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CALDERÓN–LOZANOVSKII INTERPOLATION ON QUASI-BANACH LATTICES

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Abstract. We consider the Calderón–Lozanovskii construction $\varphi(X_0, X_1)$ in the context of quasi-Banach lattices, and we provide an extension of a result by Ovchinnikov concerning the associated interpolation methods φ^c and φ^0. Our approach is based on the interpolation properties of $(\infty, 1)$-regular operators between quasi-Banach lattices.

1. Introduction

The aim of this note is to study the interpolation properties of the Calderón–Lozanovskii construction in the quasi-Banach lattice setting. Let us start by recalling this construction. Given (X_0, X_1) a compatible pair of quasi-Banach lattices and a function $\varphi : \mathbb{R}^2_+ \to \mathbb{R}_+$ which is homogeneous and nondecreasing in each argument, we consider the space $\varphi(X_0, X_1)$ of those $x \in X_0 + X_1$ such that $|x| \le \varphi(x_0, x_1)$ for some $x_0 \in X_0$ and $x_1 \in X_1$. This space becomes a quasi-Banach lattice when endowed with the quasinorm

$$\|x\|_{\varphi(X_0,X_1)} = \inf\{\lambda > 0 : |x| \le \lambda\varphi(x_0, x_1), \|x_0\|_{X_0} \le 1, \|x_1\|_{X_1} \le 1\}.$$

This space was introduced and studied by Lozanovskii [15] (see also the references therein). In particular, considerable work has been done for the case of $\varphi(s,t) = s^{1-\theta}t^\theta$ for some $\theta \in (0,1)$, which yields the Calderón product $X_0^{1-\theta}X_1^\theta$ (see [4]). The relation between this and the complex interpolation methods has

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been carefully investigated in the literature (see [4], [9], [23], [24]). There is an obvious interest in extending interpolation results which are valid in the Banach space, or Banach lattice, setting to the more general context of quasi-Banach spaces (see, e.g., [5], [6], [8], [16]).

Our interest in this note is to relate the construction $\varphi(X_0, X_1)$ with two well-known interpolation functors. In this respect, recall that, given quasinormed spaces X and Y such that there is a continuous inclusion $i : X \hookrightarrow Y$, the Gagliardo completion of X in Y is the quasinormed space whose unit ball is the closure of $i(B_X)$ in Y, where as usual B_X denotes the unit ball of X; note that when Y is complete, this clearly defines a quasi-Banach space. Let us denote $\varphi^c(X_0, X_1)$ the Gagliardo completion of the space $\varphi(X_0, X_1)$ in $X_0 + X_1$. Also, let $\varphi^0(X_0, X_1)$ denote the closure of the intersection $X_0 \cap X_1$ in $\varphi(X_0, X_1)$. We obviously have the following bounded inclusions:

$$\varphi^0(X_0, X_1) \subset \varphi(X_0, X_1) \subset \varphi^c(X_0, X_1).$$

Ovchinnikov [19] (see also [1, Theorem 4.3.11]) proved that φ^0 and φ^c are interpolation functors in the category of Banach lattices of measurable functions. Earlier attempts to extend these interpolation functors to the category of quasi-Banach lattices were made by Nilsson [18] and Ovchinnikov [20].

Our main result in this article is the extension of this fact to the category of quasi-Banach lattices with the $K_{\infty,1}$ property: that is, those spaces X for which the inequality

$$\Big\| \max_{1\le i\le n} |x_i| \Big\| \le C \max_{|a_i|\le 1} \Big\| \sum_{i=1}^n a_i x_i \Big\|$$

holds for some constant $C > 0$ independent of $(x_i)_{i=1}^n \subset X$ (see Section 4 below). It should be noted that a large class of quasi-Banach lattices, namely, that of L-convex quasi-Banach lattices, introduced by Kalton in [7], have the $K_{\infty,1}$ property (see also [18], in connection with the interpolation of L-convex lattices).

An important ingredient in our proof will be the class of (p, q)-regular operators, that is, those satisfying estimates of the form

$$\Big\| \Big(\sum_{i=1}^n |Tx_i|^p \Big)^{\frac{1}{p}} \Big\| \le K \Big\| \Big(\sum_{i=1}^n |x_i|^q \Big)^{\frac{1}{q}} \Big\|.$$

This class of operators was introduced by Bukhvalov in [2], where some interpolation results between Banach lattices were obtained. It will be shown in Theorem 3.1 that $(\infty, 1)$-regular operators have good interpolation properties with respect to the Calderón–Lozanovskii construction. This fact will allow us to further extend the interpolation functors φ^c and φ^0.

2. Definitions and preliminaries

Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \ge 0\}$. Recall that a quasi-Banach space $(X, \|\cdot\|)$ is a vector space which is complete for the metric induced by the quasinorm $\|\cdot\| : X \to \mathbb{R}_+$

that satisfies

$$\|x\| = 0 \quad \Leftrightarrow \quad x = 0,$$
$$\|\lambda x\| = |\lambda|\|x\|,$$
$$\|x+y\| \leq C\big(\|x\| + \|y\|\big),$$

where $C \geq 1$ is independent of $x, y \in X$. If, moreover, X is a vector lattice with $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, then we say that X is a *quasi-Banach lattice.*

We will denote by $\mathcal{P}$ the set of all functions $\varphi : (0,\infty)\times(0,\infty) \to \mathbb{R}_+$ satisfying

$$\varphi(\lambda s, \lambda t) = \lambda\varphi(s,t) \quad \text{for every } s,t,\lambda > 0,$$
$$\varphi(\cdot, t) \text{ is nondecreasing} \quad \text{for every } t > 0,$$
$$\varphi(s, \cdot) \text{ is nondecreasing} \quad \text{for every } s > 0.$$

We will usually make the normalization $\varphi(1,1) = 1$. Given $\varphi \in \mathcal{P}$, let us denote $\varphi_0(t) = \varphi(t,1)$ and $\varphi_1(t) = \varphi(1,t)$. Note that

$$\varphi_1(t) = t\varphi_0(1/t).$$

It follows that both φ_0 and φ_1 are quasiconcave functions (i.e., $\varphi_i(t)$ is nondecreasing and $\varphi_i(t)/t$ is nonincreasing, for $i = 0, 1$). We will make repeated use of the fact that every quasiconcave function is equivalent, up to a universal constant, to a concave function (see [1, Corollary 3.1.4]). For $0 < s < t$, we have

$$\varphi_i(s) \leq \varphi_i(t) \leq \frac{t}{s}\varphi_i(s),$$

and thus φ_i is continuous on $(0,\infty)$. It follows from the equations

$$\varphi(s,t) = t\varphi_0(s/t) = s\varphi_1(t/s)$$

that φ is continuous on $(0,\infty)\times(0,\infty)$. Since φ_i is increasing, it has a right limit $\varphi_i(0^+)$ at 0 and thus has a continuous extension $\bar{\varphi}_i$ to $\mathbb{R}_+$. Let us extend φ to a function $\bar{\varphi}$ on $\mathbb{R}^2_+$ by setting

$$\bar{\varphi}(s,0) = s\varphi_1(0^+) \qquad \text{and} \qquad \bar{\varphi}(0,t) = t\varphi_0(0^+).$$

This extension is continuous. Indeed, since $\bar{\varphi}(s,t) = s\bar{\varphi}_1(t/s)$ for $s > 0, t \geq 0$ (resp., $\bar{\varphi}(s,t) = t\bar{\varphi}_0(s/t)$ for $s \geq 0, t > 0$), $\bar{\varphi}$ is continuous on $\mathbb{R}^2_+ \setminus \{(0,0)\}$; moreover, from $\bar{\varphi}(s,t) \leq (s \vee t)\varphi(1,1)$, it follows that $\bar{\varphi}$ is also continuous at $(0,0)$. We will from now on denote simply by φ the unique continuous extension of φ to $\mathbb{R}^2_+$.

Given quasi-Banach lattices X_0, X_1, we say that (X_0, X_1) is a *compatible pair of quasi-Banach lattices* when there exists a (Hausdorff, locally solid) topological vector lattice X, along with inclusions $j_i : X_i \hookrightarrow X$ which are continuous, interval-preserving, lattice homomorphisms for $i = 0, 1$. In this way, the space

$$X_0 + X_1 = \{x \in X : x = x_0 + x_1 \text{ with } x_0 \in X_0, x_1 \in X_1\}$$

becomes a quasi-Banach lattice, endowed with the quasinorm

$$\|x\| = \inf\big\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1\big\},$$

which contains X_0 and X_1 as (nonclosed) ideals.

Note that this setting is more general than the one considered in [1] (where X is the space of measurable functions over some measure space) or in [15] (where X is a $C_\infty(Q)$-space, i.e., the space of extended continuous scalar functions with dense domain over a Stonean compact space Q). In particular, X_0 and X_1 need not be order-complete.

Now, given a compatible pair of quasi-Banach lattices (X_0, X_1) and a function $\varphi \in \mathcal{P}$, let us consider the Calderón–Lozanovskii space (see [14], [15])

$$\varphi(X_0, X_1) = \big\{x \in X_0 + X_1 : |x| \le \varphi(x_0, x_1) \text{ for some } x_0 \in X_0^+, x_1 \in X_1^+\big\}.$$

Here, for any pair of positive elements x_0, x_1 in a quasi-Banach lattice, $\varphi(x_0, x_1)$ is defined in an unambiguous way by means of Krivine's functional calculus for continuous positively 1-homogeneous functions on $\mathbb{R}^2$ (see [13, pp. 40–42], [22]). Indeed, φ may be extended to such a function (e.g., $\hat{\varphi}(s,t) = \varphi(s \vee 0, t \vee 0)$).

The space $\varphi(X_0, X_1)$ is a quasi-Banach lattice equipped with the quasinorm

$$\|x\|_{\varphi(X_0,X_1)} = \inf\big\{\lambda > 0 : |x| \le \lambda\varphi(x_0, x_1), \|x_0\|_{X_0} \le 1, \|x_1\|_{X_1} \le 1\big\}.$$

Actually, we have

$$\|x + y\|_{\varphi(X_0,X_1)} \le \max\{C_0, C_1\}\big(\|x\|_{\varphi(X_0,X_1)} + \|y\|_{\varphi(X_0,X_1)}\big),$$

where C_i is the constant appearing in the triangle inequality corresponding to X_i $(i = 0, 1)$. Given a function φ as above, there is a natural decomposition into piecewise linear functions due to Brudnyi and Kruglyak [1, Proposition 3.2.5] (see also [11]). We present next a small modification of this construction which is more suitable for our purposes.

Lemma 2.1. *Let $\varphi \in \mathcal{P}$. Given that $q > 1$, there exist $M, N \in \mathbb{N} \cup \{\infty\}$, extended sequences $(t_k)_{k=-2M}^{2N} \subset [0, +\infty]$, and $(\varepsilon_k)_{k=-M}^{N} \subset [0, 1]$ satisfying the following properties.*

1. *We have that $(t_k)_{k=-2M}^{2N}$ is increasing, $0 < \varepsilon_k < \min\{t_{2k} - t_{2k-1}, t_{2k+3} - t_{2k+2}\}$.*
2. *For every $s, t \in (0, +\infty)$, it holds that*
$$\sum_{k=-M}^{N} \varphi(1, t_{2k+1}) \min\Big(s, \frac{t}{t_{2k+1}}\Big) \le \frac{q+1}{q-1}\varphi(s,t).$$
3. *For all $t \in [t_{2k} - \varepsilon_k, t_{2k+2} + \varepsilon_k]$,*
$$\varphi(1,t) \le q\varphi(1, t_{2k+1}) \min\Big(1, \frac{t}{t_{2k+1}}\Big).$$

The notation here is consistent in the following sense.

- *If $M = \infty$, then $\lim_{k\to-\infty} t_k = 0 = \lim_{k\to-\infty} \varepsilon_k$.*
- *If $N = \infty$, then $\lim_{k\to+\infty} t_k = +\infty$, $\lim_{k\to+\infty} \varepsilon_k = 0$.*
- *If both M, N are finite, then $t_{-2M} = 0$, $t_{2N} = +\infty$, $\varepsilon_{-M} = \varepsilon_N = 0$.*

Proof. We work with the function $\varphi_1(t) = \varphi(1, t)$. Since φ_1 is quasiconcave, for every $s, t \in \mathbb{R}_+$, we have

$$\varphi_1(t) \le \max\Big(1, \frac{t}{s}\Big)\varphi_1(s).$$

Thus, we can assume without loss of generality that φ_1 is a continuous concave function on $\mathbb{R}_+$ (see [1, Corollary 3.1.4]).

According to [1, Proposition 3.2.5], for any $q' \in (1, q)$ there exist $M, N \in \mathbb{N} \cup \{\infty\}$ and an increasing sequence $(t_k)_{k=-2M}^{2N} \subset [0, +\infty]$ satisfying the following properties.

(a) If $M, N < \infty$, then $t_{-2M} = 0$ and $t_{2N} = +\infty$. Otherwise, if $M = \infty$, then $\lim_{k\to-\infty} t_k = 0$, while if $N = \infty$, then $\lim_{k\to+\infty} t_k = +\infty$.

(b) For $-M \leq k \leq N$, we have

$$\frac{\varphi_1(t_{2k})}{t_{2k}} = q' \frac{\varphi_1(t_{2k+1})}{t_{2k+1}} \qquad \text{and} \qquad \varphi_1(t_{2k+2}) = q' \varphi_1(t_{2k+1}).$$

(c) For every $s, t \in (0, +\infty)$, it holds that

$$\sum_{k=-M}^{N} \varphi_1(t_{2k+1}) \min\Big(s, \frac{t}{t_{2k+1}}\Big) \leq \frac{q'+1}{q'-1} \varphi(s, t).$$

Note that (b) yields that, for $t \in [t_{2k}, t_{2k} + 2]$, one has

$$\varphi_1(t) \leq q' \varphi_1(t_{2k+1}) \min\Big(1, \frac{t}{t_{2k+1}}\Big).$$

Now, for any $\varepsilon \in (0, \frac{q}{q'} - 1)$, using the continuity of φ_1 we can find a sequence (ε_k) with $\lim_{|k|\to+\infty} \varepsilon_k = 0$,

$$0 < \varepsilon_k < \min\{t_{2k} - t_{2k-1}, t_{2k+3} - t_{2k+2}\},$$

and such that

$$\varphi_1(t) \leq (1 + \varepsilon) q' \varphi_1(t_{2k+1}) \min\Big(1, \frac{t}{t_{2k+1}}\Big)$$

for all $t \in [t_{2k} - \varepsilon_k, t_{2k+2} + \varepsilon_k]$. These sequences satisfy the required properties. □

Throughout, we will be using the usual local representation of a quasi-Banach lattice via $C(\Omega)$-spaces (see [22]). That is, given a positive element in a quasi-Banach lattice $e \in X$, the (nonclosed) ideal generated by e is isomorphic to a space $C(\Omega)$ for a certain compact Hausdorff space Ω, and we can consider an injective lattice homomorphism $J : C(\Omega) \to X$ such that $J(1_\Omega) = e$ and $J(B_{C(\Omega)}) = [-e, e]$.

Let us briefly recall the formal meaning of an interpolation functor between quasi-Banach lattices. We use the terminology of category theory as in [1, Section 2.3]. Let $\mathcal{QBL}$ denote the category of quasi-Banach lattices and bounded linear operators between them, and let $\overrightarrow{\mathcal{QBL}}$ denote the category of compatible pairs $\vec{X} = (X_0, X_1)$ of quasi-Banach lattices and linear operators between them, where a linear operator

$$T : \vec{X} \to \vec{Y}$$

is a bounded linear mapping $T : X_0 + X_1 \to Y_0 + Y_1$ satisfying $T|_{X_0} : X_0 \to Y_0$ and $T|_{X_1} : X_1 \to Y_1$ (both being bounded, too).

A functor $F : \overrightarrow{\mathcal{QBL}} \to \mathcal{QBL}$ is called an *interpolation functor* if

(i) for every $\vec{X} = (X_0, X_1)$, we have bounded inclusions $X_0 \cap X_1 \hookrightarrow F(\vec{X}) \hookrightarrow X_0 + X_1$;

(ii) for every $T : \vec{X} \to \vec{Y}$, the operator $F(T) = T|_{F(\vec{X})} : F(\vec{X}) \to F(\vec{Y})$ is bounded.

In particular, this implies that $F(\vec{X})$ is an interpolation space for every $\vec{X}$.

3. Interpolation of $(\infty, 1)$-regular operators

Given quasi-Banach lattices E, F, and $1 \leq p, q < \infty$, a linear operator $T : E \to F$ is called (p,q)-*regular* if there is a constant $K > 0$ such that for every $\{x_i\}_{i=1}^n \subset E$, we have

$$\Big\|\Big(\sum_{i=1}^n |Tx_i|^p\Big)^{\frac{1}{p}}\Big\| \leq K\Big\|\Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}}\Big\|.$$

Similarly, T will be called (p,∞)-*regular* (resp., (∞,q)-*regular*) when

$$\Big\|\Big(\sum_{i=1}^n |Tx_i|^p\Big)^{\frac{1}{p}}\Big\| \leq K\Big\|\bigvee_{i=1}^n |x_i|\Big\| \quad \Big(\text{resp., } \Big\|\bigvee_{i=1}^n |Tx_i|\Big\| \leq K\Big\|\Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}}\Big\|\Big).$$

We will denote by $\rho_{p,q}(T)$ the smallest $K > 0$ for which the above inequalities hold for arbitrary elements in E.

The class of (p,q)-regular operators was introduced in [2] (see also [3], [12]), and has obvious connections with convexity and concavity (see [13, Section 1.d]). It is clear that a (p,q)-regular operator T is always bounded and that $\|T\| \leq \rho_{p,q}(T)$. Also, if T is (p,q)-regular, then it is (p',q')-regular for every $p' \geq p$ and $q' \leq q$, and moreover $\rho_{p',q'}(T) \leq \rho_{p,q}(T)$. In particular, among these, the largest class is that of $(\infty,1)$-regular operators, which satisfies

$$\Big\|\bigvee_{i=1}^n |Tx_i|\Big\| \leq K\Big\|\sum_{i=1}^n |x_i|\Big\|.$$

If F is Dedekind-complete and $T : E \to F$ is a regular operator (i.e., T can be written as a difference of two positive operators), then it is (p,p)-regular for every $1 \leq p \leq \infty$, and $\rho_{p,p}(T) \leq \||T|\|$. In the converse direction, if F is complemented by a positive projection in its bidual, then every $(1,1)$-regular operator $T : E \to F$ is regular (see [12, p. 307]).

In Section 4, we will consider spaces in which every linear operator is (p,q)-regular. In particular, an application of Grothendieck's inequality yields that every bounded linear operator between Banach lattices, or even L-convex quasi-Banach lattices, is $(2,2)$-regular. We state now our main result concerning the interpolation of $(\infty,1)$-regular operators with respect to the functor φ^c.

Theorem 3.1. *Let (X_0, X_1) and (Y_0, Y_1) be compatible pairs of quasi-Banach lattices, and let $T : X_0 + X_1 \to Y_0 + Y_1$ be a bounded operator such that $T|_{X_i}$:*

$X_i \to Y_i$ is $(\infty,1)$-regular for $i=0,1$. Then, for $\varphi \in \mathcal{P}$, we have that $T : \varphi^c(X_0,X_1) \to \varphi^c(Y_0,Y_1)$ is $(\infty,1)$-regular with

$$\rho_{\infty,1}(T|_{\varphi^c(X_0,X_1)}) \leq C \max\big\{\rho_{\infty,1}(T|_{X_0}), \rho_{\infty,1}(T|_{X_1})\big\}$$

for some $C>0$ which depends only on X_0, X_1, Y_0, Y_1, and φ.

Before giving our proof, we need some preliminaries.

Lemma 3.2. *Let (X_0,X_1) and (Y_0,Y_1) be interpolation couples of quasi-Banach lattices, and let $T : X_0+X_1 \to Y_0+Y_1$ be a bounded operator such that $T|_{X_i} : X_i \to Y_i$ is $(\infty,1)$-regular for $i=0,1$. Then $T : X_0+X_1 \to Y_0+Y_1$ is $(\infty,1)$-regular with*

$$\rho_{\infty,1}(T) \leq 2\max\big\{\rho_{\infty,1}(T|_{X_0}), \rho_{\infty,1}(T|_{X_1})\big\}.$$

Proof. Let us consider $(z_i)_{i=1}^n \subset X_0+X_1$ such that $\|\sum_{i=1}^n |z_i|\|_{X_0+X_1} < 1$. Hence, there exist positive $u \in X_0$, $v \in X_1$ with $\|u\|_{X_0} + \|v\|_{X_1} < 1$ and

$$\sum_{i=1}^n |z_i| \leq u+v.$$

Using the Riesz decomposition property (see [17, Theorem 1.1.1.viii]), we can write $z_i = u_i + v_i$ for $i=1,\dots,n$, with $\sum_{i=1}^n |u_i| \leq 2u$, $\sum_{i=1}^n |v_i| \leq 2v$. Now, since $T|_{X_j}$ is $(\infty,1)$-regular for $j=0,1$, we have

$$\Big\|\bigvee_{i=1}^n |Tu_i|\Big\|_{Y_0} \leq \rho_{\infty,1}(T|_{X_0})\Big\|\sum_{i=1}^n |u_i|\Big\|_{X_0} \leq 2\rho_{\infty,1}(T|_{X_0})\|u\|_{X_0},$$
$$\Big\|\bigvee_{i=1}^n |Tv_i|\Big\|_{Y_1} \leq \rho_{\infty,1}(T|_{X_1})\Big\|\sum_{i=1}^n |v_i|\Big\|_{X_1} \leq 2\rho_{\infty,1}(T|_{X_1})\|v\|_{X_1}.$$

These, together with

$$\bigvee_{i=1}^n |Tz_i| \leq \bigvee_{i=1}^n |Tu_i| + \bigvee_{i=1}^n |Tv_i|$$

yield

$$\Big\|\bigvee_{i=1}^n |Tz_i|\Big\|_{Y_0+Y_1} \leq 2\max\big\{\rho_{\infty,1}(T|_{X_0}), \rho_{\infty,1}(T|_{X_1})\big\}.$$

This finishes the proof. □

Lemma 3.3. *There is a constant $\gamma > 0$ such that, given (X_0,X_1), (Y_0,Y_1), $T : X_0+X_1 \to Y_0+Y_1$ as in Theorem 3.1, $\varphi \in \mathcal{P}$ with $\lim_{t\to 0^+}\varphi_1(t) = 0 = \lim_{t\to+\infty}\frac{\varphi_1(t)}{t}$, and $(x_i)_{i=1}^n \subset X_0+X_1$ such that $\sum_{i=1}^n |x_i| \leq \varphi(u_0,u_1)$, where $u_i \in X_i$ with $\|u_i\|_{X_i} \leq 1$ for $i=0,1$, there exist sequences $(x_i^m)_{m\in\mathbb{N}}$ for $1 \leq i \leq n$ satisfying:*

- (i) *$|x_i^m| \leq |x_i|$ for every $m \in \mathbb{N}$, $1 \leq i \leq n$,*
- (ii) *$\bigvee_{i=1}^n |x_i - x_i^m| \leq (u_0 \vee u_1)a_m$ for certain $a_m \in \mathbb{R}_+$ with $a_m \underset{m\to\infty}{\longrightarrow} 0$,*
- (iii) *$\sup_m \|\bigvee_{i=1}^n |Tx_i^m|\|_{\varphi(Y_0,Y_1)} \leq \gamma \max\{\rho_{\infty,1}(T|_{X_0}), \rho_{\infty,1}(T|_{X_1})\}$.*

Proof. By Lemma 2.1, for any $q > 1$ there exist $M, N \in \mathbb{N} \cup \{\infty\}$, an increasing sequence $(t_k)_{k=-2M}^{2N} \subset [0, +\infty]$, and $(\varepsilon_k)_{k=-M}^{N}$ such that, for every $s, t \in (0, +\infty)$, we have

$$\sum_{k=-M}^{N} \varphi_1(t_{2k+1}) \min\Big(s, \frac{t}{t_{2k+1}}\Big) \leq \frac{q+1}{q-1}\varphi(s,t), \tag{3.1}$$

and for $t \in [t_{2k} - \varepsilon_k, t_{2k+2} + \varepsilon_k]$,

$$\varphi_1(t) \leq q\varphi_1(t_{2k+1}) \min\Big(1, \frac{t}{t_{2k+1}}\Big). \tag{3.2}$$

Let us consider the ideal generated by $u_0 \vee u_1$ in $X_0 + X_1$. As usual, we can consider a compact Hausdorff space Ω and a lattice homomorphism $J : C(\Omega) \to X_0 + X_1$ such that $J(B_{C(\Omega)}) = [-u_0 \vee u_1, u_0 \vee u_1]$. Since

$$|x_i| \leq \sum_{i=1}^{n} |x_i| \leq \varphi(u_0, u_1) \leq u_0 \vee u_1,$$

there exist $(f_i)_{i=1}^n, h_0, h_1 \in B_{C(\Omega)}$ such that $J(f_i) = x_i, J(h_0) = u_0$, and $J(h_1) = u_1$.

Let $m \in \mathbb{N}$, and for $|k| \leq m$, let us consider the sets

$$U_k = \big\{\omega \in \Omega : (t_{2k} - \varepsilon_k)h_0(\omega) < h_1(\omega) < (t_{2k+2} + \varepsilon_k)h_0(\omega)\big\}$$

and

$$V_m = \Omega \setminus \big\{\omega \in \Omega : t_{-2m}h_0(\omega) \leq h_1(\omega) \leq t_{2m+2}h_0(\omega)\big\}.$$

Clearly, these are open subsets of Ω satisfying

$$\Omega = V_m \cup \bigcup_{|k|\leq m} U_k.$$

Therefore, we can consider a continuous partition of unity associated to this open covering, that is, $(\psi_k)_{|k|\leq m}$, and ξ_m positive elements in $C(\Omega)$ such that for each $|k| \leq m$, ψ_k is supported within U_k, ξ_m is supported in V_m, and for every $\omega \in \Omega$ we have

$$\sum_{|k|\leq m} \psi_k(\omega) + \xi_m(\omega) = 1.$$

Let us consider

$$f_i^m = \sum_{|k|\leq m} f_i\psi_k \in C(\Omega).$$

And denote $x_i^m = J(f_i^m)$, $y_i^k = J(f_i\psi_k)$ for $|k| \leq m$. These obviously satisfy $|y_i^k|, |x_i^m| \leq |x_i|$, for every $1 \leq i \leq n$, $m \in \mathbb{N}$ and $|k| \leq m$, and

$$x_i^m = \sum_{|k|\leq m} y_i^k.$$

We claim that the sequences (x_i^m) satisfy properties (ii) and (iii).

In order to prove (ii), given $m \in \mathbb{N}$, let us consider the sets

$$\begin{aligned} W_1^m &= \Big\{\omega \in \Omega : h_1(\omega) < \Big(t_{-2m} + \frac{\varepsilon_m}{2}\Big) h_0(\omega)\Big\}, \\ W_2^m &= \Big\{\omega \in \Omega : \Big(t_{2m+2} - \frac{\varepsilon_{m+1}}{2}\Big) h_0(\omega) < h_1(\omega)\Big\}, \\ W_3^m &= \big\{\omega \in \Omega : t_{-2m} h_0(\omega) < h_1(\omega) < t_{2m+2} h_0(\omega)\big\}. \end{aligned}$$

Since h_0 and h_1 cannot vanish simultaneously (because $h_0 \vee h_1 = 1$), for every $m \in \mathbb{N}$ these open sets W_i^m are such that $\bigcup_{l=1}^3 W_l^m = \Omega$. Let $(\vartheta_l^m)_{l=1,2,3}$ denote a continuous partition of unity associated to these sets, that is, $\vartheta_l^m \in C(\Omega)$ with each ϑ_l^m being positive and supported in W_l^m, and for every $\omega \in \Omega$ and every $m \in \mathbb{N}$,

$$\sum_{l=1}^3 \vartheta_l^m(\omega) = 1.$$

Note that for $1 \leq i \leq n$,

$$\big|(f_i - f_i^m)(\omega)\big| = \big|f_i \xi_m(\omega)\big| = \Big|f_i \xi_m \Big(\sum_{l=1}^3 \vartheta_l^m\Big)(\omega)\Big|,$$

and since ξ_m is supported in $V_m \subset \Omega \setminus W_3^m$, we have

$$|f_i - f_i^m| \leq |f_i \xi_m \vartheta_1^m| + |f_i \xi_m \vartheta_2^m|.$$

For $\omega \in \Omega$, we have

$$\begin{aligned} \big|f_i \xi_m \vartheta_1^m(\omega)\big| &\leq \varphi(h_0, h_1) \xi_m \vartheta_1^m(\omega) \\ &\leq \varphi\Big(h_0(\omega), \Big(t_{-2m} + \frac{\varepsilon_m}{2}\Big) h_0(\omega)\Big) \\ &= h_0(\omega) \varphi_1\Big(t_{-2m} + \frac{\varepsilon_m}{2}\Big). \end{aligned} \tag{3.3}$$

Similarly, we have

$$\begin{aligned} \big|f_i \xi_m \vartheta_2^m(\omega)\big| &\leq \varphi(h_0, h_1) \xi_m \vartheta_2^m(\omega) \\ &\leq \varphi\Big(\frac{h_1(\omega)}{t_{2m+2} - \frac{\varepsilon_{m+1}}{2}}, h_1(\omega)\Big) \\ &= h_1(\omega) \frac{\varphi_1(t_{2m+2} - \frac{\varepsilon_{m+1}}{2})}{t_{2m+2} - \frac{\varepsilon_{m+1}}{2}}. \end{aligned} \tag{3.4}$$

Therefore, setting

$$a_m = \varphi_1\Big(t_{-2m} + \frac{\varepsilon_m}{2}\Big) + \frac{\varphi_1(t_{2m+2} - \frac{\varepsilon_{m+1}}{2})}{t_{2m+2} - \frac{\varepsilon_{m+1}}{2}},$$

and putting together the estimates (3.3) and (3.4), we get

$$|x_i - x_i^m| \leq (u_0 \vee u_1) a_m.$$

The hypotheses on φ_1 clearly yield that $a_m \to 0$ as $m \to \infty$, so this proves (ii).

Finally, to prove (iii), note that by inequality (3.2), for every $|k| \le m$ and $\omega \in \Omega$ we have

$$\begin{aligned}\sum_{i=1}^{n} \big|f_i\psi_k(\omega)\big| &\le \big|\varphi\big(h_0(\omega), h_1(\omega)\big)\psi_k(\omega)\big| \\ &\le h_0(\omega)\varphi_1(t_{2k+2}+\varepsilon_k)\psi_k(\omega) \\ &\le q\varphi_1(t_{2k+1})h_0(\omega)\psi_k(\omega),\end{aligned}$$

and similarly

$$\sum_{i=1}^{n} \big|f_i\psi_k(\omega)\big| \le q\frac{\varphi_1(t_{2k+1})}{t_{2k+1}}h_1(\omega)\psi_k(\omega).$$

Therefore, the functions

$$F_0^m = \sum_{|k|\le m} \frac{1}{\varphi_1(t_{2k+1})}\sum_{i=1}^{n}|f_i\psi_k|, \qquad F_1^m = \sum_{|k|\le m} \frac{t_{2k+1}}{\varphi_1(t_{2k+1})}\sum_{i=1}^{n}|f_i\psi_k|$$

satisfy $F_j^m \le qh_j$ for $j = 0, 1$.

Now, let us consider

$$G_0^m = \max_{|k|\le m, 1\le i\le n}\Big\{\frac{1}{\varphi_1(t_{2k+1})}|Ty_i^k|\Big\}$$

in $Y_0 + Y_1$. Since $T|_{X_0} : X_0 \to Y_0$ is $(\infty, 1)$-regular, we have

$$\begin{aligned}\|G_0^m\|_{Y_0} &= \Big\|\max_{|k|\le m, 1\le i\le n}\Big\{\frac{1}{\varphi_1(t_{2k+1})}|Ty_i^k|\Big\}\Big\|_{Y_0} \\ &\le \rho_{\infty,1}(T|_{X_0})\Big\|\sum_{|k|\le m}\frac{1}{\varphi_1(t_{2k+1})}\sum_{i=1}^{n}|y_i^k|\Big\|_{X_0} \\ &\le \rho_{\infty,1}(T|_{X_0})\|qu_0\|_{X_0} \\ &\le q\rho_{\infty,1}(T|_{X_0}),\end{aligned}$$

while for

$$G_1^m = \max_{|k|\le N, 1\le i\le n}\Big\{\frac{t_{2k+1}}{\varphi_1(t_{2k+1})}|Ty_i^k|\Big\}$$

a similar argument yields

$$\|G_1^m\|_{Y_1} \le q\rho_{\infty,1}(T|_{X_1}).$$

Now, by equation (3.1), we have

$$\begin{aligned}\max_{1\le i\le n}|Tx_i^m| &\le \max_{1\le i\le n}\sum_{|k|\le m}|Ty_i^k| \\ &\le \sum_{|k|\le m}\varphi_1(t_{2k+1})\min\Big(G_0^m, \frac{1}{t_{2k+1}}G_1^m\Big) \\ &\le \frac{q+1}{q-1}\varphi(G_0^m, G_1^m).\end{aligned}$$

From this inequality and the fact that $\|G_j^m\|_{Y_j} \leq q\rho_{\infty,1}(T|_{X_j})$ for $j = 0,1$, it follows that

$$\Big\| \max_{1\leq i\leq n} |Tx_i^m| \Big\|_{\varphi(Y_0,Y_1)} \leq \frac{q(q+1)}{q-1} \max\big\{\rho_{\infty,1}(T|_{X_0}), \rho_{\infty,1}(T|_{X_1})\big\}.$$

This finishes the proof of (iii). □

Remark 3.4. Optimizing the estimate obtained in the previous proof for $q > 1$, we could take $\gamma = 3 + 2\sqrt{2}$.

Proof of Theorem 3.1. Let $R = \max\{\rho_{\infty,1}(T|_{X_0}), \rho_{\infty,1}(T|_{X_1})\}$. First, we claim that there is $K > 0$ such that, given $(x_i)_{i=1}^n \subset X_0 + X_1$,

$$\text{if } \Big\| \sum_{i=1}^n |x_i| \Big\|_{\varphi(X_0,X_1)} \leq 1, \quad \text{then } \Big\| \bigvee_{i=1}^n |Tx_i| \Big\|_{\varphi^c(Y_0,Y_1)} \leq KR. \tag{3.5}$$

Indeed, as before, let $\varphi_1(t) = \varphi(1,t)$. Without loss of generality, we can assume that φ_1 is a concave function (see [1, Corollary 3.1.4]). Note that if $\lim_{t\to 0^+} \varphi_1(t) = 0 = \lim_{t\to\infty} \frac{\varphi_1(t)}{t}$, then the conclusion follows directly from Lemma 3.3. Otherwise, let us consider

$$\phi_1(s) = \lim_{t\to 0^+} \varphi_1(t) \vee s \lim_{t\to\infty} \frac{\varphi_1(t)}{t} \qquad \text{and} \qquad \eta_1 = \varphi_1 - \phi_1. \tag{3.6}$$

Note that, as ϕ_1 is clearly convex, it follows that η_1 is a concave function which moreover is positive and satisfies $\lim_{t\to 0^+} \eta_1(t) = 0 = \lim_{t\to\infty} \frac{\eta_1(t)}{t}$.

Now, if we consider $\phi(s,t) = s\phi_1(\frac{t}{s})$ and $\eta(s,t) = s\eta_1(\frac{t}{s})$, it follows that

$$\phi(X_0,X_1) + \eta(X_0,X_1) = \varphi(X_0,X_1) \tag{3.7}$$

with equivalent norms (with a constant not greater than 2).

Take $(x_i)_{i=1}^n \in \varphi(X_0,X_1)$ such that $\| \sum_{i=1}^n |x_i| \|_{\varphi(X_0,X_1)} < 1$; hence, $\sum_{i=1}^n |x_i| \leq \varphi(u_0,u_1)$ for some $u_i \in X_i$ with $\|u_i\|_{X_i} \leq 1$ for $i = 0,1$. According to (3.7) and using the Riesz decomposition property, we can write $x_i = v_i + w_i$, where

$$\sum_{i=1}^n |v_i| \leq \phi(u_0,u_1) \qquad \text{and} \qquad \sum_{i=1}^n |w_i| \leq \eta(u_0,u_1).$$

On the one hand, note that $\phi(X_0,X_1)$ coincides, up to a c-equivalent norm, with X_0, X_1, or $X_0 + X_1$ for some $c > 0$. Hence, by Lemma 3.2, we have that

$$\Big\| \bigvee_{i=1}^n |Tv_i| \Big\|_{\phi(Y_0,Y_1)} \leq 2Rc. \tag{3.8}$$

On the other hand, by Lemma 3.3, there exist a constant γ and sequences $(w_i^m)_{m\in\mathbb{N}}$ for $1 \leq i \leq n$ such that

$$\sup_m \Big\| \bigvee_{i=1}^n |Tw_i^m| \Big\|_{\eta(Y_0,Y_1)} \leq \gamma R \tag{3.9}$$

and for every $i = 1,\ldots,n$ and some $(a_m)_{m\in\mathbb{N}}$ with $a_m \underset{m\to\infty}{\longrightarrow} 0$,

$$|w_i^m - w_i| \leq (u_0 \vee u_1)a_m. \tag{3.10}$$

Note, in particular, that (3.10) implies that

$$\max_{1\le i\le n} \|v_i + w_i^m - x_i\|_{X_0+X_1} \underset{m\to\infty}{\longrightarrow} 0,$$

and also that

$$\Big\| \bigvee_{i=1}^n |Tv_i + Tw_i^m| - \bigvee_{i=1}^n |Tx_i| \Big\|_{Y_0+Y_1} \underset{m\to\infty}{\longrightarrow} 0.$$

Then, putting together (3.8) and (3.9), we have

$$\begin{aligned} \Big\| \bigvee_{i=1}^n |Tv_i + Tw_i^m| \Big\|_{\varphi(Y_0,Y_1)} &\le \Big\| \bigvee_{i=1}^n |Tv_i| \Big\|_{\phi(Y_0,Y_1)} + \Big\| \bigvee_{i=1}^n |Tw_i^m| \Big\|_{\eta(Y_0,Y_1)} \\ &\le (2+\gamma)R. \end{aligned} \tag{3.11}$$

This proves claim (3.5).

Using the fact that $T : X_0+X_1 \to Y_0+Y_1$ is bounded, the following density argument will finish the proof. Given $(x_i)_{i=1}^n \subset X_0+X_1$ with $\| \sum_{i=1}^n |x_i| \|_{\varphi^c(X_0,X_1)} < 1$, we can find $(x^m)_{m\in\mathbb{N}} \subset X_0+X_1$ such that

$$\sup_m \|x^m\|_{\varphi(X_0,X_1)} < 1 \qquad \text{and} \qquad \Big\| x^m - \sum_{i=1}^n |x_i| \Big\|_{X_0+X_1} \to 0.$$

Without loss of generality, we can write $x^m = \sum_{i=1}^n |x_i^m|$ for some $(x_i^m)_{m\in\mathbb{N}}$ such that $\sum_{i=1}^n |x_i^m| \le \sum_{i=1}^n |x_i|$ and $\|x_i^m - x_i\|_{X_0+X_1} \to 0$ for every $i = 1,\dots,n$. By claim (3.5), it follows that for every $m \in \mathbb{N}$, $(Tx_i^m)_{i=1}^n \subset \varphi^c(Y_0,Y_1)$ with

$$\Big\| \bigvee_{i=1}^n |Tx_i^m| \Big\|_{\varphi(Y_0,Y_1)} \le \gamma R. \tag{3.12}$$

Now, since $T : X_0+X_1 \to Y_0+Y_1$ is bounded, we have that for every $i = 1,\dots,n$, $\|Tx_i^m - Tx_i\|_{Y_0+Y_1} \to 0$, and in particular we have that

$$\Big\| \bigvee_{i=1}^n |Tx_i^m| - \bigvee_{i=1}^n |Tx_i| \Big\|_{Y_0+Y_1} \to 0. \tag{3.13}$$

This shows that

$$\Big\| \bigvee_{i=1}^n |Tx_i| \Big\|_{\varphi^c(Y_0,Y_1)} \le \gamma R$$

and finishes the proof. □

Remark 3.5. The proof given here is heavily motivated by the one in [1, Theorem 4.3.11] and follows a similar approach. Actually, under the assumptions of Theorem 3.1, the proof of [1, Theorem 4.3.11] essentially shows that $T : \varphi^c(X_0,X_1) \to \varphi^c(Y_0,Y_1)$ is bounded as long as (X_0,X_1) and (Y_0,Y_1) are interpolation couples of Banach lattices of measurable functions on a certain measure space. However, the one given here is more general since the lattices we deal with do not necessarily consist of functions over a measure space.

4. Quasi-Banach lattices with the $K_{p,q}$ property

An application of Grothendieck's inequality due to Krivine [10] (see also [13, Theorem 1.f.14]) yields that for any Banach lattices E, F, every bounded linear operator $T : E \to F$ is $(2,2)$-regular with $\rho_{2,2}(T) \le K_G\|T\|$, where K_G denotes Grothendieck's constant. This fact was later extended by Kalton [7] to L-convex quasi-Banach lattices. Recall that a quasi-Banach lattice E is L-convex whenever its order intervals are uniformly locally convex, that is, whenever there exists $0 < \varepsilon < 1$ so that if $u \in E_+$ with $\|u\| = 1$ and $0 \le x_i \le u$ (for $i = 1, \dots, n$) satisfy

$$\frac{1}{n}(x_1 + \cdots + x_n) \ge (1 - \varepsilon)u,$$

then

$$\max_{1\le i\le n} \|x_i\| \ge \varepsilon.$$

In particular, every Banach lattice is L-convex, and so is a quasi-Banach lattice which is for an equivalent quasinorm the p-concavification of a Banach lattice. In fact, every L-convex quasi-Banach lattice is of this kind by [7, Theorem 2.2], so that L-convex quasi-Banach lattices are exactly Nilsson's quasi-Banach lattices of *type* $\mathcal{C}$ (see [18, Definition 1.7]). These include classical spaces like L_p, $\Lambda(W, p)$, and $L_{p,\infty}$ for $0 < p \le \infty$. On the other hand, examples of non-L-convex quasi-Banach lattices are the $L_p(\phi)$-spaces $(0 < p < \infty)$ with respect to pathological submeasures ϕ (see [7], [25]). Motivated by these facts, we introduce the following.

Definition 4.1. A quasi-Banach lattice F has the $K_{p,q}$ property with constant $C > 0$ if, for every quasi-Banach lattice E, every bounded linear operator $T : E \to F$ is (p,q)-regular with $\rho_{p,q}(T) \le C\|T\|$.

By [7, Theorem 3.3], every L-convex quasi-Banach lattice has the $K_{2,2}$ property. As far as we know, it is still unknown whether the converse holds. However, L-convex quasi-Banach lattices constitute a large collection of spaces for which our results hold. In particular, this includes every quasi-Banach lattice E such that ℓ_∞ is not lattice finitely representable in E. Also, if F is an L-convex quasi-Banach lattice and E is a quasi-Banach lattice which is linearly homeomorphic to a subspace of F, then E is L-convex.

Note that if a quasi-Banach lattice has the $K_{p,q}$ property for some p, q, then it has the $K_{\infty,1}$ property. Let us summarize this in the following chain of implications for a quasi-Banach lattice E:

$$\text{locally convex} \Rightarrow L\text{-convex} \Rightarrow K_{2,2} \text{ property} \Rightarrow K_{\infty,1} \text{ property}.$$

We will focus now on the $K_{\infty,1}$ property for a quasi-Banach lattice, which is the weakest among the above properties.

Proposition 4.2. *For a quasi-Banach lattice E, the following are equivalent.*

(1) *E has the $K_{\infty,1}$ property with constant C.*
(2) *Every operator $T : \ell_\infty \to E$ is $(\infty, 1)$-regular with $\rho_{\infty,1}(T) \le C\|T\|$.*

(3) *For every* $(x_i)_{i=1}^n \subset E$, *we have*

$$\big\| \max_{1\le i\le n} |x_i| \big\| \le C \max_{|a_i|\le 1} \Big\| \sum_{i=1}^n a_i x_i \Big\|.$$

Proof. The implication (1) $\Rightarrow$ (2) is trivial. Suppose that (2) holds. Then given $(x_i)_{i=1}^n \subset E$, let $T : \ell_\infty \to E$ be the operator defined by

$$T(a_i) = \sum_{i=1}^n a_i x_i$$

for $(a_i)_{i=1}^\infty \in \ell_\infty$. Let $e_i \in \ell_\infty$ denote the sequence having 1 in the ith position and 0 elsewhere. By hypothesis, the operator T is $(\infty,1)$-regular with $\rho_{\infty,1}(T) \le C\|T\|$, which in particular yields

$$\big\| \max_{1\le i\le n} |x_i| \big\| = \big\| \max_{1\le i\le n} |Te_i| \big\| \le C\|T\| \Big\| \sum_{i=1}^n |e_i| \Big\| = C \max_{|a_i|\le 1} \Big\| \sum_{i=1}^n a_i x_i \Big\|.$$

Therefore, (3) holds.

For the implication (3) $\Rightarrow$ (1), if F is a quasi-Banach lattice and $T : F \to E$ is bounded, then

$$\big\| \max_{1\le i\le n} |Tx_i| \big\| \le C \max_{|a_i|\le 1} \Big\| \sum_{i=1}^n a_i T x_i \Big\| \le C\|T\| \Big\| \sum_{i=1}^n |x_i| \Big\|.$$

Hence, $\rho_{\infty,1}(T) \le C\|T\|$. □

A modification of [7, Example 3.5] provides an example of a quasi-Banach lattice without the $K_{\infty,1}$ property.

Example 4.3. For each $n \in \mathbb{N}$, let Ω_n be the unit sphere in ℓ_∞^n; that is, $\Omega_n = \{v \in \mathbb{R}^n : \max_{1\le i\le n} |v_i| = 1\}$. Let $\mathcal{A}_n$ denote the algebra of all subsets of Ω_n. For $u \in \mathbb{R}^n \setminus \{0\}$, let

$$B_u = \Big\{ v \in \Omega_n : \sum_{i=1}^n u_i v_i \ne 0 \Big\}.$$

Let us consider the normalized submeasure defined, for $A \in \mathcal{A}_n$, by

$$\phi_n(A) = \frac{1}{n} \inf \Big\{ \#S : A \subset \bigcup_{u\in S} B_u \Big\}.$$

Given $0 < p < 1$, consider the quasi-Banach lattice $L_p(\Omega_n, \mathcal{A}_n, \phi_n)$ which is the completion of the simple $\mathcal{A}_n$-measurable functions $f : \Omega_n \to \mathbb{R}$, with respect to the quasinorm

$$\|f\|_p = \Big(\int_0^\infty \phi_n\big(|f| \ge t^{\frac{1}{p}}\big)\, dt \Big)^{\frac{1}{p}}.$$

Now, for $1 \le i \le n$, let $f_i : \Omega_n \to \mathbb{R}$ be given by $f_i(v) = v_i$. It is clear that $\max_{1\le i\le n} |f_i(v)| = 1$ for every $v \in \Omega_n$; thus,

$$\big\| \max_{1\le i\le n} |f_i| \big\|_p = 1.$$

On the other hand, for $a \in \mathbb{R}^n$ with $|a_i| \leq 1$, we have

$$\Big|\sum_{i=1}^{n} a_i f_i\Big| \leq n\chi_{B_a}.$$

Therefore, we have

$$\Big\|\sum_{i=1}^{n} a_i f_i\Big\|_p \leq n^{1-\frac{1}{p}}.$$

Taking E to be the ℓ_∞-product of the spaces $L_p(\Omega_n, \mathcal{A}_n, \phi_n)$ for $n \in \mathbb{N}$, by Proposition 4.2, we see that E cannot have the $K_{\infty,1}$ property.

5. Interpolation functors

A direct consequence of Theorem 3.1 yields that the functor φ^c is an interpolation functor in the category of quasi-Banach lattices with the $K_{\infty,1}$ property.

Corollary 5.1. *If (X_0, X_1) and (Y_0, Y_1) are compatible pairs of quasi-Banach lattices such that Y_0 and Y_1 have the $K_{\infty,1}$ property, then for every $T : (X_0, X_1) \to (Y_0, Y_1)$ and every function $\varphi \in \mathcal{P}$, we have that $T : \varphi^c(X_0, X_1) \to \varphi^c(Y_0, Y_1)$.*

Proof. Let (X_0, X_1), (Y_0, Y_1) be compatible couples of quasi-Banach lattices such that Y_0 and Y_1 have the $K_{\infty,1}$ property. Let $T : X_0 + X_1 \to Y_0 + Y_1$ be an operator which is bounded as an operator $T|_{X_0} : X_0 \to Y_0$ and $T|_{X_1} : X_1 \to Y_1$. It follows that the $T|_{X_i}$'s are $(\infty, 1)$-regular for $i = 0, 1$, so Theorem 3.1 yields that $T : \varphi^c(X_0, X_1) \to \varphi^c(Y_0, Y_1)$ is $(\infty, 1)$-regular; in particular, it is bounded, and moreover

$$\begin{aligned}\|T|_{\varphi^c(X_0,X_1)}\| &\leq \rho_{\infty,1}(T|_{\varphi^c(X_0,X_1)}) \leq \max\big\{\rho_{\infty,1}(T|_{X_0}), \rho_{\infty,1}(T|_{X_1})\big\}\\ &\leq C \max\big\{\|T|_{X_0}\|, \|T|_{X_1}\|\big\},\end{aligned}$$

where $C > 0$ depends only on the $K_{\infty,1}$ constants of Y_0 and Y_1. □

Recall that given (X_0, X_1), we can also consider $\varphi^0(X_0, X_1)$ the closure of the intersection $X_0 \cap X_1$ in $\varphi(X_0, X_1)$. Our aim is to show that this is also an interpolation functor. We will need to address some technicalities first.

Definition 5.2. A function $\varphi \in \mathcal{P}$ is called *doubly bounded* provided there exists $C > 0$ such that $\varphi_i(t) \leq C$ for $i = 0, 1$.

Lemma 5.3. *A function $\varphi \in \mathcal{P}$ is doubly bounded if and only if $\varphi(s, t) \approx \min(s, t)$.*

Proof. Suppose that there is $C > 0$ such that, for every $t \in \mathbb{R}_+$, we have $\varphi_0(t), \varphi_1(t) \leq C$. In this case, we get that

$$\begin{aligned}\varphi(s, t) &= s\varphi_1(t/s) \leq Cs,\\ \varphi(s, t) &= t\varphi_0(s/t) \leq Ct.\end{aligned}$$

Hence, it follows that $\varphi(s, t) \leq C \min(s, t)$. Since for $\varphi \in \mathcal{P}$ we have the trivial estimate $\varphi(s, t) \geq \varphi(1, 1) \min(s, t)$, the conclusion follows. The converse implication is clear. □

Lemma 5.4. *Let (X_0, X_1) be an interpolation couple of quasi-Banach lattices, and let $\varphi \in \mathcal{P}$. If φ is not doubly bounded, and $\varphi_1(t) \to 0$ as $t \to 0$, then there is $C_{\varphi,\overline{X}} > 0$, depending only on φ and the quasinorm constants of X_0, X_1, such that for every positive $x \in X_0 \cap X_1$ with $\|x\|_{\varphi(X_0,X_1)} < 1$ there exist positive $f, g \in X_0 \cap X_1$ with $\|f\|_{X_0}, \|g\|_{X_1} \leq C_\varphi$ and $x = \varphi(f, g)$.*

Proof. By symmetry of the argument, we can suppose without loss of generality that $\lim_{t\to\infty} \varphi_0(t) = \infty$. Hence, for every $\delta > 0$, there is $N > 0$ such that $\varphi_0(\frac{N}{\delta}) \geq \frac{1}{\delta}$, or in other words, $\varphi(N, \delta) \geq 1$.

Assume that $x \in (X_0 \cap X_1)^+$ with $\|x\|_{\varphi(X_0,X_1)} < 1$, and let $u \in X_0^+$, $v \in X_1^+$ with $\|u\|_{X_0} < 1$, $\|v\|_{X_1} < 1$, and

$$x \leq \varphi(u, v).$$

Let C_{X_j} be the quasinorm constant of X_j, for $j = 0, 1$, let $\delta > 0$ be small enough so that $\|v \vee \delta x\|_{X_1} < C_{X_1}$, and let $N > 0$ be such that $\varphi(N, \delta) \geq 1$. Let $u' = u \wedge Nx$ and $v' = v \vee \delta x$. Note that $u' \in X_0 \cap X_1$, $\|u'\|_{X_0} < 1$, and $v' \in X_1$, $\|v'\|_{X_1} < C_{X_1}$. Moreover,

$$\varphi(u', v') = \varphi(u, v') \wedge \varphi(Nx, v') \geq \varphi(u, v) \wedge \varphi(Nx, \delta x) = x \wedge \varphi(N, \delta)x \geq x.$$

We distinguish two cases.

Case (a): If we also have that $\lim_{t\to\infty} \varphi_1(t) = \infty$, then we can proceed in a similar way as before exchanging the roles of the variables in φ. Let $0 < \varepsilon < N$ be small enough so that $\|u' \vee \varepsilon x\|_{X_0} < C_{X_0}$, and let $M > 0$ such that $\varphi(\varepsilon, M) \geq 1$. Then, take $u'' = u' \vee \varepsilon x$ and $v'' = v' \wedge Mx$ which also satisfy $u'', v'' \in X_0 \cap X_1$ with $\|u''\|_{X_0} < C_{X_0}$, $\|v''\|_{X_1} < C_{X_1}$, and $x \leq \varphi(u'', v'')$. Moreover,

$$\varphi(u'', v'') \leq \varphi(Nx, Mx,) \leq \varphi(N, M)x.$$

Let $J_0(x)$ be the (nonclosed) ideal generated by x, which can be considered as a $C(\Omega)$-space for some compact Hausdorff space Ω. Thus, we can consider the functions $\hat{u''}, \hat{v''}, \hat{y} \in C(\Omega)$ as corresponding, respectively, to u'', v'' and $y = \varphi(u'', v'')$. Recall that in this correspondence, x is represented by $\hat{x} = \mathbb{1}_\Omega$, so

$$\hat{y} \geq \hat{x} = \mathbb{1}_\Omega.$$

Thus, $\frac{1}{\hat{y}} \in C(\Omega)$ with $\|\frac{1}{\hat{y}}\| \leq 1$. Set $\hat{f} = \frac{\hat{u''}}{\hat{y}}$ and $\hat{g} = \frac{\hat{v''}}{\hat{y}}$, which clearly correspond to elements $f, g \in J_0(x)$ such that

$$\varphi(f, g) = x.$$

This identity follows from the fact that

$$\varphi(\hat{f}, \hat{g}) = \varphi\Big(\frac{\hat{u''}}{\hat{y}}, \frac{\hat{v''}}{\hat{y}}\Big) = \frac{\varphi(\hat{u''}, \hat{v''})}{\hat{y}} = \mathbb{1}_\Omega = \hat{x}.$$

Moreover, we have

$$f \leq u'' \leq Nx, \qquad g \leq v'' \leq Mx.$$

Hence, $f, g \in X_0 \cap X_1$, with $\|f\|_{X_0} \leq \|u''\|_{X_0} < C_{X_0}$ and $\|g\|_{X_1} \leq \|v''\|_{X_1} < C_{X_1}$.

Case (b): If, on the contrary, φ_1 is bounded, then set $C_\varphi = \sup_{s>0} \varphi_1(s) < \infty$ so that

$$\varphi(s,t) = s\varphi_1\Big(\frac{t}{s}\Big) \le C_\varphi s.$$

Since $x = \varphi(u', v')$, we have $x \le C_\varphi u'$ and

$$x = \varphi(x,x) \le \varphi(C_\varphi u', x).$$

On the other hand, $x = \varphi(u', v') \le \varphi(C_\varphi u', v')$ (assuming without loss of generality that $C_\varphi \ge 1$). Thus,

$$x \le \varphi(C_\varphi u', x \wedge v').$$

Then, we can take $u'' = C_\varphi u'$ and $v'' = x \wedge v'$. Then u'', v'' belong to $J_0(x)$, the (nonclosed) ideal generated by x, which corresponds to the space $C(\Omega)$, and satisfy

$$\|u''\|_{X_0} \le C_\varphi, \qquad \|v''\|_{X_1} < C_{X_1}.$$

Hence, as before, we may find $f \le u''$ and $g \le v''$ with $x = \varphi(f,g)$. □

This fact will allow us to show that φ^0 is an interpolation functor in the category of quasi-Banach lattices with the $K_{\infty,1}$ property. More precisely, we have the following.

Theorem 5.5. *Let (X_0, X_1) and (Y_0, Y_1) be compatible pairs of quasi-Banach lattices, and let $T : X_0 + X_1 \to Y_0 + Y_1$ be such that $T|_{X_j} : X_j \to Y_j$ is $(\infty,1)$-regular for $j = 0, 1$. Then for every function $\varphi \in \mathcal{P}$, we have that $T : \varphi^0(X_0, X_1) \to \varphi^0(Y_0, Y_1)$ is $(\infty,1)$-regular with*

$$\rho_{\infty,1}(T|_{\varphi^0(X_0,X_1)}) \le C \max\big\{\rho_{\infty,1}(T|_{X_0}), \rho_{\infty,1}(T|_{X_1})\big\},$$

for some $C > 0$ depending only on X_0, X_1, Y_0, Y_1, and φ.

Proof. If φ is doubly bounded, by Lemma 5.3, then it follows that $\varphi^0(X_0, X_1) = X_0 \cap X_1$ (with an equivalent norm). Therefore, in this case the conclusion follows.

Note that we can consider a decomposition as the one given in (3.6):

$$\phi_1(s) = \lim_{t\to 0^+} \varphi_1(t) \vee s \lim_{t\to\infty} \frac{\varphi_1(t)}{t} \qquad \text{and} \qquad \eta_1 = \varphi_1 - \phi_1. \tag{5.1}$$

As before, note that ϕ_1 is convex, so η_1 is concave. Thus, taking $\phi(s,t) = s\phi_1(\frac{t}{s})$ and $\eta(s,t) = s\eta_1(\frac{t}{s})$, it holds that

$$\varphi = \phi + \eta, \tag{5.2}$$

where $\phi(s,t) \approx \max(s,t)$ and $\lim_{t\to 0} \eta_1(t) = 0 = \lim_{t\to\infty} \frac{\eta_1(t)}{t}$.

Let $(x_i)_{i=1}^n \subset X_0 \cap X_1$ be positive with $\| \sum_{i=1}^n |x_i| \|_{\varphi(X_0,X_1)} < 1$. Since $Tx_i \in Y_0 \cap Y_1$ for every $1 \le i \le n$, it will be enough to show that

$$\Big\| \max_{1\le i\le n} |Tx_i| \Big\|_{\varphi(Y_0,Y_1)} \le \gamma \max\big\{\rho_{\infty,1}(T|_{X_0}), \rho_{\infty,1}(T|_{X_1})\big\} \tag{5.3}$$

for a certain constant $\gamma > 0$ independent of T and $(x_i)_{i=1}^n$.

Note that $\sum_{i=1}^n |x_i| \le \varphi(u_0, u_1)$ with $u_j \in X_j$ and $\|u_j\|_{X_j} \le 1$. Using the Riesz decomposition property and (5.2), we can write $x_i = f_i + g_i$ with $0 \le f_i, g_i \le x_i$ in $X_0 \cap X_1$ such that $f_i \le \phi(u_0, u_1)$ and $g_i \le \eta(u_0, u_1)$.

On the one hand, since $\phi(X_0, X_1)$ coincides, up to an equivalent norm, with X_0, X_1, or $X_0 + X_1$, using Lemma 3.2, it follows that

$$\big\| \max_{1\leq i\leq n} |Tf_i| \big\|_{\phi(Y_0,Y_1)} \leq \gamma_0 \max\big\{\rho_{\infty,1}(T|_{X_0}), \rho_{\infty,1}(T|_{X_1})\big\} \tag{5.4}$$

for a certain constant γ_0. On the other hand, since we can assume that φ, and hence η, is not doubly bounded, by Lemma 5.4 there exist $C_{\eta,\overline{X}} > 0$ and $v_0, v_1 \in X_0 \cap X_1$ with $\|v_j\|_{X_j} \leq C_{\eta,\overline{X}}$ such that

$$\sum_{i=1}^{n} |g_i| = \eta(v_0, v_1).$$

Hence, Lemma 3.3 applied to $(g_i)_{i=1}^n$, v_0, and v_1 provides for $1 \leq i \leq n$ sequences $(g_i^m)_{m\in\mathbb{N}}$ in $X_0 + X_1$ such that, for $m \in \mathbb{N}$, we have

$$\max_{1\leq i\leq n} |g_i - g_i^m| \leq (v_0 \vee v_1)a_m$$

for certain $a_m \in \mathbb{R}_+$ with $a_m \underset{m\to\infty}{\longrightarrow} 0$, and

$$\sup_m \big\| \max_{1\leq i\leq n} |Tg_i^m| \big\|_{\varphi(Y_0,Y_1)} \leq \gamma \max\big\{\rho_{\infty,1}(T|_{X_0}), \rho_{\infty,1}(T|_{X_1})\big\}.$$

Hence, since $v_0, v_1 \in X_0 \cap X_1$, for every $1 \leq i \leq n$, it holds that $g_i^m \to g_i$ in $X_0 \cap X_1$. In particular, it also holds that $Tg_i^m \to Tg_i$ in $Y_0 \cap Y_1$, which yields

$$\big\| \max_{1\leq i\leq n} |Tg_i| \big\|_{\eta(Y_0,Y_1)} \leq \gamma \max\big\{\rho_{\infty,1}(T|_{X_0}), \rho_{\infty,1}(T|_{X_1})\big\}. \tag{5.5}$$

Since $Tx_i = Tf_i + Tg_i$, this finishes the proof. □

The above result immediately yields the following.

Corollary 5.6. *If (X_0, X_1) and (Y_0, Y_1) are compatible pairs of quasi-Banach lattices such that Y_0 and Y_1 have the $K_{\infty,1}$ property, then for every $T : (X_0, X_1) \to (Y_0, Y_1)$ and every function $\varphi \in \mathcal{P}$, we have $T : \varphi^0(X_0, X_1) \to \varphi^0(Y_0, Y_1)$.*

Remark 5.7. If X_0 and X_1 are quasi-Banach lattices of measurable functions over a measure space and if for some constant $M > 0$ and vectors $(x_i)_{i=1}^n \subset X_j$ it holds that

$$\big\| \max_{1\leq i\leq n} |x_i| \big\|_{X_j} \leq M \max_{t\in[0,1]} \Big\| \sum_{i=1}^{n} r_i(t)x_i \Big\|_{X_j}, \tag{5.6}$$

where r_i denotes the ith Rademacher function, and the function $\varphi \in \mathcal{P}$ satisfies the condition that $\varphi(s, t) \to 0$ as $s \to 0$ or $t \to 0$, and $\varphi(s, t) \to \infty$ as $s \to \infty$ or $t \to \infty$, then [18, Theorem 2.1] asserts that $\varphi^0(X_0, X_1)$ coincides with the $\langle\cdot\rangle_\varphi$-method introduced by Peetre in [21]. Note that by Proposition 4.2, condition (5.6) implies the $K_{\infty,1}$ property of X_j. Hence, under these somewhat stronger assumptions, the interpolation result of Theorem 5.6 also follows from this fact.

Remark 5.8. We do not know whether the $K_{\infty,1}$ property in Corollaries 5.1 and 5.6 is actually necessary.

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