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COMPLETELY RANK-NONINCREASING MULTILINEAR MAPS

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Abstract. We extend the notion of completely rank-nonincreasing (CRNI) linear maps to include the multilinear maps. We show that a bilinear map on a finite-dimensional vector space on any field is CRNI if and only if it is a skew-compression bilinear map. We also characterize CRNI continuous bilinear maps defined on the set of compact operators.

1. Introduction

Rank-preserving or rank-nonincreasing linear maps, and in particular their characterizations, have been studied extensively in recent years. Let $\mathcal{A}$ and $\mathcal{B}$ be two operator algebras, and let (P) be a property of operators such as spectrum, invertibility, class of operators, and so on. If a linear map $\phi : \mathcal{A} \to \mathcal{B}$ leaves (P) invariant, we say that it is a *linear preserver* or, more exactly, *(P)-preserving*. The linear preserver problem asks how to characterize the linear preservers.

Rank-nonincreasing linear maps and rank-preserving linear maps are examples of linear preservers that have been studied in [10]. Let $\mathcal{L}(V)$ be the space of linear maps on a vector space V. A linear map $\phi : \mathcal{S} \to \mathcal{T}$ between two linear subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{L}(V)$ is said to be *rank nonincreasing* if $\operatorname{rank}(\phi(A)) \leq \operatorname{rank}(A)$ for every A in $\mathcal{A}$, where the rank of operator A is the dimension of its range. Let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear maps on the Hilbert space $\mathcal{H}$. Suppose that $\mathcal{S}$ is a linear subspace of $\mathcal{B}(\mathcal{H})$ and that $\phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ is linear. We are not assuming that $\mathcal{S}$ is norm-closed or that ϕ is bounded. We say that ϕ is

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a *similarity* if there is an invertible operator W such that, for every $S \in \mathcal{S}$, $\phi(S) = W^{-1}SW$, and we say that ϕ is a *compression* if there is an operator V such that, for every $S \in \mathcal{S}$, $\phi(S) = V^*SV$. We say that ϕ is a *skew-compression* if there are operators A, B such that, for every $S \in \mathcal{S}$, $\phi(S) = ASB$. If $\{\phi_\lambda\}$ is a net of maps on $\mathcal{S}$, we say that $\phi_\lambda \to \phi$ *point-strongly* (resp., *point-weakly*) if, for every $S \in \mathcal{S}$, $\phi_\lambda(S) \to \phi(S)$ in the strong operator topology (resp., weak operator topology). It turns out that the characterization of limits of similarities reduces to the discussion of rank-nonincreasing and rank-preserving linear maps on $\mathcal{F}(\mathcal{H})$, the subspaces of finite-rank operators (see [8]).

Suppose that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, that $\mathcal{A}$ is a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$, that the map $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a unital *-homomorphism, and that $\phi, \psi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ are linear maps with ϕ unital and completely positive and ψ completely bounded. Two unital representations π_1, π_2 of $\mathcal{A}$ are called *approximately (unitarily) equivalent*, denoted $\pi_1 \sim_a \pi_2$, if there is a net $\{U_\lambda\}$ of unitary operators such that

$$\lim_{\lambda} \left\| U_\lambda^* \pi_1(x) U_\lambda - \pi_2(x) \right\| = 0$$

for every $x \in \mathcal{A}$. The following results relate to work by Hadwin in [4]–[6]. In the following theorem, $\mathrm{id}_{\mathcal{A}}$ denotes the identity representation on $\mathcal{A}$ and $\mathcal{F}(\mathcal{H})$ denotes the set of finite-rank operators in $\mathcal{B}(\mathcal{H})$.

Theorem 1.1 ([9, Theorem 1])**.** *Suppose that $\mathcal{A}$, $\mathcal{H}$, $\mathcal{M}$ are separable and that π, ϕ, ψ are as above.*

(1) *The following are equivalent.*
   (a) *There is a unital representation ρ of $\mathcal{A}$, with $\rho \sim_a \mathrm{id}_{\mathcal{A}}$, and an isometry V such that $\phi(x) = V^*\rho(x)V$ for every $x \in \mathcal{A}$.*
   (b) *The map ϕ is rank nonincreasing and there is a representation ρ_1 of $\mathcal{A} \cap \mathcal{F}(\mathcal{H})$, with $\rho_1 \sim_a \mathrm{id}_{\mathcal{A}\cap\mathcal{F}(\mathcal{H})}$, and an isometry W such that $\phi(x) = W^*\rho_1(x)W$ for every $x \in \mathcal{A} \cap \mathcal{F}(\mathcal{H})$.*
   (c) *There is a sequence $\{V_n\}$ of isometries such that $V_n^*AV_n \to \phi(A)$ in the weak operator topology for every $A \in \mathcal{A}$.*

(2) *The following are equivalent.*
   (a) *There is a unital representation σ of $\mathcal{A}$, with $\sigma \sim_a \mathrm{id}_{\mathcal{A}}$, and operators A, B with $\|A\|\|B\| = \|\psi\|_{cb}$ such that $\psi(x) = A\sigma(x)B$ for every $x \in \mathcal{A}$.*
   (b) *The map ψ is rank nonincreasing and there is a representation ρ_1 of $\mathcal{A} \cap \mathcal{F}(\mathcal{H})$, with $\rho_1 \sim_a \mathrm{id}_{\mathcal{A}\cap\mathcal{F}(\mathcal{H})}$, and operators A_1, B_1 such that $\psi(x) = A_1\rho_1(x)B_1$ for every $x \in \mathcal{A} \cap \mathcal{F}(\mathcal{H})$.*
   (c) *There are norm-bounded sequences $\{C_n\}$, $\{D_n\}$ such that $C_nAD_n \to \psi(A)$ in the weak operator topology for every $A \in \mathcal{A}$.*

Hadwin and Larson [9] introduced the notion of CRNI maps in order to provide a different characterization, solely in terms of rank, of the above theorem. Let $\mathcal{S}$ and $\mathcal{T}$ be subspaces of $\mathcal{B}(\mathcal{H})$, and let $\phi : \mathcal{S} \to \mathcal{T}$ be linear. We regard ϕ as CRNI if, for each $n \in \mathbb{N}$, the map $\phi_n : \mathcal{M}_n(\mathcal{S}) \to \mathcal{M}_n(\mathcal{S})$ defined by $\phi_n(s_{ij}) = (\phi(s_{ij}))$ is rank nonincreasing, where $\mathcal{M}_n(\mathcal{S})$ is the set of $n \times n$ matrices with entries from $\mathcal{S}$.

Hadwin and Larson conjectured that a linear map $\phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ is a point-strong limit of skew-compressions if and only if ϕ is CRNI. Several results in support of this conjecture have been obtained in [7] and [11]. In fact, those authors proved this conjecture for the case where $\mathcal{S}$ is a C*-algebra. More precisely, they proved the following. Suppose that $\mathcal{H}$ is a separable Hilbert space and that $\mathcal{S}$ is a separable unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Let $\phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be a linear map. Then ϕ is a point-strong limit of skew-compressions if and only if ϕ is CRNI.

The following theorem is the main result in [9].

Theorem 1.2 ([9, Theorem 2])**.** *Suppose that $\mathcal{H}$ and $\mathcal{M}$ are separable Hilbert spaces, that $\mathcal{A}$ is a separable unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$, and that $\phi, \psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ are linear maps with ϕ unital and completely positive and ψ completely bounded. Then we have the following.*

(1) *There is a unital representation ρ of $\mathcal{A}$ with $\rho \sim_a \mathrm{id}_{\mathcal{A}}$ and an isometry V such that $\phi(x) = V^*\rho(x)V$ for every $x \in \mathcal{A}$ if and only if ϕ is CRNI.*

(2) *There is a unital representation σ of $\mathcal{A}$, with $\sigma \sim_a \mathrm{id}_{\mathcal{A}}$ and operators A, B with $\|A\| = \|B\| = \|\psi\|_{cb}$ such that $\psi(x) = A\sigma(x)B$ for every $x \in \mathcal{A}$ if and only if ψ is CRNI.*

The notions of completely bounded and completely positive linear maps have already been extended to include multilinear maps by Christensen and Sinclair [2], [3]. Our goal in this article is to follow their steps by introducing and studying the notion of CRNI multilinear maps. We will prove analogues of a few results known for CRNI linear maps. We make a similar conjecture, that every CRNI bilinear map should be a point-strong limit of skew-compressions.

Most of the results in this article are generalizations of those in [7] and especially the ones in [9]. An important aspect to point out is that we present most of these results using only basic facts of linear algebra and functional analysis.

2. Definitions

Throughout this article, $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{K}$ are separable Hilbert spaces over the field of complex numbers $\mathbb{C}$; $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded operators on $\mathcal{H}$; and $\mathcal{F}(\mathcal{H})$ denotes the set of finite-rank operators in $\mathcal{B}(\mathcal{H})$. The set of all $k \times k$ matrices over $\mathbb{C}$ is denoted by $\mathcal{M}_k = \mathcal{M}_k(\mathbb{C})$, and I_k means the identity matrix in $\mathcal{M}_k(\mathbb{C})$. The $n \times n$ diagonal matrix with entries $d_1, \dots, d_n$ on its main diagonal is denoted by $\mathrm{diag}(d_1, \dots, d_n)$. And by $E_{ij} \in \mathcal{M}_k$, we mean the $k \times k$ matrix whose entries are all 0 except the (i, j)th entry, which is 1. In general, if $a \in \mathcal{B}(\mathcal{H})$, then by aE_{ij} we mean the $k \times k$ operator matrix whose entries are all zero operators except the (i, j)th entry, which is the operator a. The transpose of the matrix $A \in \mathcal{M}_k$ is denoted by A^T. For $x, y \in \mathcal{H}$, we use the notation $x \otimes y$ to denote the rank 1 operator defined by $(x \otimes y)h = \langle h, y\rangle x$. Note that if $A, B \in \mathcal{B}(\mathcal{H})$, then $A(x \otimes y)B = Ax \otimes B^*y$. If $\mathcal{H}$ is a Hilbert space, we let $\mathcal{H}^n$ denote a direct sum of n copies of $\mathcal{H}$, and we give $\mathcal{H}^n$ the ℓ^2-norm. We then have, for any $n \in \mathbb{N}$, that $\mathcal{B}(\mathcal{H}^n)$ is isomorphic to $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$, the set of all $n \times n$ matrices with entries in $\mathcal{B}(\mathcal{H})$.

Definition 2.1. Let V, W, Z be vector spaces over a field $\mathbb{F}$, and let $\mathcal{L}(V)$ denote the set of linear maps on V. Suppose that $\mathcal{A} \subseteq \mathcal{L}(V)$ and $\mathcal{B} \subseteq \mathcal{L}(W)$, and let $\phi : \mathcal{A} \times \mathcal{B} \to \mathcal{L}(Z)$ be a bilinear map. We say that ϕ is

(1) *a skew-compression* if there are linear maps $A : V \to Z$, $B : W \to V$, and $C : Z \to W$ such that $\phi(a, b) = AaBbC$ for all $a \in \mathcal{A}, b \in \mathcal{B}$,

(2) *rank nonincreasing* if $\operatorname{rank}(\phi(a, b)) \leq \min\{\operatorname{rank}(a),\ \operatorname{rank}(b)\}$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, and

(3) *CRNI* if for each $k \in \mathbb{N}$, the bilinear map ϕ_k is rank nonincreasing, where $\phi_k : \mathcal{M}_k(\mathcal{A}) \times \mathcal{M}_k(\mathcal{B}) \to \mathcal{M}_k(\mathcal{L}(Z))$ is defined by

$$\phi_k\big((a_{ij}), (b_{ij})\big) = \Big(\sum_{s=1}^{k} \phi(a_{is}, b_{sj})\Big)_{ij}.$$

Hence ϕ is a CRNI bilinear map if, for each $k \in \mathbb{N}$ and for all $(a_{ij}) \in \mathcal{M}_k(\mathcal{A})$, $(b_{ij}) \in \mathcal{M}_k(\mathcal{B})$, we have

$$\operatorname{rank}\big(\phi_k\big((a_{ij}), (b_{ij})\big)\big) \leq \min\big\{\operatorname{rank}(a_{ij}), \operatorname{rank}(b_{ij})\big\}.$$

Note that the definition of ϕ_k is intimately related to the definition of matrix multiplication.

The multilinear definition of CRNI can be similarly constructed. In this article, we focus on the more interesting case where $\phi : \mathcal{A} \times \mathcal{B} \to \mathcal{B}(\mathcal{H})$ with $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H}_2)$. In fact, for the sake of simplicity, we only consider the bilinear maps for which $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. The reader should note that most of our proofs may be trivially modified to cover the multilinear maps in the most general form.

The following example shows bilinear maps that are rank nonincreasing, but not CRNI.

Example 2.2. Define $\phi : \mathcal{M}_2 \times \mathcal{M}_2 \to \mathbb{C}$ and $\psi : \mathcal{M}_2 \times \mathcal{M}_2 \to \mathcal{M}_2$ by $\phi(A, B) = \operatorname{tr}(AB)$ and $\psi(A, B) = B^T A^T$. For

$$\widetilde{A} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = (E_{ij}) \in \mathcal{M}_2(\mathcal{M}_2),$$

we have $\phi_2(\widetilde{A}, \widetilde{A}) = 2I_2$ and $\psi_2(\widetilde{A}, \widetilde{A}) = (E_{ji})$. We have $\operatorname{rank}(\widetilde{A}) = 1$, $\operatorname{rank}(\phi_2(\widetilde{A}, \widetilde{A})) = 2$, and $\operatorname{rank}(\psi_2(\widetilde{A}, \widetilde{A})) = 4$. Then ϕ and ψ are not CRNI, but they are clearly rank-nonincreasing bilinear maps.

It is worth pointing out some very basic facts about CRNI linear and bilinear maps.

Remark 2.3. The following simple facts can be easily verified.

(1) If $\phi : \mathcal{A} \times \mathcal{B} \to \mathcal{B}(\mathcal{H})$ is a skew-compression bilinear map, then ϕ is CRNI.

(2) Let $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ be a linear map, and define the bilinear map $\phi : \mathcal{A} \times \mathbb{C} \to \mathcal{B}(K)$ by $\phi(a, c) = c\psi(a)$. Then ψ is a CRNI map if and only if ϕ is a CRNI bilinear map.

(3) If $\phi : \mathcal{A} \times \mathcal{B} \to B(\mathcal{H})$ is a CRNI bilinear map, then for fixed $A_0 \in \mathcal{A}$ and $B_0 \in \mathcal{B}$, the maps $B \to \phi(A_0, B)$ and $A \to \phi(A, B_0)$ are CRNI linear maps.

(4) If $\phi : \mathcal{A} \times \mathcal{B} \to B(\mathcal{H})$ is CRNI and $X, Y \in B(\mathcal{H})$, then the bilinear map $\psi : \mathcal{A} \times \mathcal{B} \to B(\mathcal{H})$ defined by $\psi(a,b) = X\phi(a,b)Y$ is CRNI.

If $\{\phi_\lambda\}$ is a net of maps on $\mathcal{A}$, we say that $\phi_\lambda \to \phi$ *point-strongly* if, for every $a \in \mathcal{A}$, $\phi_\lambda(a) \to \phi(a)$ in the strong operator topology. Suppose that $\phi : \mathcal{A} \times \mathcal{B} \to B(\mathcal{H})$ defined by $\phi(a,b) = \lim A_\lambda a B_\lambda b C_\lambda$ is a point-strong limit of skew-compressions. Then for each $k \in \mathbb{N}$ and each $(a_{ij}) \in \mathcal{M}_k(\mathcal{A})$, $(b_{ij}) \in \mathcal{M}_k(\mathcal{B})$, we have

$$\begin{aligned}\phi_k\big((a_{ij}),(b_{ij})\big) &= \Big(\sum_{s=1}^{k} A_\lambda a_{is} B_\lambda b_{sj} C_\lambda\Big)_{ij} \\ &= \begin{pmatrix} A_\lambda & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_\lambda \end{pmatrix} (a_{ij}) \begin{pmatrix} B_\lambda & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B_\lambda \end{pmatrix} (b_{ij}) \begin{pmatrix} C_\lambda & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & C_\lambda \end{pmatrix}.\end{aligned}$$

Therefore, ϕ is CRNI. Hence, a necessary condition for a bilinear map to be a limit of skew-compressions is that it be CRNI. We make the following conjecture, similar to that in [9].

Conjecture 2.4 ([9, Conjecture 1])**.** *A bilinear map $\phi : \mathcal{A} \times \mathcal{B} \to B(\mathcal{H})$ is CRNI if and only if ϕ is a point-strong limit of skew-compressions.*

As in [9], our results require a more general notion of CRNI.

Definition 2.5. Let $k, s \in \mathbb{N}$. A bilinear map $\phi : \mathcal{A} \times \mathcal{B} \to B(\mathcal{H})$ is said to be (k,s)*-rank nonincreasing* if

$$\operatorname{rank} \phi(a,b) \le \min\big\{k \cdot \operatorname{rank}(a), s \cdot \operatorname{rank}(b)\big\}, \quad \forall a \in \mathcal{A}, \forall b \in \mathcal{B}$$

We say that ϕ is *completely* (k,s)*-rank nonincreasing* if ϕ_n is (k,s)-rank nonincreasing for every $n \in \mathbb{N}$. Bilinear maps that are completely (k,k)-rank-nonincreasing maps are called *completely k-rank nonincreasing*.

The map ϕ defined in Example 2.2 is not CRNI, but it is easy to see that it is completely rank 2 nonincreasing.

3. Main results

Our first result reduces the above conjecture to the case of bilinear functionals (see Theorem 3.1). The process is very similar to that in [9], and the key idea is a classical identification of the set of all linear maps from a vector space V into $\mathcal{M}_N$ and the set of linear functionals on $\mathcal{M}_N(V)$. This correspondence has been used in the study of completely positive and completely bounded maps (see [1], [12]) and also in the study of CRNI maps in [9]. Let $\phi : \mathcal{A} \times \mathcal{B} \to \mathcal{M}_N(\mathbb{C})$ be a bilinear map, and for $a \in \mathcal{A}$ and $b \in \mathcal{B}$ write $\phi(a,b) = (\phi_{ij}(a,b))$. For $(a_{ij}) \in \mathcal{M}_N(\mathcal{A})$ and $(b_{ij}) \in \mathcal{M}_N(\mathcal{B})$, define $\widehat{\phi} : \mathcal{M}_N(\mathcal{A}) \times \mathcal{M}_N(\mathcal{B}) \to \mathbb{C}$ by

$$\widehat{\phi}\big((a_{ij}),(b_{ij})\big) = \frac{1}{N} \sum_{i,j=1}^{N} \Big[\sum_{s=1}^{N} \phi_{ij}(a_{is}, b_{sj})\Big].$$

If $aE_{pq} = (a'_{ij}) \in \mathcal{M}_N(\mathcal{A})$ and $bE_{kl} = (b'_{ij}) \in \mathcal{M}_N(\mathcal{B})$, then for $i \neq p$ or $j \neq l$ we have $a'_{is} = 0$ or $b'_{sj} = 0$. Hence

$$\widehat{\phi}(aE_{pq}, bE_{kl}) = \frac{1}{N}\phi_{pl}(a, b).$$

The above relation allows us to recover ϕ from $\widehat{\phi}$. In fact, for $a \in \mathcal{A}$ and $b \in \mathcal{B}$, let $\widehat{A} = [aE_{ij}] \in \mathcal{M}_{N^2}(\mathcal{A})$ and $\widehat{B} = [bE_{ij}] \in \mathcal{M}_{N^2}(\mathcal{B})$. Then

$$\begin{aligned}(\widehat{\phi})_N(\widehat{A}, \widehat{B}) &= \Big[\Big(\sum_{s=1}^{N} \widehat{\phi}(aE_{is}, bE_{sj})\Big)\Big] \\ &= \big[\phi_{ij}(a, b)\big] = \phi(a, b),\end{aligned}$$

and

$$\operatorname{rank}(\widehat{A}) = \operatorname{rank}(a), \qquad \operatorname{rank}(\widehat{B}) = \operatorname{rank}(b).$$

Now suppose that

$$A = (a_{ij}) \in \mathcal{M}_N(\mathcal{A}), \qquad B = (b_{ij}) \in \mathcal{M}_N(\mathcal{B}),$$

and that $G = (1, 1, \ldots, 1)$ is the $1 \times N^2$ matrix. Then

$$\begin{aligned}\widehat{\phi}(A, B) &= G \operatorname{diag}(E_{11}, E_{22}, \ldots, E_{NN})\phi_N(A, B) \operatorname{diag}(E_{11}, E_{22}, \ldots, E_{NN})G^T \\ &= C\phi_N(A, B)C^T,\end{aligned}$$

where $C = G \operatorname{diag}(E_{11}, E_{22}, \ldots, E_{NN})$.

We are ready to reduce our conjecture to the case of bilinear functionals.

Theorem 3.1. *Let $\phi : \mathcal{A} \times \mathcal{B} \to \mathcal{M}_N(\mathbb{C})$ be a bilinear map. Then*

(1) *ϕ is completely (p, q)-rank nonincreasing if and only if $\widehat{\phi}$ is completely (p, q)-rank nonincreasing,*
(2) *ϕ is skew-compression if and only if $\widehat{\phi}$ is skew-compression,*
(3) *ϕ is a point-SOT limit of skew-compressions if and only if $\widehat{\phi}$ is a point-SOT limit of skew-compressions.*

Proof.

(1) Suppose that ϕ is completely (p, q)-rank nonincreasing. Let $A = (a_{ij}) \in \mathcal{M}_N(\mathcal{A})$ and $B = (b_{ij}) \in \mathcal{M}_N(\mathcal{B})$. It was shown above that $\widehat{\phi}(A, B) = C\phi_N(A, B)C^T$. For $(A_{ij}) \in \mathcal{M}_N(\mathcal{M}_N(\mathcal{A}))$ and $(B_{ij}) \in \mathcal{M}_N(\mathcal{M}_N(\mathcal{B}))$, we have

$$\begin{aligned}(\widehat{\phi})_m\big((A_{ij}), (B_{ij})\big) &= \Big(\sum_{s=1}^{m} C\phi_N(A_{is}, B_{sj})C^T\Big)_{ij} \\ &= \operatorname{diag}(C, \ldots, C)(\phi_N)_m(A_{ij}, B_{ij}) \operatorname{diag}(C^T, \ldots, C^T).\end{aligned}$$

Then

$$\begin{aligned}\operatorname{rank}\big[(\widehat{\phi})_m(A_{ij}, B_{ij})\big] &\leq \operatorname{rank}\big[(\phi_N)_m(A_{ij}, B_{ij})\big] \\ &\leq \min\big\{p \operatorname{rank}(A_{ij}), q \operatorname{rank}(B_{ij})\big\}.\end{aligned}$$

Therefore, $\widehat{\phi}$ is completely (p,q)-rank nonincreasing.

Now suppose that $\widehat{\phi}$ is completely (p,q)-rank nonincreasing. Recall that for $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have $\phi(a,b) = \widehat{\phi}_N(\widehat{A}, \widehat{B})$, where $\widehat{A} = (aE_{ij})$ and $\widehat{B} = (bE_{ij})$, and that $\operatorname{rank}(\widehat{A}) = \operatorname{rank}(a)$, $\operatorname{rank}(\widehat{B}) = \operatorname{rank}(b)$. Then a similar argument as above shows that ϕ is completely (p,q)-rank nonincreasing.

(2) Suppose that $\phi(a,b) = xaybz$ for some operators x, y, z. Let $A = (a_{ij}) \in \mathcal{M}_N(\mathcal{A})$ and $B = (b_{ij}) \in \mathcal{M}_N(\mathcal{B})$. Then

$$\begin{aligned}\widehat{\phi}(A,B) &= C\phi_N(A,B)D \\ &= C\operatorname{diag}(x,\dots,x)A\operatorname{diag}(y,\dots,y)B\operatorname{diag}(z,\dots,z)D.\end{aligned}$$

Hence $\widehat{\phi}$ is skew-compression. Conversely, if $\widehat{\phi}$ is skew-compression, then it follows from the relation $\phi(a,b) = (\widehat{\phi})_N((aE_{ij}), (bE_{ij}))$ that ϕ is skew-compression.

(3) The proof is similar to statement (2) above. □

It is reasonable to think that if $\phi : \mathcal{A} \times \mathcal{B} \to \mathcal{B}(\mathcal{H})$ is a CRNI bilinear map, then for fixed $b_0 \in \mathcal{B}$, the linear map $\lambda : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ defined by $\lambda(a) = \phi(a, b_0)$ should also be CRNI. The main reason that this is true is because if $A \in \mathcal{M}_n(\mathcal{A})$ and $B_0 = \operatorname{diag}(b_0, b_0, \dots, b_0)$, then we have $\lambda_n(A) = \phi_n(A, B_0)$.

Lemma 3.2. *Let $\phi : \mathcal{A} \times \mathcal{B} \to \mathcal{B}(\mathcal{H})$ be a bilinear map, and let $n \in \mathbb{N}$. Then*

(1) *ϕ is skew-compression if and only if ϕ_n is skew-compression,*

(2) *if ϕ is completely (p,q)-rank nonincreasing, then for each $a_0 \in \mathcal{A}$ and $b_0 \in \mathcal{B}$, the maps $\lambda_{b_0}(a) = \phi(a, b_0)$ and $\mu_{a_0}(b) = \phi(a_0, b)$ are completely p-rank- and q-rank-nonincreasing linear maps, respectively.*

Proof.

(1) We only prove the backward direction. Suppose that there are matrices X, Y, and Z such that $\phi_n((a_{ij}), (b_{ij})) = X(a_{ij})Y(b_{ij})Z$. For $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we define the operator (block) matrices A and B by

$$A = aE_{11} = \begin{bmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_n(\mathcal{A})$$

and

$$B = bE_{11} = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_n(\mathcal{B}).$$

We can write X, Y, and Z as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \qquad Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

and

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}.$$

Therefore,

$$\begin{aligned}\begin{bmatrix}\phi(a,b) & 0\\ 0 & 0\end{bmatrix} &= \phi_n(A,B)\\ &= XAYBZ\\ &= \begin{bmatrix}X_{11}aY_{11}bZ_{11} & X_{11}aY_{11}bZ_{12}\\ X_{21}aY_{11}bZ_{11} & X_{21}aY_{11}bZ_{12}\end{bmatrix}.\end{aligned}$$

Then $\phi(a,b) = X_{11}aY_{11}bZ_{11}$, and hence ϕ is skew-compression.

(2) Let $b_0 \in \mathcal{B}$ be fixed, and let $A \in \mathcal{M}_n(\mathcal{A})$. For $B_0 = \operatorname{diag}(b_0, b_0, \ldots, b_0)$, we have $(\lambda_{b_0})_n(A) = \phi_n(A, B_0)$. Then

$$\begin{aligned}\operatorname{rank}\big[(\lambda_{b_0})_n(A)\big] &= \operatorname{rank}\big[\phi_n(A,B_0)\big]\\ &\le \min\{p \operatorname{rank} A, q \operatorname{rank} B_0\}\\ &\le p \operatorname{rank} A.\end{aligned}$$

Hence λ_{b_0} is completely p-rank nonincreasing. Similarly, μ is completely q-rank nonincreasing. □

It is evident that the product of two CRNI linear maps gives a CRNI bilinear map. The next corollary is a slight generalization of this.

Corollary 3.3. *Let $\psi_1 : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and $\psi_2 : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ be linear maps, and define $\phi : \mathcal{A} \times \mathcal{B} \to \mathcal{B}(\mathcal{H})$ by $\phi(A,B) = \psi_1(A)\psi_2(B)$.*

(1) *If ψ_1 and ψ_2 are completely k_1-rank- and k_2-rank-nonincreasing linear maps, respectively, then ϕ is a completely (k_1,k_2)-rank nonincreasing bilinear map.*

(2) *The converse of statement (1) holds if ψ_1 and ψ_2 have invertible operators in their ranges.*

Proof.

(1) Let ψ_1 and ψ_2 be completely k_1-rank- and k_2-rank-nonincreasing linear maps, respectively. We have

$$\begin{aligned}\phi_n\big((A_{ij}),(B_{ij})\big) &= \Big(\sum_{s=1}^{n} \psi_1(A_{is})\psi_2(B_{sj})\Big)_{ij}\\ &= \big[(\psi_1)_n(A_{ij})\big]\cdot\big[(\psi_2)_n(B_{ij})\big].\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{rank}\big[\phi_n\big((A_{ij}),(B_{ij})\big)\big] &\le \min\big\{\operatorname{rank}(\psi_1)_n(A_{ij}), \operatorname{rank}(\psi_2)_n(B_{ij})\big\}\\ &\le \min\big\{k_1 \operatorname{rank}(A_{ij})_{ij}, k_2 \operatorname{rank}(B_{ij})_{ij}\big\}.\end{aligned}$$

Therefore, ϕ is completely (k_1,k_2)-rank nonincreasing.

(2) Let ϕ be completely (k_1,k_2)-rank nonincreasing. Choose $B_0 \in \mathcal{B}$ such that $\psi_2(B_0)$ is invertible. It follows from Lemma 3.2 that $\phi(A,B_0) = \psi_1(A)\psi_2(B_0)$ is completely k_1-rank nonincreasing. Since $\psi_2(B_0)$ is invertible, then ψ_1 is completely k_1-rank nonincreasing. Similarly, ψ_2 is completely k_2-rank nonincreasing. □

To see why the extra assumption in statement (2) of Corollary 3.3 is needed, define $\psi_1, \psi_2 : \mathbb{C} \to \mathcal{M}_3(\mathbb{C})$ by

$$\psi_1(a) = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \psi_2(b) = b \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\phi(a,b) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & ab & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is CRNI, but ψ_1 and ψ_2 are not even rank nonincreasing.

The following result characterizes specific completely (p,q)-rank-nonincreasing bilinear functionals defined on the set of compact operators. In the following result, when we say that k_1 and k_2 are the smallest numbers for which the bilinear map $\phi : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ is completely (k_1,k_2)-rank nonincreasing, we mean that if ϕ is also completely (p,q)-rank nonincreasing, then $k_1 \le p$ and $k_2 \le q$.

Lemma 3.4. *Suppose that T, T_1, and T_2 are nonzero trace-class operators.*

(1) *Let $\phi : \mathcal{K}(H) \times \mathcal{K}(H) \to \mathbb{C}$ be defined by $\phi(A,B) = \operatorname{tr}(TAB)$. Then the smallest positive integers k_1 and k_2 for which ϕ is completely (k_1,k_2)-rank nonincreasing is when $k_1 = k_2 = \operatorname{rank}(T)$.*

(2) *Let $\psi : \mathcal{K}(H) \times \mathcal{K}(H) \to \mathbb{C}$ be defined by $\psi(A,B) = \operatorname{tr}(T_1A)\operatorname{tr}(T_2B)$. Then the smallest positive integers k_1 and k_2 for which ψ is completely (k_1,k_2)-rank nonincreasing is when $k_1 = \operatorname{rank}(T_1)$ and $k_2 = \operatorname{rank}(T_2)$.*

(3) *Let $\phi : \mathcal{K}(H) \times \mathcal{K}(H) \to \mathbb{C}$ be defined by $\phi(A,B) = \operatorname{tr}(T_1AT_2B)$. Then the smallest positive integers k_1 and k_2 for which ϕ is completely (k_1,k_2)-rank nonincreasing is when $k_1 = k_2 = \min\{\operatorname{rank}(T_1), \operatorname{rank}(T_2)\}$.*

Proof.

(1) Suppose that $\phi(A,B) = \operatorname{tr}(TAB)$ is completely (k_1,k_2)-rank nonincreasing, and fix $B_0 \in \mathcal{K}(H)$. Then the map $\alpha : \mathcal{K}(H) \to \mathbb{C}$ defined by $\alpha(A) = \operatorname{tr}(TAB_0) = \operatorname{tr}(B_0TA)$ is completely k_1-rank nonincreasing. By [9, Lemma 1], we have that $\operatorname{rank}(B_0T) \le k_1$ for any $B_0 \in \mathcal{K}(H)$. Therefore, $\operatorname{rank}(T) \le k_1$. Similarly, $\operatorname{rank}(T) \le k_2$. On the other hand, if $\operatorname{rank}(T) = 1$, then $T = e \otimes f$; hence

$$\begin{aligned} \phi(A,B) &= \operatorname{tr}\big((e \otimes f)AB\big) \\ &= \operatorname{tr}(ABe \otimes f) \\ &= \langle ABe, f\rangle \end{aligned}$$

is a skew-compression, which is CRNI. If $\operatorname{rank}(T) = k$, then T is the sum of k rank 1 transformations. So ϕ is the sum of k many CRNI maps, and hence it is completely (k,k)-rank nonincreasing.

(2) This follows from Lemma 1 in [9] and Corollary 3.3.

(3) Suppose that ϕ is completely (k_1,k_2)-rank nonincreasing. Then for fixed A_0 and B_0, the linear maps $\alpha, \beta : \mathcal{K}(\mathcal{H}) \to \mathbb{C}$ defined by $\alpha(A) =$

$\operatorname{tr}(T_1AT_2B_0) = \operatorname{tr}(T_2B_0T_1A)$ and $\beta(B) = \operatorname{tr}(T_1A_0T_2B)$ are completely k_1-rank and k_2-rank nonincreasing, respectively. By [9, Lemma 1], $\operatorname{rank}(T_2B_0T_1) \leq k_1$ and $\operatorname{rank}(T_1A_0T_2) \leq k_2$ for every $A_0 \in \mathcal{K}(\mathcal{H})$ and $B_0 \in \mathcal{K}(K)$. Then

$$\min\bigl\{\operatorname{rank}(T_1), \operatorname{rank}(T_2)\bigr\} \leq k_1$$

and

$$\min\bigl\{\operatorname{rank}(T_1), \operatorname{rank}(T_2)\bigr\} \leq k_2.$$

On the other hand, suppose that $\min\{\operatorname{rank}(T_1), \operatorname{rank}(T_2)\} = 1$. Without loss of generality, assume that $\operatorname{rank}(T_2) = 1$. So $T_2 = e \otimes f$ and

$$\begin{aligned}\phi(A, B) &= \operatorname{tr}\bigl(BT_1A(e \otimes f)\bigr)\\ &= \langle BT_1Ae, f\rangle\end{aligned}$$

is therefore a skew-compression map. Hence ϕ is CRNI. If $\min\{\operatorname{rank}(T_1), \operatorname{rank}(T_2)\} = k = \operatorname{rank}(T_2)$, then T_2 is the sum of k rank 1 transformations. So $T_2 = \sum_{i=1}^{k} e_i \otimes f_i$ and

$$\begin{aligned}\phi(A, B) &= \sum_{i=1}^{k} \operatorname{tr}\bigl(BT_1A(e_i \otimes f_i)\bigr)\\ &= \sum_{i=1}^{k} \langle BT_1Ae_i, f_i\rangle\end{aligned}$$

is the sum of k CRNI maps. Hence ϕ is completely (k, k)-rank nonincreasing. □

As an immediate consequence of the preceding lemma, the nonzero bilinear maps $\phi : \mathcal{M}_r \times \mathcal{M}_s \to \mathbb{C}$ defined by $\phi(A, B) = \operatorname{tr}(XA)\operatorname{tr}(YB)$, $\phi(A, B) = \operatorname{tr}(XAB)$, or $\phi(A, B) = \operatorname{tr}(XAYB)$ are CRNI if and only if $\operatorname{rank}(X) = 1 = \operatorname{rank}(Y)$.

It is a well-known fact that a continuous linear map $\psi : \mathcal{K}(\mathcal{H}) \to \mathbb{C}$ can be written as $\psi(A) = \operatorname{tr}(AK)$ for some trace-class operator K. Peter Šemrl pointed out that, for a continuous bilinear map $\phi : \mathcal{K}(\mathcal{H}) \times \mathcal{K}(\mathcal{H}) \to \mathbb{C}$, there exists a bounded linear map $\alpha : \mathcal{K}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ such that $\phi(A, B) = \operatorname{tr}(A\alpha(B))$, where $\mathcal{T}(\mathcal{H})$ denotes the ideal of the trace-class operators. This sparked the idea of the proof of Theorem 3.5. Another key idea in the proof of the following theorem is the fact that if $\mathcal{M}$ is a subspace of $\mathcal{B}(\mathcal{H})$ that contains only elements of rank 0 or 1, then $\mathcal{M} \subseteq z_0 \otimes \mathcal{H}$ or $\mathcal{M} \subseteq \mathcal{H} \otimes z_0$ for some $z_0 \in \mathcal{H}$.

Theorem 3.5. *Let $\phi : \mathcal{K}(\mathcal{H}) \times \mathcal{K}(\mathcal{H}) \to \mathbb{C}$ be a CRNI continuous bilinear map. Then ϕ is CRNI if and only if there exist an operator $D \in \mathcal{K}(\mathcal{H})$ and a rank 1 operator F such that for all $A, B \in \mathcal{K}(\mathcal{H})$, either $\phi(A, B) = \operatorname{tr}(ADBF)$ or $\phi(A, B) = \operatorname{tr}(AFBD)$. In particular, ϕ is skew-compression.*

Proof. The backward direction follows from Lemma 3.4. For the forward direction, suppose that ϕ is a non-identically zero CRNI continuous map, and fix $B \in$

$\mathcal{K}(\mathcal{H})$. Since the map $A \to \phi(A,B)$ is a continuous linear functional, then there exists a unique trace-class operator C_B such that $\phi(A,B) = \operatorname{tr}(AC_B)$. Define $\alpha : \mathcal{K}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ by $\alpha(B) = C_B$. Clearly α is linear and bounded, and since ϕ is not the zero map, then α is not identically zero. It follows from Lemma 3.4 that $\operatorname{rank}(\alpha(B)) \le 1$ for all $B \in \mathcal{K}(\mathcal{H})$. Since every element of $\alpha(\mathcal{K}(\mathcal{H}))$ has rank 0 or 1, then by the remark made above, we have $\alpha(B) = f(B) \otimes z_0$ or $\alpha(B) = z_0 \otimes f(B)$ for some $z_0 \in \mathcal{H}$ and some bounded linear or conjugate linear map $f : \mathcal{K}(\mathcal{H}) \to \mathcal{H}$. First, assume the case where $\alpha(B) = f(B) \otimes z_0$. The fact that α is not identically zero allows us to choose $x_1, y_1 \in \mathcal{H}$ and $A_1 \in \mathcal{K}(\mathcal{H})$ such that $\langle f(x_1 \otimes y_1), A_1^* z_0\rangle = 1$. For $x, y \in \mathcal{H}$ and $A_2 \in \mathcal{K}(\mathcal{H})$, let

$$S = \begin{bmatrix} x_1 \otimes y_1 & x_1 \otimes y \\ x \otimes y_1 & x \otimes y \end{bmatrix} \qquad \text{and} \qquad \widetilde{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathcal{M}_2\big(\mathcal{K}(\mathcal{H})\big).$$

Then we have

$$\phi_2(\widetilde{A}, S) = \begin{bmatrix} 1 & \langle f(x_1 \otimes y), A_1^* z_0\rangle \\ \langle f(x \otimes y_1), A_2^* z_0\rangle & \langle f(x \otimes y), A_2^* z_0\rangle \end{bmatrix}.$$

Since ϕ is CRNI and $\operatorname{rank}(S) = 1$, then the columns of $\phi_2(\widetilde{A}, S)$ must be linearly dependent. Hence $\langle f(x \otimes y), A_2^* z_0\rangle = \gamma(y)\langle f(x \otimes y_1), A_2^* z_0\rangle$, where $\gamma(y) = \langle f(x_1 \otimes y), A_1^* z_0\rangle \in \mathbb{C}$. It follows that $f(x \otimes y) = \gamma(y) f(x \otimes y_1)$. Since $\gamma : \mathcal{H} \to \mathbb{C}$ is continuous and conjugate linear, then $\exists h_0 \in \mathcal{H}$ such that $\gamma(y) = \langle h_0, y\rangle$. Define the map $D \in \mathcal{B}(\mathcal{H})$ by $D(x) = f(x \otimes y_1)$. Then

$$\begin{aligned} \alpha(x \otimes y) &= \gamma(y) D(x) \otimes z_0 \\ &= D\big(\langle h_0, y\rangle x \otimes z_0\big) \\ &= D(x \otimes y)(h_0 \otimes z_0). \end{aligned}$$

Consequently, for every finite-rank operator F, we have $\alpha(F) = DF(h_0 \otimes z_0)$. It follows from the continuity of α and density of $\mathcal{F}(\mathcal{H})$ in $\mathcal{K}(\mathcal{H})$ that $\alpha(B) = DB(h_0 \otimes z_0)$ for all $B \in \mathcal{K}(\mathcal{H})$. Thus α is skew-compression and

$$\begin{aligned} \phi(A,B) &= \operatorname{tr}\big(A\alpha(B)\big) \\ &= \operatorname{tr}\big(ADB(h_0 \otimes z_0)\big) \\ &= \langle ADBh_0, z_0\rangle. \end{aligned}$$

Therefore, ϕ is skew-compression.

Now assume the case $\alpha(B) = z_0 \otimes f(B)$, where f is bounded and conjugate linear. Choose $x_1, y_1 \in \mathcal{H}$ and $A_1 \in \mathcal{K}(\mathcal{H})$ such that $\langle A_1 z_0, f(x_1 \otimes y_1)\rangle = 1$. For $x, y \in \mathcal{H}$ and $A_2 \in \mathcal{K}(\mathcal{H})$, let

$$S = \begin{bmatrix} x_1 \otimes y_1 & x_1 \otimes y \\ x \otimes y_1 & x \otimes y \end{bmatrix} \qquad \text{and} \qquad \widetilde{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathcal{M}_2\big(\mathcal{K}(\mathcal{H})\big).$$

Then we have

$$\phi_2(\widetilde{A}, S) = \begin{bmatrix} 1 & \langle A_1 z_0, f(x_1 \otimes y)\rangle \\ \langle A_2 z_0, f(x \otimes y_1)\rangle & \langle A_2 z_0, f(x \otimes y)\rangle \end{bmatrix}.$$

Since $\operatorname{rank}(S) = 1$, then the rows of $\phi_2(\widetilde{A}, S)$ must be linearly dependent. Then $\langle A_2 z_0, f(x \otimes y)\rangle = \gamma(x)\langle A_1 z_0, f(x_1 \otimes y)\rangle$, where $\gamma(x) = \langle A_2 z_0, f(x \otimes y_1)\rangle \in \mathbb{C}$. Hence $f(x \otimes y) = \overline{\gamma}(x) f(x_1 \otimes y)$. Since the map $\gamma : \mathcal{H} \to \mathbb{C}$ is continuous and linear, then $\exists h_0 \in \mathcal{H}$ such that $\gamma(x) = \langle x, h_0\rangle$. Since f is conjugate linear, then the map $D : \mathcal{H} \to \mathcal{H}$ defined by $D(y) = f(x_1 \otimes y)$ is in $\mathcal{B}(\mathcal{H})$. Then we have

$$\begin{aligned}\alpha(x \otimes y) &= z_0 \otimes \overline{\gamma(x)} D(y)\\ &= z_0 \otimes \langle h_0, x\rangle D(y)\\ &= z_0 \otimes (Dy \otimes x) h_0\\ &= (z_0 \otimes h_0)(Dy \otimes x)^*\\ &= (z_0 \otimes h_0)(x \otimes y) D^*.\end{aligned}$$

It follows again that $\alpha(B) = (z_0 \otimes h_0) B D^*$ for all $B \in \mathcal{K}(\mathcal{H})$. □

Let $\phi : \mathcal{A} \times \mathcal{B} \to \mathcal{B}(\mathcal{H})$ be bilinear, and let

$$\mathcal{A}^{(p)} = \big\{\operatorname{diag}(a, \ldots, a) \in \mathcal{M}_p(\mathcal{A}) : a \in \mathcal{A}\big\}.$$

Define $\mathcal{B}^{(q)}$ similarly, and define $\phi_{(p,q)} : \mathcal{A}^{(p)} \times \mathcal{B}^{(q)} \to \mathcal{M}_r(\mathcal{B}(\mathcal{H}))$ by $\phi_{(p,q)}(A, B) = \phi(a, b)$. It is clear that ϕ is completely (p, q)-rank nonincreasing if and only if $\phi_{(p,q)}$ is CRNI.

Theorem 3.6. *Suppose that $\phi : \mathcal{K}(\mathcal{H}) \times \mathcal{K}(\mathcal{H}) \to \mathcal{M}_n$ is a continuous bilinear map. Then ϕ is completely (p, q)-rank nonincreasing if and only if there are operators R, S, and T such that*

$$\phi(A, B) = R A^{(p)} S B^{(q)} T$$

for every $A, B \in \mathcal{K}(\mathcal{H})$.

Proof. Suppose that ϕ is completely (p, q)-rank nonincreasing. Then the bilinear map $\phi_{(p,q)} : \mathcal{K}(\mathcal{H})^{(p)} \times \mathcal{K}(\mathcal{H})^{(q)} \to \mathcal{M}_r(\mathbb{C})$ defined above is CRNI. The result follows from Theorem 3.5. The other direction of the theorem is easy to prove. □

The following theorem is a special case of Theorem 3.5 when $\mathbb{F} = \mathbb{C}$ and $d = c$. An interesting aspect of the next theorem is its elementary constructive proof.

Theorem 3.7. *Suppose that $\mathbb{F}$ is a field and that $\phi : \mathcal{M}_c(\mathbb{F}) \times \mathcal{M}_d(\mathbb{F}) \to \mathbb{C}$ is a CRNI bilinear map. Then ϕ is skew-compression.*

Proof. We assume that ϕ is not identically zero. We prove the statement for the case where $c = d = k$; the general statement is proved similarly. We also assume that $\phi(E_{11}, E_{11}) = 1$. Since $\operatorname{rank}((E_{ij})) = 1$, then $\operatorname{rank}(\phi_n((E_{ij}), \widehat{G})) \le 1$ and $\operatorname{rank}(\phi_n(\widehat{G}, (E_{ij}))) \le 1$ for any $\widehat{G} \in \mathcal{M}_n(\mathcal{M}_k(\mathbb{C}))$. Let $\widehat{G} \in \mathcal{M}_n(\mathcal{M}_k(\mathbb{C}))$ be the matrix that has E_{11} in its $(1,1)$-position, let matrices $G_1, G_2, \ldots, G_n$ be in its second row, and let the matrix $0_{k\times k}$ be elsewhere. Since ϕ is CRNI, then the following matrix has rank 1:

$$\phi_n\left(\begin{bmatrix} E_{11} & 0 & \cdots & 0 \\ G_1 & G_2 & & G_n \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & E_{22} & & E_{2n} \\ \vdots & & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{bmatrix}\right)$$
$$= \begin{bmatrix} \phi(E_{11}, E_{11}) & \phi(E_{11}, E_{12}) & \cdots & \phi(E_{11}, E_{1k}) \\ \sum_{i=1}^n \phi(G_i, E_{i1}) & \sum_{i=1}^n \phi(G_i, E_{i2}) & & \sum_{i=1}^n \phi(G_i, E_{in}) \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Let $p_j = \phi(E_{11}, E_{1j})$ for $j = 1, \dots, n$. Since the jth column of the above matrix is p_j times the first column (for any choice of G_i), then we have

$$\phi(G, E_{ij}) = p_j \phi(G, E_{i1}), \quad \forall G \in \mathcal{M}_k(\mathbb{C}), \forall i, \forall j.$$

Also, the rank of the following matrix is 1:

$$\phi_n\left(\begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & E_{22} & & E_{2n} \\ \vdots & & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{bmatrix}, \begin{bmatrix} E_{11} & G_1 & \cdots & 0 \\ 0 & G_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & G_n & \cdots & 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} \phi(E_{11}, E_{11}) & \sum_{j=1}^n \phi(E_{1j}, G_j) & \cdots & 0 \\ \phi(E_{21}, E_{11}) & \sum_{j=1}^n \phi(E_{2j}, G_j) & & 0 \\ \vdots & & \ddots & \\ \phi(E_{n1}, E_{11}) & \sum_{j=1}^n \phi(E_{nj}, G_j) & \cdots & 0 \end{bmatrix}.$$

Let $q_i = \phi(E_{i1}, E_{11})$ for $i = 1, \dots, n$. Since the ith row of the above matrix is q_i times the first row (for any choice of G_j), we have

$$\phi(E_{ij}, G) = q_i \phi(E_{1j}, G), \quad \forall G \in \mathcal{M}_k(\mathbb{C}), \forall i, j.$$

Then for arbitrary $A = (a_{ij}) \in \mathcal{M}_k(\mathbb{C})$ and $B = (b_{ij}) \in \mathcal{M}_k(\mathbb{C})$, we have

$$\begin{aligned}
\phi(A, B) &= \sum_{i=1}^n \sum_{j=1}^n b_{ij} \phi(A, E_{ij}) \\
&= \sum_{i=1}^n \sum_{j=1}^n b_{ij} p_j \phi(A, E_{i1}) \\
&= \sum_{i=1}^n \Big\{ \phi(A, E_{i1}) \Big(\sum_{j=1}^n b_{ij} p_j \Big) \Big\} \\
&= \sum_{i=1}^n \Big\{ \Big(\sum_{s=1}^n \sum_{r=1}^n a_{rs} \phi(E_{rs}, E_{i1}) \Big) \Big(\sum_{j=1}^n b_{ij} p_j \Big) \Big\}
\end{aligned}$$

$$= \sum_{i=1}^{n} \Big\{ \Big(\sum_{s=1}^{n} \sum_{r=1}^{n} a_{rs} q_r \phi(E_{1s}, E_{i1}) \Big) \Big(\sum_{j=1}^{n} b_{ij} p_j \Big) \Big\}$$
$$= \sum_{i=1}^{n} \Big\{ \Big[\sum_{s=1}^{n} \phi(E_{1s}, E_{i1}) \Big(\sum_{r=1}^{n} a_{rs} q_r \Big) \Big] \Big(\sum_{j=1}^{n} b_{ij} p_j \Big) \Big\}.$$

Let $Y = (y_{is})$, where $y_{is} = \phi(E_{1s}, E_{i1})$. Then

$$\begin{aligned} \phi(A,B) &= \sum_{i=1}^{n} \Big\{ \Big[\sum_{s=1}^{n} y_{is} \Big(\sum_{r=1}^{n} a_{rs} q_r \Big) \Big] \Big(\sum_{j=1}^{n} b_{ij} p_j \Big) \Big\} \\ &= \sum_{i=1}^{n} \sum_{s=1}^{n} \Big[\Big(\sum_{r=1}^{n} a_{rs} q_r \Big) y_{is} \Big(\sum_{j=1}^{n} b_{ij} p_j \Big) \Big] \\ &= \begin{bmatrix} 1 & q_2 & \cdots & q_n \end{bmatrix} AYB \begin{bmatrix} 1 & p_2 & \cdots & p_n \end{bmatrix}^T . \qquad \square \end{aligned}$$

Theorem 1.2 is the main result of [9] that gives a characterization for the CRNI completely bounded linear maps $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ for the case where $\mathcal{H}$, $\mathcal{K}$, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ are separable. In an attempt to generalize this result to bilinear maps, we noticed that the proof of [9, Lemma 2], which is the main tool in proving Theorem 1.2, was incomplete. This was discussed and confirmed by Hadwin and Larson [9]. Below we present our slightly different and complete proof. We are still unable to extend this lemma to bilinear maps.

Lemma 3.8 ([9, Lemma 2])**.** *Suppose that $\phi : \mathcal{K}(\mathcal{H}) \to \mathcal{M}_N$ is linear, continuous, and completely k-rank nonincreasing, that k is minimal, that m is a cardinal, and that A and B are matrices such that*

$$\phi(T) = AT^{(m)}B.$$

Then there exists a projection P such that

1. *P is in the commutant of $\mathcal{K}(\mathcal{H})^{(m)} = \{T^{(m)} : T \in \mathcal{K}(\mathcal{H})\}$,*
2. *$\mathcal{K}(\mathcal{H})^{(m)}|_{\operatorname{ran}(P)} = \{T^{(m)}|_{\operatorname{ran}(P)} : T \in \mathcal{K}(\mathcal{H})\}$ is unitarily equivalent to $\mathcal{K}(\mathcal{H})^{(k)}$, that is, there is a unitary U such that $T^{(m)}|_{\operatorname{ran}(P)} = U^*T^{(k)}U$ for every $T \in \mathcal{K}(\mathcal{H})$,*
3. *$\mathcal{K}(\mathcal{H})^{(m)}|_{\operatorname{ran}(P)}$ has a cyclic vector, and*
4. *for every $T \in \mathcal{K}(\mathcal{H})$,*

$$\begin{aligned} \phi(T) &= APT^{(m)}PB \\ &= APU^*T^{(k)}UPB. \end{aligned}$$

Proof. Assume that ϕ is not identically zero. We first consider the case where $N = 1$. Since $\phi(T) = AT^{(m)}B \in \mathbb{C}$, then $B : \mathbb{C} \to \mathcal{H}^{(m)}$ and $A : \mathcal{H}^{(m)} \to \mathbb{C}$. Let $v = A^*(1)$ and $u = B(1)$. Let P' be the orthogonal projection onto $[\mathcal{K}(\mathcal{H})^{(m)}(u)]^-$, and let P be the orthogonal projection onto $[\mathcal{K}(\mathcal{H})^{(m)}(P'v)]^-$. Then P commutes with $\mathcal{K}(\mathcal{H})^{(m)}$ and since the identity map is in the weak operator closure of $\mathcal{K}(\mathcal{H})^{(m)}$, we have that $P'u = u$ and $P(P'v) = P'v$. We know that the restriction of $\mathcal{K}(\mathcal{H})^{(m)}$ to a nontrivial reducing subspace (here $\operatorname{ran}(P)$) is unitarily equivalent to $\mathcal{K}(\mathcal{H})^{(t)}$ for some $t \leq m$. Then clearly (3) holds. Since $\mathcal{K}(H)^{(m)}|_{\operatorname{ran}(P)}$ is unitarily equivalent to $\mathcal{K}(\mathcal{H})^{(t)}$, then there is a unitary U such that $T^{(m)}|_{\mathrm{ran}(P)} = U^*T^{(t)}U$ for every $T \in \mathcal{K}(\mathcal{H})$. Therefore, (2) and (4) hold if we show that $t = k$. Write

$$U(Pu) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} \in \mathcal{H}^{(t)} \qquad \text{and} \qquad U\big(P'(v)\big) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix} \in \mathcal{H}^{(t)}.$$

Since $\phi : \mathcal{K}(\mathcal{H}) \to \mathbb{C}$ is continuous, then there exists a unique trace-class operator K such that $\phi(T) = \mathrm{tr}(TK)$ for all $T \in \mathcal{K}(\mathcal{H})$. Since ϕ is completely k-rank nonincreasing and $\phi(T) = \mathrm{tr}(TK)$, then [9, Lemma 1] implies that $\mathrm{rank}(K) \le k$. In fact, the minimality of k implies that $\mathrm{rank}(K) = k$. We have

$$\begin{aligned}
\phi(T) &= \langle T^{(m)}u, v\rangle \\
&= \langle P'T^{(m)}P'u, v\rangle \\
&= \langle T^{(m)}u, PP'v\rangle \\
&= \langle PT^{(m)}u, P'v\rangle \\
&= \langle T^{(m)}Pu, P'v\rangle \\
&= \big\langle U^*T^{(t)}U(Pu), P'v\big\rangle \\
&= \big\langle T^{(t)}U(Pu), UP'v\big\rangle \\
&= \sum_i \langle Tu_i, v_i\rangle \\
&= \mathrm{tr}\Big(T\sum_i u_i \otimes v_i\Big).
\end{aligned}$$

Then $K = \sum_i u_i \otimes v_i$ and $\mathrm{rank}(\sum_i u_i \otimes v_i) = k$; hence $k \le t \le m$. The set $\{v_1, v_2, \ldots\}$ is linearly independent because $U(P'v)$ is a cyclic vector for $\mathcal{K}(\mathcal{H})^{(t)}$. Since $\mathrm{rank}(\sum_i u_i \otimes v_i) = k$, we must have $\dim(\mathrm{span}\{u_1, u_2, \ldots\}) = k$.

Similarly, since $\mathcal{K}(\mathcal{H})^{(m)}|_{\mathrm{ran}(P')}$ is unitarily equivalent to $\mathcal{K}(\mathcal{H})^{(r)}$, then there is a unitary W such that $T^{(m)}|_{\mathrm{ran}(P')} = W^*T^{(r)}W$ for every $T \in \mathcal{K}(\mathcal{H})$. Write

$$W(u) = \begin{bmatrix} u'_1 \\ u'_2 \\ \vdots \end{bmatrix} \in \mathcal{H}^{(r)} \qquad \text{and} \qquad W\big(P'(v)\big) \begin{bmatrix} v'_1 \\ v'_2 \\ \vdots \end{bmatrix} \in \mathcal{H}^{(r)}.$$

Since $W(u)$ is a cyclic vector for $\mathcal{K}(\mathcal{H})^{(r)}$, then $\{u'_1, u'_2, \ldots\}$ must be linearly independent. A similar argument to the one above shows that $\dim(\mathrm{span}\{v'_1, v'_2, \ldots\}) = k$. Therefore, $\dim(\mathrm{span}\{v, v_2, \ldots\}) = k$ and since $\{v_1, v_2, \ldots\}$ is linearly independent, then $t = k$.

For the general case, we refer the reader to the second part of the proof of [9, Lemma 2]. □

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