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ON A GENERALIZED UNIFORM ZERO-TWO LAW FOR POSITIVE CONTRACTIONS OF NONCOMMUTATIVE L_1-SPACES AND ITS VECTOR-VALUED EXTENSION

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Abstract. Ornstein and Sucheston first proved that for a given positive contraction $T : L_1 \to L_1$ there exists $m \in \mathbb{N}$ such that if $\|T^{m+1} - T^m\| < 2$, then $\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0$. This result was referred to as the *zero-two law*. In the present article, we prove a generalized uniform zero-two law for the multiparametric family of positive contractions of noncommutative L_1-spaces. Moreover, we also establish a vector-valued analogue of the uniform zero-two law for positive contractions of $L_1(M, \Phi)$—the noncommutative L_1-spaces associated with center-valued traces.

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a measure space with a positive σ-additive measure μ, and let $L_1(X, \mathcal{F}, \mu)$ be the usual associated real L_1-space. A linear operator $T : L_1(X, \mathcal{F}, \mu) \to L_1(X, \mathcal{F}, \mu)$ is called a *positive contraction* if $Tf \geq 0$ whenever $f \geq 0$ and $\|T\| \leq 1$. Some examples of positive contractions associated with positive kernels can be found in [24].

In [21, Theorem 1.1], the following was proved.

Theorem 1.1. *Let $T : L_1 \to L_1$ be a positive contraction. Then either*

$$\sup_{\|f\|_1 \leq 1} \lim_{n\to\infty} \|T^{n+1} f - T^n f\| = 2 \tag{1}$$

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or $\|T^{n+1}f - T^n f\| \to 0$ *for every* $f \in L^1$.

We note that this result first appeared in [13], but Ornstein and Sucheston [21] were able to obtain its analytical proof. Later, the formulated theorem became known as the *strong zero-two law*. Consequently, by [21, Theorem 1.3], if T is ergodic with $T^*\mathbf{1} = \mathbf{1}$ (e.g., T is ergodic and conservative), then either (1) holds or $\|T^n g\|_1 \to 0$ for every $g \in L_1$ with $\int g\, d\mu = 0$.

By interchanging "sup" and "lim" in the strong zero-two law we have the following *uniform zero-two law*, proved by Foguel in [4, Theorem I] using ideas from [21].

Theorem 1.2. *Let* $T : L_1 \to L_1$ *be a positive contraction. If for some* $m \in \mathbb{N} \cup \{0\}$ *we have* $\|T^{m+1} - T^m\| < 2$*, then*

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0.$$

Zaharopol provided another proof of Theorem 1.2, which is reduced to the following results.

Theorem 1.3 ([27, Section 2])**.** *Let* $T : L_1 \to L_1$ *be a positive contraction. Then for the following statements,*

(i) *there is some* $m \in \mathbb{N}$ *such that* $\|T^{m+1} - T^m\| < 2$*,*
(ii) *there is some* $m \in \mathbb{N}$ *such that* $\|T^{m+1} - (T^{m+1} \wedge T^m)\| < 1$*,*
(iii) *we have*

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0,$$

the implications (i) $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii) hold.*

To establish the implication (ii) $\Rightarrow$ (iii), the following auxiliary fact was established in [27].

Theorem 1.4 ([27, Theorem 1])**.** *Let* $T, S : L_1 \to L_1$ *be two positive contractions such that* $T \le S$*. If* $\|S - T\| < 1$*, then* $\|S^n - T^n\| < 1$ *for all* $n \in \mathbb{N}$*.*

In [17], the last result for Jordan algebras was extended. Therefore, the natural next step is to find an analogue of Theorem 1.3 in the noncommutative setting.

The aim of this article is to prove a noncommutative version of a generalized uniform "zero-two" law for the multiparametric family of positive contractions of L_1-spaces associated with von Neumann algebras. In the case when the algebra is commutative, we recover a result of [16, Theorem 3.1]. Moreover, we emphasize that Theorem 1.2 will be included in the main result in the present article as a particular case.

On the other hand, development of the theory of integration for measures μ with values in ordered spaces has inspired the study of lattice-normed L_p-spaces (see, e.g., [15]). The existence of center-valued traces on finite von Neumann algebras naturally leads to the development of the theory of integration for this kind of trace. In [2] and [6], noncommutative L_p-spaces associated with central-valued traces were investigated. More recently, in [3], a module approach in a somewhat different direction was taken by choosing modules with L^∞-spaces as the ring. These works have succeeded in extending many important results in functional analysis to topological L^∞-modules. In this article, we follow an approach that is based on measurable bundles of Banach lattices (see [11], [12]), which differs from the approach used in [3].

Another main aim of this paper is to establish the uniform zero-two law for noncommutative L_1-spaces associated with central-valued traces. In [6], it was established that L_p-spaces associated with central-valued traces are Banach–Kantorovich spaces, and the theory of Banach–Kantorovich spaces is now well-developed (see, e.g., [15]). One of the important approaches to studying Banach–Kantorovich spaces is provided by the theory of continuous and measurable Banach bundles (see [11]). In this approach, the representation of a Banach–Kantorovich lattice as a space of measurable sections of a measurable Banach bundle makes it possible to obtain the needed properties of the lattice by means of the corresponding stalkwise verification of the properties. In [6], as an application of this approach, noncommutative $L_p(M,\Phi)$-spaces associated with center-valued traces are represented as a bundle of noncommutative L_p-spaces associated with numerical traces.

In the second part of this article, we prove a vector-valued analogue of the main result for positive contractions of noncommutative L_1-spaces associated with central-valued traces. To do this, we mainly employ the theory of continuous and measurable Banach bundles for the existence of vector-valued lifting, which allows us to prove the required result.

The present article is organized as follows. In Section 2, we collect some necessary well-know facts about noncommutative L_1-spaces. In Section 3, we prove an auxiliary result (a noncommutative analogue of Theorem 1.4) about dominant operators. Section 4 is devoted to the proof of a generalized uniform zero-two law for a multiparametric family of positive contractions of the noncommutative L_1-spaces. In Section 5, we recall necessary definitions about $L_1(M,\Phi)$—the noncommutative L_1-spaces associated with center-valued traces. Finally, in Section 6, by means of the result of Section 5, we first prove that every positive contraction of $L_1(M,\Phi)$ can be represented as a measurable bundle of positive contractions of noncommutative L_1-spaces, and this allows us to establish a vector-valued analogue of the uniform zero-two law for positive contractions of $L_1(M,\Phi)$.

2. Preliminaries

Throughout this article, let M be a von Neumann algebra with the unit $\mathbb{1}$ and let τ be a faithful, normal, semifinite trace on M. We therefore omit this condition from the formulation of the theorems. Recall that an element $x \in M$ is called *self-adjoint* if $x = x^*$. The set of all self-adjoint elements is denoted by M_{sa}. By M_* we denote a predual space to M (see [23] for definitions).

Let $\mathfrak{N} = \{x \in M : \tau(|x|) < \infty\}$. Here $|x|$ denotes the modulus of an element x; that is, $|x| = \sqrt{x^*x}$. The map $\|\cdot\|_1 : \mathfrak{N} \to [0,\infty)$ given by the formula $\|x\|_1 = \tau(|x|)$ defines a norm (for details, see [19]). The completion of $\mathfrak{N}$ with respect to the norm $\|\cdot\|_1$ is denoted by $L_1(M,\tau)$. It is known from [19] that the spaces $L_1(M,\tau)$ and M_* are isometrically isomorphic and therefore can be identified. In what follows, we will use this fact without noting it.

Theorem 2.1 ([19, Theorem 5]). *The space $L_1(M,\tau)$ coincides with the set*

$$L_1 = \Big\{x = \int_{-\infty}^{\infty} \lambda\, de_\lambda : \int_{-\infty}^{\infty} |\lambda|\, d\tau(e_\lambda) < \infty\Big\}.$$

Moreover,

$$\|x\|_1 = \int_{-\infty}^{\infty} |\lambda|\, d\tau(e_\lambda).$$

It is known from [19] that the equality

$$L_1(M,\tau) = L_1(M_{\text{sa}},\tau) + iL_1(M_{\text{sa}},\tau) \tag{2}$$

is valid. Note that $L_1(M_{\text{sa}},\tau)$ is a predual to M_{sa}.

Let $T : L_1(M,\tau) \to L_1(M,\tau)$ be any bounded linear operator, and let $\tilde{T}$ denote its restriction to $L_1(M_{\text{sa}},\tau)$. Then due to (2) we have $T(x+iy) = \tilde{T}(x) + i\tilde{T}(y)$, where $x, y \in L_1(M_{\text{sa}},\tau)$. This means that any linear bounded operator is uniquely defined by its restriction to $L_1(M_{\text{sa}},\tau)$. Therefore, in what follows, we only consider linear operators on $L_1(M_{\text{sa}},\tau)$ over real numbers.

Recall that a linear operator T is called *positive* if $Tx \geq 0$ whenever $x \geq 0$. A linear operator T is said to be a *contraction* if $\|T(x)\|_1 \leq \|x\|_1$ for all $x \in L_1(M_{\text{sa}},\tau)$. Denote

$$\|T\| = \sup\big\{\|Tx\|_1 : \|x\|_1 = 1, x \in L_1(M_{\text{sa}},\tau)\big\}.$$

Let $T, S : L_1 \to L_1$ be two positive contractions. In what follows, we write $T \leq S$ if $S - T$ is a positive operator.

The following auxiliary facts are well known (see, e.g., [17]).

Lemma 2.2. *Let $T : L_1(M_{\text{sa}},\tau) \to L_1(M_{\text{sa}},\tau)$ be a positive operator. Then*

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1, x\geq 0} \|Tx\|.$$

Lemma 2.3. *Let $T, S : L_1(M_{\text{sa}},\tau) \to L_1(M_{\text{sa}},\tau)$ be two positive contractions such that $T \leq S$. Then, for every $x \in L_1(M_{\text{sa}},\tau)$, $x \geq 0$, the following equality holds:*

$$\|Sx - Tx\| = \|Sx\| - \|Tx\|.$$

3. Dominant operators

In this section, we prove an auxiliary result related to dominant operators. A similar result was proved in [17], but for the sake of completeness we show the following.

Theorem 3.1. *Let $Z, T, S : L_1(M_{\text{sa}},\tau) \to L_1(M_{\text{sa}},\tau)$ be positive contractions such that $T \leq S$ and $ZS = SZ$. If there is an $n_0 \in \mathbb{N}$ such that $\|Z(S^{n_0} - T^{n_0})\| < 1$, then $\|Z(S^n - T^n)\| < 1$ for every $n \geq n_0$.*

Proof. Assume the contrary; that is, $\|Z(S^n - T^n)\| = 1$ for some $n > n_0$. Denote

$$m = \min\big\{n \in \mathbb{N} : \big\|Z(S^{n_0+n} - T^{n_0+n})\big\| = 1\big\}.$$

It is clear that $m \geq 1$. The positivity of Z with $T \leq S$ implies that $Z(S^{n_0+n} - T^{n_0+n})$ is a positive operator. Then, according to Lemma 2.2, there exists a sequence $\{x_n\} \in L_1(M_{\mathrm{sa}}, \tau)$ such that $x_n \geq 0$, $\|x_n\| = 1, \forall n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \big\|Z(S^{n_0+m} - T^{n_0+m})x_n\big\| = 1. \tag{3}$$

The positivity of $Z(S^{n_0+m} - T^{n_0+m})$ and $x_n \geq 0$, together with Lemma 2.3, yield that

$$\big\|Z(S^{n_0+m} - T^{n_0+m})x_n\big\| = \|ZS^{n_0+m}x_n\| - \|ZT^{n_0+m}x_n\| \tag{4}$$

for every $n \in \mathbb{N}$. It then follows from (3) and (4) that

$$\lim_{n\to\infty} \|ZS^{n_0+m}x_n\| = 1, \tag{5}$$

$$\lim_{n\to\infty} \|ZT^{n_0+m}x_n\| = 0. \tag{6}$$

Thanks to the contractivity of S and Z together with $ZS = SZ$ we obtain

$$\|ZS^{n_0+m}x_n\| = \big\|S(ZS^{n_0+m-1}x_n)\big\| \leq \|ZS^{n_0+m-1}x_n\| \leq \|S^m x_n\|.$$

Hence, the last inequalities together with (5) imply

$$\lim_{n\to\infty} \|ZS^{n_0+m-1}x_n\| = 1, \qquad \lim_{n\to\infty} \|S^m x_n\| = 1. \tag{7}$$

Moreover, the contractivity of Z, S, and T $(i = 1, 2)$ implies that $\|ZT^{n_0+m-1}x_n\| \leq 1$, $\|T^m x_n\| \leq 1$, and $\|ZS^{n_0}T^m x_n\| \leq 1$ for every $n \in \mathbb{N}$. Therefore, we may choose a subsequence $\{y_k\}$ of $\{x_n\}$ such that the sequences $\{\|ZT^{n_0+m-1}y_k\|\}$, $\{\|T^m y_k\|\}$, and $\{\|ZS^{n_0}T^m y_k\|\}$ converge. Hence, let us denote their limits as follows:

$$\alpha = \lim_{k\to\infty} \|ZT^{n_0+m-1}y_k\|, \tag{8}$$

$$\beta = \lim_{k\to\infty} \|ZS^{n_0}T^m y_k\|, \tag{9}$$

$$\gamma = \lim_{k\to\infty} \|T^m y_k\|. \tag{10}$$

The inequality $\|Z(S^{n_0+m-1} - T^{n_0+m-1})\| < 1$ with (7) implies that $\alpha > 0$. Hence we may choose a subsequence $\{z_k\}$ of $\{y_k\}$ such that $\|ZT^{n_0+m-1}z_k\| \neq 0$ for all $k \in \mathbb{N}$.

From $\|ZT^{n_0+m-1}z_k\| \leq \|T^m z_k\|$, together with (8) and (10), we find $\alpha \leq \gamma$, and hence $\gamma > 0$.

Now, using Lemma 2.3, we get

$$\begin{aligned}
\|ZS^{n_0}T^m z_k\| &= \big\|ZS^{n_0+m}z_k - Z(S^{n_0+m} - S^{n_0}T^m)z_k\big\| \\
&= \|ZS^{n_0+m}z_k\| - \big\|ZS^{n_0}(S^m - T^m)z_k\big\| \\
&\geq \|ZS^{n_0+m}z_k\| - \|S^m z_k - T^m z_k\| \\
&= \|ZS^{n_0+m}z_k\| - \|S^m z_k\| + \|T^m z_k\|.
\end{aligned} \tag{11}$$

Due to (5) and (7) we have

$$\lim_{k\to\infty} \big(\|ZS^{n_0+m}z_k\| - \|S^m z_k\|\big) = 0,$$

which with (11) implies that

$$\lim_{k\to\infty} \|ZS^{n_0}T^m z_k\| \geq \lim_{k\to\infty} \|T^m z_k\|.$$

This means $\beta \geq \gamma$.

On the other hand, from $\|ZS^{n_0}T^m z_k\| \leq \|T_2^m z_k\|$ it follows that $\gamma \geq \beta$, and so $\gamma = \beta$.

Let us denote

$$u_k = \frac{T^m z_k}{\|T^m z_k\|}, \quad k \in \mathbb{N}.$$

Then from $\gamma = \beta$, together with (6), we obtain

$$\lim_{k\to\infty} \|ZS^{n_0}u_k\| = \lim_{k\to\infty} \frac{\|ZS^{n_0}T^m z_k\|}{\|T^m z_k\|} = 1,$$
$$\lim_{k\to\infty} \|ZT^{n_0}u_k\| = \lim_{k\to\infty} \frac{\|ZT^{n_0+m} z_k\|}{\|T^m z_k\|} = 0.$$

So, keeping in mind Lemma 2.3 and the positivity of $Z(S^{n_0} - T^{n_0})$, we find that

$$\lim_{k\to\infty} \big\|Z(S^{n_0} - T^{n_0})u_k\big\| = 1.$$

Since $\|u_k\| = 1$, $u_k \geq 0$ (for all $k \in \mathbb{N}$), from Lemma 2.2 we infer that $\|Z(S^{n_0} - T^{n_0})\| = 1$, which is a contradiction. This completes the proof. □

We note that the proved theorem extends a main result (Theorem 3.3) of the paper [17], which can be seen in the following corollary.

Corollary 3.2. *Let $T, S : L_1(M_{\text{sa}}, \tau) \to L_1(M_{\text{sa}}, \tau)$ be positive contractions such that $T \leq S$. If there is an $n_0 \in \mathbb{N}$ such that $\|S^{n_0} - T^{n_0}\| < 1$, then $\|S^n - T^n\| < 1$ for every $n \geq n_0$.*

The proof immediately follows if we take $Z = \text{Id}$. Note that if $n_0 = 1$ and M is a commutative von Neumann algebra, then from Corollary 3.2 we immediately get Zaharopol's result (see Theorem 1.4).

4. A multiparametric generalization of the zero-two law

In this section, we prove a multiparametric generalization of the zero-two law for positive contractions of noncommutative L_1-spaces.

Let $T : L_1(M_{\text{sa}}, \tau) \to L_1(M_{\text{sa}}, \tau)$ be a positive contraction. Then its dual T^* acts on M_{sa}, and it is also positive and enjoys $T^*\mathbb{1} \leq \mathbb{1}$. If we have $T^*\mathbb{1} = \mathbb{1}$, then T is called a *unital positive contraction*.

Let us first introduce some notation. Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $\mathbf{m} = (m_1, \dots, m_d), \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}_0^d$ $(d \geq 1)$ by the usual way, we define $\mathbf{m} + \mathbf{n} = (m_1 + n_1, \dots, m_d + n_d)$, $\ell\mathbf{n} = (\ell n_1, \dots, \ell n_d)$, where $\ell \in \mathbb{N}_0$. We write $\mathbf{n} \leq \mathbf{k}$ if and only if $n_i \leq k_i$ $(i = 1, 2, \dots, d)$.

Let us formulate our main result.

Theorem 4.1. *Let $Z : L_1(M_{\text{sa}}, \tau) \to L_1(M_{\text{sa}}, \tau)$ be a unital positive contraction. Assume that $T_k : L_1(M_{\text{sa}}, \tau) \to L_1(M_{\text{sa}}, \tau)$, $(k = 1, \dots, d)$ be unital positive contractions such that $ZT_i = T_iZ$, $T_iT_j = T_jT_i$, for every $i, j \in \{1, \dots, d\}$. If there are $\mathbf{m} \in \mathbb{N}_0^d$, $\mathbf{k} \in \mathbb{N}_0^d$ and a positive contraction $S : L_1(M_{\text{sa}}, \tau) \to L_1(M_{\text{sa}}, \tau)$ such that $SZ = ZS$ with*

$$Z\mathbf{T}^{\mathbf{m}+\mathbf{k}} \geq ZS, \qquad Z\mathbf{T}^{\mathbf{m}} \geq ZS \quad \textit{with} \tag{12}$$

$$\left\|Z(\mathbf{T}^{\mathbf{m}+\mathbf{k}} - S)\right\| < 1, \qquad \left\|Z(\mathbf{T}^{\mathbf{m}} - S)\right\| < 1, \tag{13}$$

then for any $\varepsilon > 0$ there are $M \in \mathbb{N}$ and $\mathbf{n}_0 \in \mathbb{N}_0^d$ such that

$$\left\|Z^M(\mathbf{T}^{\mathbf{n}+\mathbf{k}} - \mathbf{T}^{\mathbf{n}})\right\| < \varepsilon \quad \textit{for all } \mathbf{n} \geq \mathbf{n}_0.$$

Here $\mathbf{T}^{\mathbf{n}} := T_1^{n_1} \cdots T_d^{n_d}$, $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}_0^d$.

Proof. First we note that for any positive contraction $\mathbf{T}$ on L_1-spaces (see [26, p. 310]), there is $\gamma > 0$ such that

$$\left\|\Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^{\ell} - \mathbf{T}^{\mathbf{k}}\Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^{\ell}\right\| \leq \frac{\gamma}{\sqrt{\ell}}, \quad \text{for all } \ell \in \mathbb{N}. \tag{14}$$

Now take any $\varepsilon > 0$ and fix $\ell_\varepsilon \in \mathbb{N}$ such that $\gamma/\sqrt{\ell_\varepsilon} < \varepsilon/2$.
Define

$$Q_1 = \frac{1}{2}(\mathbf{T}^{\mathbf{m}+\mathbf{k}} - S) + \frac{1}{2}\mathbf{T}^{\mathbf{k}}(\mathbf{T}^{\mathbf{m}} - S).$$

It then follows from (12) and (13) that ZQ_1 is positive and that $\|ZQ_1\| < 1$. Moreover, one has

$$\mathbf{T}^{\mathbf{m}+\mathbf{k}} = \Big(\frac{I + \mathbf{T}^{k}}{2}\Big)S + Q_1,$$

where I stands for the identity mapping.

For each $\ell \in \mathbb{N}$ let us define

$$Q_{\ell+1} = \Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^{\ell} Q_1 S^{\ell} + \mathbf{T}^{\mathbf{m}+\mathbf{k}} Q_{\ell}, \quad \ell \in \mathbb{N}.$$

Taking into account the positivity of S and Q_1, one can see that Q_ℓ is a positive operator on $L_1(M_{\text{sa}}, \tau)$ and $ZQ_\ell = Q_\ell Z$. Moreover, one has

$$\mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} = \Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^{\ell} S^{\ell} + Q_{\ell}, \quad \ell \in \mathbb{N}. \tag{15}$$

Let us prove (15) by induction. Clearly, it is valid for $\ell = 1$. Assume that (15) is true for ℓ, and we will prove it for $\ell + 1$. Indeed, we have

$$\begin{aligned}
\mathbf{T}^{(\ell+1)(\mathbf{m}+\mathbf{k})} &= \mathbf{T}^{m+k}\mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} = \Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^{\ell} \mathbf{T}^{\mathbf{m}+\mathbf{k}} S^{\ell} + \mathbf{T}^{\mathbf{m}+\mathbf{k}} Q_{\ell} \\
&= \Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^{\ell} \Big(\Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)S + Q_1\Big) S^{\ell} + \mathbf{T}^{\mathbf{m}+\mathbf{k}} Q_{\ell} \\
&= \Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^{\ell+1} S^{\ell+1} + \Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^{\ell} Q_1 S^{\ell} + \mathbf{T}^{\mathbf{m}+\mathbf{k}} Q_{\ell} \\
&= \Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^{\ell+1} S^{\ell+1} + Q_{\ell+1},
\end{aligned}$$

which proves the required equality.

Now let us put $V_\ell^{(1)} = S^\ell$, and

$$V_\ell^{(d+1)} = \mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} V_\ell^{(d)} + V_\ell^{(1)} Q_\ell^d, \quad d \in \mathbb{N}.$$

One can see that, for every $d, \ell \in \mathbb{N}$, the operator $ZV_\ell^{(d)}$ is positive since Z and S are commuting. Moreover, one has

$$\mathbf{T}^{d\ell(\mathbf{m}+\mathbf{k})} = \Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^\ell V_\ell^{(d)} + Q_\ell^d, \quad d, \ell \in \mathbb{N}. \tag{16}$$

Again, let us prove the last equality by induction. Keeping in mind that (16) is true for d, it is enough to establish (16) for $d+1$. Indeed, we have

$$\begin{aligned}
\mathbf{T}^{(d+1)\ell(\mathbf{m}+\mathbf{k})} = \mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} T^{d(\mathbf{m}+\mathbf{k})} &= \mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})}\Big(\Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^\ell V_\ell^{(d)} + Q_\ell^d\Big) \\
&= \Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^\ell \mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} V_\ell^{(d)} + \Big(\Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^\ell S^\ell + Q_\ell\Big) Q_\ell^d \\
&= \Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^\ell (\mathbf{T}^{\ell(\mathbf{m}+\mathbf{k})} V_\ell^{(d)} + V_\ell^{(1)} Q_\ell^d) + Q_\ell^{d+1} \\
&= \Big(\frac{I + \mathbf{T}^{\mathbf{k}}}{2}\Big)^\ell V_\ell^{(d+1)} + Q_\ell^{d+1},
\end{aligned}$$

which proves (16).

From $Z^*(\mathbb{1}) = \mathbf{T}^*(\mathbb{1}) = \mathbb{1}$, it follows from (16) that

$$V_\ell^{(d)*}(\mathbb{1}) + Q_\ell^{*d}(\mathbb{1}) = \mathbb{1}.$$

Now the positivity of $ZV_\ell^{(d)}$ and ZQ_ℓ imply that $\|ZV_\ell^{(d)}\| \le 1$ and $\|ZQ_\ell\| \le 1$.

From (12) and (13), due to Theorem 3.1, one finds that $\|Z(\mathbf{T}^{\ell\mathbf{m}} - S^\ell)\| < 1$ for all $\ell \in \mathbb{N}$. Using this inequality with $\mathbf{T}^*(\mathbb{1}) = \mathbb{1}$ and the positivity of $Z(\mathbf{T}^{\ell\mathbf{m}} - S^\ell)$, we find that

$$\big\|Z(\mathbf{T}^{\ell\mathbf{m}} - S^\ell)\big\| = \big\|\big((\mathbf{T}^*)^{\ell\mathbf{m}} - S^{*\ell}\big)Z^*\big\| = \big\|\mathbb{1} - S^{*\ell}(\mathbb{1})\big\| < 1. \tag{17}$$

The equality (15) yields that

$$Q_\ell^*(\mathbb{1}) = \mathbb{1} - S^{*\ell}(\mathbb{1}).$$

Hence, from (17), with the positivity of ZQ_ℓ, we obtain

$$\|ZQ_\ell\| = \big\|Q_\ell^*(\mathbb{1})\big\| = \big\|\mathbb{1} - S^{*\ell}(\mathbb{1})\big\| < 1$$

for all $\ell \in \mathbb{N}$.

Therefore, there is a number $d_\varepsilon \in \mathbb{N}$ such that $\|(ZQ_{\ell_\varepsilon})^{d_\varepsilon}\| < \frac{\varepsilon}{4}$. From the commutativity of Z and Q_ℓ one finds

$$\|Z^{d_\varepsilon} Q_{\ell_\varepsilon}^{d_\varepsilon}\| < \frac{\varepsilon}{4}. \tag{18}$$

Now putting $\mathbf{n}_0 = d_\varepsilon \ell_\varepsilon(\mathbf{m}+\mathbf{k})$, from (16) with (18) we obtain

$$\begin{aligned}\big\|Z^{d_\varepsilon}(\mathbf{T}^{\mathbf{n}_0+\mathbf{k}} - \mathbf{T}^{\mathbf{n}_0})\big\| &= \big\|Z^{d_\varepsilon}(\mathbf{T}^{d_\varepsilon\ell_\varepsilon(\mathbf{m}+\mathbf{k})+\mathbf{k}} - \mathbf{T}^{d_\varepsilon\ell_\varepsilon(\mathbf{m}+\mathbf{k})})\big\| \\ &\le \Big\|Z^{d_\varepsilon}\Big(\mathbf{T}^{\mathbf{k}}\Big(\frac{I+\mathbf{T}^{\mathbf{k}}}{2}\Big)^{\ell_\varepsilon} - \Big(\frac{I+\mathbf{T}^{\mathbf{k}}}{2}\Big)^{\ell_\varepsilon}\Big)V^{(d_\varepsilon)}_{\ell_\varepsilon}\Big\| \\ &\quad + \big\|Z^{d_\varepsilon}Q^{d_\varepsilon}_{\ell_\varepsilon}(\mathbf{T}^{\mathbf{k}} - I)\big\| \\ &\le \Big\|\mathbf{T}^{\mathbf{k}}\Big(\frac{I+\mathbf{T}^{\mathbf{k}}}{2}\Big)^{\ell_\varepsilon} - \Big(\frac{I+\mathbf{T}^{\mathbf{k}}}{2}\Big)^{\ell_\varepsilon}\Big\| \\ &\quad + 2\|Z^{d_\varepsilon}Q^{d_\varepsilon}_{\ell_\varepsilon}\| \\ &\le \frac{\gamma}{\sqrt{\ell_\varepsilon}} + 2\cdot\frac{\varepsilon}{4} < \varepsilon.\end{aligned}$$

Take any $\mathbf{n} \ge \mathbf{n}_0$. Then from the last inequality, one gets

$$\big\|Z^{d_\varepsilon}(\mathbf{T}^{\mathbf{n}+\mathbf{k}} - \mathbf{T}^{\mathbf{n}})\big\| = \big\|\mathbf{T}^{\mathbf{n}-\mathbf{n}_0}Z^{d_\varepsilon}(\mathbf{T}^{\mathbf{n}_0+\mathbf{k}} - \mathbf{T}^{\mathbf{n}_0})\big\| \le \big\|Z^{d_\varepsilon}(\mathbf{T}^{\mathbf{n}_0+\mathbf{k}} - \mathbf{T}^{\mathbf{n}_0})\big\| < \varepsilon,$$

which completes the proof. □

Corollary 4.2. *Assume that $T_k : L_1(M_{\mathrm{sa}}, \tau) \to L_1(M_{\mathrm{sa}}, \tau)$ $(k = 1, \ldots, d)$ are unital positive contractions such that $T_iT_j = T_jT_i$, for every $i, j \in \{1, \ldots, d\}$. If there are $\mathbf{m} \in \mathbb{N}_0^d$, $\mathbf{k} \in \mathbb{N}_0^d$, and a positive contraction $S : L_1(M_{\mathrm{sa}}, \tau) \to L_1(M_{\mathrm{sa}}, \tau)$ such that*

$$\mathbf{T}^{\mathbf{m}+\mathbf{k}} \ge S, \qquad \mathbf{T}^{\mathbf{m}} \ge S \quad \textit{with} \tag{19}$$

$$\|\mathbf{T}^{\mathbf{m}+\mathbf{k}} - S\| < 1, \qquad \|\mathbf{T}^{\mathbf{m}} - S\| < 1, \tag{20}$$

then one has

$$\lim_{\mathbf{n}\to\infty} \|\mathbf{T}^{\mathbf{n}+\mathbf{k}} - \mathbf{T}^{\mathbf{n}}\| = 0.$$

The proof immediately follows from Theorem 4.1 if one takes $Z = \mathrm{Id}$.

Remark 4.3. We note that in [20] a similar kind of result, for a single contractions of C^*-algebras, was proved. Our main result extends it for more general multi-parametric contractions. We point out that if the algebra becomes commutative, then the proved theorems cover the main results of [16, Theorem 3.1].

Corollary 4.4. *Let $T, P : L_1(M_{\mathrm{sa}}, \tau) \to L_1(M_{\mathrm{sa}}, \tau)$ be two commuting unital positive contractions. If for some $m_0 \in \mathbb{N}$ and a positive contraction $S : L_1(M_{\mathrm{sa}}, \tau) \to L_1(M_{\mathrm{sa}}, \tau)$ we have*

$$T^{m_0+k}P^{m_0} \ge S, \qquad T^{m_0}P^{m_0} \ge S \quad \textit{with}$$

$$\|T^{m_0+k}P^{m_0} - S\| < 1, \qquad \|T^{m_0}P^{m_0} - S\| < 1,$$

then

$$\lim_{n,m\to\infty} \|T^{n+k}P^m - T^nP^m\| = 0.$$

The proof immediately follows from Corollary 4.2 if we take $\mathbf{m} = (m_0, m_0)$ and $\mathbf{k} = (k, 0)$.

Remark 4.5. Since the dual of $L_1(M_{\mathrm{sa}}, \tau)$ is M_{sa}, we have that, due to the duality theory, the proved Theorem 4.1 holds true if we replace the L_1-space with M_{sa}.

Recall that for a given linear operator $T : L_1(M_{\text{sa}}, \tau) \to L_1(M_{\text{sa}}, \tau)$, an element $x \in L_1(M_{\text{sa}}, \tau)$ is a *a fixed point* of T, if $Tx = x$. The set of all fixed points of T is denoted by $\text{Fix}(T)$.

Corollary 4.6. *Let $T : L_1(M_{\text{sa}}, \tau) \to L_1(M_{\text{sa}}, \tau)$ be a unital positive contraction. If for some $m_0 \in \mathbb{N}$ and a positive contraction $S : L_1(M_{\text{sa}}, \tau) \to L_1(M_{\text{sa}}, \tau)$ we have*

$$T^{m_0+1} \geq S, \qquad T^{m_0} \geq S \quad \text{with}$$
$$\|T^{m_0+1} - S\| < 1, \qquad \|T^{m_0} - S\| < 1,$$

and

$$L_1(M_{\text{sa}}, \tau) = \text{Fix}(T) \oplus \overline{(I - T)\big(L_1(M_{\text{sa}}, \tau)\big)}^{\|\cdot\|_1}, \tag{21}$$

then there exists a projection $P : L_1(M_{\text{sa}}, \tau) \to L_1(M_{\text{sa}}, \tau)$, ($\|P\| \leq 1$) such that

$$\lim_{n\to\infty} T^n x = Px.$$

Proof. Due to (21) and the density argument, it is enough to prove the assertion for $x \in \text{Fix}(T)$ and $x \in (I - T)(L_1(M_{\text{sa}}, \tau))$. For $x \in \text{Fix}(T)$, we immediately have

$$\lim_{n\to\infty} T^n x = x.$$

According to Corollary 4.2, we find $\|T^n(I - T)\| \to 0$ as $n \to \infty$. Hence, for $x \in (I - T)(L_1(M_{\text{sa}}, \tau))$, we get $\lim_{n\to\infty} T^n x = 0$, and hence P is a projection onto $\text{Fix}(T)$. This completes the proof. □

We note that in [22] some regularity conditions more general than ergodicity are used to obtain different versions of the Esterle–Katznelson–Tzafriri theorem [14].

5. Noncommutative L_1-spaces associated with a center-valued trace

In this section, we recall some necessary notions and facts about the noncommutative L_1-spaces associated with a center-valued trace.

Let M be any finite von Neumann algebra, and let $S(M)$ be the set of all measurable operators affiliated to M (see [18] for definitions). Let Z be some subalgebra of the center $Z(M)$. Then we may identify Z with the $*$-algebra $L_\infty(\Omega, \Sigma, m)$ and $S(Z)$ with $L_0(\Omega, \Sigma, m)$. Recall that a *center-valued (i.e., Z-valued) trace* on the von Neumann algebra M is a Z-linear mapping $\Phi : M \to Z$ with $\Phi(x^*x) = \Phi(xx^*) \geq 0$ for all $x \in M$. It is clear that $\Phi(M_+) \subset Z_+$. A trace Φ is said to be *faithful* if the equality $\Phi(x^*x) = 0$ implies $x = 0$ and *normal* if $\Phi(x_\alpha) \uparrow \Phi(x)$ for every $x_\alpha, x \in M_{\text{sa}}$, $x_\alpha \uparrow x$. Note that the existence of such traces was studied in [1].

Let M be an arbitrary finite von Neumann algebra, and let Φ be a center-valued trace on M. The local measure topology $t(M)$ on $S(M)$ is the linear (Hausdorff) topology whose fundamental system of neighborhoods of 0 is given by

$$\begin{aligned} V(B, \varepsilon, \delta) = \big\{&x \in S(M)\colon \text{ there exists } p \in P(M), z \in P\big(Z(M)\big) \\ &\text{such that } xp \in M, \|xp\|_M \leq \varepsilon, z^\perp \in W(B, \varepsilon, \delta), \Phi_M(zp^\perp) \leq \varepsilon z\big\}, \end{aligned}$$

where $\|\cdot\|_M$ is the C^*-norm in M. It is known that $(S(M), t(M))$ is a complete topological $*$-algebra (see [25]).

From [18, Section 3.5], we have the following criterion for convergence in the topology $t(M)$.

Proposition 5.1. *A net $\{x_\alpha\}_{\alpha\in A} \subset S(M)$ converges to zero in the topology $t(M)$ if and only if $\Phi_M(E_\lambda^\perp(|x_\alpha|) \xrightarrow{t(M)} 0$ for any $\lambda > 0$.*

Following [2], an operator $x \in S(M)$ is said to be Φ-*integrable* if there exists a sequence $\{x_n\} \subset M$ such that $x_n \overset{t(M)}{\to} x$ and $\|x_n - x_m\|_\Phi \xrightarrow{t(Z)} 0$ as $n, m \to \infty$.

Let x be a Φ-integrable operator from $S(M)$. Then there exists a $\widehat{\Phi}(x) \in S(Z)$ such that $\Phi(x_n) \xrightarrow{t(Z)} \widehat{\Phi}(x)$. In addition, $\widehat{\Phi}(x)$ does not depend on the choice of a sequence $\{x_n\} \subset M$, for which $x_n \xrightarrow{t(M)} x$, $\Phi(|x_n - x_m|) \xrightarrow{t(Z)} 0$ (see [2]). It is clear that each operator $x \in M$ is Φ-integrable and that $\widehat{\Phi}(x) = \Phi(x)$.

Denote by $L_1(M, \Phi)$ the set of all Φ-integrable operators from $S(M)$. If $x \in S(M)$, then $x \in L_1(M, \Phi)$ if and only if $|x| \in L_1(M, \Phi)$; in addition, $|\widehat{\Phi}(x)| \le \widehat{\Phi}(|x|)$ (see [1]). For any $x \in L_1(M, \Phi)$, set $\|x\|_{1,\Phi} = \widehat{\Phi}(|x|)$. It is known that $L_1(M, \Phi)$ is a linear subspace of $S(M)$, $ML_1(M, \Phi)M \subset L_1(M, \Phi)$, and $x^* \in L_1(M, \Phi)$ for all $x \in L_1(M, \Phi)$ (see [1]).

Now let us recall some facts about Banach–Kantorovich spaces over the ring of measurable functions (see [12]).

Let X be a mapping that maps every point $\omega \in \Omega$ to some Banach space $(X(\omega), \|\cdot\|_{X(\omega)})$. In what follows, we assume that $X(\omega) \neq \{0\}$ for all $\omega \in \Omega$. A function u is said to be a *section* of X if it is defined almost everywhere in Ω and takes its value $u(\omega) \in X(\omega)$ for $\omega \in \mathrm{dom}(u)$, where $\omega \in \mathrm{dom}(u)$ is the domain of u. Let L be some set of sections.

Definition 5.2. (see [12]). A pair (X, L) is said to be a *measurable bundle of Banach spaces* over Ω if

1. $\lambda_1 c_1 + \lambda_2 c_2 \in L$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $c_1, c_2 \in L$, where $\lambda_1 c_1 + \lambda_2 c_2 : \omega \in \mathrm{dom}(c_1) \cap \mathrm{dom}(c_2) \to \lambda_1 c_1(\omega) + \lambda_2 c_2(\omega)$;
2. the function $\|c\| : \omega \in \mathrm{dom}(c) \to \|c(\omega)\|_{X(\omega)}$ is measurable for all $c \in L$;
3. for every $\omega \in \Omega$ the set $\{c(\omega) : c \in L, \omega \in \mathrm{dom}(c)\}$ is dense in $X(\omega)$.

A section s is a step section, if there are pairwise disjoint sets $A_1, A_2, \ldots, A_n \in \Sigma$ and sections $c_1, c_2, \ldots, c_n \in L$ such that $\bigcup_{i=1}^n A_i = \Omega$ and $s(\omega) = \sum_{i=1}^n \chi_{A_i}(\omega) \times c_i(\omega)$ for almost all $\omega \in \Omega$.

A section u is measurable if for any $A \in \Sigma$ there is a sequence s_n of step sections such that $s_n(\omega) \to u(\omega)$ for almost all $\omega \in A$.

Let $M(\Omega, X)$ be the set of all measurable sections. By $L_0(\Omega, X)$ we denote the factorization of $M(\Omega, X)$ with respect to equality almost everywhere. Usually, by $\hat{u}$ we denote a class from $L_0(\Omega, X)$ containing a section $u \in M(\Omega, X)$, and by $\|\hat{u}\|$ we denote an element of $L_0(\Omega)$ containing $\|u(\omega)\|_{X(\omega)}$. Let $\mathcal{L}^\infty(\Omega, X) = \{u \in M(\Omega, X) : \|u(\omega)\|_{X(\omega)} \in \mathcal{L}^\infty(\Omega)\}$ and $L^\infty(\Omega, X) = \{\widehat{u} \in L_0(\Omega, X) : \|\widehat{u}\| \in L^\infty(\Omega)\}$. In what follows, by $\mathbf{1}$ we denote the identity of the algebra $L^\infty(\Omega, X)$.

We notice that one can define the spaces $\mathcal{L}^\infty(\Omega, X)$ and $L^\infty(\Omega, X)$ with the real-valued norms $\|u\|_{\mathcal{L}^\infty(\Omega,X)} = \sup_{\omega\in\Omega} |u(\omega)|_{X(\omega)}$ and $\|\widehat{u}\|_\infty = \|\|\widehat{u}\|\|_{L^\infty(\Omega)}$, respectively.

Definition 5.3. Let X, Y be measurable bundles of Banach spaces. A set of linear operators $\{T(\omega) : X(\omega) \to Y(\omega)\}$ is called a *measurable bundle of linear operators* if $T(\omega)(u(\omega))$ is a measurable section for any measurable section u.

Let (X, L) be a measurable bundle of Banach spaces. If each $X(\omega)$ is a noncommutative L_1-space (i.e., $X(\omega) = L_1(M(\omega), \tau_\omega)$) associated with finite von Neumann algebras $M(\omega)$ and with a strictly normal numerical trace τ_ω on $M(\omega)$, then the measurable bundle (X, L) of Banach spaces is called a *measurable bundle of noncommutative L_1-spaces*.

Theorem 5.4 ([6, Theorem 1]). *There exists a measurable bundle (X, L) of noncommutative L_1-spaces $L_1(M(\omega), \tau_\omega)$ such that $L_0(\Omega, X)$ is a Banach–Kantorovich $*$-algebroid that is isometrically and order $*$-isomorphic to $L_1(M, \Phi)$. Moreover, the isometric and order $*$-isomorphism $H : L_1(M, \Phi) \to L_0(\Omega, X)$ can be chosen with the following properties:*

(a) *$\Phi(x)(\omega) = \tau_\omega(H(x)(\omega))$ for all $x \in M$ and for almost all $\omega \in \Omega$.*
(b) *$x \in M$ if and only if $H(x)(\omega) \in M(\omega)$ almost everywhere, and there exists a positive number $\lambda > 0$ such that $\|H(x)(\omega)\|_{M(\omega)} \leq \lambda$ for almost all ω.*
(c) *$z \in Z$ if and only if $H(z) = (\widehat{z(\omega)\mathbb{1}_\omega})$ for some $\widehat{z(\omega)} \in L_\infty(\Omega)$, where $\mathbf{1}_\omega$ is $\mathbb{1}$, the unit algebra $M(\omega)$; in particular, $H(\mathbf{1})(\omega) = \mathbb{1}_\omega$ for almost all ω.*
(d) *The section $(H(x)(\omega))^*$ is measurable for all $x \in L_1(M, \Phi)$.*
(e) *The section $H(x)(\omega) \cdot H(y)(\omega)$ is measurable for all $x, y \in M$.*

Let M be a finite von Neumann algebra with a center-valued trace Φ on M. Then M can be identified with a linear subspace of $L^\infty(\Omega, X)$ by the isomorphism H, since if $x \in M$, then one has

$$\big\|H(x)\big\|_{L_0(\Omega,X)} = \|x\|_1 = \Phi\big(|x|\big) \in L^\infty(\Omega).$$

The existence of the lifting in a noncommutative setting was proved in [10].

Theorem 5.5 ([10, Theorem 3.1]). *There exists a mapping $\ell : M \to \mathcal{L}^\infty(\Omega, X)$ with the following properties:*

(a) *For every $x \in M$ one has $\ell(x) \in x, \operatorname{dom} \ell(x) = \Omega$.*
(b) *If $x_1, x_2 \in M$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, then $\ell(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \ell(x_1) + \lambda_2 \ell(x_2)$.*
(c) *$\|\ell(x)(\omega)\|_{L_p(M(\omega),\tau_\omega)} = p(\|x\|_p)(\omega)$ for all $x \in M$ and for all $\omega \in \Omega$.*
(d) *If $x \in M, \lambda \in L^\infty(\Omega)$, then $\ell(ex) = p(e)\ell(x)$.*
(e) *If $x \in M$, then $\ell(x^*) = \ell(x)^*$.*
(f) *If $x, y \in M$, then $\ell(xy) = \ell(x)\ell(y)$.*
(g) *The set $\{\ell(x)(\omega) : x \in M\}$ is dense in $L_1(M(\omega), \tau_\omega)$ for all $\omega \in \Omega$.*

Remark 5.6. We note that, in the case of C^*-algebras, the existence of the lifting was given in [7].

Definition 5.7. The defined map ℓ in Theorem 5.5 is called a *noncommutative vector-valued lifting associated with the lifting ρ*.

6. Vector-valued analogue of the noncommutative zero-two law

In this section, we prove the existence of a vector-valued analogue of Theorem 4.1.

Let M be any finite von Neumann algebra, and let $L_1(M,\Phi)$ be the noncommutative L_1-space associated with M and the center-valued trace Φ. Let (X,L) be a measurable bundle of noncommutative L_1-spaces $L_1(M(\omega),\tau_\omega)$ associated with finite von Neumann algebras $M(\omega)$ and with strictly normal numerical traces τ_ω on $M(\omega)$, corresponding to $L_1(M,\Phi)$. In what follows, we denote by $\mathbf{1}$, as before, the identity of the algebra $L^\infty(\Omega,X)$.

Theorem 6.1. *Let $T : L_1(M,\Phi) \to L_1(M,\Phi)$ be a positive contraction with $T(\mathbf{1}) \le \mathbf{1}$. Then there exists a measurable bundle of positive contractions T_ω : $L_1(M(\omega),\tau_\omega) \to L_1(M(\omega),\tau_\omega)$ such that*

$$T_\omega\big(x(\omega)\big) = (Tx)(\omega),$$

for all $x \in L_1(M,\Phi)$ and for almost all $\omega \in \Omega$, and

$$\|T\|(\omega) = \|T_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)}.$$

Proof. Let $x \in M_{\mathrm{sa}}$. Then

$$|Tx| \le T\big(|x|\big) \le \|x\|_M T(\mathbf{1}) \le \|x\|_M \mathbf{1};$$

that is, $Tx \in M$. If $x \in M$, then there exist $y,z \in M_{\mathrm{sa}}$ such that $x = y + iz$. Then $Tx = Ty + iTz$. As $Ty, Tz \in M$, we have $Tx \in M$.

Let $\ell : M(\subset L^\infty(\Omega,X)) \to \mathcal{L}^\infty(\Omega,X)$ be the noncommutative vector-valued lifting associated with the lifting p (see Theorem 5.5).

We define the linear operator φ_ω from $\{\ell(x)(\omega) : x \in M\}$ into $L_1(M(\omega),\tau_\omega)$ by

$$\varphi_\omega\big(\ell(x)(\omega)\big) = \ell(Tx)(\omega).$$

The contractivity of T implies that

$$\begin{aligned}\big\|\varphi_\omega\big(\ell(x)(\omega)\big)\big\|_{L_1(M(\omega),\tau_\omega)} &= \big\|\ell(Tx)(\omega)\big\|_{L_1(M(\omega),\tau_\omega)} = \rho\big(\|Tx\|_1\big)(\omega)\\ &\le \rho\big(\|x\|_1\big)(\omega) = \big\|\ell(x)(\omega)\big\|_{L_1(M(\omega),\tau_\omega)}.\end{aligned}$$

This means that φ_ω is bounded and well defined. Moreover, one has

$$\|\varphi_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)} \le 1.$$

The positivity of T yields that φ_ω is positive as well.

Since the set $\{\ell(x)(\omega) : x \in M\}$ is dense in $L_1(M(\omega),\tau_\omega)$, we can extend φ_ω by continuity to a linear positive contraction $T_\omega : L_1(M(\omega),\tau_\omega) \to L_1(M(\omega),\tau_\omega)$ by $T_\omega(x(\omega)) = \lim_{n\to\infty} \varphi_\omega(\ell(x_n)(\omega))$.

From $\varphi_\omega(\ell(x)(\omega)) \in \mathcal{L}^\infty(\Omega,X)$, for any $x \in M$, we obtain $T_\omega(x(\omega)) \in M(\Omega,X)$ for any $x \in M(\Omega,X)$. Therefore, $\{T_\omega\}$ is a measurable bundle of positive operators.

Using the same argument as in the proof of [9, Theorem 4.5], one can prove

$$T_\omega\big(x(\omega)\big) = (Tx)(\omega)$$

for all $x \in L_1(M,\Phi)$ and for almost all $\omega \in \Omega$.

Now let us establish $\|T\|(\omega) = \|T_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)}$.

Let $x \in M$. Then

$$\begin{aligned}\big\|\varphi_\omega\big(\ell(x)(\omega)\big)\big\|_{L_1(M(\omega),\tau_\omega)} &= \big\|\ell(Tx)(\omega)\big\|_{L_1(M(\omega),\tau_\omega)} = \rho\big(\|Tx\|_1\big)(\omega)\\ &\le \rho\big(\|T\|\|x\|_1\big)(\omega) = \rho\big(\|T\|\big)(\omega)p\big(\|x\|_1\big)(\omega)\\ &= \rho\big(\|T\|\big)(\omega)\big\|\ell(x)(\omega)\big\|_{L_1(M(\omega),\tau_\omega)}.\end{aligned}$$

If $x(\omega) \in L_1(M(\omega), \tau_\omega)$, then one finds

$$\begin{aligned}\big\|T_\omega x(\omega)\big\|_{L_1(M(\omega),\tau_\omega)} &= \lim_{n\to\infty}\big\|\varphi_\omega\big(\ell(x_n)(\omega)\big)\big\|_{L_1(M(\omega),\tau_\omega)}\\ &\le \rho\big(\|T\|\big)(\omega)\lim_{n\to\infty}\big\|\ell(x_n)(\omega)\big\|_{L_1(M(\omega),\tau_\omega)}\\ &= \rho\big(\|T\|\big)(\omega)\big\|x(\omega)\big\|_{L_1(M(\omega),\tau_\omega)}.\end{aligned}$$

Hence, $\|T_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)} \le \rho(\|T\|)(\omega)$.

By [8, Proposition 2], for any $\varepsilon > 0$ there exists $x \in L_1(M, \Phi)$ with $\|x\|_1 = \mathbf{1}$ such that

$$\|Tx\|_1 \ge \|T\| - \varepsilon\mathbf{1}.$$

Then

$$\begin{aligned}\rho\big(\|T\|\big)(\omega) - \varepsilon \le \rho\big(\|Tx\|_1\big)(\omega) &= \big\|\ell(Tx)(\omega)\big\|_{L_1(M(\omega),\tau_\omega)}\\ &= \big\|T_\omega\ell(x)(\omega)\big\|_{L_1(M(\omega),\tau_\omega)}\\ &\le \|T_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)}\big\|\ell(x)(\omega)\big\|_{L_1(M(\omega),\tau_\omega)}\\ &= \|T_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)}p\big(\|x\|_1\big)(\omega)\\ &= \|T_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)}.\end{aligned}$$

The arbitrariness of ε yields

$$p\big(\|T\|\big)(\omega) \le \|T_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)}.$$

Hence

$$p\big(\|T\|\big)(\omega) = \|T_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)}$$

for all $\omega \in \Omega$, or, equivalently, we have

$$\|T\|(\omega) = \|T_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)}$$

for almost all $\omega \in \Omega$. This completes the proof. □

Corollary 6.2. *Let $T : L_1(M, \Phi) \to L_1(M, \Phi)$ be a positive contraction with $T(\mathbf{1}) = \mathbf{1}$. Then there exists a measurable bundle of positive contractions $T_\omega : L_1(M(\omega), \tau_\omega) \to L_1(M(\omega), \tau_\omega)$ such that*

$$T_\omega\big(x(\omega)\big) = (Tx)(\omega)$$

for all $x \in L_1(M, \Phi)$ and for almost all $\omega \in \Omega$ and

$$\|T\|(\omega) = \|T_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)}.$$

Let $T : L_1(M,\Phi) \to L_1(M,\Phi)$ be a positive contraction with $T(\mathbf{1}) = \mathbf{1}$, and let $T_\omega : L_1(M(\omega),\tau_\omega) \to L_1(M(\omega),\tau_\omega)$ be a measurable bundle of positive contractions. Then T is called *unital positive contraction* if one has $T^*_\omega(\mathbb{1}_\omega) = \mathbb{1}_\omega$ for almost all $\omega \in \Omega$.

Theorem 6.3. *Assume that $T : L_1(M,\Phi) \to L_1(M,\Phi)$ is a unital positive contraction. If there are $m, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and a positive contraction $S : L_1(M,\Phi) \to L_1(M,\Phi)$ with $S(\mathbf{1}) = \mathbf{1}$ such that*

$$T^{m+k} \geq S, \qquad T^m \geq S \quad \text{with}$$
$$\|T^{m+k} - S\| < \mathbf{1}, \qquad \|T^m - S\| < \mathbf{1},$$

then

$$(o) - \lim_{n\to\infty} \|T^{n+k} - T^n\| = 0.$$

Proof. By Corollary 6.2, there exist $T_\omega : L_1(M(\omega),\tau_\omega) \to L_1(M(\omega),\tau_\omega)$ and $S_\omega : L_1(M(\omega),\tau_\omega) \to L_1(M(\omega),\tau_\omega)$ such that $T_\omega(x(\omega)) = (Tx)(\omega)$ and $S_\omega(x(\omega)) = (Sx)(\omega)$ for all $x \in L_1(M,\Phi)$ and for almost all $\omega \in \Omega$.

From $T^{m+k} \geq S, T^m \geq S$ we get $T^{m+k}_\omega \geq S_\omega, T^m_\omega \geq S_\omega$ for almost all $\omega \in \Omega$. Since $\|T^{m+k} - S\| < \mathbf{1}, \|T^m - S\| < \mathbf{1}$, we find

$$\|T^{m+k}_\omega - S_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)} < 1, \qquad \|T^m_\omega - S_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)} < 1$$

for almost all $\omega \in \Omega$. Then, by using $T^*_\omega(\mathbb{1}_\omega) = \mathbb{1}_\omega$, we determine that the positive contraction T_ω satisfies all conditions of Corollary 4.2 for almost all $\omega \in \Omega$. Therefore,

$$\lim_{n\to\infty} \|T^{n+k}_\omega - T^n_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)} = 0$$

for almost all $\omega \in \Omega$.

According to

$$\|T^{n+k} - T^n\|(\omega) = \|T^{n+k}_\omega - T^n_\omega\|_{L_1(M(\omega),\tau_\omega)\to L_1(M(\omega),\tau_\omega)}, \quad \text{a.e.}$$

we obtain $\lim_{n\to\infty} \|T^{n+k} - T^n\|(\omega) = 0$ for almost all $\omega \in \Omega$, which means that

$$(o) - \lim_{n\to\infty} \|T^{n+k} - T^n\| = 0.$$

This completes the proof. □

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