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ROTATION OF GAUSSIAN PATHS ON WIENER SPACE WITH APPLICATIONS

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Abstract. In this paper we first develop the rotation theorem of the Gaussian paths on Wiener space. We next analyze the generalized analytic Fourier–Feynman transform. As an application of our rotation theorem, we represent the multiple generalized analytic Fourier–Feynman transform as a single generalized Fourier–Feynman transform.

1. Introduction and preliminaries

Given a positive real $T > 0$, let $C_0[0,T]$ denote a 1-parameter Wiener space, that is, the space of all real-valued continuous functions x on the compact interval $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener-measurable subsets of $C_0[0,T]$, and let $\mathfrak{m}$ denote the Wiener measure which is a Gaussian measure on $C_0[0,T]$ with mean zero and covariance function $r(s,t) = \min\{s,t\}$. Then, as is well known, $(C_0[0,T], \mathcal{M}, \mathfrak{m})$ is a complete measure space. Throughout this article, we will denote the Wiener integral of a Wiener-measurable functional F by

$$E[F] \equiv E_x\big[F(x)\big] = \int_{C_0[0,T]} F(x)\, d\mathfrak{m}(x).$$

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A subset B of $C_0[0,T]$ is said to be *scale-invariant measurable* (SIM) (see [16]) provided that $\rho B \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be *scale-invariant null* provided that $\mathfrak{m}(\rho N) = 0$ for all $\rho > 0$. A property that holds except in the case of a scale-invariant null set is said to hold *scale-invariant almost everywhere* (SI-a.e.). A functional F is considered SIM provided that F is defined on a SIM set and $F(\rho\,\cdot)$ is Wiener-measurable for every $\rho > 0$. If two functionals F and G are equal SI-a.e., then we write $F \approx G$.

The Paley–Wiener–Zygmund (PWZ) stochastic integral (see [19]) plays a key role throughout this paper. Let $\{\phi_n\}_{n=1}^{\infty}$ be a complete orthonormal set in $L_2[0,T]$, each of whose elements is of bounded variation on $[0,T]$. Then for each $v \in L_2[0,T]$, the PWZ stochastic integral $\langle v, x\rangle$ is defined by the formula

$$\langle v, x\rangle = \lim_{n\to\infty} \int_0^T \sum_{j=1}^n (v, \phi_j)_2 \phi_j(t)\, dx(t)$$

for all $x \in C_0[0,T]$ for which the limit exists, where $(\cdot,\cdot)_2$ denotes the L_2-inner product. For each $v \in L_2[0,T]$, the limit defining the PWZ stochastic integral $\langle v, x\rangle$ is essentially independent of the choice of the complete orthonormal set $\{\phi_n\}_{n=1}^{\infty}$, and it exists for SI-a.e. $x \in C_0[0,T]$.

It is well known that for each v in $L_2[0,T]$, $\langle v,\cdot\rangle$ is a Gaussian random variable on $C_0[0,T]$ with mean zero and variance $\|v\|_2^2$, and also that if $\{\alpha_1,\dots,\alpha_n\}$ is an orthogonal set of functions in $L_2[0,T]$, then the random variables $\langle \alpha_j, x\rangle$ are independent. (For a more detailed study of the PWZ stochastic integral, see [18], [20].)

Given a function $h \in L_2[0,T]$ with $\|h\|_2 > 0$, let $\mathcal{Z}_h : C_0[0,T]\times[0,T] \to \mathbb{R}$ be the stochastic process given by

$$\mathcal{Z}_h(x,t) = \langle h\chi_{[0,t]}, x\rangle, \tag{1.1}$$

where $\chi_{[0,t]}$ denotes the indicator function of the set $[0,t]$. Next, let

$$\beta_h(t) = \int_0^t h^2(u)\, du. \tag{1.2}$$

Then the stochastic process $\mathcal{Z}_h$ on $C_0[0,T]\times[0,T]$ is a Gaussian process with mean zero and covariance function

$$E_x\big[\mathcal{Z}_h(x,s)\mathcal{Z}_h(x,t)\big] = \beta_h\big(\min\{s,t\}\big).$$

In addition, by [23, Theorem 21.1] (we present this theorem in the Appendix), $\mathcal{Z}_h(\cdot,t)$ is stochastically continuous in t on $[0,T]$ (see the Appendix for the precise concept of the stochastic continuity). If h is of bounded variation on $[0,T]$, then $\mathcal{Z}_h$ is a continuous process. Also, for any nonzero functions h_1 and h_2 in $L_2[0,T]$,

$$E_x\big[\mathcal{Z}_{h_1}(x,s)\mathcal{Z}_{h_2}(x,t)\big] = \int_0^{\min\{s,t\}} h_1(u)h_2(u)\, du.$$

Of course if $h(t) \equiv 1$ on $[0,T]$, then the process $\mathcal{W}$ on $C_0[0,T]\times[0,T]$ given by $(w,t) \mapsto \mathcal{W}_t(x) = \mathcal{Z}_1(x,t) = x(t)$ is a Wiener process (standard Brownian motion).

We note that the coordinate process $\mathcal{Z}_1$ is stationary in time, whereas the stochastic process $\mathcal{Z}_h$ generally is not. (For more detailed studies on the stochastic process $\mathcal{Z}_h$, see [5], [8], [12], [21].)

If u and h are functions in $L_2[0,T]$, then uh is in $L_1[0,T]$. Also there exist two functions u and h so that $uh \in L_1[0,T]\setminus L_2[0,T]$. In this case, the PWZ stochastic integral

$$\big\langle u, \mathcal{Z}_h(x,\cdot)\big\rangle \equiv \int_0^T u(s)\, d\mathcal{Z}_h(x,t) = \int_0^T u(t)h(t)\, dx(t)$$

may not be well defined. But, from [8, Lemma 1], it follows that, for each $u \in L_2[0,T]$ and $h \in L_\infty[0,T]$ with $\|h\|_2 > 0$,

$$\big\langle u, \mathcal{Z}_h(x,\cdot)\big\rangle = \langle uh, x\rangle \tag{1.3}$$

for SI-a.e. $x \in C_0[0,T]$. Thus, in Sections 3, 4, and 5 below, we require h to be in $L_\infty[0,T]$ rather than simply in $L_2[0,T]$.

Throughout this article, we will assume that each functional F (or G) we consider satisfies the conditions

$$F : C_0[0,T] \to \mathbb{C} \quad \text{is SI-a.e. defined and SIM,} \tag{1.4}$$

and for all $h \in L_2[0,T]$ with $\|h\|_2 > 0$ and each $\rho > 0$,

$$E_x\big[\big|F\big(\rho\mathcal{Z}_h(x,\cdot)\big)\big|\big] < +\infty. \tag{1.5}$$

One of the essential structures of Wiener measure $\mathfrak{m}$ is the *rotation invariant property*. Strictly speaking, given real numbers a and b with $a^2 + b^2 = 1$, the Gaussian process $\{ax_1 + bx_2 : (x_1, x_2) \in C_0[0,T] \times C_0[0,T]\}$ is equivalent to the ordinary Wiener process $\mathcal{W}$. On the other hand, in his seminal paper [1], Bearman studied a rotation theorem for the double Wiener integrals. Bearman's theorem was further developed by Cameron and Storvick [3] and by Johnson and Skoug [17] in their studies of Wiener integral equations. Bearman's rotation theorem for the Wiener integral has played an important role in various research areas in mathematics and physics involving the Wiener integration theory. The result is summarized as follows: for a Wiener integrable functional F on $C_0[0,T]$ which satisfies the condition (1.4) and for any $a, b \in \mathbb{R}$, $F(aw + bz)$ is integrable on $(C_0[0,T])^2$, the product of two copies of $C_0[0,T]$, and

$$E_z\big[E_w\big[F(aw+bz)\big]\big] = E_x\big[F(\sqrt{a^2+b^2}x)\big].$$

These results have produced many applications. For instance, in [9]–[12], Huffman, Park, and Skoug used the rotation theorem to obtain fundamental relationships between the analytic Fourier–Feynman transform (FFT) and the convolution product of functionals on $C_0[0,T]$ (for instance, see the proof of Theorem 3.3 in [10], the proof of Lemma 4.1 in [11], and the proof of Theorem 2.1 in [12]). Also in [13], [14], the authors established a Fubini theorem via Bearman's theorem in order to establish various integration formulas for the analytic Feynman integrals and the FFTs on $C_0[0,T]$. The rotation theorem was further developed in [6] to study a behavior of the conditional FFTs on the product Wiener spaces.

Recently in [5], the present authors and Skoug used another rotation theorem of Wiener measure $\mathfrak{m}$ to study a multiple generalized FFT (MGFFT) associated with the Gaussian processes $\mathcal{Z}_h$ on $C_0[0,T]$. The rotation form used in [5] is a generalization of Bearman's celebrated result and is intended to interpret behaviors of the nonstationary Gaussian processes $\mathcal{Z}_h$ given by (1.1). The authors also investigated various relationships which exist between the MGFFT and the corresponding convolution product. The rotation theorem which was introduced in [4] and used in [5] is as follows.

Theorem 1.1. *Let F be a Wiener-measurable functional which satisfies the conditions (1.4) and (1.5) above. Then for any nonzero functions h_1 and h_2 in $L_2[0,T]$,*

$$E_z\big[E_w\big[F\big(\mathcal{Z}_{h_1}(w,\cdot)+\mathcal{Z}_{h_2}(z,\cdot)\big)\big]\big]=E_x\big[F\big(\mathcal{Z}_{\mathbf{r}(h_1,h_2)}(x,\cdot)\big)\big], \tag{1.6}$$

where h_1,h_2, and $\mathbf{r}(h_1,h_2)$ are related by

$$\mathbf{r}(h_1,h_2)(t)=\sum_{n=1}^{\infty}\sqrt{(h_1,\phi_n)_2^2+(h_2,\phi_n)_2^2}\phi_n(t) \tag{1.7}$$

for some complete orthonormal set $\{\phi_n\}$ in $L_2[0,T]$, each of whose elements is of bounded variation on $[0,T]$.

However, in [4], the techniques used in the proof of our Theorem 1.1 are not clear and are too tedious for the reader. Also, representation (1.7) of the function $\mathbf{r}(h_1,h_2)$ appearing in the right-hand side of (1.6) is very complicated.

The purpose of the present article is, first of all, to establish the rotation theorem involving the Gaussian paths (see Theorem 2.1 below) under a milder condition, and secondly, as an application of our rotation theorem, to analyze the MGFFT of functionals F on $C_0[0,T]$ as a single generalized FFT.

2. Rotation property of Gaussian paths

The purpose of this section is to establish a rotation theorem for Gaussian processes on the product Wiener spaces $(C_0[0,T])^2$. The rotation theorem in this section contains weaker conditions than those found in Theorem 1.1. Also, the proof of the rotation theorem is not as limited as those steps of the proof in [4].

Given nonzero functions h_1 and h_2 in $L_2[0,T]$, let $\mathcal{Z}_{h_1}$ and $\mathcal{Z}_{h_2}$ be the Gaussian processes given by (1.1) with h replaced with h_1 and h_2, respectively. Then the process

$$\mathfrak{Z}_{h_1,h_2}:\big(C_0[0,T]\big)^2\times[0,T]\to\mathbb{R}$$

given by

$$\mathfrak{Z}_{h_1,h_2}(x_1,x_2,t)=\mathcal{Z}_{h_1}(x_1,t)+\mathcal{Z}_{h_2}(x_2,t)$$

is also a Gaussian process with mean zero and covariance function

$$\begin{aligned}\mathfrak{v}_{h_1,h_2}\big(\min\{s,t\}\big)&\equiv E_{x_1}\big[E_{x_2}\big[\mathfrak{Z}_{h_1,h_2}(x_1,x_2,s)\mathfrak{Z}_{h_1,h_2}(x_1,x_2,t)\big]\big]\\&=\int_0^{\min\{s,t\}}h_1^2(u)\,du+\int_0^{\min\{s,t\}}h_2^2(u)\,du.\end{aligned}$$

Let h_1 and h_2 be functions in $L_2[0,T]$. Then there exists a function $\mathbf{s} \in L_2[0,T]$ such that

$$\mathbf{s}^2(t) = h_1^2(t) + h_2^2(t) \tag{2.1}$$

for m_L-a.e. $t \in [0,T]$, where m_L denotes the Lebesgue measure on $[0,T]$. Note that the function $\mathbf{s}$ satisfying (2.1) is not unique. We will use the symbol $\mathbf{s}(h_1,h_2)$ for the functions $\mathbf{s}$ that satisfy (2.1) above. Given functions h_1 and h_2 in $L_2[0,T]$, infinitely many functions $\mathbf{s}(h_1,h_2)$ exist in $L_2[0,T]$. Thus $\mathbf{s}(h_1,h_2)$ can be considered as an equivalence class of the equivalence relation $\sim$ on $L_2[0,T]$ given by

$$\mathbf{s}_1 \sim \mathbf{s}_2 \Longleftrightarrow \mathbf{s}_1^2 = \mathbf{s}_2^2 \quad m_L\text{-a.e.}$$

We observe that for every function $\mathbf{s}$ in the equivalence class $\mathbf{s}(h_1,h_2)$, the Gaussian random variable $\langle \mathbf{s}, x\rangle$ has the normal distribution $N(0, \|h_1\|_2^2 + \|h_2\|_2^2)$.

Inductively, given a set $\mathcal{H} = \{h_1,\dots,h_n\}$ of functions in $L_2[0,T]$, let $\mathbf{s}(\mathcal{H}) \equiv \mathbf{s}(h_1,h_2,\dots,h_n)$ be the equivalence class of the functions $\mathbf{s}$ which satisfy the relation

$$\mathbf{s}^2(t) = h_1^2(t) + \cdots + h_n^2(t). \tag{2.2}$$

For convenience, throughout the rest of this paper we will regard $\mathbf{s}(\mathcal{H})$ as a function in $L_2[0,T]$. We note that if the functions $h_1,\dots,h_n$ are in $L_\infty[0,T]$, then we can take $\mathbf{s}(\mathcal{H})$ to be in $L_\infty[0,T]$. By mathematical induction, it follows that

$$\mathbf{s}\big(\mathbf{s}(h_1,h_2,\dots,h_{k-1}),h_k\big) = \mathbf{s}(h_1,h_2,\dots,h_k) \tag{2.3}$$

for all $k \in \{2,\dots,n\}$.

Next, given nonzero functions h_1 and h_2 in $L_2[0,T]$, we consider the stochastic process $\mathcal{Z}_{\mathbf{s}(h_1,h_2)}$. Then $\mathcal{Z}_{\mathbf{s}(h_1,h_2)}$ is a Gaussian process with mean zero and covariance function

$$\begin{aligned}
&E_x\big[\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,s)\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,t)\big]\\
&\quad= \int_0^{\min\{s,t\}} \mathbf{s}^2(h_1,h_2)(u)\,du\\
&\quad= \int_0^{\min\{s,t\}} \big(h_1^2(u)+h_2^2(u)\big)\,du\\
&\quad= \int_0^{\min\{s,t\}} h_1^2(u)\,du + \int_0^{\min\{s,t\}} h_2^2(u)\,du\\
&\quad= \mathfrak{v}_{h_1,h_2}\big(\min\{s,t\}\big).
\end{aligned}$$

Given two functions h_1 and h_2 in $L_2[0,T]$, the function $\mathbf{r}(h_1,h_2)$ given by (1.7) is strongly dependent on the Fourier coefficients of the functions h_1 and h_2 with respect to a complete orthonormal set $\{\phi_n\}$ in the Hilbert space $L_2[0,T]$. In our next theorem, we develop the rotation theorem with a weaker condition rather than the condition (1.7). The condition (2.1) below does not depend on a complete orthonormal set in $L_2[0,T]$.

Theorem 2.1. *Let F be a Wiener-measurable functional which satisfies the conditions (1.4) and (1.5) above. Then for any nonzero functions h_1 and h_2 in $L_2[0,T]$,*

$$E_{x_2}\big[E_{x_1}\big[F\big(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot)\big)\big]\big] = E_x\big[F\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot)\big)\big], \tag{2.4}$$

where h_1, h_2, and $\mathbf{s}(h_1,h_2)$ are related by (2.1).

Proof. From the observation above, one can see that $\mathfrak{Z}_{h_1,h_2} : (C_0[0,T])^2 \to \mathbb{R}$ and $\mathcal{Z}_{\mathbf{s}(h_1,h_2)} : C_0[0,T] \to \mathbb{R}$ are equivalent processes; that is, for each $t \in [0,T]$, the random variables $\mathfrak{Z}_{h_1,h_2}(\cdot,\cdot,t)$ and $\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(\cdot,t)$ have the same Gaussian distribution $N(0,\beta_{h_1}(t)+\beta_{h_2}(t))$, where β_h is given by (1.2) above. Thus, applying the change-of-variable theorem, equation (2.4) follows as desired. □

Corollary 2.2. *Let F be a Wiener-measurable functional which satisfies the conditions (1.4) and (1.5) above. Then for all $a,b \in \mathbb{R}$, $F(ax_1+ax_2)$ is integrable on $(C_0[0,T])^2$ and*

$$E_{x_2}\big[E_{x_1}\big[F(ax_1+bx_2)\big]\big] = E_x\big[F\big(\mathcal{Z}_{\mathbf{s}(a,b)}(x,\cdot)\big)\big]. \tag{2.5}$$

In particular,

$$E_{x_2}\big[E_{x_1}\big[F(ax_1+bx_2)\big]\big] = E_x\big[F(\sqrt{a^2+b^2}x)\big]. \tag{2.6}$$

Proof. Choosing $h_1 = a$ and $h_2 = b$, as constant functions on $[0,T]$, in (2.4) above, equation (2.5) follows immediately. Also we can choose $\mathbf{s}(a,b)$ as a constant function. In this case one can see that either $\mathbf{s}(a,b) = \sqrt{a^2+b^2}$ or $\mathbf{s}(a,b) = -\sqrt{a^2+b^2}$. But we know that for all Wiener-measurable functionals F, $E_x[F(x)] = E_x[F(-x)]$. Thus equation (2.6) is established. □

Remark 2.3. Equation (2.6) can be obtained from the Bearman's result (see [1, p. 130]). In [3], Cameron and Storvick used equation (2.6) to study a Wiener integral equation which is equivalent to a diffusion equation. In [17], Johnson and Skoug also used equation (2.6) to study various behaviors of the Wiener measure and related topics.

Using equation (2.3), the Fubini theorem, and an induction argument, we obtain the following theorem.

Theorem 2.4. *Let F be a Wiener-measurable functional which satisfies the conditions (1.4) and (1.5) above. Then for any finite sequence $\mathcal{H} = \{h_1,\dots,h_n\}$ of nonzero functions in $L_2[0,T]$,*

$$E_{x_n}\Big[\cdots\Big[E_{x_1}\Big[F\Big(\sum_{j=1}^{n}\mathcal{Z}_{h_j}(x_j,\cdot)\Big)\Big]\Big]\cdots\Big] = E_x\big[F\big(\mathcal{Z}_{\mathbf{s}(h_1,\dots,h_n)}(x,\cdot)\big)\big],$$

where $\mathcal{H} = \{h_1,\dots,h_n\}$ and $\mathbf{s}(\mathcal{H})$ are related by (2.2).

3. Cylinder functionals

Functionals that involve PWZ stochastic integrals are quite common. In this section, we apply our results to the cylinder functionals F on $C_0[0,T]$ given by

$$F(x) = f\big(\langle g_1, x\rangle, \dots, \langle g_n, x\rangle\big), \tag{3.1}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a Lebesgue-measurable function and $\{g_1, \dots, g_n\}$ is an independent set of functions in $L_2[0,T]$. It is well known [7] that the functional F given by (3.1) is Wiener-measurable if and only if f is Lebesgue-measurable.

Given a finite set $\mathcal{G} = \{g_1, \dots, g_n\}$ of nonzero functions in $L_2[0,T]$, let $\mathcal{P}_{\mathcal{G}} : C_0[0,T] \to \mathbb{R}^n$ be given by

$$\mathcal{P}_{\mathcal{G}}(x) = \big(\langle g_1, x\rangle, \dots, \langle g_n, x\rangle\big). \tag{3.2}$$

In order to simplify our expressions here, we use the following conventions: for $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ and $\mathcal{P}_{\mathcal{G}}(x)$ given by equation (3.2), let

$$f(\vec{u}) \equiv f(u_1, \dots, u_n),$$
$$f\big(\vec{u} + \mathcal{P}_{\mathcal{G}}(x)\big) \equiv f\big(u_1 + \langle g_1, x\rangle, \dots, u_n + \langle g_n, x\rangle\big),$$

and for measurable functionals G on $(C_0[0,T])^n$, let

$$E_{\vec{x}}\big[G(x_1, \dots, x_n)\big] \equiv \int_{(C_0[0,T])^n} G(x_1, \dots, x_n)\, d\mathfrak{m}^n(x_1, \dots, x_n).$$

Lemma 3.1. *Let $\mathcal{G} = \{g_1, \dots, g_n\}$ be an orthogonal set of nonzero functions in $L_2[0,T]$, and let $\mathcal{P}_{\mathcal{G}}$ be given by equation (3.2) above. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Lebesgue-measurable function. Then*

$$E_x\big[f\big(\mathcal{P}_{\mathcal{G}}(x)\big)\big] \stackrel{*}{=} \Big(\prod_{j=1}^{n} 2\pi\|g_j\|_2^2\Big)^{-1/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\Big\{-\sum_{j=1}^{n} \frac{u_j^2}{2\|g_j\|_2^2}\Big\}\, d\vec{u}, \tag{3.3}$$

where by $\stackrel{*}{=}$ we mean that if either side exists, then both sides exist and equality holds.

Proof. As we commented above, it is known (see [7]) that $f(\vec{u})$ is Lebesgue-measurable if and only if $f \circ \mathcal{P}_{\mathcal{G}}(x) \equiv f(\mathcal{P}_{\mathcal{G}}(x))$ is Wiener-measurable. Also, we see that the PWZ stochastic integrals $\langle g_j, x\rangle$, $j \in \{1, \dots, n\}$, are independent Gaussian random variables. Thus, by the change-of-variable theorem, equation (3.3) follows at once. □

To establish our second rotation theorem for cylinder functionals (Theorem 3.4 below), we will consider the pairs $(\mathcal{A}, \mathcal{H})$ of finite sequences $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$ and $\mathcal{H} = \{h_1, \dots, h_m\}$ of nonzero functions in $L_2[0,T]$ which satisfy the requirements that:

(c1) $\mathcal{A}$ is an orthogonal set of nonzero functions in $L_2[0,T]$,
(c2) $h_k \in L_\infty[0,T]$ for each $k \in \{1, \dots, m\}$, and
(c3) $\mathcal{A}h_k \equiv \{\alpha_1 h_k, \dots, \alpha_n h_k\}$ is orthonormal in $L_2[0,T]$ for all $k \in \{1, \dots, m\}$.

Remark 3.2. Let the pair of $\mathcal{A}=\{\alpha_1,\dots,\alpha_n\}$ and $\mathcal{H}=\{h_1,\dots,h_m\}$ satisfy the preceding requirements (c1), (c2), and (c3). We then observe that $\|h_k\|_2>0$ for each $k\in\{1,\dots,m\}$ and, in view of the requirement (c3), the PWZ stochastic integrals $\langle\alpha_j\mathbf{s}(\mathcal{H}),x\rangle=\langle\alpha_j,\mathcal{Z}_{\mathbf{s}(\mathcal{H})}(x,\cdot)\rangle$, $j=1,\dots,n$, form a set of independent Gaussian random variables on $C_0[0,T]$ with mean zero and variance

$$\begin{aligned}\big\|\alpha_j\mathbf{s}(\mathcal{H})\big\|_2^2&=\int_0^T\alpha_j^2(t)\mathbf{s}^2(\mathcal{H})(t)\,dt\\&=\int_0^T\alpha_j^2(t)\Big[\sum_{k=1}^m h_k^2(t)\Big]\,dt=\sum_{k=1}^m\|\alpha_jh_k\|_2^2=m\end{aligned}\tag{3.4}$$

for each $j\in\{1,\dots,n\}$, where $\mathbf{s}(\mathcal{H})$ is a function in $L_\infty[0,T]$ satisfying equation (2.2) above.

Example 3.3. For each $j\in\mathbb{N}$, let $\alpha_j(t)=\frac{\sqrt{2}}{\sqrt[4]{T}}\cos(\frac{(2j-1)\pi}{2T}t)$ on $[0,T]$. Then $\Theta=\{\alpha_j\}_{j=1}^\infty$ is an orthogonal sequence of functions in $L_2[0,T]$. In addition, $\widetilde{\Theta}=\{\alpha_j/\sqrt[4]{T}\}_{j=1}^\infty$ is a complete orthonormal sequence in $L_2[0,T]$. In this case, we have the following assertions.

(p1) For every $j\in\mathbb{N}$, $\|\alpha_j\|_2^2=\sqrt{T}>0$ and $\alpha_j\in L_\infty[0,T]$.

(p2) Let $n\in\mathbb{N}$ be fixed and let k be a positive integer with $k>n$. Then for any $i,j\in\{1,\dots,n\}$,

$$\int_0^T\alpha_i(t)\alpha_j(t)\alpha_k^2(t)\,dt=\begin{cases}0, & i\neq j,\\ 1, & i=j.\end{cases}$$

These facts tell us that for each integer k with $k>n$, the set $\{\alpha_1\alpha_k,\dots,\alpha_n\alpha_k\}$ is an orthonormal set of functions in $L_2[0,T]$. In view of the observations (p1) and (p2), we can take various pairs $(\mathcal{A},\mathcal{H})$ from Θ that satisfy the conditions (c1), (c2), and (c3).

Given an orthogonal set $\mathcal{A}=\{\alpha_1,\dots,\alpha_n\}$, let $\mathcal{N}_\infty(\mathcal{A})$ be the space of functions $h\in L_\infty[0,T]$ such that $\mathcal{A}h=\{\alpha h:\alpha\in\mathcal{A}\}$ is orthonormal in $L_2[0,T]$. Then for any finite subset $\mathcal{H}$ of $\mathcal{N}_\infty(\mathcal{A})$, the pair $(\mathcal{A},\mathcal{H})$ satisfies the condition (c1), (c2), and (c3) above. For every $h\in\mathcal{N}_\infty[0,T]$, let

$$\mathcal{P}_{\mathcal{A}h}(x)=\big(\langle\alpha_1h,x\rangle,\dots,\langle\alpha_nh,x\rangle\big).\tag{3.5}$$

By equation (1.3), it follows that

$$\mathcal{P}_{\mathcal{A}h}(x)=\big(\big\langle\alpha_1,\mathcal{Z}_h(x,\cdot)\big\rangle,\dots,\big\langle\alpha_n,\mathcal{Z}_h(x,\cdot)\big\rangle\big).$$

We are now ready to present our rotation theorem for the multiple Wiener integral of cylinder functionals.

Theorem 3.4. *Let $\mathcal{A}=\{\alpha_1,\dots,\alpha_n\}$ be an orthogonal set of nonzero functions in $L_2[0,T]$, and let $\mathcal{H}=\{h_1,\dots,h_m\}$ be a finite sequence in $\mathcal{N}_\infty(\mathcal{A})$. Let $f:\mathbb{R}^n\to\mathbb{C}$ be a Lebesgue integrable function and, for each $k\in\{1,\dots,m\}$, let $\mathcal{P}_{\mathcal{A}h_k}$ be given by equation (3.5) with h replaced with h_k, respectively. Then*

$$E_{\vec{x}}\big[f\big(\mathcal{P}_{\mathcal{A}h_1}(x_1)+\cdots+\mathcal{P}_{\mathcal{A}h_m}(x_m)\big)\big]=E_x\big[f\big(\mathcal{P}_{\mathcal{A}\mathbf{s}(\mathcal{H})}(x)\big)\big],\tag{3.6}$$

where $\mathbf{s}(\mathcal{H})$ *is a function in* $L_\infty[0,T]$ *satisfying equation (2.2) above. Also, both of the expressions in (3.6) are given by one of the following expressions:*

$$\begin{aligned}
&\Big(\prod_{j=1}^{n} 2\pi\big\|\alpha_j\mathbf{s}(\mathcal{H})\big\|_2^2\Big)^{-1/2}\int_{\mathbb{R}^n} f(\vec{r})\exp\Big\{-\sum_{j=1}^{n}\frac{r_j^2}{2\|\alpha_j\mathbf{s}(\mathcal{H})\|_2^2}\Big\}\,d\vec{r}\\
&\quad=\Big(\prod_{j=1}^{n} 2\pi\Big(\sum_{k=1}^{m}\|\alpha_j h_k\|_2^2\Big)\Big)^{-1/2}\int_{\mathbb{R}^n} f(\vec{r})\exp\Big\{-\sum_{j=1}^{n}\frac{r_j^2}{2(\sum_{k=1}^{m}\|\alpha_j h_k\|_2^2)}\Big\}\,d\vec{r}\\
&\quad=(2\pi m)^{-n/2}\int_{\mathbb{R}^n} f(\vec{r})\exp\Big\{-\sum_{j=1}^{n}\frac{r_j^2}{2m}\Big\}\,d\vec{r}.
\end{aligned}\tag{3.7}$$

Proof. We first note that

$$\begin{aligned}
&\mathcal{P}_{\mathcal{A}h_1}(x_1)+\cdots+\mathcal{P}_{\mathcal{A}h_m}(x_m)\\
&\quad=\big(\langle\alpha_1h_1,x_1\rangle,\dots,\langle\alpha_nh_1,x_1\rangle\big)+\cdots+\big(\langle\alpha_1h_m,x_m\rangle,\dots,\langle\alpha_nh_m,x_m\rangle\big)\\
&\quad=\big(\langle\alpha_1h_1,x_1\rangle+\cdots+\langle\alpha_1h_m,x_m\rangle,\dots,\langle\alpha_nh_1,x_1\rangle+\cdots+\langle\alpha_nh_m,x_m\rangle\big)\\
&\quad=\big(\big\langle\alpha_1,\mathcal{Z}_{h_1}(x_1,\cdot)\big\rangle+\cdots+\big\langle\alpha_1,\mathcal{Z}_{h_m}(x_m,\cdot)\big\rangle,\\
&\qquad\dots,\big\langle\alpha_n,\mathcal{Z}_{h_1}(x_1,\cdot)\big\rangle+\cdots+\big\langle\alpha_n,\mathcal{Z}_{h_m}(x_m,\cdot)\big\rangle\big)\\
&\quad=\Big(\Big\langle\alpha_1,\sum_{k=1}^{m}\mathcal{Z}_{h_1}(x_k,\cdot)\Big\rangle,\dots,\Big\langle\alpha_n,\sum_{k=1}^{m}\mathcal{Z}_{h_k}(x_k,\cdot)\Big\rangle\Big)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{P}_{\mathcal{A}\mathbf{s}(\mathcal{H})}(x)&=\big(\big\langle\alpha_1\mathbf{s}(\mathcal{H}),x\big\rangle,\dots,\big\langle\alpha_n\mathbf{s}(\mathcal{H}),x\big\rangle\big)\\
&=\big(\big\langle\alpha_1,\mathcal{Z}_{\mathbf{s}(\mathcal{H})}(x,\cdot)\big\rangle,\dots,\big\langle\alpha_n,\mathcal{Z}_{\mathbf{s}(\mathcal{H})}(x,\cdot)\big\rangle\big).
\end{aligned}$$

Thus equation (3.6) can be rewritten as

$$E_{\vec{x}}\big[F\big(\mathcal{Z}_{h_1}(x_1,\cdot)+\cdots+\mathcal{Z}_{h_m}(x_m,\cdot)\big)\big]=E_x\big[F\big(\mathcal{Z}_{\mathbf{s}(\mathcal{H})}(x,\cdot)\big)\big]$$

with F given by equation (3.1) above. Therefore, by Theorem 2.4, the equality in (3.6) holds.

Next, we note that for $i,j\in\{1,\dots,n\}$ with $i\neq j$,

$$\begin{aligned}
\big(\alpha_i\mathbf{s}(\mathcal{H}),\alpha_j\mathbf{s}(\mathcal{H})\big)_2&=\int_0^T\alpha_i(t)\alpha_j(t)\mathbf{s}^2(\mathcal{H})(t)\,dt\\
&=\sum_{k=1}^{m}\int_0^T\alpha_i(t)\alpha_j(t)h_k^2(t)\,dt=0.
\end{aligned}\tag{3.8}$$

From equations (3.4) and (3.8), we see that $\mathcal{A}\mathbf{s}(\mathcal{H})=\{\alpha_1\mathbf{s}(\mathcal{H}),\dots,\alpha_n\mathbf{s}(\mathcal{H})\}$ is an orthogonal set of functions in $L_2[0,T]$ with $\|\alpha_j\mathbf{s}(\mathcal{H})\|_2^2=m$ for all $j\in\{1,\dots,n\}$ and $\langle\alpha_j\mathbf{s}(\mathcal{H}),x\rangle$ is a Gaussian random variable with mean zero and variance m for each $j\in\{1,\dots,n\}$. Hence using equation (3.3) with $\mathcal{P}_{\mathcal{G}}$ replaced with $\mathcal{P}_{\mathcal{A}\mathbf{s}(\mathcal{H})}$, we see that the right-hand side of equation (3.6) is given by the expressions in (3.7) above. □

Remark 3.5. In Theorem 3.4, we assumed that the finite sequence $\mathcal{H}$ is in $\mathcal{N}_\infty(\mathcal{A})$ so that the pair $(\mathcal{A},\mathcal{H})$ satisfies the requirements (c1), (c2), and (c3) above. However, it is still true that equation (3.6) holds under only the two conditions (c1) and (c2) above. In this case, to obtain an explicit expression like (3.7), we might apply the Gram–Schmidt process to the set $\mathcal{A}\mathbf{s}(\mathcal{H})$.

On the other hand, under two conditions (c1) and (c2), and in view of the second expression of (3.7), we observe that

$$E_{\vec{x}}\big[f\big(\mathcal{P}_{\mathcal{A}h_1}(x_1)+\cdots+\mathcal{P}_{\mathcal{A}h_m}(x_m)\big)\big]=E_x\big[f\big(\mathcal{P}_{\mathcal{U}}(x)\big)\big],$$

where $\mathcal{U}=\{u_1,\dots,u_n\}$ is an orthogonal set of functions in $L_2[0,T]$ such that

$$\|u_j\|_2^2=\sum_{k=1}^m\|\alpha_j h_k\|_2^2. \tag{3.9}$$

Given any orthonormal set $\{e_1,\dots,e_n\}$ of functions in $L_2[0,T]$, let

$$u_j(t)=\Big(\sum_{k=1}^m\|\alpha_j h_k\|_2^2\Big)^{1/2}e_j(t),\quad t\in[0,T]$$

for each $j\in\{1,\dots,n\}$. Then the set $\mathcal{U}=\{u_1,\dots,u_n\}$ is an orthogonal set of functions in $L_2[0,T]$ and satisfies equation (3.9) above. In this case, using (3.3) with $\mathcal{G}$ replaced with $\mathcal{U}$, we see that $E_x[f(\mathcal{P}_{\mathcal{U}}(x))]$ is given by the second expression of (3.7).

4. Generalized analytic Fourier–Feynman transform

In defining various analytic Feynman integrals of functionals F on $C_0[0,T]$, one usually starts, for $\lambda>0$, with the Wiener integral $E_x[F(\lambda^{-1/2}x)]$ and then extends analytically in λ to the right-half complex plane. In this paper, we start with the generalized Wiener integral

$$E_x\big[F\big(\lambda^{-1/2}\mathcal{Z}_h(x,\cdot)\big)\big]=J_F(h;\lambda), \tag{4.1}$$

where $\mathcal{Z}_h$ is the Gaussian process given by equation (1.1) above. But in order to present our results involving the generalized analytic FFT and the analytic MGFFT, we follow the exposition of [5], [9], [12].

Throughout this and the next section, let $\mathbb{C}_+$ and $\widetilde{\mathbb{C}}_+$ denote the set of the complex numbers with positive real part and the nonzero complex numbers with nonnegative real part, respectively. For each $\lambda\in\widetilde{\mathbb{C}}_+$, let $\lambda^{1/2}$ denote the principal square root of λ (i.e., $\lambda^{1/2}$ is always chosen to have positive real part, so that $\lambda^{-1/2}=(\lambda^{-1})^{1/2}$ is in $\mathbb{C}_+$ for all $\lambda\in\widetilde{\mathbb{C}}_+$).

Let F be a $\mathbb{C}$-valued SIM functional on $C_0[0,T]$ such that the generalized Wiener integral $J_F(h;\lambda)$ given by (4.1) exists and is finite for all $\lambda>0$. If there exists a function $J_F^*(h;\lambda)$ analytic on $\mathbb{C}_+$ such that $J_F^*(h;\lambda)=J_F(h;\lambda)$ for all $\lambda>0$, then $J_F^*(h;\lambda)$ is defined to be the analytic $\mathcal{Z}_h$-Wiener integral (namely, the generalized analytic Wiener integral associated with the Gaussian paths $\mathcal{Z}_h(x,\cdot)$) of F over $C_0[0,T]$ with parameter λ, and for $\lambda\in\mathbb{C}_+$ we write

$$E_x^{\mathrm{anw}_\lambda}\big[F\big(\mathcal{Z}_h(x,\cdot)\big)\big]=J_F^*(h;\lambda).$$

Let $q \neq 0$ be a real number and let F be a functional such that $E_x^{\mathrm{anw}_\lambda}[F(\mathcal{Z}_h(x,\cdot))]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic $\mathcal{Z}_h$-Feynman integral (namely, the generalized analytic Feynman integral associated with the Gaussian paths $\mathcal{Z}_h(x,\cdot)$) of F with parameter q and we write

$$E_x^{\mathrm{anf}_q}\big[F\big(\mathcal{Z}_h(x,\cdot)\big)\big] = \lim_{\lambda\to -iq} E_x^{\mathrm{anw}_\lambda}\big[F\big(\mathcal{Z}_h(x,\cdot)\big)\big],$$

where λ approaches $-iq$ through values in $\mathbb{C}_+$. Next (see [5], [12]), we state the definition of the analytic FFT associated with the Gaussian process $\mathcal{Z}_h$ ($\mathcal{Z}_h$-FFT).

Definition 4.1. Given a nonzero function h in $L_2[0,T]$, let $\mathcal{Z}_h$ be given by (1.1). For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$, let

$$T_{\lambda,h}(F)(y) \equiv E_x^{\mathrm{anw}_\lambda}\big[F\big(y+\mathcal{Z}_h(x,\cdot)\big)\big] = J^*_{F(y+\cdot)}(h;\lambda)$$

be the analytic $\mathcal{Z}_h$-Wiener transform of F. Let q be a nonzero real number. For $p \in (1,2]$, we define the L_p analytic $\mathcal{Z}_h$-FFT, $T^{(p)}_{q,h}(F)$ of F, by the formula

$$T^{(p)}_{q,h}(F)(y) = \underset{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}{\mathrm{l.i.m.}}\,(w_s^{p'})T_{\lambda,h}(F)(y)$$

if it exists; that is, for each $\rho > 0$,

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} E_y\Big[\big|T_{\lambda,h}(F)(\rho y) - T^{(p)}_{q,h}(F)(\rho y)\big|^{p'}\Big] = 0,$$

where $1/p + 1/p' = 1$. We define the L_1 analytic $\mathcal{Z}_h$-FFT, $T^{(1)}_{q,h}(F)$ of F, by the formula (if it exists)

$$T^{(1)}_{q,h}(F)(y) = \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} T_{\lambda,h}(F)(y)$$

for SI-a.e. $y \in C_0[0,T]$.

For $p \in [1,2]$, we note that $T^{(p)}_{q,h}(F)$ is defined only SI-a.e. We also note that if $T^{(p)}_{q,h}(F)$ exists and if $F \approx G$, then $T^{(p)}_{q,h}(G)$ exists and $T^{(p)}_{q,h}(G) \approx T^{(p)}_{q,h}(F)$. One can see that for each $h \in L_2[0,T]$, $T^{(p)}_{q,h}(F) \approx T^{(p)}_{q,-h}(F)$ since $E_x[F(x)] = E_x[F(-x)]$.

In this section we will show that the L_p analytic $\mathcal{Z}_h$-FFT, $T^{(p)}_{q,h}(F)$, exists for cylinder functionals F on $C_0[0,T]$. Let n be a positive integer (fixed throughout this and the next section), and let an orthogonal set $\mathcal{A} = \{\alpha_1,\dots,\alpha_n\}$ of nonzero functions in $L_2[0,T]$ be given. For $p \in [1,\infty)$, let $\mathcal{B}^{(p)}_{\mathcal{A}}$ be the space of all functionals on $C_0[0,T]$ of the form

$$F(x) = f\big(\langle\alpha_1,x\rangle,\dots,\langle\alpha_n,x\rangle\big) \equiv f\big(\mathcal{P}_{\mathcal{A}}(x)\big) \tag{4.2}$$

for SI-a.e. $x \in C_0[0,T]$, where f is in $L_p(\mathbb{R}^n)$. Let $\mathcal{B}^{(\infty)}_{\mathcal{A}}$ be the space of all functionals having the form (4.2) with f in $C_0(\mathbb{R}^n)$, the space of bounded continuous functions on $\mathbb{R}^n$ that vanish at infinity. It is quite easy to see that if F is in

$\mathcal{B}_{\mathcal{A}}^{(p)}$, then F is SIM. Also, using equation (1.3), the linearity of the PWZ stochastic integral, the change-of-variable theorem, and the Hölder inequality, it follows that, for each $p \in [1,+\infty]$, any $\rho > 0$, and each nonzero function $h \in L_\infty[0,T]$,

$$\begin{aligned} &E_x\big[\big|F\big(\rho\mathcal{Z}_h(x,\cdot)\big)\big|\big] \\ &\quad= E_x\big[\big|f\big(\rho\langle\alpha_1 h, x\rangle,\ldots,\rho\langle\alpha_1 h, x\rangle\big)\big|\big] \\ &\quad= \big[(2\pi)^n \det V_\rho\big]^{-1/2}\int_{\mathbb{R}^n}\big|f(u_1,\ldots,u_n)\big|\exp\Big\{-\frac{1}{2}V_\rho^{-1}\vec{u}\cdot\vec{u}\Big\}\,d\vec{u} \\ &\quad< +\infty, \end{aligned} \tag{4.3}$$

where V_ρ denotes the covariance matrix of the Gaussian random variables $\{\rho\langle\alpha_1 h, x\rangle, \ldots, \rho\langle\alpha_1 h, x\rangle\}$ and $\vec{u}\cdot\vec{v}$ denotes the standard inner product of $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$. In (4.3) above, without loss of generality, we may assume that the set $\{\alpha_1 h,\ldots,\alpha_n h\}$ is linearly independent in $L_2[0,T]$. Thus, the functional F given by (4.2) satisfies the condition (1.5).

In Theorem 4.3 below, we establish the existence of the analytic $\mathcal{Z}_h$-Wiener transform $T_{\lambda,h}(F)(y)$ of F in $\mathcal{B}_{\mathcal{A}}^{(p)}$. To do this, we use the following notation. For $\lambda \in \widetilde{\mathbb{C}}_+$ and real $\sigma^2 > 0$, let $K_n(\lambda;\sigma^2) \equiv (\frac{\lambda}{2\pi\sigma^2})^{n/2}$. Next, let

$$H(\lambda;\sigma^2;\vec{u}) \equiv \exp\Big\{-\frac{\lambda}{2\sigma^2}\sum_{j=1}^{n}u_j^2\Big\}. \tag{4.4}$$

In particular, if $\sigma^2 = 1$, then let $K_n(\lambda;1) \equiv K_n(\lambda)$ and $H(\lambda;1;\vec{u}) \equiv H(\lambda;\vec{u})$.

Remark 4.2. We note that for each $(\lambda,\sigma^2) \in \mathbb{C}_+ \times (0,+\infty)$ and $\vec{\xi} \in \mathbb{R}^n$, $H(\lambda;\sigma^2;\vec{u}-\vec{\xi})$, as a function of $\vec{u}$, is an element of $L_p(\mathbb{R}^n)$ for all $p \in [1,+\infty]$; in fact, $H(\lambda;\sigma^2;\vec{u}-\vec{\xi})$ also belongs to $C_0(\mathbb{R}^n)$. In addition, for all $(\lambda,\sigma^2) \in \widetilde{\mathbb{C}}_+ \times (0,+\infty)$, $|H(\lambda;\sigma^2;\vec{u}-\vec{\xi})| \le 1$.

Theorem 4.3. *Let $\mathcal{A} = \{\alpha_1,\ldots,\alpha_n\}$ be an orthogonal set of nonzero functions in $L_2[0,T]$. Let $p \in [1,+\infty]$, and let $F \in \mathcal{B}_{\mathcal{A}}^{(p)}$ be given by equation (4.2). Then for all $\lambda \in \mathbb{C}_+$, every $h \in \mathcal{N}_\infty(\mathcal{A})$ and SI-a.e. $y \in C_0[0,T]$, the analytic $\mathcal{Z}_h$-Wiener transform $T_{\lambda,h}(F)(y)$ exists and has the form*

$$T_{\lambda,h}(F)(y) = (\Psi_\lambda f)\big(\mathcal{P}_{\mathcal{A}}(y)\big), \tag{4.5}$$

where

$$(\Psi_\lambda f)(\xi_1,\ldots,\xi_n) \equiv (\Psi_\lambda f)(\vec{\xi}) = K_n(\lambda)\int_{\mathbb{R}^n} f(\vec{u})H(\lambda;\vec{u}-\vec{\xi})\,d\vec{u}. \tag{4.6}$$

Proof. For $\lambda > 0$, using (4.2), (1.3), (3.2) with $\mathcal{G}$ and x replaced by $\mathcal{A}$ and y, respectively, and using (3.5) and (3.3) with $\mathcal{G}$ replaced by $\mathcal{A}h$, and using (4.4) with $\sigma^2 = 1$, we obtain

$$\begin{aligned}
&J_{F(y+\cdot)}(h;\lambda)\\
&\quad= E_x\big[F\big(y+\lambda^{-1/2}\mathcal{Z}_h(x,\cdot)\big)\big]\\
&\quad= E_x\big[f\big(\langle\alpha_1,y\rangle+\lambda^{-1/2}\big\langle\alpha_1,\mathcal{Z}_h(x,\cdot)\big\rangle,\dots,\langle\alpha_n,y\rangle+\lambda^{-1/2}\big\langle\alpha_n,\mathcal{Z}_h(x,\cdot)\big\rangle\big)\big]\\
&\quad= E_x\big[f\big(\langle\alpha_1,y\rangle+\lambda^{-1/2}\langle\alpha_1 h,x\rangle,\dots,\langle\alpha_n,y\rangle+\lambda^{-1/2}\langle\alpha_n h,x\rangle\big)\big]\\
&\quad= E_x\big[f\big(\mathcal{P}_{\mathcal{A}}(y)+\lambda^{-1/2}\mathcal{P}_{\mathcal{A}h}(x)\big)\big]\\
&\quad= K_n(\lambda)\int_{\mathbb{R}^n} f\big(\mathcal{P}_{\mathcal{A}}(y)+\vec{u}\big)\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^n u_j^2\Big\}\,d\vec{u}\\
&\quad= K_n(\lambda)\int_{\mathbb{R}^n} f(\vec{u})H\big(\lambda;\vec{u}-\mathcal{P}_{\mathcal{A}}(y)\big)\,d\vec{u}\\
&\quad= (\Psi_\lambda f)\big(\mathcal{P}_{\mathcal{A}}(y)\big),
\end{aligned}$$

where $(\Psi_\lambda f)(\vec{\xi})$ is given by equation (4.6).

For $\vec{u},\vec{\xi}\in\mathbb{R}^n$, let

$$(\psi_\lambda f)(\vec{u}) = K_n(\lambda)f(\vec{u})H(\lambda;\vec{u}-\vec{\xi}), \tag{4.7}$$

and given $\lambda\in\mathbb{C}_+$, let $\{\lambda_l\}_{l=1}^\infty$ be any sequence in $\mathbb{C}_+$ such that $\lambda_l\to\lambda$. Then for all $l\in\mathbb{N}$,

$$\big|(\psi_{\lambda_l} f)(\vec{u})\big| \le K_n(\mathfrak{a})\big|f(\vec{u})\big|H(\mathfrak{b};\vec{u}-\vec{\xi}), \tag{4.8}$$

where $\mathfrak{a}=\sup\{|\lambda_l|\}_{l=1}^\infty$ and $\mathfrak{b}=\inf\{\mathrm{Re}(\lambda_l)\}_{l=1}^\infty$. Applying Remark 4.2 and using Hölder's inequality, we can see that $|f(\vec{u})|H(\mathfrak{b};\vec{u}-\vec{\xi})$, as a function of $\vec{u}$, is an element of $L_1(\mathbb{R}^n)$ whenever $f\in L_p(\mathbb{R}^n)$ for every $p\in[1,+\infty]$. Hence, using the dominated convergence theorem, it follows that $(\Psi_\lambda f)(\mathcal{P}_{\mathcal{A}}(y))$ is a continuous function of λ on $\mathbb{C}_+$. Clearly, $H(\lambda;\vec{u}-\vec{\xi})$ is analytic on $\mathbb{C}_+$ as a function of λ. Hence, by the Fubini theorem and the Cauchy theorem, we obtain that

$$\int_\Delta(\Psi_\lambda f)\big(\mathcal{P}_{\mathcal{A}}(y)\big)\,d\lambda = K_n(\lambda)\int_{\mathbb{R}^n} f(\vec{u})\int_\Delta H(\lambda;\vec{u}-\vec{\xi})\,d\lambda\,d\vec{u} = 0$$

for any rectifiable simple closed curve Δ lying in $\mathbb{C}_+$. Thus by the Morera theorem,

$$T_{\lambda,h}(F)(y) \equiv E_x^{\mathrm{anw}_\lambda}\big[F\big(y+\mathcal{Z}_h(x,\cdot)\big)\big] = (\Psi_\lambda f)\big(\mathcal{P}_{\mathcal{A}}(y)\big)$$

is an analytic function of λ throughout $\mathbb{C}_+$. Therefore, we obtain the desired result. □

In order to prove our corollary and theorems below, we will use two lemmas introduced in [15, pp. 98–102]. These two lemmas are true without the dimension restriction $\nu<(2p/(2-p))$ (in our notation, $\nu=n$); in fact, for each $p\in[1,2]$, these two lemmas are valid for all integers $\nu>0$. We now restate the lemmas in [15] using our notation.

Lemma 4.4. *Let σ^2 be a positive real number and let $p \in [1,2]$ be given. Given a nonzero complex number $\lambda \in \widetilde{\mathbb{C}}_+$, f in $L_p(\mathbb{R}^n)$ and $\vec{\xi} \in \mathbb{R}^n$, let*

$$(C_\lambda f)(\vec{\xi}) \equiv K_n(\lambda;\sigma^2) \int_{\mathbb{R}^n} f(\vec{\xi}) H(\lambda;\sigma^2;\vec{u}-\vec{\xi})\, d\vec{u}.$$

Then C_λ is a bounded linear operator from $L_p(\mathbb{R}^n)$ to $L_{p'}(\mathbb{R}^n)$ and

$$\begin{aligned} \|C_\lambda\| &= \big\{K_n(\lambda;\sigma^2)\big\}^{2/p-1} \\ &= \big\{K_n(\lambda;\sigma^2)\big\}^{1-2/p'}. \end{aligned}$$

Lemma 4.5. *Let $p \in [1,2]$ be given, let f be in $L_p(\mathbb{R}^n)$, and let q be a nonzero real number. Then*

$$\|C_\lambda f - C_{-iq} f\|_{p'} \to 0$$

as $\lambda \to -iq$ through $\mathbb{C}_+$.

With Lemma 4.4, we now state the following corollary to Theorem 4.3.

Corollary 4.6. *Let $\mathcal{A}$ and $F \in \mathcal{B}_{\mathcal{A}}^{(p)}$ be as in Theorem 4.3. Next let $T_{\lambda,h}(F)$ and $\Psi_\lambda f$ be related by (4.5). Then the following hold.*

(i) *If $F \in \mathcal{B}_{\mathcal{A}}^{(1)}$, then for all $\lambda \in \mathbb{C}_+$ and every $h \in \mathcal{N}_\infty(\mathcal{A})$, $T_{\lambda,h}(F) \in \mathcal{B}_{\mathcal{A}}^{(\infty)}$ and for all $\lambda \in \widetilde{\mathbb{C}}_+$,*

$$\|\Psi_\lambda f\|_\infty \le K_n\big(|\lambda|\big)\|f\|_1. \tag{4.9}$$

(ii) *If $F \in \mathcal{B}_{\mathcal{A}}^{(p)}$ with $p \in (1,2]$, then for all $\lambda \in \mathbb{C}_+$ and every $h \in \mathcal{N}_\infty(\mathcal{A})$, $T_{\lambda,h}(F) \in \mathcal{B}_{\mathcal{A}}^{(p')}$ where $p' = \frac{p}{p-1}$, and for all $\lambda \in \widetilde{\mathbb{C}}_+$,*

$$\|\Psi_\lambda f\|_{p'} \le K_n\big(|\lambda|\big)^{(2/p-1)}\|f\|_p. \tag{4.10}$$

In (4.9) and (4.10), $\Psi_\lambda f$ is given by (4.6) above.

Proof. (i) If $p = 1$, then for all $(\lambda,\vec{\xi}) \in \widetilde{\mathbb{C}}_+ \times \mathbb{R}^n$, $|(\Psi_\lambda f)(\vec{\xi})| \le K_n(|\lambda|)\|f\|_1$ because $|H(\lambda;\vec{u}-\vec{\xi})| \le 1$ for all $\lambda \in \widetilde{\mathbb{C}}_+$. Hence (4.9) holds by the definition of the L_∞-norm of functions on $\mathbb{R}^n$. Using (4.6) and (4.4) with $\sigma^2 = 1$, we obtain that, for $\lambda \in \widetilde{\mathbb{C}}_+$,

$$\begin{aligned} \big|(\Psi_\lambda f)(\vec{\xi})\big| &\le K_n\big(|\lambda|\big) \int_{\mathbb{R}^n} \big|f(\vec{u})H(\lambda;\vec{u}-\vec{\xi})\big|\, d\vec{u} \\ &\le K_n\big(|\lambda|\big) \int_{\mathbb{R}^n} \big|f(\vec{u})\big| \exp\Big\{-\frac{\mathrm{Re}(\lambda)}{2}\sum_{j=1}^n (u_j-\xi_j)^2\Big\}\, d\vec{u} \\ &\le K_n\big(|\lambda|\big) \int_{\mathbb{R}^n} \big|f(\vec{u})\big|\, d\vec{u} \\ &= K_n\big(|\lambda|\big)\|f\|_1. \end{aligned} \tag{4.11}$$

Also, using the dominated convergence theorem and the fact that $(\Psi_\lambda f)(\vec{\xi})$ belongs to $C_0(\mathbb{R}^n)$, we obtain

$$\lim_{|\vec{\xi}|\to\infty}(\Psi_\lambda f)(\vec{\xi}) = K_n(\lambda)\int_{\mathbb{R}^n} f(\vec{u}) \lim_{|\vec{\xi}|\to\infty} H(\lambda;\vec{u}-\vec{\xi})\,d\vec{u} = 0.$$

Thus $T_{\lambda,h}(F) \in \mathcal{B}_{\mathcal{A}}^{(\infty)}$ for all $\lambda \in \mathbb{C}_+$.

(ii) Now let $p \in (1,2]$. Then Lemma 4.4 tells us that for all $\lambda \in \widetilde{\mathbb{C}}_+$, $\Psi_\lambda f$ is an element of $L_{p'}(\mathbb{R}^n)$ which satisfies (4.10), and so $T_{\lambda,h}(F) \in \mathcal{B}_{\mathcal{A}}^{(p')}$ for $\lambda \in \mathbb{C}_+$. □

In Theorems 4.7 and 4.8 below, we establish the existence of the L_p analytic $\mathcal{Z}_h$-FFT, $T_{q,h}^{(p)}(F)$, of functionals F in the class $\mathcal{B}_{\mathcal{A}}^{(p)}$.

Theorem 4.7. *Let $F \in \mathcal{B}_{\mathcal{A}}^{(1)}$ be given by equation (4.2). Then for every nonzero real number q and every $h \in \mathcal{N}_\infty(\mathcal{A})$, the L_1 analytic $\mathcal{Z}_h$-FFT of F, $T_{q,h}^{(1)}(F)(y)$, exists as an element of $\mathcal{B}_{\mathcal{A}}^{(\infty)}$ and is given by*

$$T_{q,h}^{(1)}(F)(y) = (\Psi_{-iq} f)\big(\mathcal{P}_{\mathcal{A}}(y)\big) \tag{4.12}$$

for SI-a.e. $y \in C_0[0,T]$, where $\Psi_{-iq}f$ is given by equation (4.6) with λ replaced with $-iq$.

Proof. Let $F \in \mathcal{B}_{\mathcal{A}}^{(1)}(\mathcal{A})$. From the inequality (4.8), one can see that the function $(\psi_\lambda f)$ given by (4.7) is an element of $L_1(\mathbb{R}^n)$ for all $\lambda \in \mathbb{C}_+$. Hence $(\Psi_\lambda f)(\vec{\xi})$ converges pointwise to $(\Psi_{-iq} f)(\vec{\xi})$, as $\lambda \to -iq$ through $\mathbb{C}_+$, by the use of the dominated convergence theorem. Now let $\mu \in \mathcal{M}(\mathbb{R}^n)$ denote the dual of $C_0(\mathbb{R}^n)$. Since $|(\Psi_\lambda f)(\vec{\xi})| \le K_n(|\lambda|)\|f\|_1$ for all $\lambda \in \widetilde{\mathbb{C}}_+$, by (4.11) we have

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} \int_{\mathbb{R}^n} (\Psi_\lambda f)(\vec{\xi})\,d\mu(\vec{\xi}) = \int_{\mathbb{R}^n} (\Psi_{-iq} f)(\vec{\xi})\,d\mu(\vec{\xi})$$

by the dominated convergence theorem. Thus, as elements of $C_0(\mathbb{R}^n)$, $(\Psi_\lambda f)(\vec{\xi})$ converges weakly to $(\Psi_{-iq} f)(\vec{\xi})$ as $\lambda \to -iq$ through values in $\mathbb{C}_+$. Hence, $T_{q,h}^{(1)}(F)(y)$ exists as an element of $\mathcal{B}_{\mathcal{A}}^{(\infty)}$ and is given by equation (4.12). □

Theorem 4.8. *Let $p \in (1,2]$ and let $F \in \mathcal{B}_{\mathcal{A}}^{(p)}$ be given by equation (4.2). Then for every nonzero real number q and every $h \in \mathcal{N}_\infty(\mathcal{A})$, the L_p analytic $\mathcal{Z}_h$-FFT, $T_{q,h}^{(p)}(F)(y)$ exists as an element of $\mathcal{B}_{\mathcal{A}}^{(p')}$ and is given by*

$$T_{q,h}^{(p)}(F)(y) = (\Psi_{-iq} f)\big(\mathcal{P}_{\mathcal{A}}(y)\big) \tag{4.13}$$

for SI-a.e. $y \in C_0[0,T]$, where $\Psi_{-iq}f$ is given by equation (4.6) with λ replaced with $-iq$.

Proof. From (ii) of Corollary 4.6, we know that for each $\lambda \in \widetilde{\mathbb{C}}_+$, $\Psi_\lambda f$ is in $L_{p'}(\mathbb{R}^n)$. Using Lemma 4.5, we obtain that for $f \in L_p(\mathbb{R}^n)$,

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} \|\Psi_\lambda f - \Psi_{-iq} f\|_{p'} = 0. \tag{4.14}$$

Now to show that $T_{q,h}^{(p)}(F)(y)$ exists and is given by the formula (4.13), it will suffice to show that for each $\rho > 0$,

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} E_y\big[\big|(\Psi_\lambda f)\big(\mathcal{P}_{\mathcal{A}}(\rho y)\big) - (\Psi_{-iq} f)\big(\mathcal{P}_{\mathcal{A}}(\rho y)\big)\big|^{p'}\big] = 0.$$

But for all $\rho > 0$,

$$\begin{aligned} &E_y\big[\big|(\Psi_\lambda f)\big(\mathcal{P}_{\mathcal{A}}(\rho y)\big) - (\Psi_{-iq} f)\big(\mathcal{P}_{\mathcal{A}}(\rho y)\big)\big|^{p'}\big] \\ &\quad = \Big(\prod_{j=1}^{n} 2\pi\rho^2\|\alpha_j\|_2^2\Big)^{-1/2} \int_{\mathbb{R}^n} \big|(\Psi_\lambda f)(\vec{\xi}) - (\Psi_{-iq} f)(\vec{\xi})\big|^{p'} \\ &\qquad \times \exp\Big\{-\sum_{j=1}^{n} \frac{\xi_j^2}{2\rho^2\|\alpha_j\|_2^2}\Big\}\, d\vec{\xi} \\ &\quad \le \Big(\prod_{j=1}^{n} 2\pi\rho^2\|\alpha_j\|_2^2\Big)^{-1/2} \|\Psi_\lambda f - \Psi_{-iq} f\|_{p'}^{p'} \end{aligned}$$

which goes to zero as $\lambda \to -iq$ through $\mathbb{C}_+$ by (4.14). Thus $T_{q,h}^{(p)}(F)(y)$ exists, belongs to $\mathcal{B}_{\mathcal{A}}^{(p')}$, and is given by equation (4.13). □

We finish this section by obtaining inverse transform theorems for functionals F in $\mathcal{B}_{\mathcal{A}}^{(p)}$.

Theorem 4.9. *Let $p \in [1, 2]$ and let $F \in \mathcal{B}_{\mathcal{A}}^{(p)}$. Let q be a nonzero real number and let $h \in \mathcal{N}_\infty(\mathcal{A})$. Then we have the following.*

(i) *For each $\rho > 0$,*

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} E_y\big[\big|T_{\bar{\lambda},h}\big(T_{\lambda,h}(F)\big)(\rho y) - F(\rho y)\big|^{p}\big] = 0.$$

(ii) *$T_{\bar{\lambda},h}(T_{\lambda,h}(F)) \to F$ SI-a.e. as $\lambda \to -iq$ through $\mathbb{C}_+$.*

It will be helpful to establish the following lemma before giving the proof of Theorem 4.9. Equation (4.15) below can be found in [2, p. 525].

Lemma 4.10. *Let $\lambda \in \mathbb{C}_+$ be given. Then*

$$\begin{aligned} &\int_{\mathbb{R}} \exp\Big\{-\frac{\lambda}{2}(u-w)^2 - \frac{\bar{\lambda}}{2}(w-v)^2\Big\}\, dw \\ &\quad = \Big(\frac{\pi}{\operatorname{Re}(\lambda)}\Big)^{1/2} \exp\Big\{-\frac{|\lambda|^2}{4\operatorname{Re}(\lambda)}(u-v)^2\Big\}. \end{aligned} \tag{4.15}$$

Proof of Theorem 4.9. Proceeding as in the proof of Theorem 4.3, we obtain, for all $\lambda > 0$,

$$\begin{aligned}
&T_{\bar{\lambda},h}\big(T_{\lambda,h}(F)\big)(y)\\
&\quad= K_n(\bar{\lambda})\int_{\mathbb{R}^n}(\Psi_\lambda f)(\vec{w})H\big(\lambda;\vec{w}-\mathcal{P}_{\mathcal{A}}(y)\big)\,d\vec{w}\\
&\quad= K_n(\bar{\lambda})\int_{\mathbb{R}^n}\Big[K_n(\lambda)\int_{\mathbb{R}^n}f(\vec{u})H(\lambda;\vec{u}-\vec{w})\,d\vec{u}\Big]H\big(\lambda;\vec{w}-\mathcal{P}_{\mathcal{A}}(y)\big)\,d\vec{w}\\
&\quad= K_n(\bar{\lambda})K_n(\lambda)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}f(\vec{u})H(\lambda;\vec{u}-\vec{w})H\big(\lambda;\vec{w}-\mathcal{P}_{\mathcal{A}}(y)\big)\,d\vec{u}\,d\vec{w}\\
&\quad\equiv \Theta\big(\lambda,\bar{\lambda};\mathcal{P}_{\mathcal{A}}(y)\big),
\end{aligned}$$

where $\Psi_\lambda f$ is given by (4.6) and to compute $\Theta(\lambda,\bar{\lambda};\mathcal{P}_{\mathcal{A}}(y))$, we begin by writing

$$\begin{aligned}
&\Theta(\lambda,\bar{\lambda};\vec{v})\\
&\quad= \Big|\frac{\lambda}{2\pi}\Big|^n\int_{\mathbb{R}^{2n}}f(\vec{u})H(\lambda;\vec{u}-\vec{w})H(\lambda;\vec{w}-\vec{v})\,d\vec{u}\,d\vec{w}\\
&\quad= \Big|\frac{\lambda}{2\pi}\Big|^n\int_{\mathbb{R}^{2n}}f(\vec{u})\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^n(u_j-w_j)^2-\frac{\bar{\lambda}}{2}\sum_{j=1}^n(w_j-v_j)^2\Big\}\,d\vec{w}\,d\vec{u}.
\end{aligned}$$

But, using equation (4.15), we can write the expression just above as

$$\begin{aligned}
\Theta(\lambda,\bar{\lambda};\vec{v}) &= \Big|\frac{\lambda}{2\pi}\Big|^n\int_{\mathbb{R}^n}f(\vec{u})\Big(\frac{\pi}{\mathrm{Re}(\lambda)}\Big)^{n/2}\exp\Big\{-\frac{|\lambda|^2}{4\,\mathrm{Re}(\lambda)}\sum_{j=1}^n(u_j-v_j)^2\Big\}\,d\vec{u}\\
&= (f*\phi_\varepsilon)(\vec{v}),
\end{aligned}$$

where

$$\phi(\vec{v})\equiv\phi(v_1,\dots,v_n)=(2\pi)^{-n/2}\exp\Big\{-\frac{1}{2}\sum_{j=1}^n v_j^2\Big\},\qquad \varepsilon=\frac{\sqrt{2\,\mathrm{Re}(\lambda)}}{|\lambda|},$$

and

$$\phi_\varepsilon(\vec{v})=\frac{1}{\varepsilon^n}\phi\Big(\frac{v_1}{\varepsilon},\dots,\frac{v_n}{\varepsilon}\Big).$$

Now ϕ is nonnegative and $\int_{\mathbb{R}^n}\phi(\vec{v})\,d\vec{v}=1$, so using [22, Theorem 1.18] it follows that

$$\begin{aligned}
\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}\int_{\mathbb{R}^n}\big|\Theta(\lambda,\bar{\lambda};\vec{v})-f(\vec{v})\big|^p\,d\vec{v} &= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n}\big|(f*\phi_\varepsilon)(\vec{v})-f(\vec{v})\big|^p\,d\vec{v}\\
&= \lim_{\varepsilon\to 0^+}\|f*\phi_\varepsilon-f\|_p^p\\
&= 0
\end{aligned}$$

since $\varepsilon \to 0^+$ as $\lambda \to -iq$ through $\mathbb{C}_+$. But now (i) of the theorem follows easily since for each fixed $\rho > 0$,

$$\begin{aligned} E_y\big[\big|T_{\bar\lambda,h}\big(T_{\lambda,h}(F)\big)(\rho y) - F(\rho y)\big|\big] &= (\rho^2\pi)^{-n/2}\int_{\mathbb{R}^n}\big|\Theta(\lambda,\bar\lambda;\vec v) - f(\vec v)\big|^p \exp\Big\{-\frac{1}{2\rho^2}\sum_{j-1}^{n} v_j^2\Big\}\,d\vec v \\ &\le (\rho^2\pi)^{-n/2}\|f*\phi_\varepsilon - f\|_p^p. \end{aligned}$$

Finally, item (ii) of the theorem follows because, by [22, Theorem 1.25], it follows that the function $\Theta(\lambda,\bar\lambda;\vec v) = (f*\phi_\varepsilon)(\vec v)$ converges pointwise to the function $f(\vec v)$ as $\lambda \to -iq$ through $\mathbb{C}_+$. □

Theorem 4.11. *Let $F \in \mathcal{B}_{\mathcal{A}}^{(2)}$. Then for all nonzero real numbers q_1 and q_2, and every $h \in \mathcal{N}_\infty(\mathcal{A})$, $T_{q_2,h}^{(2)}(T_{q_1,h}^{(2)}(F))$ belongs to $\mathcal{B}_{\mathcal{A}}^{(2)}$. Furthermore, for all nonzero real q,*

$$T_{-q,h}^{(2)}\big(T_{q,h}^{(2)}(F)\big) \approx F. \tag{4.16}$$

Proof. Note that for $p = 2$, $p' = 2$, and so for $F \in \mathcal{B}_{\mathcal{A}}^{(2)}$, $h \in \mathcal{N}_\infty(\mathcal{A})$, and $q \in \mathbb{R}\setminus\{0\}$, $T_{q,h}^{(2)}(F)$ is in $\mathcal{B}_{\mathcal{A}}^{(2)}$ by Theorem 4.8. Thus for all $q_1, q_2 \in \mathbb{R}\setminus\{0\}$, $T_{q_2,h}^{(2)}(T_{q_1,h}^{(2)}(F))$ is well defined and is in $\mathcal{B}_{\mathcal{A}}^{(2)}$. Equation (4.16) now follows for each nonzero real number q by letting $\lambda \to -iq$ in (ii) of Theorem 4.9. □

5. Multiple generalized Fourier–Feynman transforms

The MGFFT of functionals on $C_0[0,T]$ was introduced in [5]. In this section, we evaluate the MGFFT of functionals F in the class $\mathcal{B}_{\mathcal{A}}^{(p)}$ using our rotation theorem, Theorem 3.4. We start this section with the definition of the MGFFT of functionals F on $C_0[0,T]$. (For a detailed study of the MGFFT, see [5].)

Definition 5.1. Let F be a SIM functional on $C_0[0,T]$. For positive real $t > 0$, define a transform $\mathcal{M}_{t,\mathcal{H}}(F)$ of F as follows: for a finite sequence $\mathcal{H} = \{h_1,\dots,h_m\}$ of nonzero functions in $L_2[0,T]$, let

$$\mathcal{M}_{t,\mathcal{H}}(F)(y) \equiv E_{\vec x}\Big[F\Big(y + t^{-1/2}\sum_{k=1}^{m}\mathcal{Z}_{h_k}(x_k,\cdot)\Big)\Big], \quad y \in C_0[0,T]. \tag{5.1}$$

Let $\mathcal{M}_{\lambda,\mathcal{H}}(F)(y)$ be an analytic extension of $\mathcal{M}_{t,\mathcal{H}}(F)(y)$ as a function of $\lambda \in \mathbb{C}_+$ and let q be a nonzero real number. For $p \in [1,2]$, we define the L_p analytic MGFFT, $\mathcal{M}_{q,\mathcal{H}}^{(p)}(F)(y)$ of F, by the formula (if it exists)

$$\mathcal{M}_{q,\mathcal{H}}^{(p)}(F)(y) = \begin{cases} \operatorname{l.i.m.}_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}_+}}(w_s^{p'})\mathcal{M}_{\lambda,\mathcal{H}}(F)(y), & 1 < p \le 2, \\ \lim_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}_+}}\mathcal{M}_{\lambda,\mathcal{H}}(F)(y), & p = 1. \end{cases}$$

Clearly, it follows that, for all nonzero functions h in $L_2[0,T]$,

$$\mathcal{M}_{\lambda,\{h\}}(F)(y) = T_{\lambda,h}(F)(y) \qquad \text{and} \qquad \mathcal{M}_{q,\{h\}}^{(p)}(F)(y) = T_{q,h}^{(p)}(F)(y)$$

if the transforms exist.

In our next theorem, we analyze the MGFFT of functionals F in $\mathcal{B}_{\mathcal{A}}^{(p)}$ as the single FFT.

Theorem 5.2. *Let $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$ be an orthogonal set of nonzero functions in $L_2[0,T]$. Let $p \in [1,2]$ and let $F \in \mathcal{B}_{\mathcal{A}}^{(p)}$ be given by equation (4.2). Then for all real $q \neq 0$ and any finite sequence $\mathcal{H} = \{h_1, \dots, h_m\}$ in $\mathcal{N}_\infty(\mathcal{A})$, the L_p analytic MGFFT, $\mathcal{M}_{q,\mathcal{H}}^{(p)}(F)$, exists, belongs to $\mathcal{B}_{\mathcal{A}}^{(p')}$, and is given by*

$$\mathcal{M}_{q,\mathcal{H}}^{(p)}(F)(y) = (\Psi_\lambda^m f)\big(\mathcal{P}_{\mathcal{A}}(y)\big) = T_{q,\mathbf{s}(\mathcal{H})}^{(p)}(F)(y)$$

for SI-a.e. $y \in C_0[0,T]$, where $\Psi_\lambda^m f$ is given by (5.2) below and $\mathbf{s}(\mathcal{H})$ is a function in $L_\infty[0,T]$ satisfying equation (2.2) above.

Remark 5.3. Throughout Section 4, we assumed that given an orthogonal set $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$ and a function $h \in L_\infty[0,T]$, $\mathcal{A}h = \{\alpha_1 h, \dots, \alpha_n h\}$ is an orthonormal set of functions in $L_2[0,T]$. However, all of our results in Section 4 hold if $\mathcal{A}h$ is an orthogonal set such that $\|\alpha_1 h\|_2 = \cdots = \|\alpha_n h\|_2 \equiv \sigma > 0$. In this case, equations (4.6) and (4.5) are rewritten by

$$(\Psi_\lambda^{\sigma^2} f)(\vec{\xi}) \equiv K_n(\lambda;\sigma^2) \int_{\mathbb{R}^n} f(\vec{u}) H(\lambda;\sigma^2;\vec{u}-\vec{\xi})\, d\vec{u} \tag{5.2}$$

and

$$T_{\lambda,h}(F)(y) = (\Psi_\lambda^{\sigma^2} f)\big(\mathcal{P}_{\mathcal{A}}(y)\big). \tag{5.3}$$

Proof of Theorem 5.2. Recall that for each $h \in \mathcal{H}$, $\mathcal{A}h = \{\alpha_1 h, \dots, \alpha_n h\}$ is an orthonormal set in $L_2[0,T]$ and $\|\alpha_j \mathbf{s}(\mathcal{H})\|_2^2 = m$ for all $j \in \{1, \dots, n\}$ (see Remark 3.2 above). For $t > 0$, using (5.1), (4.2), (3.6), the last expression of (3.7), (4.4), (5.2), and (5.3) with λ and σ^2 replaced with t and m, respectively, we obtain

$$\begin{aligned}
\mathcal{M}_{t,\mathcal{H}}(F)(y) &= E_{\vec{x}}\Big[f\Big(\mathcal{P}_{\mathcal{A}}(y) + t^{-1/2}\sum_{k=1}^m \mathcal{P}_{\mathcal{A}h_k}(x_k)\Big)\Big] \\
&= E_x\big[f\big(\mathcal{P}_{\mathcal{A}}(y) + t^{-1/2}\mathcal{P}_{\mathcal{A}\mathbf{s}(\mathcal{H})}(x)\big)\big] \\
&= (2\pi m)^{-n/2}\int_{\mathbb{R}^n} f\big(\mathcal{P}_{\mathcal{A}}(y) + t^{-1/2}\vec{r}\big)\exp\Big\{-\sum_{j=1}^n \frac{r_j^2}{2m}\Big\}\, d\vec{r} \\
&= \Big(\frac{t}{2\pi m}\Big)^{n/2}\int_{\mathbb{R}^n} f\big(\mathcal{P}_{\mathcal{A}}(y) + \vec{u}\big)\exp\Big\{-\frac{t}{2m}\sum_{j=1}^n u_j^2\Big\}\, d\vec{u} \\
&= K_n(t;m)\int_{\mathbb{R}^n} f(\vec{u}) H\big(t;m;\vec{u}-\mathcal{P}_{\mathcal{A}}(y)\big)\, d\vec{u} \\
&= (\Psi_t^m f)\big(\mathcal{P}_{\mathcal{A}}(y)\big) \\
&= T_{t,\mathbf{s}(\mathcal{H})}(F)(y)
\end{aligned}$$

for SI-a.e. $y \in C_0[0,T]$. Applying Theorem 4.3 and Remark 5.3, we next observe that for all $\lambda \in \mathbb{C}_+$, $\mathcal{M}_{\lambda,\mathcal{H}}(F)(y) = T_{\lambda,\mathbf{s}(\mathcal{H})}(F)(y)$ for SI-a.e. $y \in C_0[0,T]$. Now by analytic continuation in λ and Theorem 4.8, we have the desired results. □

Given a sequence $\mathcal{H} = \{h_1, \dots, h_m\}$ in $\mathcal{N}_\infty(\mathcal{A})$, consider the case $h_1 = \cdots = h_m \equiv h$. In this case, for notational convenience, let $\mathcal{H} \equiv \wedge^m(h)$. Then, in view of equation (2.2), we can choose $\mathbf{s}(\wedge^m(h))$ to be $\sqrt{m}h$.

Corollary 5.4. *Let $\mathcal{A}$ and $F \in \mathcal{B}_{\mathcal{A}}^{(p)}$ be as in Theorem 5.2. Then for all real $q \neq 0$ and every $h \in \mathcal{N}_\infty(\mathcal{A})$, $\mathcal{M}_{q,\wedge^m(h)}^{(p)}(F)(y) = T_{q,\mathbf{s}(\wedge^m(h))}^{(p)}(F)(y)$ for SI-a.e. $y \in C_0[0,T]$. In particular,*

$$\mathcal{M}_{q,\wedge^m(h)}^{(p)}(F) \approx T_{q,\sqrt{m}h}^{(p)}(F).$$

In view of Theorems 4.11 and 5.2, we obtain the following result for L_2 analytic MGFFT for functionals in $\mathcal{B}_{\mathcal{A}}^{(2)}$.

Theorem 5.5. *Let $\mathcal{A}$ and $F \in \mathcal{B}_{\mathcal{A}}^{(2)}$ be as in Theorem 5.2. Then for all real $q \neq 0$ and any finite sequence $\mathcal{H} = \{h_1, \dots, h_m\}$ in $\mathcal{N}_\infty(\mathcal{A})$,*

(i) $\mathcal{M}_{q,\mathcal{H}}^{(2)}(F)(y) = T_{q,h_m}^{(2)}(T_{q,h_{m-1}}^{(2)}(\cdots(T_{q,h_1}^{(2)}(F))\cdots))(y)$ *for SI-a.e.* $y \in C_0[0,T]$, *and*

(ii) $\mathcal{M}_{-q,\mathcal{H}}^{(2)}(\mathcal{M}_{q,\mathcal{H}}^{(2)}(F)) \approx F$, *that is to say, the L_2 analytic MGFFT, $\mathcal{M}_{q,\mathcal{H}}^{(2)}$, has the inverse transform $\{\mathcal{M}_{q,\mathcal{H}}^{(2)}\}^{-1} = \mathcal{M}_{-q,\mathcal{H}}^{(2)}$.*

Appendix A

In this appendix we present the concept of the stochastic continuity of stochastic processes and related theorems commented in Section 1 above. For the detailed proofs of the theorems below, see [23, Section 21].

Definition A.1. A stochastic process X on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and an interval $D \subset \mathbb{R}$ is said to be stochastically continuous (or continuous in probability) at $t_0 \in D$ if $X(t,\cdot)$ converges to $X(t_0,\cdot)$ in probability as $t \to t_0$ in the sense that

$$\lim_{t\to t_0} \mathrm{P}\big(\{\omega \in \Omega : \big|X(t,\omega) - X(t_0,\omega)\big| \geq \varepsilon\}\big) = 0 \quad \forall \varepsilon > 0.$$

We say that X is *stochastically continuous* if this holds for all $t_0 \in D$.

Theorem A.2 ([23, Theorem 21.1])**.** *Let X be a Gaussian process on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and an interval $D \subset \mathbb{R}$. If the mean function $m(t) = \mathbb{E}[X(t,\cdot)]$, $t \in D$, is continuous at $t_0 \in D$ and the covariance function $r(s,t) = \mathbb{C}\mathrm{ov}[X(s,\cdot), X(t,\cdot)]$, $s, t, \in D$, is continuous at $(t_0,t_0) \in D \times D$, then X is stochastically continuous at t_0.*

Theorem A.3 ([23, Theorem 21.2])**.** *Let X be a Gaussian process on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and an interval $D \subset \mathbb{R}$. If*

$$\mathbb{E}\big[X(t,\cdot)\big] = 0 \quad \text{for } t \in D,$$
$$\mathbb{V}\mathrm{ar}\big[X(t,\cdot) - X(s,\cdot)\big] \leq \alpha|t-s|^\beta \quad \text{for } s,t \in D,$$

for some positive real numbers α and β, then X is stochastically continuous at every $t \in D$.

Appendix B

In this appendix we present the theorems which was used in the proof of Theorem 4.9. For the detailed proofs of the theorems below, see [22, Chapter 1].

Theorem B.1 ([22, Theorem 1.18]). *Suppose that* $\varphi \in L_1(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} \varphi(\vec{u})\, d\vec{u} = 1$ *and, for* $\varepsilon > 0$, *let* $\varphi_\varepsilon(\vec{u}) = \varepsilon^{-n}\varphi(\varepsilon^{-1}\vec{u})$. *If* $f \in L_p(\mathbb{R}^n)$, $1 \le p < +\infty$, *or* $f \in C_0(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n)$, *then* $\|f * \varphi_\varepsilon - f\|_p \to 0$ *as* $\varepsilon \to 0$, *where* $(f * g)$ *denotes the convolution of* f *and* g, *i.e.,*

$$(f * g)(\vec{v}) = \int_{\mathbb{R}^n} f(\vec{v} - \vec{u}) g(\vec{u})\, d\vec{u}.$$

Theorem B.2 ([22, Theorem 1.25]). *Suppose that* $\varphi \in L_1(\mathbb{R}^n)$. *Let*

$$\psi(\vec{v}) = \operatorname*{ess.\,sup.}_{|\vec{t}| \ge |\vec{v}|} \big|\varphi(\vec{v})\big|$$

and, for $\varepsilon > 0$, *let* $\varphi_\varepsilon(\vec{v}) = \varepsilon^{-n}\varphi(\varepsilon^{-1}\vec{v})$. *If* $\psi \in L_1(\mathbb{R}^n)$ *and* $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, *then*

$$\lim_{\varepsilon \to 0} (f * \varphi_\varepsilon)(\vec{v}) = f(\vec{v}) \int_{\mathbb{R}^n} \varphi(\vec{t})\, d\vec{t}$$

whenever $\vec{v}$ *belongs to the set* L_f *given by*

$$L_f \equiv \Big\{ \vec{v} \in \mathbb{R}^n : \frac{1}{r^n} \int_{\{\vec{t}: |\vec{t}| < r\}} \big| f(\vec{v} - \vec{t}) - f(\vec{v}) \big|\, d\vec{t} \to 0 \text{ as } r \to 0 \Big\}.$$

In fact, for given f, *it follows that* $m_L(\mathbb{R}^n \setminus L_f) = 0$, *where* m_L *denotes the Lebesgue measure on* $\mathbb{R}^n$.

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