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A VARIANT OF THE HANKEL MULTIPLIER

SAIFALLAH GHOBBER[1,2]

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Abstract. The first aim of this article is to survey and revisit some uncertainty principles for the Hankel transform by means of the Hankel multiplier. Then we define the *wavelet* Hankel multiplier and study its boundedness and Schatten-class properties. Finally, we prove that the wavelet Hankel multiplier is unitary equivalent to a scalar multiple of the phase space restriction operator, for which we deduce a trace formula.

1. Introduction

Let $d \geq 1$ be the dimension, and let us denote by $\langle \cdot, \cdot \rangle$ the scalar product and by $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$. Then the Fourier transform is defined for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i\langle x, \xi \rangle} \frac{\mathrm{d}x}{(2\pi)^{d/2}},$$

and it is extended from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ in the usual way. With this normalization, if $f(x) = \tilde{f}(|x|)$ is a radial function on $\mathbb{R}^d$, then

$$\mathcal{F}(f)(\xi) = \mathcal{H}_{d/2-1}(\tilde{f})\big(|\xi|\big),$$

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where for $\alpha \geq -1/2$, $\mathcal{H}_\alpha$ is the Hankel transform (also known as the *Fourier–Bessel transform*) defined by

$$\mathcal{H}_\alpha(f)(\xi) = \int_0^\infty f(x) j_\alpha(x\xi)\, \mathrm{d}\mu_\alpha(x), \quad \xi \in \mathbb{R}_+ = (0, \infty).$$

Here $\mathrm{d}\mu_\alpha(x) = \frac{x^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)}\, \mathrm{d}x$ and j_α is the *spherical* Bessel function given by

$$j_\alpha(x) := \Gamma(\alpha+1) \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \Big(\frac{x}{2}\Big)^{2n}.$$

For $1 \leq p < \infty$, we denote by $L^p_\alpha(\mathbb{R}_+)$ the Banach space consisting of measurable functions f on $\mathbb{R}_+$ equipped with the norms

$$\|f\|_{p,\alpha} = \Big(\int_0^\infty \big|f(x)\big|^p \, \mathrm{d}\mu_\alpha(x)\Big)^{1/p}.$$

Note that $\mathcal{H}_{-1/2}$ is the usual Fourier cosine transform defined on $L^2_\alpha(\mathbb{R}_+)$, which is just the Fourier transform $\mathcal{F}$ restricted to even functions on $L^2(\mathbb{R})$. Thus throughout this article, α will be a real number such that $\alpha > -1/2$. The Hankel inversion formula gives us back the signal f via

$$f(x) = \int_0^\infty \mathcal{H}_\alpha(f)(\xi) j_\alpha(x\xi)\, \mathrm{d}\mu_\alpha(\xi), \quad x > 0.$$

This is the basis for pseudodifferential operators on $\mathbb{R}_+$. Indeed if σ is a suitable function on $\mathbb{R}_+$, then we define the pseudodifferential operator F_σ by

$$F_\sigma f(x) = \int_0^\infty \sigma(\xi) \mathcal{H}_\alpha(f)(\xi) j_\alpha(x\xi)\, \mathrm{d}\mu_\alpha(\xi).$$

Pseudodifferential operators F_σ are known as the *Hankel multipliers*. It is well known that F_σ is a bounded linear operator which has been used in quantization and time-frequency analysis. In the case where σ is identically equal to 1, $F_\sigma : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ is the identity in view of the Hankel inversion formula.

In this article, we survey and revisit some known results on the uncertainty principles. The first result is the following well-known Heisenberg uncertainty inequality for the Hankel transform.

Theorem 1.1. *For every $f \in L^2_\alpha(\mathbb{R}_+)$, we have*

$$\|xf\|_{2,\alpha} \big\|\xi \mathcal{H}_\alpha(f)\big\|_{2,\alpha} \geq (\alpha+1)\|f\|^2_{2,\alpha} \tag{1.1}$$

with equality if and only if $f(x) = ce^{-\mu x^2/2}$ for some $c \in \mathbb{C}$ and $\mu > 0$.

It is also well known that by using a dilation argument, the last inequality is equivalent to the sharp inequality

$$\|xf\|^2_{2,\alpha} + \big\|\xi \mathcal{H}_\alpha(f)\big\|^2_{2,\alpha} \geq (2\alpha+2)\|f\|^2_{2,\alpha} \tag{1.2}$$

with equality if and only if $f(x) = ce^{-x^2/2}$ for some $c \in \mathbb{C}$.

Thus, time and frequency energy concentrations are restricted by the Heisenberg uncertainty principle (1.2). This principle has a particularly important interpretation in quantum mechanics as an uncertainty regarding the position and momentum of a free particle. The Heisenberg inequality (1.2) was first proved by Bowie [3] and then by Rösler and Voit [19]. Moreover, in [8], we proved a stronger version that shows that Laguerre functions $\{\ell_n^\alpha\}_{n=0}^\infty$ are successive optimal on Heisenberg's uncertainty principle.

Theorem 1.2. *For every $f \in L^2_\alpha(\mathbb{R}_+)$ such that f is orthogonal to the sequence $\{\ell_k^\alpha\}_{k=0}^{n-1}$, we have*

$$\|xf\|^2_{2,\alpha} + \big\|\xi \mathcal{H}_\alpha(f)\big\|^2_{2,\alpha} \geq (4n + 2\alpha + 2)\|f\|^2_{2,\alpha} \tag{1.3}$$

with equality if and only if $f = c_n \ell_n^\alpha$ for some $c_n \in \mathbb{C}$.

The sequence of Laguerre functions $\{\ell_n^\alpha\}_{n=0}^\infty$ forms an orthonormal basis for $L^2_\alpha(\mathbb{R}_+)$, and each ℓ_n^α is an eigenfunction for the Hankel transform associated to the eigenvalue $(-1)^n$. More generally (see [7], [11]), we recall the following result.

Theorem 1.3. *Let $s, \beta > 0$.*

(1) *There exists a constant $c_{s,\alpha,\beta}$ such that, for all $f \in L^2_\alpha(\mathbb{R}_+)$,*

$$\|x^s f\|^\beta_{2,\alpha} \big\|\xi^\beta \mathcal{H}_\alpha(f)\big\|^s_{2,\alpha} \geq c_{s,\alpha,\beta}\|f\|^{s+\beta}_{2,\alpha}. \tag{1.4}$$

(2) *There exists a constant $c(s,\alpha,\beta)$ such that, for all $f \in L^1_\alpha(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$,*

$$\|x^s f\|^{\alpha+\beta+1}_{1,\alpha} \big\|\xi^\beta \mathcal{H}_\alpha(f)\big\|^{\alpha+s+1}_{2,\alpha} \geq c(s,\alpha,\beta)\|f\|^{\alpha+s+1}_{1,\alpha}\|f\|^{\alpha+\beta+1}_{2,\alpha}. \tag{1.5}$$

The proof of (1.5) can be obtained by combining a Nash-type inequality and a Carlson-type inequality. The proof of (1.4) is based on the orthogonal projection $F_\Sigma = \mathcal{H}_\alpha \chi_\Sigma \mathcal{H}_\alpha$, which is a special case of the Hankel multiplier $F_\sigma = \mathcal{H}_\alpha \sigma \mathcal{H}_\alpha$ defined on $L^2_\alpha(\mathbb{R}_+)$, where χ_Σ is the characteristic function on the subset $\Sigma \subset \mathbb{R}_+$. In Section 3, we deal with the Hankel multiplier F_Σ and its applications on the uncertainty principles, first on the subspace of ε_1-concentrated and ε_2-bandlimited signals in $L^2_\alpha(\mathbb{R}_+)$,

$$L^2_\alpha(\varepsilon_1,\varepsilon_2,S,\Sigma) = \big\{f \in L^2_\alpha(\mathbb{R}_+) : \|\chi_{S^c} f\|_{2,\alpha} \leq \varepsilon_1\|f\|_{2,\alpha}; \|F_{\Sigma^c} f\|_{2,\alpha} \leq \varepsilon_2\|f\|_{2,\alpha}\big\},$$

and then on the subspace of ε_1-time-limited and ε_2-band-limited signals in $L^1_\alpha(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$,

$$\begin{aligned} &L^1_\alpha \cap L^2_\alpha(\varepsilon_1,\varepsilon_2,S,\Sigma) \\ &\quad = \big\{f \in L^1_\alpha \cap L^2_\alpha(\mathbb{R}_+) : \|\chi_{S^c} f\|_{1,\alpha} \leq \varepsilon_1\|f\|_{2,\alpha}; \|F_{\Sigma^c} f\|_{2,\alpha} \leq \varepsilon_2\|f\|_{2,\alpha}\big\}, \end{aligned}$$

where $\varepsilon_1, \varepsilon_2 \in [0,1)$ and $\Omega^c = \mathbb{R}_+ \backslash \Omega$ is the complement of Ω in $\mathbb{R}_+$. In the case where $\varepsilon_1 = \varepsilon_2 = 0$, we have that S and Σ are the exact supports of f and $\mathcal{H}_\alpha(f)$, respectively. However, in the case where $\varepsilon_1, \varepsilon_2 \in (0,1)$, the subsets S and Σ are considered as *essential* supports of f and $\mathcal{H}_\alpha(f)$, respectively. It is well known that if a nonzero function f has support of finite measure $0 < \mu_\alpha(\operatorname{supp} f) < \infty$, then its Hankel transform has support of infinite measure (see [9]). That is why Donoho and Stark [5] replaced the exact support by the essential support. In this direction, we recall the following Donoho–Stark-type uncertainty inequality in the Hankel setting (see [6], [23]).

(1) Let $\varepsilon_1, \varepsilon_2 \in (0,1)$ such that $\varepsilon_1 + \varepsilon_2 < 1$. If $f \in L^2_\alpha(\varepsilon_1, \varepsilon_2, S, \Sigma)$, then

$$\mu_\alpha(S)\mu_\alpha(\Sigma) \geq (1 - \varepsilon_1 - \varepsilon_2)^2.$$

(2) If $f \in L^1_\alpha \cap L^2_\alpha(\varepsilon_1, \varepsilon_2, S, \Sigma)$, then

$$\mu_\alpha(S)\mu_\alpha(\Sigma) \geq (1 - \varepsilon_1)^2(1 - \varepsilon_2^2).$$

The second inequality improves the first one since $(1-\varepsilon_1)^2(1-\varepsilon_2^2) > (1-\varepsilon_1-\varepsilon_2)^2$. On the other hand, on the second inequality, we can obtain lower bounds of $\mu_\alpha(S)$ and $\mu_\alpha(\Sigma)$ separately, which give more information than the lower band of the product $\mu_\alpha(S)\mu_\alpha(\Sigma)$.

In Section 3, we use the local uncertainty principle (see [9], [17]) and the Nash and Carlson inequalities (see [7]) in the Hankel setting to obtain new Heisenberg-type inequalities for functions in $L^2_\alpha(\varepsilon_1, \varepsilon_2, S, \Sigma)$ or $L^1_\alpha \cap L^2_\alpha(\varepsilon_1, \varepsilon_2, S, \Sigma)$ with constants that depend on ε_1, ε_2, S, and Σ. More precisely, we prove the following theorem.

Theorem A. *Let $\varepsilon_1, \varepsilon_2 \in (0,1)$.*

(1) *Let $s, \beta > \alpha + 1$. Then there exists a constant $c_1(s, \alpha, \beta)$ such that for all $f \in L^2_\alpha(\varepsilon_1, \varepsilon_2, S, \Sigma)$,*

$$\|x^s f\|^\beta_{2,\alpha} \big\|\xi^\beta \mathcal{H}_\alpha(f)\big\|^s_{2,\alpha} \geq c_1(s, \alpha, \beta)\Big(\frac{(1-\varepsilon_1^2)(1-\varepsilon_2^2)}{\mu_\alpha(S)\mu_\alpha(\Sigma)}\Big)^{\frac{s\beta}{2\alpha+2}} \|f\|^{s+\beta}_{2,\alpha}. \tag{1.6}$$

(2) *Let $s, \beta > 0$. Then there exists a constant $c = c_2(s, \alpha, \beta)$ such that for all $f \in L^1_\alpha \cap L^2_\alpha(\varepsilon_1, \varepsilon_2, S, \Sigma)$,*

$$\|x^s f\|^{\alpha+\beta+1}_{1,\alpha} \big\|\xi^\beta \mathcal{H}_\alpha(f)\big\|^{\alpha+s+1}_{2,\alpha} \geq c\Big(\frac{(1-\varepsilon_1)^2(1-\varepsilon_2^2)}{\mu_\alpha(S)\mu_\alpha(\Sigma)}\Big)^{\frac{(\alpha+\beta+1)(\alpha+s+1)}{2\alpha+2}} \times \|f\|^{\alpha+s+1}_{1,\alpha}\|f\|^{\alpha+\beta+1}_{2,\alpha}. \tag{1.7}$$

Note that (1.6) holds also for $s, \beta \leq \alpha + 1$, but not necessarily with the same constant. Furthermore, from the last two inequalities one can easily deduce a lower bound of the product $\mu_\alpha(S)\mu_\alpha(\Sigma)$, with constants depending on the signal f, and this can be viewed as the ε-concentration version of the Donoho–Stark theorem (see [2]).

Now let ϕ and ψ be two bounded functions in $L^2_\alpha(\mathbb{R}_+)$ such that $\|\phi\|_{2,\alpha} = \|\psi\|_{2,\alpha}$. The aims of Section 4 are to make precise the definition of the pseudo-differential operator $\bar{\psi}F_\sigma\phi : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$, where σ is a symbol in $L^p_\alpha(\mathbb{R}_+)$, $1 \leq p \leq \infty$, and to prove that the resulting bounded linear operator is in the Schatten–von Neumann class S_p. More precisely, we use the Riesz–Thorin theorem to prove the following.

Theorem B. *Let $\sigma \in L^p_\alpha(\mathbb{R}_+)$, $1 \leq p \leq \infty$. Then the linear operator $\bar{\psi}F_\sigma\phi : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ is in S_p and*

$$\|\bar{\psi}F_\sigma\phi\|_{S_p} \leq \|\phi\|^{\frac{1}{p'}}_\infty \|\psi\|^{\frac{1}{p'}}_\infty \|\sigma\|_{p,\alpha}, \tag{1.8}$$

where p' is the conjugate index of p, and by convention $S_\infty = B(L^2_\alpha(\mathbb{R}_+))$ is the space of bounded operators from $L^2_\alpha(\mathbb{R}_+)$ into itself.

The bounded linear operator $\bar{\psi}F_\sigma\phi : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ can be considered as a variant of the localization operator corresponding to the symbol σ and the *admissible wavelets* ϕ and ψ studied by Wong [25]. Thus, it is reasonable to call the linear operator $\bar{\psi}F_\sigma\phi : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ a *wavelet Hankel multiplier*. Finally, for an appropriate choice of ϕ and σ, the wavelet Hankel multiplier $\bar{\phi}F_\sigma\phi : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ is unitary equivalent to a scalar multiple of the phase-space-limiting operator on $L^2_\alpha(\mathbb{R}_+)$ arising from the Landau–Pollak–Slepian theory in signal analysis (see the fundamental papers [15], [16], [20], [21]).

2. Preliminaries

2.1. **Generalities.** Let X be a separable and complex Hilbert space (of infinite dimension) in which the inner product and the norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let $\mathcal{A} : X \to X$ be a compact operator for which we denote by $\mathcal{A}^* : X \to X$ its adjoint. Then the linear operator $|\mathcal{A}| = \sqrt{\mathcal{A}^*\mathcal{A}} : X \to X$ is positive and compact. The singular values $\{e_n(\mathcal{A})\}_{n=1}^\infty$ of $\mathcal{A}$ are the eigenvalues of the self-adjoint operator $|\mathcal{A}|$. For $1 \le p < \infty$, the Schatten-class S_p is the space of all compact operators whose singular values lie in ℓ_p. In particular, S_2 is the space of Hilbert–Schmidt operators, and S_1 is the space of trace-class operators. Moreover, from [18, Section VI.6] and [25, Proposition 2.6], we have the following criterion for a bounded linear operator to be in the trace class.

Proposition 2.1. *Let $\mathcal{A} : X \to X$ be a bounded linear operator such that, for all orthonormal bases $\{\varphi_n\}_{n=1}^\infty$ for X,*

$$\sum_{n=1}^{\infty} \big|\langle \mathcal{A}\varphi_n, \varphi_n\rangle\big| < \infty. \tag{2.1}$$

Then $\mathcal{A} : X \to X$ is in the trace class S_1 with

$$\mathrm{tr}(\mathcal{A}) = \sum_{n=1}^{\infty} \langle \mathcal{A}\varphi_n, \varphi_n\rangle, \tag{2.2}$$

where $\{\varphi_n\}_{n=1}^\infty$ is any orthonormal basis for X.

If, in addition, $\mathcal{A}$ is positive, then (see [25, Proposition 2.7])

$$\|\mathcal{A}\|_{S_1} = \sum_{n=1}^{\infty} e_n(\mathcal{A}) = \mathrm{tr}(\mathcal{A}). \tag{2.3}$$

Moreover, from [25, Proposition 2.8], we have the following criterion for a bounded linear operator $\mathcal{A} : X \to X$ to be in the Hilbert–Schmidt class S_2.

Proposition 2.2. *Let $\mathcal{A} : X \to X$ be a bounded linear operator such that, for all orthonormal bases $\{\varphi_n\}_{n=1}^\infty$ for X,*

$$\sum_{n=1}^{\infty} \|\mathcal{A}\varphi_n\|^2 < \infty. \tag{2.4}$$

Then $\mathcal{A}: X \to X$ *is in the Hilbert–Schmidt class* S_2 *with*

$$\|\mathcal{A}\|_{S_2}^2 = \sum_{n=1}^{\infty} e_n(\mathcal{A})^2 = \sum_{n=1}^{\infty} \|\mathcal{A}\varphi_n\|^2, \tag{2.5}$$

where $\{\varphi_n\}_{n=1}^{\infty}$ *is any orthonormal basis for* X*.*

Finally, if the compact operator $\mathcal{A} : X \to X$ is Hilbert–Schmidt, then the positive operator $\mathcal{A}^*\mathcal{A}$ is in the space of trace class S_1 and

$$\|\mathcal{A}\|_{\mathrm{HS}}^2 := \|\mathcal{A}\|_{S_2}^2 = \|\mathcal{A}^*\mathcal{A}\|_{S_1} = \mathrm{tr}(\mathcal{A}^*\mathcal{A}) = \sum_{n=1}^{\infty} \|\mathcal{A}\varphi_n\|^2, \tag{2.6}$$

for any orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ for X.

For consistency, we define $S_\infty := B(X)$ to be the space of bounded operators from X into X, equipped with norm

$$\|\mathcal{A}\|_{S_\infty} = \sup_{f:\|f\|\leq 1} \|\mathcal{A}f\|. \tag{2.7}$$

It is obvious that $S_p \subseteq S_q$, $1 \leq p \leq q \leq \infty$.

2.2. **The Hankel transform.** For $\alpha > -1/2$, let us recall the *Poisson representation formula*:

$$j_\alpha(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} (1-s^2)^{\alpha-1/2} \cos(sx)\,\mathrm{d}x.$$

Therefore, j_α is bounded with $|j_\alpha(x)| \leq j_\alpha(0) = 1$. As a consequence, if $f \in L^1_\alpha(\mathbb{R}_+)$, then its Hankel transform is bounded and

$$\big\|\mathcal{H}_\alpha(f)\big\|_\infty \leq \|f\|_{1,\alpha}, \tag{2.8}$$

where $\|\cdot\|_\infty$ is the usual essential supremum norm, and $L^\infty(\mathbb{R}_+)$ will denote the usual space of essentially bounded functions.

It is also well known that the Hankel transform extends from $L^1_\alpha(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$ to an isometry on $L^2_\alpha(\mathbb{R}_+)$ with $\mathcal{H}_\alpha^{-1} = \mathcal{H}_\alpha$ and

$$\big\|\mathcal{H}_\alpha(f)\big\|_{2,\alpha} = \|f\|_{2,\alpha}. \tag{2.9}$$

Moreover, $\mathcal{H}_\alpha$ satisfies a Parseval-type relation

$$\big\langle \mathcal{H}_\alpha(f), \mathcal{H}_\alpha(g)\big\rangle_{\mu_\alpha} = \langle f, g\rangle_{\mu_\alpha}, \tag{2.10}$$

where $\langle \cdot, \cdot \rangle_{\mu_\alpha}$ is the inner product on the Hilbert space $L^2_\alpha(\mathbb{R}_+)$ defined by

$$\langle f, g\rangle_{\mu_\alpha} = \int_0^\infty f(x)\overline{g(x)}\,\mathrm{d}\mu_\alpha(x).$$

Furthermore, we will make use of a few formulas involving the functions j_α (see, e.g., [24, pp. 132–134]):

$$j'_\alpha(x) = -\frac{x}{2(\alpha+1)} j_{\alpha+1}(x) \tag{2.11}$$

and

$$\int_0^b j_\alpha(t)^2 t^{2\alpha+1}\,\mathrm{d}t = \frac{b^{2\alpha+2}}{2}\Big(j_\alpha'(b)^2 + \frac{2\alpha}{b} j_\alpha'(b) j_\alpha(b) + j_\alpha(b)^2\Big), \tag{2.12}$$

while for $u \neq v$, we have

$$\int_0^b j_\alpha(ut) j_\alpha(vt) t^{2\alpha+1}\,\mathrm{d}t = \frac{b^{2\alpha+1}}{u^2 - v^2}\big(v j_\alpha'(vb) j_\alpha(ub) - u j_\alpha'(ub) j_\alpha(vb)\big). \tag{2.13}$$

2.3. **Wavelet Hankel multipliers.** For $\sigma \in L^\infty(\mathbb{R}_+)$, we define the linear operator $F_\sigma : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ by

$$F_\sigma f = \ddot{H}_\alpha\big[\sigma \mathcal{H}_\alpha(f)\big]. \tag{2.14}$$

This operator is known as the *Hankel multiplier*, and if $\sigma = 1$, then $F_\sigma = I$, where I is the identity operator. Moreover, from Plancherel's formula (2.9), it is clear that F_σ is bounded with

$$\|F_\sigma\|_{S_\infty} \leq \|\sigma\|_\infty,$$

and from Parseval's formula (2.10), we obtain for all $\phi, \psi \in L^\infty(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$,

$$\langle \phi f, \psi g\rangle_{\mu_\alpha} = \big\langle \mathcal{H}_\alpha(\phi f), \mathcal{H}_\alpha(\psi g)\big\rangle_{\mu_\alpha}, \quad f, g \in L^2_\alpha(\mathbb{R}_+).$$

Definition 2.3. Let $\sigma \in L^1_\alpha(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+)$, and let $\phi, \psi \in L^\infty(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$ such that $\|\phi\|_{2,\alpha} = \|\psi\|_{2,\alpha} = 1$. We define the wavelet Hankel multiplier $P_{\sigma,\phi,\psi} : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ by

$$\langle P_{\sigma,\phi,\psi} f, g\rangle_{\mu_\alpha} = \big\langle \sigma \mathcal{H}_\alpha(\phi f), \mathcal{H}_\alpha(\psi g)\big\rangle_{\mu_\alpha}. \tag{2.15}$$

Then $P_{\sigma,\phi,\psi} : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ and $\bar{\psi} F_\sigma \phi : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ are unitary equivalent.

Proposition 2.4. *Let* $\sigma \in L^1_\alpha(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+)$, *and let* $\phi, \psi \in L^\infty(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$ *such that* $\|\phi\|_{2,\alpha} = \|\psi\|_{2,\alpha} = 1$. *Then*

$$\langle P_{\sigma,\phi,\psi} f, g\rangle_{\mu_\alpha} = \langle \bar{\psi} F_\sigma \phi, g\rangle_{\mu_\alpha}. \tag{2.16}$$

Proof. From (2.14) and Parseval's formula (2.10), we have

$$\begin{aligned}
\langle P_{\sigma,\phi,\psi} f, g\rangle_{\mu_\alpha} &= \big\langle \sigma \mathcal{H}_\alpha(\phi f), \mathcal{H}_\alpha(\psi g)\big\rangle_{\mu_\alpha} \\
&= \big\langle \mathcal{H}_\alpha\big(F_\sigma(\phi f)\big), \mathcal{H}_\alpha(\psi g)\big\rangle_{\mu_\alpha} \\
&= \big\langle F_\sigma(\phi f), \psi g\big\rangle_{\mu_\alpha} \\
&= \big\langle (\bar{\psi} F_\sigma \phi) f, g\big\rangle_{\mu_\alpha}.
\end{aligned}$$

The proof is complete. □

The linear operator $P_{\sigma,\phi,\psi}$ is a variant of a localization operator corresponding to the symbol σ and the admissible wavelets ϕ and ψ, which were studied first in [4] and later more extensively in [25]. If $\phi = \psi = 1$, then $P_{\sigma,\phi,\psi}$ will be a Hankel multiplier. As discussed in [25], the function ϕ (and ψ) occasionally plays the role of an admissible wavelet that satisfies the admissibility condition

$$c_\phi = \int_0^\infty \left|\left\langle \phi, j_\alpha(\cdot,\xi)\phi\right\rangle_{\mu_\alpha}\right|^2 \mathrm{d}\mu_\alpha(\xi) < \infty,$$

which, by Plancherel's formula, gives

$$c_\phi = \int_0^\infty \left|\mathcal{H}_\alpha\big(|\phi|^2\big)(\xi)\right|^2 \mathrm{d}\mu_\alpha(\xi) = \|\phi\|_{4,\alpha}^4.$$

Hence, if $\phi, \psi \in L^2_\alpha(\mathbb{R}_+) \cap L^4_\alpha(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ are two admissible wavelets such that $\|\phi\|_{2,\alpha} = \|\psi\|_{4,\alpha} = 1$, then (2.15) can be written as

$$\begin{aligned}\frac{1}{\sqrt{c_\phi c_\psi}}&\langle P_{\sigma,\phi,\psi} f, g\rangle_{\mu_\alpha} \\ &= \frac{1}{\|\phi\|_{4,\alpha}^2\|\psi\|_{4,\alpha}^2}\int_0^\infty \sigma(\xi)\left\langle f, \bar{\phi} j_\alpha(\cdot,\xi)\right\rangle_{\mu_\alpha}\overline{\left\langle g, \bar{\psi} j_\alpha(\cdot,\xi)\right\rangle_{\mu_\alpha}}\,\mathrm{d}\mu_\alpha(\xi). \quad (2.17)\end{aligned}$$

This is why we can refer to the localization operator type $P_{\sigma,\phi,\psi}$ as the *wavelet Hankel multiplier*. For the linear operators $P_{\sigma,\phi,\psi}$ studied in this article, we use functions ϕ and ψ in $L^2_\alpha(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, which are not necessarily admissible wavelets, but there is no problem in still calling $P_{\sigma,\phi,\psi}$ the wavelet Hankel multiplier.

Note that if $\sigma = \chi_\Omega$ is the characteristic function on the subset $\Omega \subset \mathbb{R}_+$, then we write F_σ as F_Ω, and if in addition $\phi = \psi$, we also write $P_{\sigma,\phi,\psi}$ as $P_{\Omega,\phi}$. The Hankel multiplier F_Ω is known as the *frequency-limiting operator* on $L^2_\alpha(\mathbb{R}_+)$ and we will prove in the last section that $P_{\Omega,\phi}$ can be viewed as the *phase space* (or *time-frequency*) *limiting operator*.

3. Uncertainty principles for the Hankel multiplier

First, it is easy to see that $F_\Sigma : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ is a self-adjoint projection. Now, let $\mathrm{PW}_\alpha(\Sigma)$ be its range; that is,

$$\mathrm{PW}_\alpha(\Sigma) = \mathrm{Im}(F_\Sigma) = \left\{f \in L^2_\alpha(\mathbb{R}_+) : \operatorname{supp}\mathcal{H}_\alpha(f) \subset \Sigma\right\},$$

which is the Paley–Wiener-type subspace of $L^2_\alpha(\mathbb{R}_+)$ consisting of band-limited functions in $L^2_\alpha(\mathbb{R}_+)$. Then $\mathrm{PW}_\alpha(\Sigma)$ is a reproducing kernel Hilbert space with kernel

$$k_\alpha(x,\xi) = \mathcal{H}_\alpha\big(\chi_\Sigma j_\alpha(x,\cdot)\big)(\xi) = \int_\Sigma j_\alpha(xt) j_\alpha(\xi t)\,\mathrm{d}\mu_\alpha(t);$$

that is, for all $f \in \mathrm{PW}_\alpha(\Sigma)$,

$$f(\xi) = F_\Sigma f(\xi) = \left\langle f, k_\alpha(\cdot,\xi)\right\rangle_{\mu_\alpha} = \int_0^\infty f(x) k_\alpha(x,\xi)\,\mathrm{d}\mu_\alpha(x).$$

Clearly, $k_\alpha(x,\xi) = k_\alpha(\xi, x)$, and if $\Sigma = [0, b]$, then $\mathrm{PW}_\alpha(\Sigma)$ is a space of entire functions of exponential type, and a straightforward computation using (2.11) and (2.13) shows that, for $x \neq \xi$,

$$k_\alpha(x,\xi) = \frac{b^{2\alpha+2}}{2^{\alpha+2}\Gamma(\alpha+2)} \frac{x^2 j_\alpha(b\xi) j_{\alpha+1}(bx) - \xi^2 j_\alpha(bx) j_{\alpha+1}(b\xi)}{x^2-\xi^2}. \tag{3.1}$$

Hence, given a measurable subset $S \subset \mathbb{R}_+$, we can define $F_\Sigma E_S$, the so-called *concentration operator* on S for functions of $\mathrm{PW}_\alpha(\Sigma)$, where E_S is the *time-limiting operator* on $L^1_\alpha(\mathbb{R}_+) \cup L^2_\alpha(\mathbb{R}_+)$ defined by

$$E_S f = \chi_S f.$$

Clearly, E_S is a self-adjoint projection on $L^2_\alpha(\mathbb{R}_+)$. Therefore, for all $f \in L^2_\alpha(\mathbb{R}_+)$,

$$F_\Sigma E_S f(\xi) = \big\langle f, k_\alpha(\cdot,\xi)\chi_S \big\rangle_{\mu_\alpha} = \int_S f(x) k_\alpha(x,\xi)\,\mathrm{d}\mu_\alpha(x).$$

Thus, $F_\Sigma E_S$ is a Hilbert–Schmidt operator with norm

$$\|F_\Sigma E_S\|^2_{\mathrm{HS}} = \int_0^\infty \int_0^\infty \chi_S(x) \big|k_\alpha(x,\xi)\big|^2 \,\mathrm{d}\mu_\alpha(x)\,\mathrm{d}\mu_\alpha(\xi).$$

Consequently,

$$\|F_\Sigma E_S\|_{\mathrm{HS}} \le \sqrt{\mu_\alpha(S)\mu_\alpha(\Sigma)}. \tag{3.2}$$

In particular,

$$\|E_S F_\Sigma\|_{S_\infty} = \|F_\Sigma E_S\|_{S_\infty} = \sup_{f \in \mathrm{PW}_\alpha(\Sigma)} \frac{\|E_S f\|_{2,\alpha}}{\|f\|_{2,\alpha}} \le \sqrt{\mu_\alpha(S)\mu_\alpha(\Sigma)}.$$

But, since E_S and F_Σ are two orthogonal projections, then $\|E_S F_\Sigma\|_{S_\infty} \le 1$, where $S_\infty = B(L^2_\alpha(\mathbb{R}_+))$. The condition

$$\|E_S F_\Sigma\|_{S_\infty} < 1 \tag{3.3}$$

ensures that

$$\mathrm{Im}(E_S) \cap \mathrm{Im}(F_\Sigma) = \{0\},$$

or, equivalently, for all $f \in L^2_\alpha(\mathbb{R}_+)$ (see [9]),

$$\|f\|^2_{2,\alpha} \le \big(1 - \|E_S F_\Sigma\|_{S_\infty}\big)^{-2} \big(\|E_{S^c} f\|^2_{2,\alpha} + \|F_{\Sigma^c} f\|^2_{2,\alpha}\big).$$

This means that f and $\mathcal{H}_\alpha(f)$ cannot be simultaneously supported on the subsets S and Σ, respectively. In this case, the pair (S,Σ) is called *strongly annihilating*. It is of critical importance to be able to estimate as accurately as possible the quantity $\|E_S F_\Sigma\|_{S_\infty}$ which controls both the invertibility of $I - E_S F_\Sigma$ and the annihilating constant $C(S,\Sigma) = (1 - \|E_S F_\Sigma\|_{S_\infty})^{-2}$. Unfortunately, it is not easy to find a pair of subsets that is strongly annihilating (see [13] for more discussion and history) and to give a good estimation of $\|E_S F_\Sigma\|_{S_\infty}$. For example, the author and Jöricke proved in [10] that any pair of sets of finite measure or (ε,α)-thin are strongly annihilating. Moreover, if Ω is relatively dense, then the pair $(\Omega^c, [0,b])$ is strongly annihilating. More generally, it is very interesting to find orthogonal projections $\mathcal{P}$ and $\mathcal{Q}$ that satisfy $\|\mathcal{P}\mathcal{Q}\|_{S_\infty} < 1$; these can be useful when applied to the problem of stable signal recovery (see, e.g., [5], [12]).

3.1. **Uncertainty principle on $L^2_\alpha(\mathbb{R}_+)$.** In [5], Donoho and Stark replaced the exact support by the *essential support*, which can be measured as follows.

Definition 3.1. Let $0 \le \varepsilon < 1$, and let f be a nonzero function in $L^2_\alpha(\mathbb{R}_+)$. Then we say that

(1) f is *ε-concentrated* on S if $\|E_{S^c} f\|_{2,\alpha} \le \varepsilon \|f\|_{2,\alpha}$, and
(2) f is *ε-band-limited* on Σ if $\|F_{\Sigma^c} f\|_{2,\alpha} \le \varepsilon \|f\|_{2,\alpha}$.

It is clear that if f is ε-band-limited on Σ, then, by Plancherel's theorem (2.9), its Hankel transform $\mathcal{H}_\alpha(f)$ is ε-concentrated on Σ. If $\varepsilon = 0$, then S and Σ are, respectively, the exact supports of f and $\mathcal{H}_\alpha(f)$; moreover, when $\varepsilon \in (0,1)$, S and Σ may be considered as the essential supports of f and $\mathcal{H}_\alpha(f)$, respectively. In this way, Donoho and Stark obtained a quantitative version of the uncertainty principle about the essential supports (see also [2]). Its counterpart in the Hankel setting was obtained in [23].

Theorem 3.2. *Let $0 \le \varepsilon_1, \varepsilon_2 < 1$ such that $\varepsilon_1 + \varepsilon_2 < 1$. Then if a nonzero function $f \in L^2_\alpha(\mathbb{R}_+)$ is ε_1-concentrated on S and ε_2-band-limited on Σ, then we have*

$$\mu_\alpha(S)\mu_\alpha(\Sigma) \ge (1 - \varepsilon_1 - \varepsilon_2)^2. \tag{3.4}$$

This means that the *essential support* of f and $\mathcal{H}_\alpha(f)$ cannot be too small. Moreover, we recall the following local uncertainty principle (see [9], [17]).

Theorem 3.3.

(1) *If $s > \alpha + 1$, then there exists a constant $c_1(s,\alpha)$ such that for every $f \in L^2_\alpha(\mathbb{R}_+)$ and every subset $\Sigma \subset \mathbb{R}_+$ of finite measure $\mu_\alpha(\Sigma) < \infty$,*

$$\|F_\Sigma f\|^2_{2,\alpha} < c_1(s,\alpha)\mu_\alpha(\Sigma)\|f\|^{2-\frac{2\alpha+2}{s}}_{2,\alpha}\|x^s f\|^{\frac{2\alpha+2}{s}}_{2,\alpha}. \tag{3.5}$$

Moreover, the constant

$$c_1(s,\alpha) = \frac{\Gamma(\frac{\alpha+1}{s})\Gamma(1-\frac{\alpha+1}{s})(s-\alpha-1)^{\frac{\alpha+1}{s}-1}}{2^{\alpha+1}(\alpha+1)^{\frac{\alpha+1}{s}}\Gamma(\alpha+1)}$$

is optimal, and equality in (3.5) is never attained.

(2) *If $0 < s < \alpha + 1$, there exists a constant $c_2(s,\alpha)$ such that for every $f \in L^2_\alpha(\mathbb{R}_+)$ and every subset $\Sigma \subset \mathbb{R}_+$ of finite measure $\mu_\alpha(\Sigma) < \infty$,*

$$\|F_\Sigma f\|^2_{2,\alpha} < c_2(s,\alpha)\big[\mu_\alpha(\Sigma)\big]^{\frac{s}{\alpha+1}}\|x^s f\|^2_{2,\alpha}, \tag{3.6}$$

where

$$c_2(s,\alpha) = \Big(\frac{\alpha+1}{\alpha+1-s}\Big)^2\Big(\frac{\alpha+1-s}{s^2 2^{\alpha+1}\Gamma(\alpha+1)}\Big)^{\frac{s}{\alpha+1}}$$

and equality in (3.6) is never attained.

(3) *If $s = \alpha+1$, then there exists a constant c_α such that for every $f \in L^2_\alpha(\mathbb{R}_+)$ and every subset $\Sigma \subset \mathbb{R}_+$ of finite measure $\mu_\alpha(\Sigma) < \infty$,*

$$\|F_\Sigma f\|^2_{2,\alpha} < c_\alpha \mu_\alpha(\Sigma)^{\frac{1}{2\alpha+2}}\|f\|^{2-\frac{1}{\alpha+1}}_{2,\alpha}\|x^{\alpha+1} f\|^{\frac{1}{\alpha+1}}_{2,\alpha}, \tag{3.7}$$

where $c_\alpha = 2(\alpha+1)(2\alpha+1)^{\frac{1}{2(\alpha+1)}-1}c_2(1/2,\alpha)$.

As an immediate consequence, we obtain the following result, which compares the measure of the support of $\mathcal{H}_\alpha(f)$ and the generalized dispersion of f.

Corollary 3.4. *For all $s > 0$ and all $f \in \mathrm{PW}_\alpha(\Sigma)$,*

$$\mu_\alpha\big(\operatorname{supp}\mathcal{H}_\alpha(f)\big)\|x^s f\|_{2,\alpha}^{\frac{2\alpha+2}{s}} > c(s,\alpha)\|f\|_{2,\alpha}^{\frac{2\alpha+2}{s}}, \tag{3.8}$$

where $c(s,\alpha) = \min\big(\frac{1}{c_\alpha^{2\alpha+2}}, \frac{1}{c_1(s,\alpha)}, \frac{1}{c_2(s,\alpha)^{\frac{\alpha+1}{s}}}\big)$.

By interchanging the roles of f and $\mathcal{H}_\alpha(f)$, and by replacing Σ by S and s by β in (3.5) and (3.6), we obtain the following result.

Theorem 3.5.

(1) *If $\beta > \alpha + 1$, then for all $f \in L^2_\alpha(\mathbb{R}_+)$,*

$$\|E_S f\|_{2,\alpha}^2 \le c_1(\beta,\alpha)\mu_\alpha(S)\|f\|_{2,\alpha}^{2-\frac{2\alpha+2}{\beta}}\big\|\xi^\beta\mathcal{H}_\alpha(f)\big\|_{2,\alpha}^{\frac{2\alpha+2}{\beta}}. \tag{3.9}$$

(2) *If $\beta < \alpha + 1$, then for all $f \in L^2_\alpha(\mathbb{R}_+)$,*

$$\|E_S f\|_{2,\alpha}^2 \le c_2(\beta,\alpha)\big[\mu_\alpha(S)\big]^{\frac{\beta}{\alpha+1}}\big\|\xi^\beta\mathcal{H}_\alpha(f)\big\|_{2,\alpha}^2. \tag{3.10}$$

(3) *If $\beta = \alpha + 1$, then for all $f \in L^2_\alpha(\mathbb{R}_+)$,*

$$\|E_S f\|_{2,\alpha}^2 < c_\alpha\mu_\alpha(S)^{\frac{1}{2\alpha+2}}\|f\|_{2,\alpha}^{2-\frac{1}{\alpha+1}}\big\|\xi^{\alpha+1}\mathcal{H}_\alpha(f)\big\|_{2,\alpha}^{\frac{1}{\alpha+1}}. \tag{3.11}$$

(4) *For all $\beta > 0$ and all $f \in \mathrm{Im}(E_S) = \{f \in L^2_\alpha(\mathbb{R}_+) : \operatorname{supp} f \subset S\}$,*

$$\mu_\alpha(\operatorname{supp} f)\big\|\xi^\beta\mathcal{H}_\alpha(f)\big\|_{2,\alpha}^{\frac{2\alpha+2}{\beta}} > c(\beta,\alpha)\|f\|_{2,\alpha}^{\frac{2\alpha+2}{\beta}}. \tag{3.12}$$

Clearly, from (3.3), the left-hand sides of (3.8) and (3.12) cannot be finite together, except for $f = 0$, because a nonzero function f and its Hankel transform $\mathcal{H}_\alpha(f)$ cannot simultaneously have support of finite measure.

Let S, Σ be two measurable subsets of $\mathbb{R}_+$ such that $0 < \mu_\alpha(S), \mu_\alpha(\Sigma) < \infty$, and let $L^2_\alpha(\varepsilon_1, \varepsilon_2, S, \Sigma)$ be the subspace of $L^2_\alpha(\mathbb{R}_+)$ consisting of all nonzero functions that are ε_1-concentrated on S and ε_2-band-limited on Σ. Now we can formulate our new version of the Heisenberg-type uncertainty principle for functions in $L^2_\alpha(\varepsilon_1, \varepsilon_2, S, \Sigma)$ with constant depending on $\varepsilon_1, \varepsilon_2$ and S, Σ.

Theorem 3.6. *Let $\varepsilon_1, \varepsilon_2 \in (0,1)$, and let $f \in L^2_\alpha(\varepsilon_1, \varepsilon_2, S, \Sigma)$.*

(1) *If $s, \beta > \alpha + 1$, then*

$$\|x^s f\|_{2,\alpha} \ge \Big(\frac{1-\varepsilon_2^2}{c_1(s,\alpha)\mu_\alpha(\Sigma)}\Big)^{\frac{s}{2\alpha+2}}\|f\|_{2,\alpha} \tag{3.13}$$

and

$$\big\|\xi^\beta\mathcal{H}_\alpha(f)\big\|_{2,\alpha} \ge \Big(\frac{1-\varepsilon_1^2}{c_1(\beta,\alpha)^2\mu_\alpha(S)}\Big)^{\frac{\beta}{2\alpha+2}}\|f\|_{2,\alpha}. \tag{3.14}$$

(2) *If* $0 < s, \beta < \alpha + 1$, *then*

$$\|x^s f\|_{2,\alpha} \geq \frac{\sqrt{1-\varepsilon_2^2}}{\sqrt{c_2(s,\alpha)}\mu_\alpha(\Sigma)^{\frac{s}{2\alpha+2}}}\|f\|_{2,\alpha} \tag{3.15}$$

and

$$\left\|\xi^\beta \mathcal{H}_\alpha(f)\right\|_{2,\alpha} \geq \frac{\sqrt{1-\varepsilon_1^2}}{\sqrt{c_2(\beta,\alpha)}\mu_\alpha(S)^{\frac{\beta}{2\alpha+2}}}\|f\|_{2,\alpha}. \tag{3.16}$$

(3) *If* $s = \beta = \alpha + 1$, *then*

$$\|x^{\alpha+1} f\|_{2,\alpha} \geq \frac{(1-\varepsilon_2^2)^{\alpha+1}}{c_\alpha^{\alpha+1}\sqrt{\mu_\alpha(\Sigma)}}\|f\|_{2,\alpha} \tag{3.17}$$

and

$$\left\|\xi^{\alpha+1} \mathcal{H}_\alpha(f)\right\|_{2,\alpha} \geq \frac{(1-\varepsilon_1^2)^{\alpha+1}}{c_\alpha^{\alpha+1}\sqrt{\mu_\alpha(S)}}\|f\|_{2,\alpha}. \tag{3.18}$$

Proof. From (3.5) and (3.9), we have

$$\left\|\xi^\beta \mathcal{H}_\alpha(f)\right\|_{2,\alpha}^{\frac{2\alpha+2}{\beta}} \geq \frac{\|E_S f\|_{2,\alpha}^2}{c_1(\beta,\alpha)\|f\|_{2,\alpha}^{2-\frac{2\alpha+2}{\beta}}\mu_\alpha(S)}$$

and

$$\|x^s f\|_{2,\alpha}^{\frac{2\alpha+2}{s}} \geq \frac{\|F_\Sigma f\|_{2,\alpha}^2}{c_1(s,\alpha)\|f\|_{2,\alpha}^{2-\frac{2\alpha+2}{s}}\mu_\alpha(\Sigma)}.$$

Now, since $f \in L_\alpha^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$, then

$$\|F_\Sigma f\|_{2,\alpha}^2 = \|f\|_{2,\alpha}^2 - \|F_{\Sigma^c} f\|_{2,\alpha}^2 \geq (1-\varepsilon_2^2)\|f\|_{2,\alpha}^2$$

and

$$\|E_S f\|_{2,\alpha}^2 = \|f\|_{2,\alpha}^2 - \|E_{S^c} f\|_{2,\alpha}^2 \geq (1-\varepsilon_1^2)\|f\|_{2,\alpha}^2.$$

This proves the first result. Analogously, we obtain the second and third results. □

The last theorem gives lower bounds for the measures of the two dispersions $\|x^s f\|_{2,\alpha}$ and $\|\xi^s \mathcal{H}_\alpha(f)\|_{2,\alpha}$ separately. This gives more information than a lower bound of the product between them.

Corollary 3.7. *Let* $\varepsilon_1, \varepsilon_2 \in (0,1)$. *Then for all* $f \in L_\alpha^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$, *we have the following.*

(1) *If* $s, \beta > \alpha + 1$, *then*

$$\|x^s f\|_{2,\alpha}^\beta \left\|\xi^\beta \mathcal{H}_\alpha(f)\right\|_{2,\alpha}^s \geq \Big(\frac{(1-\varepsilon_1^2)(1-\varepsilon_2^2)}{c_1(s,\alpha)c_1(\beta,\alpha)\mu_\alpha(S)\mu_\alpha(\Sigma)}\Big)^{\frac{s\beta}{2\alpha+2}}\|f\|_{2,\alpha}^{s+\beta}. \tag{3.19}$$

(2) *If* $0 < s, \beta < \alpha + 1$, *then*

$$\|x^s f\|_{2,\alpha}^\beta \left\|\xi^\beta \mathcal{H}_\alpha(f)\right\|_{2,\alpha}^s \geq \frac{(1-\varepsilon_1^2)^{s/2}(1-\varepsilon_2^2)^{\beta/2}}{c_2(s,\alpha)^{\beta/2}c_2(\beta,\alpha)^{s/2}(\mu_\alpha(S)\mu_\alpha(\Sigma))^{\frac{s\beta}{2\alpha+2}}}\|f\|_{2,\alpha}^{s+\beta}. \tag{3.20}$$

(3) *If $s=\beta=\alpha+1$, then*

$$\|x^{\alpha+1}f\|_{2,\alpha}\big\|\xi^{\alpha+1}\mathcal{H}_\alpha(f)\big\|_{2,\alpha}\geq\frac{((1-\varepsilon_1^2)(1-\varepsilon_2^2))^{\alpha+1}}{c_\alpha^{2\alpha+2}\sqrt{\mu_\alpha(S)\mu_\alpha(\Sigma)}}\|f\|_{2,\alpha}^2. \tag{3.21}$$

Remark 3.8. For simplicity, suppose that $s=\beta$. First, we remark that

$$(1-\varepsilon_1^2)(1-\varepsilon_2^2)=1-(\varepsilon_1-\varepsilon_2)^2+(\varepsilon_1\varepsilon_2)^2+2\varepsilon_1\varepsilon_2>(1-\varepsilon_1-\varepsilon_2)^2. \tag{3.22}$$

Since the constant in Heisenberg's inequality (1.1) is optimal for all functions in $L_\alpha^2(\mathbb{R}_+)$, then Corollary 3.7 is of interest if the constants in (3.19), (3.20), and (3.21) exceed $\alpha+1$ in (1.1). This implies that

$$\mu_\alpha(S)\mu_\alpha(\Sigma)\leq\begin{cases}\frac{(1-\varepsilon_1^2)(1-\varepsilon_2^2)}{(\alpha+1)^{\frac{2\alpha+2}{s}}c_1(\alpha,s)^2} & s>\alpha+1,\\ \left(\frac{(1-\varepsilon_1^2)(1-\varepsilon_2^2)}{(\alpha+1)^2c_2(\alpha,s)^2}\right)^{\frac{\alpha+1}{s}} & s<\alpha+1,\\ \frac{((1-\varepsilon_1^2)(1-\varepsilon_2^2))^{2\alpha+2}}{(\alpha+1)^2c_\alpha^{4\alpha+4}} & s=\alpha+1.\end{cases} \tag{3.23}$$

This can be possible for some s, α, $\varepsilon_1,\varepsilon_2$, and S,Σ since from (3.4),

$$(1-\varepsilon_1-\varepsilon_2)^2\leq\mu_\alpha(S)\mu_\alpha(\Sigma).$$

From Theorems 3.3 and 3.5, we can also deduce lower bounds for the measures of S and Σ separately.

Theorem 3.9. *Let $\varepsilon_1,\varepsilon_2\in(0,1)$, and let $f\in L_\alpha^2(\varepsilon_1,\varepsilon_2,S,\Sigma)$. Then we have the following.*

(1) *If $s,\beta>\alpha+1$, then*

$$\mu_\alpha(S)\geq\left(\frac{\|f\|_{2,\alpha}}{\|\xi^\beta\mathcal{H}_\alpha(f)\|_{2,\alpha}}\right)^{\frac{2\alpha+2}{\beta}}\frac{1-\varepsilon_1^2}{c_1(\beta,\alpha)} \tag{3.24}$$

and

$$\mu_\alpha(\Sigma)\geq\left(\frac{\|f\|_{2,\alpha}}{\|x^sf\|_{2,\alpha}}\right)^{\frac{2\alpha+2}{s}}\frac{1-\varepsilon_2^2}{c_1(s,\alpha)}. \tag{3.25}$$

(2) *If $0<s,\beta<\alpha+1$, then*

$$\mu_\alpha(S)\geq\left(\frac{\|f\|_{2,\alpha}}{\|\xi^\beta\mathcal{H}_\alpha(f)\|_{2,\alpha}}\sqrt{\frac{1-\varepsilon_1^2}{c_2(\beta,\alpha)}}\right)^{\frac{2\alpha+2}{\beta}} \tag{3.26}$$

and

$$\mu_\alpha(\Sigma)\geq\left(\frac{\|f\|_{2,\alpha}}{\|x^sf\|_{2,\alpha}}\sqrt{\frac{1-\varepsilon_2^2}{c_2(s,\alpha)}}\right)^{\frac{2\alpha+2}{s}}. \tag{3.27}$$

(3) *If $s=\beta=\alpha+1$, then*

$$\mu_\alpha(S)\geq\frac{\|f\|_{2,\alpha}^2}{\|\xi^{\alpha+1}\mathcal{H}_\alpha(f)\|_{2,\alpha}^2}\frac{(1-\varepsilon_1^2)^{2\alpha+2}}{c_\alpha^{2\alpha+2}} \tag{3.28}$$

and

$$\mu_\alpha(\Sigma) \geq \frac{\|f\|_{2,\alpha}^2}{\|x^{\alpha+1}f\|_{2,\alpha}^2} \frac{(1-\varepsilon_2^2)^{2\alpha+2}}{c_\alpha^{2\alpha+2}}. \tag{3.29}$$

These lower bounds give more information than the lower bound of the following ε-concentration version of the Donoho–Stark-type uncertainty principle (3.4), with a new constant depending on the signal f.

Corollary 3.10. *Let $\varepsilon_1, \varepsilon_2 \in (0,1)$. If $f \in L_\alpha^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$, then*

$$\mu_\alpha(S)\mu_\alpha(\Sigma) \geq \begin{cases} \frac{C_f(s,\alpha,\beta)}{c_1(s,\alpha)c_1(\beta,\alpha)}(1-\varepsilon_1^2)(1-\varepsilon_2^2) & s,\beta > \alpha+1, \\ \frac{C_f(s,\alpha,\beta)}{(c_2(\beta,\alpha)^{\frac{1}{\beta}} c_2(s,\alpha)^{\frac{1}{s}})^{\alpha+1}}((1-\varepsilon_1^2)^{\frac{1}{\beta}}(1-\varepsilon_2^2)^{\frac{1}{s}})^{\alpha+1} & 0 < s,\beta < \alpha+1, \\ \frac{C_f(\alpha+1,\alpha,\alpha+1)}{c_\alpha^{4\alpha+4}}((1-\varepsilon_1^2)(1-\varepsilon_2^2))^{2\alpha+2} & s = \beta = \alpha+1, \end{cases}$$

where

$$C_f(s,\alpha,\beta) = \Big(\frac{\|f\|_{2,\alpha}^{s+\beta}}{\|x^s f\|_{2,\alpha}^{\beta} \|\xi^\beta \mathcal{H}_\alpha(f)\|_{2,\alpha}^{s}}\Big)^{\frac{2\alpha+2}{s\beta}}.$$

3.2. **Uncertainty principle on $L_\alpha^1(\mathbb{R}_+) \cap L_\alpha^2(\mathbb{R}_+)$.** In this section, a function $f \in L_\alpha^1(\mathbb{R}_+) \cap L_\alpha^2(\mathbb{R}_+)$ is ε-time-limited on S if

$$\|E_{S^c} f\|_{1,\alpha} \leq \varepsilon \|f\|_{1,\alpha}$$

and is ε-band-limited on Σ if

$$\|F_{\Sigma^c} f\|_{2,\alpha} \leq \varepsilon \|f\|_{2,\alpha}.$$

From [6] and [7], we recall the following results.

Theorem 3.11. *Let $s, \beta > 0$. Then we have the following.*

(1) A Carlson-type inequality*: there exists a constant $C_1(\alpha, s)$ such that for all $f \in L_\alpha^1(\mathbb{R}_+) \cap L_\alpha^2(\mathbb{R}_+)$,*

$$\|f\|_{1,\alpha}^{1+\frac{s}{\alpha+1}} \leq C_1(\alpha,s) \|f\|_{2,\alpha}^{\frac{s}{\alpha+1}} \|x^s f\|_{1,\alpha}. \tag{3.30}$$

(2) A Nash-type inequality*: there exists a constant $C_2(\alpha,\beta)$ such that for all $f \in L_\alpha^1(\mathbb{R}_+) \cap L_\alpha^2(\mathbb{R}_+)$,*

$$\|f\|_{2,\alpha}^{1+\frac{\beta}{\alpha+1}} \leq C_2(\alpha,\beta) \|f\|_{1,\alpha}^{\frac{\beta}{\alpha+1}} \big\|\xi^\beta \mathcal{H}_\alpha(f)\big\|_{2,\alpha}. \tag{3.31}$$

In the last theorem, the constants $C_1(\alpha, s)$ and $C_2(\alpha,\beta)$ can be computed (see [7]), but they are not optimal, which is why we omit the computations. Combining the Nash-type inequality (3.31) and the Carlson-type inequality (3.30), we obtain a variation on the Heisenberg uncertainty inequality.

Corollary 3.12. *Let $s, \beta > 0$. Then there exists a constant $C = C(\alpha, \beta, s)$ such that for all $f \in L_\alpha^1(\mathbb{R}_+) \cap L_\alpha^2(\mathbb{R}_+)$,*

$$\|x^s f\|_{1,\alpha}^{\alpha+\beta+1} \big\|\xi^\beta \mathcal{H}_\alpha(f)\big\|_{2,\alpha}^{\alpha+s+1} \geq C \|f\|_{1,\alpha}^{\alpha+s+1} \|f\|_{2,\alpha}^{\alpha+\beta+1}, \tag{3.32}$$

where

$$C = C_1(\alpha,s)^{-\alpha-\beta-1} C_2(\alpha,\beta)^{-\alpha-s-1}.$$

In particular,

$$\|x^s f\|_{1,\alpha}\big\|\xi^s \mathcal{H}_\alpha(f)\big\|_{2,\alpha} \geq C(\alpha,s)\|f\|_{1,\alpha}\|f\|_{2,\alpha}. \tag{3.33}$$

The advantage of Heisenberg-type inequality (3.32) compared to (1.4) is that in (3.32), we can from (3.30) and (3.31) estimate separately the time and frequency dispersions $\|x^s f\|_{1,\alpha}$, $\|\xi^s \mathcal{H}_\alpha(f)\|_{2,\alpha}$ around zero.

Moreover, (3.31) and (3.30) imply the following variation on the local uncertainty principle.

Theorem 3.13. *Let $s, \beta > 0$. Then*

(1) *there exists a constant $\tilde{C}_1(\alpha,s)$ such that for all $f \in L^1_\alpha(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$ and all measurable subsets Σ of finite measure,*

$$\|F_\Sigma f\|^2_{2,\alpha} \leq \tilde{C}_1(\alpha,s)\mu_\alpha(\Sigma)\|f\|_{2,\alpha}^{\frac{2s}{\alpha+s+1}}\|x^s f\|_{1,\alpha}^{\frac{2\alpha+2}{\alpha+s+1}}, \tag{3.34}$$

where

$$\tilde{C}_1(\alpha,s) = C_1(\alpha,s)^{\frac{2\alpha+2}{\alpha+s+1}};$$

(2) *there exists a constant $\tilde{C}_2(\alpha,\beta)$ such that for all $f \in L^1_\alpha(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$ and all measurable subsets S of finite measure,*

$$\|E_S f\|^2_{1,\alpha} \leq \tilde{C}_2(\alpha,\beta)\mu_\alpha(S)\|f\|_{1,\alpha}^{\frac{2\beta}{\alpha+\beta+1}}\big\|\xi^\beta \mathcal{H}_\alpha(f)\big\|_{2,\alpha}^{\frac{2\alpha+2}{\alpha+\beta+1}}, \tag{3.35}$$

where

$$\tilde{C}_2(\alpha,\beta) = C_2(\alpha,\beta)^{\frac{2\alpha+2}{\alpha+\beta+1}}.$$

Proof. By Plancherel's formula (2.9) and (2.8),

$$\|F_\Sigma f\|^2_{2,\alpha} = \big\|\chi_\Sigma \mathcal{H}_\alpha(f)\big\|^2_{2,\alpha} \leq \mu_\alpha(\Sigma)\big\|\mathcal{H}_\alpha(f)\big\|^2_\infty \leq \mu_\alpha(\Sigma)\|f\|^2_{1,\alpha}.$$

Then the first result follows from the Carlson-type inequality (3.30). Now by the Cauchy–Schwarz inequality, we have

$$\|E_S f\|^2_{1,\alpha} \leq \mu_\alpha(S)\|f\|^2_{2,\alpha},$$

and by the Nash-type inequality (3.31) we deduce the second result. □

Corollary 3.14. *Let $s, \beta > 0$. Then*

(1) *for all $f \in L^1_\alpha(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$ such that* $\operatorname{supp}\mathcal{H}_\alpha(f) \subset \Sigma$,

$$\mu_\alpha\big(\operatorname{supp}\mathcal{H}_\alpha(f)\big)\|x^s f\|_{1,\alpha}^{\frac{2\alpha+2}{\alpha+s+1}} \geq \tilde{C}_1(\alpha,s)^{-1}\|f\|_{2,\alpha}^{\frac{2\alpha+2}{\alpha+s+1}}; \tag{3.36}$$

(2) *for all $f \in L^1_\alpha(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$ such that* $\operatorname{supp} f \subset S$,

$$\mu_\alpha(\operatorname{supp} f)\big\|\xi^\beta \mathcal{H}_\alpha(f)\big\|_{2,\alpha}^{\frac{2\alpha+2}{\alpha+\beta+1}} \geq \tilde{C}_2(\alpha,\beta)^{-1}\|f\|_{1,\alpha}^{\frac{2\alpha+2}{\alpha+\beta+1}}. \tag{3.37}$$

Now let $L^1_\alpha \cap L^2_\alpha(\varepsilon_1,\varepsilon_2,S,\Sigma)$ be the set of all functions in $L^1_\alpha(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$ that are ε_1-time-limited on S and ε_2-band-limited on Σ, where $0 < \mu_\alpha(S), \mu_\alpha(\Sigma) < \infty$. Then from [6, Proposition 3.5], if $f \in L^1_\alpha \cap L^2_\alpha(\varepsilon_1,\varepsilon_2,S,\Sigma)$, then we have the following Donoho–Stark-type uncertainty inequality:

$$\mu_\alpha(S)\mu_\alpha(\Sigma) \geq (1-\varepsilon_1)^2(1-\varepsilon_2^2). \tag{3.38}$$

Moreover, from Theorem 3.13 we obtain the following result.

Theorem 3.15. *Let $s, \beta > 0$. Then for all $f \in L^1_\alpha \cap L^2_\alpha(\varepsilon_1, \varepsilon_2, S, \Sigma)$,*

$$\|x^s f\|_{1,\alpha} \geq \frac{(1-\varepsilon_2^2)^{\frac{\alpha+s+1}{2\alpha+2}}}{C_1(\alpha, s)\mu_\alpha(\Sigma)^{\frac{\alpha+s+1}{2\alpha+2}}} \|f\|_{2,\alpha} \tag{3.39}$$

and

$$\left\|\xi^\beta \mathcal{H}_\alpha(f)\right\|_{2,\alpha} \geq \frac{(1-\varepsilon_1)^{\frac{\alpha+\beta+1}{\alpha+1}}}{C_2(\alpha, \beta)\mu_\alpha(S)^{\frac{\alpha+\beta+1}{2\alpha+2}}} \|f\|_{1,\alpha}. \tag{3.40}$$

Proof. Let $f \in L^1_\alpha \cap L^2_\alpha(\varepsilon_1, \varepsilon_2, S, \Sigma)$. Then

$$\|F_\Sigma f\|^2_{2,\alpha} = \|f\|^2_{2,\alpha} - \|F_{\Sigma^c} f\|^2_{2,\alpha} \geq (1-\varepsilon_2^2)\|f\|^2_{2,\alpha}$$

and

$$\|E_S f\|_{1,\alpha} \geq \|f\|_{1,\alpha} - \|E_{S^c} f\|_{1,\alpha} \geq (1-\varepsilon_1)\|f\|_{1,\alpha}.$$

Then desired result follows from (3.34) and (3.35). ☐

Consequently, we obtain the following variation on Heisenberg's inequality with constant depending on $s, \beta, \varepsilon_1, \varepsilon_2, S, \Sigma$.

Corollary 3.16. *For all $s, \beta > 0$ and all $f \in L^1_\alpha \cap L^2_\alpha(\varepsilon_1, \varepsilon_2, S, \Sigma)$,*

$$\begin{aligned} \|x^s f\|^{\alpha+\beta+1}_{1,\alpha} \left\|\xi^\beta \mathcal{H}_\alpha(f)\right\|^{\alpha+s+1}_{2,\alpha} \geq C\Big(\frac{(1-\varepsilon_1)^2(1-\varepsilon_2^2)}{\mu_\alpha(S)\mu_\alpha(\Sigma)}\Big)^{\frac{(\alpha+\beta+1)(\alpha+s+1)}{2\alpha+2}} \\ \times \|f\|^{\alpha+s+1}_{1,\alpha}\|f\|^{\alpha+\beta+1}_{2,\alpha}. \end{aligned} \tag{3.41}$$

Remark 3.17. From (3.34) and (3.35), we also obtain that if $f \in L^1_\alpha \cap L^2_\alpha(\varepsilon_1, \varepsilon_2, S, \Sigma)$, then

$$\mu_\alpha(\Sigma) \geq \frac{1-\varepsilon_2^2}{\tilde{C}_1(\alpha, s)}\Big(\frac{\|f\|_{2,\alpha}}{\|x^s f\|_{1,\alpha}}\Big)^{\frac{2\alpha+2}{\alpha+s+1}} \tag{3.42}$$

and

$$\mu_\alpha(S) \geq \frac{(1-\varepsilon_1)^2}{\tilde{C}_2(\alpha, s)}\Big(\frac{\|f\|_{1,\alpha}}{\|\xi^\beta \mathcal{H}_\alpha(f)\|_{2,\alpha}}\Big)^{\frac{2\alpha+2}{\alpha+\beta+1}}, \tag{3.43}$$

which imply the following variation on the Donoho–Stark uncertainty inequality with the constant depending on $s, \beta, \varepsilon_1, \varepsilon_2$, and f:

$$\begin{aligned} \mu_\alpha(S)\mu_\alpha(\Sigma) \geq \Big(C(\alpha, \beta, s)\frac{\|f\|^{\alpha+s+1}_{1,\alpha}\|f\|^{\alpha+\beta+1}_{2,\alpha}}{\|x^s f\|^{\alpha+\beta+1}_{1,\alpha}\|\xi^\beta \mathcal{H}_\alpha(f)\|^{\alpha+s+1}_{2,\alpha}}\Big)^{\frac{2\alpha+2}{(\alpha+\beta+1)(\alpha+s+1)}} \\ \times (1-\varepsilon_1)^2(1-\varepsilon_2^2). \end{aligned} \tag{3.44}$$

4. The wavelet Hankel multiplier

In this section, let ϕ and ψ be two functions in $L^\infty(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$ such that $\|\phi\|_{2,\alpha} = \|\psi\|_{2,\alpha} = 1$.

4.1. **Boundedness.** The aim of this section is to prove that we can also define $P_{\sigma,\phi,\psi}$ for the symbol $\sigma \in L^p_\alpha(\mathbb{R}_+)$, $1 < p < \infty$. First, if $\sigma \in L^\infty(\mathbb{R}_+)$, then we have the following result.

Proposition 4.1. *Let $\sigma \in L^\infty(\mathbb{R}_+)$. Then $P_{\sigma,\phi,\psi}$ is in S_∞ and*

$$\|P_{\sigma,\phi,\psi}\|_{S_\infty} \le \|\phi\|_\infty \|\psi\|_\infty \|\sigma\|_\infty. \tag{4.1}$$

Proof. By the Cauchy–Schwarz inequality,

$$\left|\langle P_{\sigma,\phi,\psi} f, g\rangle_{\mu_\alpha}\right| \le \|\sigma\|_\infty \left\|\mathcal{H}_\alpha(\phi f)\right\|_{2,\alpha} \left\|\mathcal{H}_\alpha(\psi g)\right\|_{2,\alpha}.$$

Then by Plancherel's formula (2.9), we obtain

$$\begin{aligned}\left|\langle P_{\sigma,\phi,\psi} f, g\rangle_{\mu_\alpha}\right| &\le \|\sigma\|_\infty \|\phi f\|_{2,\alpha} \|\psi g\|_{2,\alpha}\\ &\le \|\sigma\|_\infty \|\phi\|_\infty \|\psi\|_\infty \|f\|_{2,\alpha} \|g\|_{2,\alpha}.\end{aligned}$$

This completes the proof. □

Now, if we consider $\sigma \in L^1_\alpha(\mathbb{R}_+)$, then we obtain the following result.

Proposition 4.2. *Let $\sigma \in L^1_\alpha(\mathbb{R}_+)$. Then $P_{\sigma,\phi,\psi}$ is in S_∞ and*

$$\|P_{\sigma,\phi,\psi}\|_{S_\infty} \le \|\sigma\|_{1,\alpha}. \tag{4.2}$$

Proof. Since $\mathcal{H}_\alpha(\phi f)(\xi) = \langle f, \bar{\phi} j_\alpha(\cdot,\xi)\rangle_{\mu_\alpha}$, then by the Cauchy–Schwarz inequality,

$$\left\|\mathcal{H}_\alpha(\phi f)\right\|_\infty \le \|f\|_{2,\alpha} \|\phi\|_{2,\alpha}.$$

Therefore, since $\|\phi\|_{2,\alpha} = \|\psi\|_{2,\alpha} = 1$, we obtain

$$\begin{aligned}\left|\langle P_{\sigma,\phi,\psi} f, g\rangle_{\mu_\alpha}\right| &\le \|\sigma\|_{1,\alpha} \left\|\mathcal{H}_\alpha(\phi f)\right\|_\infty \left\|\mathcal{H}_\alpha(\psi g)\right\|_\infty\\ &\le \|\sigma\|_{1,\alpha} \|f\|_{2,\alpha} \|g\|_{2,\alpha}.\end{aligned} \tag{4.3}$$

This completes the proof. □

Thus, by (4.1), (4.2), and the Riesz–Thorin interpolation argument in [22, Theorem 2] (see also [25, Theorem 12.4]) we obtain the following theorem.

Theorem 4.3. *Let $\sigma \in L^p_\alpha(\mathbb{R}_+)$, $1 < p < \infty$. Then the linear operator $P_{\sigma,\phi,\psi} : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ is bounded and*

$$\|P_{\sigma,\phi,\psi}\|_{S_\infty} \le \|\phi\|_\infty^{\frac{1}{p'}} \|\psi\|_\infty^{\frac{1}{p'}} \|\sigma\|_{p,\alpha}. \tag{4.4}$$

Hence we can define the operator $(\bar{\psi} F_\sigma \phi) : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$, where $\sigma \in L^p_\alpha(\mathbb{R}_+)$, $1 \le p \le \infty$ by

$$\langle P_{\sigma,\phi,\psi} f, g\rangle_{\mu_\alpha} = \left\langle (\bar{\psi} F_\sigma \phi) f, g\right\rangle_{\mu_\alpha}. \tag{4.5}$$

4.2. **Schatten-class properties.** Let us begin with the following theorem.

Theorem 4.4. *Let σ be symbol in $L^1_\alpha(\mathbb{R}_+)$. Then the wavelet Hankel multiplier $P_{\sigma,\phi,\psi}$ is Hilbert–Schmidt and*

$$\|P_{\sigma,\phi,\psi}\|^2_{S_2} = \int_0^\infty \sigma(\xi)\big\langle P_{\bar{\sigma},\psi,\phi}\bar{\psi}, \bar{\phi}j^2_\alpha(\cdot,\xi)\big\rangle_{\mu_\alpha} \mathrm{d}\mu_\alpha(\xi) \le \|\sigma\|^2_{1,\alpha}. \tag{4.6}$$

Proof. First, by (2.15) it follows immediately that the adjoint of $P_{\sigma,\phi,\psi}$ is $P_{\bar{\sigma},\psi,\phi} : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$. Now, let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis for $L^2_\alpha(\mathbb{R}_+)$. Then by (2.15) and Fubini's theorem, we obtain

$$\begin{aligned}
\sum_{n=1}^\infty \|P_{\sigma,\phi,\psi}\varphi_n\|^2_{2,\alpha} &= \sum_{n=1}^\infty \langle P_{\sigma,\phi,\psi}\varphi_n, P_{\sigma,\phi,\psi}\varphi_n\rangle_{\mu_\alpha} \\
&= \sum_{n=1}^\infty \big\langle \sigma\mathcal{H}_\alpha(\phi\varphi_n), \mathcal{H}_\alpha(\psi P_{\sigma,\phi,\psi}\varphi_n)\big\rangle_{\mu_\alpha} \\
&= \sum_{n=1}^\infty \int_0^\infty \sigma(\xi)\big\langle \varphi_n, \bar{\phi}j_\alpha(\cdot,\xi)\big\rangle_{\mu_\alpha} \overline{\big\langle P_{\sigma,\phi,\psi}\varphi_n, \bar{\psi}j_\alpha(\cdot,\xi)\big\rangle_{\mu_\alpha}} \mathrm{d}\mu_\alpha(\xi) \\
&= \int_0^\infty \sigma(\xi) \sum_{n=1}^\infty \big\langle P_{\bar{\sigma},\psi,\phi}\bar{\psi}j_\alpha(\cdot,\xi), \varphi_n\big\rangle_{\mu_\alpha} \big\langle \varphi_n, \bar{\phi}j_\alpha(\cdot,\xi)\big\rangle_{\mu_\alpha} \mathrm{d}\mu_\alpha(\xi) \\
&= \int_0^\infty \sigma(\xi)\big\langle P_{\bar{\sigma},\psi,\phi}\bar{\psi}j_\alpha(\cdot,\xi), \bar{\phi}j_\alpha(\cdot,\xi)\big\rangle_{\mu_\alpha} \mathrm{d}\mu_\alpha(\xi),
\end{aligned}$$

where we used Parseval's identity in the last line. Therefore, from Proposition 4.2 and since $j_\alpha \le 1$, we have

$$\begin{aligned}
\sum_{n=1}^\infty \|P_{\sigma,\phi,\psi}\varphi_n\|^2_{2,\alpha} &\le \|P_{\bar{\sigma},\psi,\phi}\|_{S_\infty}\|\phi\|_{2,\alpha}\|\psi\|_{2,\alpha}\|\sigma\|_{1,\alpha} \\
&\le \|\sigma\|^2_{1,\alpha}.
\end{aligned}$$

Thus from Proposition 2.2, the operator $P_{\sigma,\phi,\psi}$ is in S_2 and $\|P_{\sigma,\phi,\psi}\|_{S_2} \le \|\sigma\|_{1,\alpha}$. The proof is complete. □

Consequently, the operator $P_{\sigma,\phi,\psi}$ is also compact for symbols in $L^p_\alpha(\mathbb{R}_+)$.

Corollary 4.5. *Let σ be a symbol in $L^p_\alpha(\mathbb{R}_+)$, $1 \le p < \infty$. Then the wavelet Hankel multiplier $P_{\sigma,\phi,\psi}$ is compact.*

Proof. Let $\{\sigma_n\}_{n=1}^\infty$ be a sequence of functions in $L^1_\alpha(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ such that $\sigma_n \to \sigma$ in $L^p_\alpha(\mathbb{R}_+)$ as $n \to \infty$. Then by Theorem 4.3,

$$\|P_{\sigma_n,\phi,\psi} - P_{\sigma,\phi,\psi}\|_{S_\infty} \le \|\phi\|_\infty^{\frac{1}{p'}}\|\psi\|_\infty^{\frac{1}{p'}}\|\sigma_n - \sigma\|_{p,\alpha}. \tag{4.7}$$

Therefore, $P_{\sigma_n,\phi,\psi} \to P_{\sigma,\phi,\psi}$ in S_∞ as $n \to \infty$. Now, since by Theorem 4.4 the operators $P_{\sigma_n,\phi,\psi}$ are in S_2 and hence compact, and since the set of compact operators is a closed subspace of S_∞, then the operator $P_{\sigma,\phi,\psi}$ is also compact. □

More precisely, we will prove that the operator $P_{\sigma,\phi,\psi}$ is in fact in the Schatten class S_p, $1 \le p < \infty$. Of particular interest is the Schatten–von Neumann class S_1.

Theorem 4.6. *Let $\sigma \in L^1_\alpha(\mathbb{R}_+)$. Then the wavelet Hankel multiplier $P_{\sigma,\phi,\psi} : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ is trace-class with*

$$\|P_{\sigma,\phi,\psi}\|_{S_1} \le \|\sigma\|_{1,\alpha}, \tag{4.8}$$

and we have the following trace formula:

$$\operatorname{tr}(P_{\sigma,\phi,\psi}) = \int_0^\infty \sigma(\xi)\big\langle \bar{\psi} j_\alpha(\cdot,\xi), \bar{\phi} j_\alpha(\cdot,\xi)\big\rangle_{\mu_\alpha} \,\mathrm{d}\mu_\alpha(\xi). \tag{4.9}$$

Proof. Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis for $L^2_\alpha(\mathbb{R}_+)$. Then

$$\begin{aligned}\sum_{n=1}^\infty \langle P_{\sigma,\phi,\psi}\varphi_n, \varphi_n\rangle_{\mu_\alpha} &= \sum_{n=1}^\infty \int_0^\infty \sigma(\xi)\mathcal{H}_\alpha(\phi\varphi_n)(\xi)\overline{\mathcal{H}_\alpha(\psi\varphi_n)(\xi)}\,\mathrm{d}\mu_\alpha(\xi)\\ &= \sum_{n=1}^\infty \int_0^\infty \sigma(\xi)\big\langle \bar{\psi} j_\alpha(\cdot,\xi), \varphi_n\big\rangle_{\mu_\alpha}\big\langle \varphi_n, \bar{\phi} j_\alpha(\cdot,\xi)\big\rangle_{\mu_\alpha}\,\mathrm{d}\mu_\alpha(\xi).\end{aligned}$$

Thus by Fubini's theorem,

$$\begin{aligned}&\sum_{n=1}^\infty \langle P_{\sigma,\phi,\psi}\varphi_n, \varphi_n\rangle_{\mu_\alpha}\\ &\quad= \int_0^\infty \sigma(\xi)\sum_{n=1}^\infty \big\langle \bar{\psi} j_\alpha(\cdot,\xi), \varphi_n\big\rangle_{\mu_\alpha}\big\langle \varphi_n, \bar{\phi} j_\alpha(\cdot,\xi)\big\rangle_{\mu_\alpha}\,\mathrm{d}\mu_\alpha(\xi).\end{aligned} \tag{4.10}$$

Therefore, by Parseval's identity and the fact that j_α is bounded by 1,

$$\begin{aligned}\sum_{n=1}^\infty \big|\langle P_{\sigma,\phi,\psi}\varphi_n, \varphi_n\rangle_{\mu_\alpha}\big| &\le \frac{1}{2}\int_0^\infty |\sigma(\xi)|\\ &\qquad\times \sum_{n=1}^\infty \Big(\big|\big\langle \bar{\phi} j_\alpha(\cdot,\xi), \varphi_n\big\rangle_{\mu_\alpha}\big|^2 + \big|\big\langle \bar{\psi} j_\alpha(\cdot,\xi), \varphi_n\big\rangle_{\mu_\alpha}\big|^2\Big)\,\mathrm{d}\mu_\alpha(\xi)\\ &= \frac{1}{2}\int_0^\infty |\sigma(\xi)|\big(\|\phi j_\alpha(\cdot,\xi)\|^2_{2,\alpha} + \|\psi j_\alpha(\cdot,\xi)\|^2_{2,\alpha}\big)\,\mathrm{d}\mu_\alpha(\xi)\\ &\le \|\sigma\|_{1,\alpha}.\end{aligned}$$

By Proposition 2.1, the operator $P_{\sigma,\phi,\psi}$ is in S_1, and with (4.10) and Parseval's identity,

$$\operatorname{tr}(P_{\sigma,\phi,\psi}) = \sum_{n=1}^\infty \langle P_{\sigma,\phi,\psi}\varphi_n, \varphi_n\rangle_{\mu_\alpha} = \int_0^\infty \sigma(\xi)\big\langle \bar{\psi} j_\alpha(\cdot,\xi), \bar{\phi} j_\alpha(\cdot,\xi)\big\rangle_{\mu_\alpha}\,\mathrm{d}\mu_\alpha(\xi).$$

This completes the proof. □

Moreover, by (4.1), (4.8), and the interpolation argument, we deduce the following result.

Corollary 4.7. *Let $\sigma \in L^p_\alpha(\mathbb{R}_+)$, $1 < p < \infty$. Then the linear operator $P_{\sigma,\phi,\psi} : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ is in S_p and*

$$\|P_{\sigma,\phi,\psi}\|_{S_p} \le \|\phi\|_\infty^{\frac{1}{p'}}\|\psi\|_\infty^{\frac{1}{p'}}\|\sigma\|_{p,\alpha}. \tag{4.11}$$

4.3. **An uncertainty inequality.** In this section, we will assume that ϕ and ψ satisfy $\|\phi\|_\infty\|\psi\|_\infty = 1$. Now let $\sigma_1 = \chi_S$ and $\sigma_2 = \chi_\Sigma$, and let $L_1 = P_{\sigma_1,\phi,\psi}$ and $L_2 = P_{\sigma_2,\phi,\psi}$.

From [1], we recall the following definition of ε-localization, which has been introduced and used to refine the degrees-of-freedom estimate of Landau and Pollak [16].

Definition 4.8. Let $\varepsilon \in (0,1)$. Then, a nonzero function $f \in L^2_\alpha(\mathbb{R}_+)$ is *ε-localized* with respect to an operator $L : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ if

$$\|Lf - f\|_{2,\alpha} \le \varepsilon\|f\|_{2,\alpha}. \tag{4.12}$$

Landau in [14] introduced the notion of ε-approximated eigenvalues and eigenfunctions. That is, ρ is said to be an *ε-approximated eigenvalue of* L if there exists a unit L^2_α-norm function f in $L^2_\alpha(\mathbb{R}_+)$ such that

$$\|Lf - \rho f\|_{2,\alpha} \le \varepsilon. \tag{4.13}$$

Then f is called an *ε-approximated eigenfunction corresponding to ρ*. So a function $f \in L^2_\alpha(\mathbb{R}_+)$ that is ε-localized with respect to L is an ε-approximated eigenfunction of L corresponding to 1.

Theorem 4.9. *Let $\varepsilon_1, \varepsilon_2 \in (0,1)$ such that $\varepsilon_1 + \varepsilon_2 < 1$. If $f \in L^2_\alpha(\mathbb{R}_+)$ is ε_1-localized with respect to $P_{\sigma_1,\phi,\psi}$ and ε_2-localized with respect to $P_{\sigma_2,\phi,\psi}$, then for every $p \ge 1$,*

$$\mu_\alpha(S)\mu_\alpha(\Sigma) \ge (1 - \varepsilon_1 - \varepsilon_2)^p. \tag{4.14}$$

Proof. From Proposition 4.1,

$$\begin{aligned}\|f - L_2L_1f\|_{2,\alpha} &\le \|f - L_2f\|_{2,\alpha} + \|L_2f - L_2L_1f\|_{2,\alpha}\\ &\le \|L_2f - f\|_{2,\alpha} + \|L_2\|_{S_\infty}\|L_1f - f\|_{2,\alpha}\\ &\le (\varepsilon_1 + \varepsilon_2)\|f\|_{2,\alpha}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|L_2L_1f\|_{2,\alpha} &\ge \|f\|_{2,\alpha} - \|f - L_2L_1f\|_{2,\alpha}\\ &\ge (1 - \varepsilon_1 - \varepsilon_2)\|f\|_{2,\alpha}.\end{aligned}$$

Thus, from Theorem 4.3 it follows that

$$\begin{aligned}1 - (\varepsilon_1 + \varepsilon_2) &\le \|L_2L_1\|_{S_\infty}\\ &\le \|L_1\|_{S_\infty}\|L_2\|_{S_\infty}\\ &\le \big(\mu_\alpha(S)\mu_\alpha(\Sigma)\big)^{1/p}.\end{aligned}$$

This proves the desired result. □

Note that $1 - \varepsilon_1 - \varepsilon_2 \ge (1 - \varepsilon_1 - \varepsilon_2)^2$. Thus, for $p = 1$, (4.14) improves the classical Donoho–Stark inequality (3.4).

4.4. **The phase-space restriction operator.** We define the phase-space restriction operator by

$$F_\Sigma E_S F_\Sigma = (E_S F_\Sigma)^* E_S F_\Sigma.$$

Then from (2.6), the phase-space restriction operator $F_\Sigma E_S F_\Sigma$ is positive and trace-class with

$$\|F_\Sigma E_S F_\Sigma\|_{S_1} = \|E_S F_\Sigma\|_{S_2}^2 \le \mu_\alpha(S)\mu_\alpha(\Sigma). \tag{4.15}$$

The linear operator $F_\Sigma E_S F_\Sigma : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ is bounded and self-adjoint, and it can be called the *generalized Landau–Pollak–Slepian operator* (see the fundamental papers by Landau and Pollak [15], [16], Slepian [20], and Slepian and Pollak [21] for more detailed information). Moreover,

$$\|E_S F_\Sigma\|_{S_\infty}^2 = \|F_\Sigma E_S\|_{S_\infty}^2 = \|F_\Sigma E_S F_\Sigma\|_{S_\infty} = \lambda_0,$$

where $\lambda_0 \le 1$ is the first eigenvalue corresponding to the first eigenfunction φ_0 of the compact operator $F_\Sigma E_S$, when considered as an operator on $\mathrm{PW}_\alpha(\Sigma)$. This eigenfunction is in $\mathrm{PW}_\alpha(\Sigma)$ and realizes the maximum of concentration on the set S.

Following Wong's point of view in [25], we will show that the phase-space restriction operator $F_\Sigma E_S F_\Sigma : L^2_\alpha(\mathbb{R}_+) \to L^2_\alpha(\mathbb{R}_+)$ can be viewed as a wavelet Hankel operator.

Theorem 4.10. *Let $\phi = \psi$ be the function on $\mathbb{R}_+$ defined by $\phi = \frac{1}{\sqrt{\mu_\alpha(\Sigma)}}\chi_\Sigma$, and let $\sigma = \chi_S$. Then*

$$F_\Sigma E_S F_\Sigma = \mu_\alpha(\Sigma)\mathcal{H}_\alpha P_{S,\phi}\mathcal{H}_\alpha = \mu_\alpha(\Sigma)\mathcal{H}_\alpha(\phi F_S \phi)\mathcal{H}_\alpha. \tag{4.16}$$

Proof. Clearly, the function ϕ belongs to $L^2_\alpha(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, with $\|\phi\|_{2,\alpha} = 1$. Since, for any function $f \in L^2_\alpha(\mathbb{R}_+)$,

$$F_\Sigma \mathcal{H}_\alpha(f) = \mathcal{H}_\alpha(f\chi_\Sigma),$$

we have that

$$\begin{aligned}\mathcal{H}_\alpha(\phi f) &= \frac{1}{\sqrt{\mu_\alpha(\Sigma)}}\mathcal{H}_\alpha(f\chi_\Sigma)\\ &= \frac{1}{\sqrt{\mu_\alpha(\Sigma)}}F_\Sigma\mathcal{H}_\alpha(f).\end{aligned}$$

Thus, for all $f, g \in L^2_\alpha(\mathbb{R}_+)$,

$$\begin{aligned}\langle P_{S,\phi} f, g\rangle_{\mu_\alpha} &= \big\langle \chi_S \mathcal{H}_\alpha(\phi f), \mathcal{H}_\alpha(\phi g)\big\rangle_{\mu_\alpha}\\ &= \frac{1}{\mu_\alpha(\Sigma)}\big\langle \chi_S F_\Sigma \mathcal{H}_\alpha(f), F_\Sigma\mathcal{H}_\alpha(g)\big\rangle_{\mu_\alpha}\\ &= \frac{1}{\mu_\alpha(\Sigma)}\big\langle E_S F_\Sigma \mathcal{H}_\alpha(f), F_\Sigma\mathcal{H}_\alpha(g)\big\rangle_{\mu_\alpha}\\ &= \frac{1}{\mu_\alpha(\Sigma)}\big\langle F_\Sigma E_S F_\Sigma \mathcal{H}_\alpha(f), \mathcal{H}_\alpha(g)\big\rangle_{\mu_\alpha}.\end{aligned}$$

Therefore, by Parseval's equality (2.10), we obtain

$$\langle P_{S,\phi} f, g\rangle_{\mu_\alpha} = \frac{1}{\mu_\alpha(\Sigma)} \big\langle \mathcal{H}_\alpha F_\Sigma E_S F_\Sigma \mathcal{H}_\alpha(f), g\big\rangle_{\mu_\alpha}.$$

Hence, $\mu_\alpha(\Sigma) P_{S,\phi} = \mathcal{H}_\alpha F_\Sigma E_S F_\Sigma \mathcal{H}_\alpha$. □

From Theorems 4.6 and 4.10, we deduce the following corollary.

Corollary 4.11. *The phase-space operator $P_\Sigma E_S P_\Sigma$ is trace-class with*

$$\operatorname{tr}(F_\Sigma E_S F_\Sigma) = \mu_\alpha(\Sigma) \operatorname{tr}(P_{S,\phi}) = \int_S \int_\Sigma j_\alpha^2(x\xi)\, \mathrm{d}\mu_\alpha(x)\, \mathrm{d}\mu_\alpha(\xi). \tag{4.17}$$

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[1]Department of Mathematics, College of Education Al Majmaah, Majmaah University, P.O. Box 66, 11952 Al Majmaah, Kingdom of Saudi Arabia.

[2]LR11ES11 Analyse Mathématiques et Applications, Faculté des Sciences de Tunis, Université de Tunis El Manar, 2092 Tunis, Tunisie.

E-mail address: s.ghobber@mu.edu.sa