - uu^\sharp)xuu^\sharp \neq 0$. It is clear that $vu = u$, $wu \in \mathrm{F}_1^0(A)$ and $z = (v + w)u \in \mathrm{F}_1(A)$.

By the additivity of the trace, $\tau(z) = \tau(vu) + \tau(wu) = \tau(u)$. Hence

$$\begin{aligned}\mu(\lambda)T\big((v + w)u\big) &= \mu(\lambda)T(vu) + \mu(\lambda)T(wu)\\ &= \mu(\lambda)T(u) + \mu(\lambda)T(wu)\end{aligned}$$

and

$$\begin{aligned}\mu(\lambda)T\big((v + w)u\big) &= T\big(\lambda(v + w)u\big) = T(\lambda vu) + T(\lambda wu)\\ &= T(\lambda u) + T(\lambda wu) = \mu(\lambda)T(u) + \frac{\sigma(\lambda)}{\sigma(1)}T(wu).\end{aligned}$$

This shows that

$$\mu(\lambda) = \frac{\sigma(\lambda)}{\sigma(1)}.$$

We have proved that $T(\lambda u) = \frac{\sigma(\lambda)}{\sigma(1)}T(u)$, for all $u \in \mathrm{F}_1(A)$. Taking into account the additivity of T and that every element of the socle is the sum of rank 1 elements,

$$T(\lambda x) = \frac{\sigma(\lambda)}{\sigma(1)}T(x), \quad \text{for all } x \in \mathrm{Soc}(A).$$

The map $\rho : \mathbb{C} \to \mathbb{C}$, defined by $\rho(\lambda) = \frac{\sigma(\lambda)}{\sigma(1)}$ for every $\lambda \in \mathbb{C}$, is a ring automorphism of the complex field. Note that ρ is clearly additive and bijective. Besides, for every $\lambda, \mu \in \mathbb{C}$ and $u \in \mathrm{F}_1(A)$, we have

$$\rho(\lambda\mu)T(u) = T(\lambda\mu u) = \rho(\lambda)T(\mu u) = \rho(\lambda)\rho(\mu)T(u),$$

which implies that ρ is multiplicative. $\square$

We now present the main result of this section.

Theorem 3.7. *Let A be a unital semiprime Banach algebra with essential socle. Assume that $\mathrm{Z}(A) = \mathbb{C}1$. Let $T : A \to A$ be a bimonotone additive surjective map. Then T is a scalar multiple of either a linear or conjugate-linear Jordan isomorphism.*

Proof. From Propositions 3.5 and 3.6, $\pi(T(p)) \in \mathrm{Min}(A)$ for every $p \in \mathrm{Min}(A)$, $T(\mathrm{Soc}(A)) = \mathrm{Soc}(A)$, there exists $\sigma(\lambda) \in \mathbb{C}$ such that $T(\lambda 1) = \sigma(\lambda)1$, and the map $\rho(\lambda) = \frac{\sigma(\lambda)}{\sigma(1)}$ is an automorphism of $\mathbb{C}$ such that $T(\lambda x) = \rho(\lambda)T(x)$, for every $\lambda \in \mathbb{C}$ and $x \in \mathrm{Soc}(A)$.

We can assume that $T(1) = 1$ (equivalently, $\sigma(1) = 1$) since if $\sigma(1) \neq 1$, then the mapping $S(x) = \sigma^{-1}(1)T(x)$ is a unital bimonotone additive surjective map, and the same arguments can be applied to S. We claim that T is a linear or conjugate-linear Jordan isomorphism. As the mapping $T|_{\mathrm{Soc}(A)} : \mathrm{Soc}(A) \to \mathrm{Soc}(A)$ is surjective, additive, and preserves rank 1 idempotents and their linear spans in both directions, by [12, Theorem 3.4], $T|_{\mathrm{Soc}(A)}$ is a real linear Jordan isomorphism (note that by assumption, A has no nonzero central elements in its socle). In particular, for every $\lambda \in \mathbb{R}$ and $x \in \mathrm{Soc}(A)$, it follows that

$$\rho(\lambda)T(x) = T(\lambda x) = \lambda T(x).$$

That is, $\rho(\lambda) = \lambda$ for every $\lambda \in \mathbb{R}$. Moreover, with ρ being an automorphism of $\mathbb{C}$, $\rho(i)^2 = \rho(i^2) = \rho(-1) = -1$. That is, either $\rho(i) = i$ or $\rho(i) = -i$. In the first case, $\rho(\lambda) = \lambda$ for every $\lambda \in \mathbb{C}$; in the second case, $\rho(\lambda) = \overline{\lambda}$ for every $\lambda \in \mathbb{C}$. This shows that $T|_{\mathrm{Soc}(A)}$ is either a linear or conjugate-linear Jordan isomorphism.

For every $p \in \mathrm{Min}(A)$ and $a \in A$, it is clear that $p \perp (1-p)a(1-p)$. By Proposition 3.3(iii), it follows that $T(p) \perp T((1-p)a(1-p))$. Hence,

$$\begin{aligned} T(p)T(a) &= T(p)T(ap + pa - pap), \\ T(a)T(p) &= T(ap + pa - pap)T(p). \end{aligned}$$

Taking into account the fact that $T|_{\mathrm{Soc}(A)}$ is a Jordan isomorphism, from these identities we deduce that

$$T(p) \circ T(a) = T(p) \circ T(ap + pa - pap) = T\big(p \circ (ap + pa - pap)\big) = T(p \circ a).$$

Given $\lambda \in \mathbb{C}$, $p \in \mathrm{Min}(A)$ and $a \in A$,

$$T(\lambda p \circ a) = \rho(\lambda)T(p \circ a) = \rho(\lambda)T(p) \circ T(a) = T(\lambda p) \circ T(a).$$

Keeping in mind that every element of the socle of a semisimple Banach algebra is a linear combination of minimal idempotents, we deduce that

$$T(x \circ a) = T(x) \circ T(a), \quad \text{for all } x \in \mathrm{Soc}(A) \text{ and } a \in A. \tag{3.4}$$

Let $x, y \in \mathrm{Soc}(A)$ and $a \in A$. From (3.4) and (3.1), we get

$$\begin{aligned} \big\{T(x), T(a), T(y)\big\} &= T(x) \circ \big(T(a) \circ T(y)\big) + T(y) \circ \big(T(a) \circ T(x)\big) \\ &\quad - \big(T(x) \circ T(y)\big) \circ T(a) \\ &= T(x) \circ T(a \circ y) + T(y) \circ T(a \circ x) - T(x \circ y) \circ T(a) \\ &= \big(T(x) \circ T(a \circ y) + T(y) \circ T(a \circ x) - T(x \circ y) \circ T(a)\big) \\ &= \big(T\big(x \circ (a \circ y)\big) + T\big(y \circ (a \circ x)\big) - T\big((x \circ y) \circ a\big)\big) \\ &= T\big(x \circ (a \circ y) + y \circ (a \circ x) - (x \circ y) \circ a\big) = T\big(\{x, a, y\}\big). \end{aligned}$$

The identity just proved and (3.2) yield that

$$\begin{aligned}\bigl\{T(x), T(a)^2, T(y)\bigr\} &= \bigl\{T(x)\circ T(a), T(a), T(y)\bigr\} + \bigl\{T(x), T(a), T(a)\circ T(y)\bigr\} \\ &\quad - \bigl\{T(x), T(a), T(y)\bigr\}\circ T(a) \\ &= \bigl\{T(x\circ a), T(a), T(y)\bigr\} \\ &\quad + \bigl\{T(x), T(a), T(a\circ y)\bigr\} - T\bigl(\{x, a, y\}\bigr)\circ T(a) \\ &= \bigl(T\bigl(\{x\circ a, a, y\}\bigr) + T\bigl(\{x, a, a\circ y\}\bigr) - T\bigl(\{x, a, y\}\circ a\bigr)\bigr) \\ &= T\bigl(\{x, a^2, y\}\bigr).\end{aligned}$$

In particular,

$$T(x)T(a)^2T(x) = T(xa^2x) = T(x)T(a^2)T(x), \quad \text{for all } x \in \mathrm{Soc}(A), a \in A. \tag{3.5}$$

Besides, for every $x \in \mathrm{Soc}(A)$, $a \in A$ and $\lambda \in \mathbb{C}$,

$$T(x)T(\lambda a)T(x) = T(\lambda xax) = \rho(\lambda)T(xax) = \rho(\lambda)T(x)T(a)T(x). \tag{3.6}$$

Finally, given $a \in A$, as T is surjective there exists $b \in A$ such that $T(b) = T(a)^2 - T(a^2)$. For every $x \in \mathrm{Soc}(A)$, (3.5) entails that

$$0 = T(x)\bigl(T(a)^2 - T(a^2)\bigr)T(x) = T(x)T(b)T(x) = T(xbx),$$

and with T injective, it follows that $xbx = 0$ for every $x \in \mathrm{Soc}(A)$. Taking into account Remark 3.4(3), we deduce that $b = 0$, and hence $T(a^2) = T(a)^2$. The same arguments allow us to conclude from (3.6) that $T(\lambda a) = \rho(\lambda)T(a)$, for every $\lambda \in \mathbb{C}$ and $a \in A$. Therefore, T is a linear or conjugate-linear Jordan isomorphism, as claimed. □

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