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APPROXIMATE UNIQUENESS FOR MAPS FROM $C(X)$ INTO SIMPLE REAL RANK 0 C*-ALGEBRAS

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Abstract. Let X be a finite CW-complex, and let $\mathcal{A}$ be a unital separable simple finite $\mathcal{Z}$-stable C*-algebra with real rank 0. We prove an approximate uniqueness theorem for almost multiplicative contractive completely positive linear maps from $C(X)$ into $\mathcal{A}$. We also give conditions for when such a map can, within a certain "error," be approximated by a finite-dimensional *-homomorphism.

1. Introduction

A basic result in linear algebra states the following. Let A, B be two n by n normal matrices; then A and B are unitarily equivalent (i.e., there exists an n by n unitary matrix U such that $A = UBU^*$) if and only if A and B have the same spectrum, counting multiplicities. This fundamental result is a starting point for spectral theory with vast generalizations and implications, especially infinite-dimensional ones.

In C*-algebra theory, one important class of generalizations is the class of uniqueness theorems which dot the landscape of extension theory as well as the Elliott classification program. One important such generalization was the work of Brown, Douglas, and Fillmore (whom we hereafter designate BDF), who classified the essentially normal operators using Fredholm indices. One of the items that they proved was the following. Let X be a compact metric space, and let $\phi, \psi : C(X) \to \mathbb{B}(l^2)/\mathcal{K}$ be two unital *-monomorphisms. Then ϕ and ψ are unitarily

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equivalent if and only if $[\phi] = [\psi]$ in $KK(C(X), \mathbb{B}(l^2)/\mathcal{K})$ ([1]). The work of BDF resulted in very intense interest in extension theory for C*-algebras with implications to K-theory, KK-theory, and noncommutative topology. Their work, together with Elliott's, served as the starting point of the powerful *uniqueness* and *stable uniqueness* theorems of classification theory (with connections to many interesting subjects, including the theory of absorbing extensions).

Since the literature on this subject is too big, we mention only a few uniqueness theorems, with an emphasis on those which are most closely related to the contents of this paper. Let X be a compact metric space, and let $\mathcal{A}$ be a unital C*-algebra. Recall that two maps $\phi, \psi : C(X) \to \mathcal{A}$ are *approximately unitarily equivalent* if there exists a sequence $\{u_n\}$ of unitaries in $\mathcal{A}$ such that for all $f \in C(X)$, $u_n \phi(f) u_n^* \to \psi(f)$ in the norm topology.

While there are many precedents, perhaps a good place to begin is the following uniqueness result of Gong and Lin. Let X be a compact metric space, and let $\mathcal{A}$ be a unital simple C*-algebra with real rank 0, stable rank 1, weakly unperforated K_0 group, and with a unique tracial state τ. Suppose that $\phi, \psi : C(X) \to \mathcal{A}$ are two unital *-monomorphisms. Then ϕ and ψ are approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $KL(C(X), \mathcal{A})$ and $\tau \circ \phi = \tau \circ \psi$ (see [5, Theorem 2.15]).

In [14], the above restriction on the tracial simplex was removed on the assumption that $\mathcal{A}$ is tracially approximate finite-dimensional (TAF). We note that Lin used this result to generalize a result of Kishimoto. Specifically, Lin showed in [14] that if $\mathcal{A}$ is a simple unital approximately homogeneous (AH) algebra with real rank 0 and bounded dimension growth, and if α is an approximately inner *-automorphism of $\mathcal{A}$ with the tracial Rokhlin property, then the crossed product $\mathcal{A} \times_\alpha \mathbb{Z}$ is a unital simple AH-algebra with bounded dimension growth and real rank 0. (In [18], the restriction on the tracial simplex in the result of [5] was again removed with the additional assumption that $\mathcal{A}$ is $\mathcal{Z}$-stable; here, $\mathcal{Z}$ is the Jiang–Su algebra.)

Finally, we briefly mention that, in recent years, with the many breakthroughs in classification theory, the real rank 0 condition in the codomain algebra $\mathcal{A}$ was removed, although with additional assumptions like rationally tracially approximate interval and $\mathcal{Z}$-stability. This has allowed for interesting uniqueness results, such as the case where the codomain algebra is the projectionless algebra $\mathcal{Z}$ (see, e.g., [15] and the references therein).

In the present article, we prove an approximate and almost multiplicative version of the result in [18]. In fact, approximate, almost multiplicative versions of the corresponding uniqueness result are often present in the articles cited above. That is because such a result is often more useful in applications. We also prove a result which gives natural K-theoretic and "injectivity"-type conditions for when such an almost multiplicative map can be, within an "error," approximated by a finite-dimensional *-homomorphism. This is actually part of a long line of results related to many issues, including, for example, Lin's famous result that almost commuting self-adjoint matrices are uniformly close to commuting self-adjoint matrices (see, e.g., [2], [4]–[6], [8], [9], [11], [16], and the references in those papers). The author intends to use the results reached here in a future paper to classify all extensions of the form

$$0 \to \mathcal{B} \to \mathcal{E} \to C(X) \to 0,$$

where $\mathcal{B}$ is a simple, real rank 0, $\mathcal{Z}$-stable C*-algebra with continuous scale and where X is a finite CW-complex.

Some good basic references for the subject matter of this note can be found in [5], [6], [13], [14], and the references therein. We refer to those references for the basic results, definitions, and notation that we will use in this paper.

2. Main result

For a C*-algebra $\mathcal{C}$, we let $T(\mathcal{C})$ denote the *tracial state space* of $\mathcal{C}$. Next, throughout this paper, whenever we have a stably finite simple unital separable C*-algebra $\mathcal{C}$, we will assume that every quasitrace of $\mathcal{C}$ is a trace. Recall that $\mathcal{Z}$ denotes the Jiang–Su algebra (see [7], [3]). Also, recall that a C*-algebra $\mathcal{C}$ is said to be *$\mathcal{Z}$-stable* if $\mathcal{C} \otimes \mathcal{Z} \cong \mathcal{C}$.

Proposition 2.1. *Let X be a compact metric space with finite covering dimension, and let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of unital simple separable finite $\mathcal{Z}$-stable real rank 0 C*-algebras. Suppose that $\phi : C(X) \to \prod_{k\in\mathbb{Z}_+} \mathcal{A}_k / \bigoplus_{k\in\mathbb{Z}_+} \mathcal{A}_k$ is a unital *-homomorphism, and let $\epsilon > 0$ be given. Then, there are a unital commutative AF-subalgebra $\mathcal{D} \subseteq \prod_{k\in\mathbb{Z}_+} \mathcal{A}_k / \bigoplus_{k\in\mathbb{Z}_+} \mathcal{A}_k$, a sequence $(p_k)_{k\in\mathbb{Z}_+} \in \prod_{k\in\mathbb{Z}_+} \mathcal{A}_k$ of projections lifting $p =_{df} 1_{\mathcal{D}} \in \prod_{k\in\mathbb{Z}_+} \mathcal{A}_k / \bigoplus_{k\in\mathbb{Z}_+} \mathcal{A}_k$, and an integer $L \geq 1$ such that*

1. $p \in \phi(C(X))' \cap (\prod_{k\in\mathbb{Z}_+} \mathcal{A}_k / \bigoplus_{k\in\mathbb{Z}_+} \mathcal{A}_k)$,
2. $p\phi(C(X)) \subseteq \mathcal{D}$,
3. $\tau(p_k) > 1 - \epsilon$ *for all* $\tau \in T(\mathcal{A}_k)$ *and for all* $k \geq L$.

Proof. The proof is exactly the same as that of [18, Proof of Proposition 2.6.]. $\square$

Let $\mathcal{C}, \mathcal{D}$ be C*-algebras, and let $\mathcal{G} \subseteq \mathcal{C}$ and $\delta > 0$. Recall that a map $\phi : \mathcal{C} \to \mathcal{D}$ is said to be *$\mathcal{G}$-δ-multiplicative* if for all $a, b \in \mathcal{G}$,

$$\left\|\phi(ab) - \phi(a)\phi(b)\right\| < \delta.$$

Also, throughout this article, we often use "c.p.c." to abbreviate "completely positive contractive." All our c.p.c. maps are assumed to be linear.

Lemma 2.2. *Let X be a compact metric space. Then for every $\epsilon > 0$, for every finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ such that the following will hold: for any unital simple separable finite real rank 0 $\mathcal{Z}$-stable C*-algebra $\mathcal{A}$, for any $\mathcal{G}$-δ-multiplicative unital c.p.c. map $\phi : C(X) \to \mathcal{A}$, there exists a commutative finite-dimensional C*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ where*

1. $\|\phi(f)1_{\mathcal{B}} - 1_{\mathcal{B}}\phi(f)\| < \epsilon$ *for all* $f \in \mathcal{F}$,
2. $\mathrm{dist}(1_{\mathcal{B}}\phi(f)1_{\mathcal{B}}, \mathcal{B}) < \epsilon$ *for all* $f \in \mathcal{F}$,
3. $\tau(1_{\mathcal{B}}) > 1 - \epsilon$ *for all* $\tau \in T(\mathcal{A})$.

Proof. Suppose, to the contrary, that X is a compact metric space, $\epsilon > 0$, and suppose that $\mathcal{F} \subset C(X)$ is a finite subset for which the statement fails. We can replace $C(X)$ with the (unital) C*-subalgebra generated by $\mathcal{F} \cup \{1_X\}$. This C*-subalgebra is a unital commutative C*-algebra with spectrum having finite covering dimension. Hence, we may assume that X has finite covering dimension.

Hence, let $\{\mathcal{G}_n\}_{n=1}^\infty$ be an increasing sequence of finite subsets of $C(X)$, let $\{\mathcal{A}_n\}_{n=1}^\infty$ be a sequence of unital simple separable finite real rank 0 $\mathcal{Z}$-stable C*-algebras, and for each $n \geq 1$, let $\phi_n : C(X) \to \mathcal{A}_n$ be a unital c.p.c. $\mathcal{G}_n$-$\frac{1}{n}$-multiplicative map such that

(a) $\mathcal{F} \subseteq \mathcal{G}_1$,
(b) $\bigcup_{n=1}^\infty \mathcal{G}_n$ is dense in $C(X)$, and
(c) for all $n \geq 1$, there is no commutative finite-dimensional C*-subalgebra $\mathcal{B} \subseteq \mathcal{A}_n$ such that the following three conditions hold:
  (i) $\|\phi_n(f)1_\mathcal{B} - 1_\mathcal{B}\phi_n(f)\| < \epsilon$ for all $f \in \mathcal{F}$,
  (ii) $\mathrm{dist}(1_\mathcal{B}\phi_n(f)1_\mathcal{B}, \mathcal{B}) < \epsilon$ for all $f \in \mathcal{F}$, and
  (iii) $\tau(1_\mathcal{B}) > 1 - \epsilon$ for all $\tau \in T(\mathcal{A}_n)$.

(We denote by "(*)" the above statements in the remainder of this article.)

The unital c.p.c. map $(\phi_n)_{n\in\mathbb{Z}_+} : C(X) \to \prod_{n\in\mathbb{Z}_+} \mathcal{A}_n$ naturally induces a unital *-homomorphism $\phi : C(X) \to \prod_{n\in\mathbb{Z}_+} \mathcal{A}_n / \bigoplus_{n\in\mathbb{Z}_+} \mathcal{A}_n$. To simplify notation, let us denote $\mathcal{C} =_{df} \prod_{n\in\mathbb{Z}_+} \mathcal{A}_n / \bigoplus_{n\in\mathbb{Z}_+} \mathcal{A}_n$. By Proposition 2.1, let $\mathcal{D} \subseteq \mathcal{C}$ be a unital commutative AF-subalgebra, $(p_n)_{n\in\mathbb{Z}_+} \in \prod_{n\in\mathbb{Z}_+} \mathcal{A}_n$ an ∞-tuple of projections lifting $p =_{df} 1_\mathcal{D} \in \mathcal{C}$, and $L_0 \geq 1$ such that

(i) $p \in \phi(C(X))' \cap \mathcal{C}$,
(ii) $p\phi(C(X)) \subseteq \mathcal{D}$,
(iii) $\tau(p_k) > 1 - \epsilon/10$ for all $\tau \in T(\mathcal{A}_k)$ and for all $k \geq L_0$.

Hence, there exists a finite-dimensional unital C*-subalgebra $\mathcal{B}_0 \subseteq \mathcal{D}$ such that $\mathrm{dist}(1_{\mathcal{B}_0}\phi(f)1_{\mathcal{B}_0}, \mathcal{B}_0) = \mathrm{dist}(p\phi(f), \mathcal{B}_0) < \epsilon/2$ for all $f \in \mathcal{F}$.

Note that $\mathcal{B}_0$ is commutative and that $1_{\mathcal{B}_0} = 1_\mathcal{D} = p$. Since $(p_n)_{n\in\mathbb{Z}_+}$ lifts $1_\mathcal{D} = 1_{\mathcal{B}_0}$, $1_{\mathcal{B}_0}\mathcal{C}1_{\mathcal{B}_0}$ is a quotient of $\prod_{n\in\mathbb{Z}_+} p_n\mathcal{A}_n p_n$. Since finite-dimensional C*-algebras are semiprojective, there exists a *-homomorphism

$$\sigma : \mathcal{B}_0 \to \prod_{n\in\mathbb{Z}_+} p_n \mathcal{A}_n p_n$$

which lifts the natural inclusion map $\mathcal{B}_0 \hookrightarrow 1_{\mathcal{B}_0}\mathcal{C}1_{\mathcal{B}_0}$. Now $\sigma = (\sigma_n)_{n\in\mathbb{Z}_+}$ where for all n, $\sigma_n : \mathcal{B}_0 \to p_n\mathcal{A}_n p_n$ is a *-homomorphism. All but finitely many of the σ_n's are unital. We can find a large enough integer $L \geq L_0$ such that if $\mathcal{B} =_{df} \sigma_L(\mathcal{B}_0)$, then $\mathcal{B} \subseteq \mathcal{A}_L$ is a commutative finite-dimensional C*-subalgebra such that $\mathrm{dist}(p_L\phi_L(f)p_L, \mathcal{B}) < \epsilon$ for all $f \in \mathcal{F}$, $1_\mathcal{B} = p_L$, and $\|\phi_L(f)1_\mathcal{B} - 1_\mathcal{B}\phi_L(f)\| < \epsilon$ for all $f \in \mathcal{F}$. Since $\tau(1_\mathcal{B}) = \tau(p_L) > 1 - \epsilon$ for all $\tau \in T(\mathcal{A}_L)$, this contradicts part (c) of (*). □

We need a lemma concerning the c.p.c. almost multiplicative maps from $C(X)$ into a matrix algebra (see [13, Lemma 6.2.7]), as follows.

Lemma 2.3. *Let X be a compact metric space. Let $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$ be given. Then there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$*

*satisfying the following. For every positive integer $n \geq 1$, if $L : C(X) \to \mathbb{M}_n(\mathbb{C})$ is a $\mathcal{G}$-δ-multiplicative c.p.c. map, then there exists a *-homomorphism $h : C(X) \to \mathbb{M}_n$ with $h(1_{C(X)}) = r$ such that*

$$\mathrm{tr}(1-r) < \epsilon$$

and

$$\big\|(1-r)L(f)(1-r) + h(f) - L(f)\big\| < \epsilon$$

for all $f \in \mathcal{F}$. Moreover, the map $C(X) \to (1-r)\mathbb{M}_n(1-r) : f \mapsto (1-r)L(f) \times (1-r)$ is $\mathcal{F}$-ϵ-multiplicative. (In the above, tr is the unique tracial state on $\mathbb{M}_n$.)

The proof of the next lemma is a variation on [18, Theorem 3.1] and also [14, Lemma 4.4] (see also [17]).

Proposition 2.4. *Let X be a compact metric space, $\epsilon > 0$, and let $\mathcal{F} \subset C(X)$ be a finite subset. Let $N \geq 1$ be an integer, and let $\eta > 0$ be such that for all $f \in \mathcal{F}$, $|f(x) - f(x')| < \frac{\epsilon}{8}$ if $\mathrm{dist}(x,x') < 2\eta$. Then for any integer $s \geq 2$, any finite $\eta/2$-dense subset $\{x_1, x_2, \ldots, x_n\} \subset X$ for which $\overline{O_j} \cap \overline{O_k} = \emptyset$ for $j \neq k$, where for all j,*

$$O_j =_{df} B(x_j, \eta/s) = \big\{x \in X : \mathrm{dist}(x, x_j) < \eta/s\big\},$$

and for any $1/(2s) > \sigma > 0$, there exist a finite subset $\mathcal{G} \subset C(X)$ and $\delta > 0$ satisfying the following.

For any unital separable simple finite $\mathcal{Z}$-stable C^*-algebra $\mathcal{A}$ with real rank 0, and any unital $\mathcal{G}$-δ-multiplicative c.p.c. map $\phi : C(X) \to \mathcal{A}$ with

$$\mu_{\tau\circ\phi}(O'_j) > \sigma\eta$$

for all $\tau \in T(\mathcal{A})$ and for all $1 \leq j \leq n$, where

$$O'_j =_{df} B\big(x_j, \eta/(2s)\big) = \big\{x \in X : \mathrm{dist}(x, x_j) < \eta/(2s)\big\}$$

for $1 \leq j \leq n$, there exist a projection $p \in \mathcal{A}$ and an $\mathcal{F}$-ϵ-multiplicative c.p.c. map $L_1 : C(X) \to p\mathcal{A}p$ such that

1. *there exist pairwise orthogonal projections $p_0, p_1, p_2, \ldots, p_n, t \in \mathcal{A}$ with $p_0 = p$, $\sum_{j=0}^n p_j + t = 1_{\mathcal{A}}$, and $Np \preceq p_j$ for $j \neq 0$,[1] and*
2. *there exists a finite-dimensional *-homomorphism $h_1 : C(X) \to t\mathcal{A}t$ such that $\phi(f)$ is within ϵ of $L_1(f) + \sum_{j=1}^n f(x_j)p_j + h_1(f)$ for all $f \in \mathcal{F}$.*

Proof. Let X, ϵ, $\mathcal{F}$, $N \geq 1$, $\eta > 0$, $s \geq 2$, $\{x_1, x_2, \ldots, x_n\}$, $O_1, O_2, \ldots, O_n$, and σ be given as in the hypotheses. For simplicity, we may assume that $\epsilon < 1$ and all the elements of $\mathcal{F}$ have norm less than or equal to 1. For $1 \leq j \leq n$, let $f_j : X \to [0,1]$ be a continuous function such that $f_j(x) = 1$ for all $x \in \overline{O'_j}$, $f_j(x) = 0$ for all $x \in X - O_j$, and $f_j f_k = 0$ for all $j \neq k$. Let $\mathcal{G}_1 =_{df} \mathcal{F} \cup \{1_{C(X)}\} \cup \{f_j : 1 \leq j \leq n\}$. Let

$$\epsilon_1 =_{df} \frac{1}{10(N+1)} \min\{\epsilon, \sigma\eta\} > 0.$$

[1] "Np" abbreviates "$\bigoplus^N p$".

Plug X, ϵ_1 and $\mathcal{G}_1$ into Lemma 2.3 to get $\mathcal{G}_2$ and $\epsilon_2 > 0$. Contracting ϵ_2 if necessary, we may assume that all the elements of $\mathcal{G}_2$ have norm less than or equal to 1, and also that $\epsilon_2 < \epsilon_1$. We may also assume that $\mathcal{G}_1 \subset \mathcal{G}_2$. Let

$$\mathcal{G}_3 =_{df} \mathcal{G}_2 \cup \{fg : f, g \in \mathcal{G}_2\}.$$

Let $\epsilon_3 > 0$ be such that if $\mathcal{C}$ is a C*-algebra, if $\psi_1 : C(X) \to \mathcal{C}$ is a c.p.c. $\mathcal{G}_3$-ϵ_3-multiplicative map, and if $q \in \mathcal{A}$ is a nonzero projection such that $\|\psi_1(f)q - q\psi_1(f)\| < \epsilon_3$ for all $f \in \mathcal{G}_3$, then the map $C(X) \to q\mathcal{C}q : f \mapsto q\psi_1(f)q$ is a c.p.c. $\mathcal{G}_2$-ϵ_2-multiplicative map.

Contracting ϵ_3 if necessary, we may assume that if $\mathcal{C}$ is a C*-algebra, $\mathcal{C}_1 \subseteq \mathcal{C}$ is a finite-dimensional C*-subalgebra, and $\psi_2 : C(X) \to \mathcal{C}$ is a c.p.c. $\mathcal{G}_3$-ϵ_3-multiplicative map such that

$$\left\| 1_{\mathcal{C}_1}\psi_2(f) - \psi_2(f)1_{\mathcal{C}_1} \right\| < \epsilon_3$$

and

$$\operatorname{dist}\left(1_{\mathcal{C}_1}\psi_2(f)1_{\mathcal{C}_1}, \mathcal{C}_1\right) < \epsilon_3$$

for all $f \in \mathcal{G}_3$, then there exists a $\mathcal{G}_2$-ϵ_2-multiplicative c.p.c. map $\psi_3 : C(X) \to \mathcal{C}_1$ such that

$$\left\| \psi_3(f) - 1_{\mathcal{C}_1}\psi_2(f)1_{\mathcal{C}_1} \right\| < \epsilon_2$$

for all $f \in \mathcal{G}_2$.

Contracting ϵ_3 further if necessary, we may assume that $\epsilon_3 < \epsilon_1$. Plug X, $\epsilon_3/2$, and $\mathcal{G}_3$ into Lemma 2.2 to get δ and a finite subset $\mathcal{G} \subset C(X)$. Again, contracting δ if necessary, we may assume that $\delta < \epsilon_3$ and that the elements of $\mathcal{G}$ have norm less than or equal to 1. We may also assume that $\mathcal{G}_3 \subseteq \mathcal{G}$. Now suppose that $\mathcal{A}$ is a unital separable simple finite real rank 0 $\mathcal{Z}$-stable C*-algebra. Suppose that $\phi : C(X) \to \mathcal{A}$ is a unital $\mathcal{G}$-δ-multiplicative c.p.c. map such that

$$\mu_{\tau\circ\phi}(O'_j) > \sigma\eta \tag{2.1}$$

for all $\tau \in T(\mathcal{A})$ and for all $1 \leq j \leq n$. By Lemma 2.2 and by the definition of $\mathcal{G}$ and δ, let $\mathcal{B} \subseteq \mathcal{A}$ be a finite-dimensional C*-subalgebra where

(i) $\|\phi(f)1_{\mathcal{B}} - 1_{\mathcal{B}}\phi(f)\| < \epsilon_3/2$ for all $f \in \mathcal{G}_3$,
(ii) $\operatorname{dist}(1_{\mathcal{B}}\phi(f)1_{\mathcal{B}}, \mathcal{B}) < \epsilon_3/2$ for all $f \in \mathcal{G}_3$,
(iii) $\|1_{\mathcal{B}}\phi(f)1_{\mathcal{B}} + (1_{\mathcal{A}} - 1_{\mathcal{B}})\phi(f)(1_{\mathcal{A}} - 1_{\mathcal{B}}) - \phi(f)\| < \epsilon_3$ for all $f \in \mathcal{G}_3$, and
(iv) $\tau(1_{\mathcal{B}}) > 1 - \epsilon_3/2$ for all $\tau \in T(\mathcal{A})$.

By the definition of ϵ_3, the map

$$L : C(X) \to (1_{\mathcal{A}} - 1_{\mathcal{B}})\mathcal{A}(1_{\mathcal{A}} - 1_{\mathcal{B}}) : f \mapsto (1_{\mathcal{A}} - 1_{\mathcal{B}})\phi(f)(1_{\mathcal{A}} - 1_{\mathcal{B}})$$

is a c.p.c. $\mathcal{G}_2$-ϵ_2-multiplicative map. Also, by the definition of ϵ_3, there exists a c.p.c. $\mathcal{G}_2$-ϵ_2-multiplicative map

$$L' : C(X) \to \mathcal{B}$$

such that

$$\left\| L'(f) - 1_{\mathcal{B}}\phi(f)1_{\mathcal{B}} \right\| < \epsilon_2$$

for all $f \in \mathcal{G}_2$. Hence,

$$\big\| L(f) + L'(f) - \phi(f) \big\| < \epsilon_2 + \epsilon_3$$

for all $f \in \mathcal{G}_2$. Also,

$$\tau\big(L(1_{C(X)})\big) = \tau(1_{\mathcal{A}} - 1_{\mathcal{B}}) < \epsilon_3$$

for all $\tau \in T(\mathcal{A})$.

Since $L' : C(X) \to \mathcal{B}$ is $\mathcal{G}_2$-ϵ_2-multiplicative and c.p.c., by the definitions of $\mathcal{G}_2$ and ϵ_2 and by Lemma 2.3 there exists a projection $r \in \mathcal{B}$ and there exists a *-homomorphism $h : C(X) \to \mathcal{B}$ with $h(1_{C(X)}) = r$ such that

$$\nu(1_{\mathcal{B}} - r) < \epsilon_1$$

for all $\nu \in T(\mathcal{B})$, and

$$\big\| (1_{\mathcal{B}} - r)L'(f)(1_{\mathcal{B}} - r) + h(f) - L'(f) \big\| < \epsilon_1$$

for all $f \in \mathcal{G}_1$. Note that from the above, for all $\tau \in T(\mathcal{A})$,

$$\tau(1_{\mathcal{B}} - r) < \epsilon_1 \tau(1_{\mathcal{B}}) < \epsilon_1.$$

Moreover, we have that the map $L'' : C(X) \to (1_{\mathcal{B}} - r)\mathcal{B}(1_{\mathcal{B}} - r) : f \mapsto (1_{\mathcal{B}} - r)L'(f)(1_{\mathcal{B}} - r)$ is a c.p.c. $\mathcal{G}_1$-ϵ_1-multiplicative map. Let $L_1 = L + L'' : C(X) \to (1_{\mathcal{A}} - r)\mathcal{A}(1_{\mathcal{A}} - r)$. Hence, L_1 is a unital c.p.c. $\mathcal{G}_1$-ϵ_1-multiplicative map. (Recall that $\mathcal{G}_1 \subseteq \mathcal{G}_2$ and $\epsilon_2 < \epsilon_1$.) Also,

$$\big\| L_1(f) + h(f) - \phi(f) \big\| < \epsilon_1 + \epsilon_2 + \epsilon_3 < 3\epsilon_1 \leq \frac{3\epsilon}{10(N+1)}$$

for all $f \in \mathcal{G}_1$ and

$$\tau\big(L_1(1_{C(X)})\big) = \tau(1_{\mathcal{A}} - r) = \tau(1_{\mathcal{A}} - 1_{\mathcal{B}}) + \tau(1_{\mathcal{B}} - r) < \epsilon_3 + \epsilon_1 < 2\epsilon_1 \leq \frac{2\epsilon}{10(N+1)}$$

for all $\tau \in T(\mathcal{A})$. We let

$$p =_{df} L_1(1_{C(X)}).$$

Now, for all $1 \leq j \leq n$, for all $\tau \in T(\mathcal{A})$,

$$\begin{aligned}
\tau\big(h(f_j)\big) &= \big(\tau\big(h(f_j)\big) + \tau\big(L_1(f_j)\big)\big) - \tau\big(L_1(f_j)\big) \\
&> \tau\big(\phi(f_j)\big) - 3\epsilon_1 - \tau\big(L_1(f_j)\big) \\
&> \tau\big(\phi(f_j)\big) - 5\epsilon_1 \\
&> \sigma\eta - 5\epsilon_1 \quad \text{(by (2.1))} \\
&> 2N\epsilon_1 \quad \text{(by the definition of } \epsilon_1\text{)} \\
&> N\tau\big(L_1(1_{C(X)})\big).
\end{aligned}$$

Let $r_1, r_2, \ldots, r_m \in \mathcal{B}$ be projections, and let $y_1, y_2, \ldots, y_m \in X$ be such that for all $f \in \mathcal{G}_1$,

$$h(f) = \sum_{k=1}^{m} f(y_k) r_k.$$

Recall that for all $1 \leq j \leq n$, we have $f_j \in \mathcal{G}_1$. Hence, for all $1 \leq j \leq n$, for all $\tau \in T(\mathcal{A})$, we have

$$\sum_{k=1}^{m} f_j(y_k)\tau(r_k) > N\tau\big(L_1(1_{C(X)})\big) > 0.$$

For all $1 \leq j \leq n$, let

$$S_j =_{df} \{k : y_k \in O_j\} = \big\{k : \mathrm{dist}(y_k, x_j) < \eta/s\big\} \supseteq \big\{k : f_j(y_k) > 0\big\}.$$

Let $p_j \in \mathcal{A}$ be the projection given by

$$p_j =_{df} \sum_{k \in S_j} r_k.$$

Also, let

$$t =_{df} 1_{\mathcal{B}} - \sum_{j=1}^{n} p_j,$$

and let $h_1 : C(X) \to \mathcal{B}$ be the *-homomorphism

$$h_1(f) =_{df} th(f)t$$

for all $f \in C(X)$. Finally, let $p_0 =_{df} p$.

Then we have

$$1_{\mathcal{A}} = t + \sum_{j=0}^{n} p_j,$$

$$\Big\| L_1(f) + \sum_{j=1}^{n} f(x_j)p_j + h_1(f) - \phi(f) \Big\| < \frac{3\epsilon}{10} + \frac{\epsilon}{8} < \epsilon,$$

for all $f \in \mathcal{F}$ (recall the definition of η in the hypotheses), and

$$\tau(p_j) > N\tau\big(L_1(1_{C(X)})\big) = N\tau(p)$$

for all $1 \leq j \leq n$ and for all $\tau \in T(\mathcal{A})$. $\square$

For a unital C*-algebra $\mathcal{C}$, recall that $\underline{K}(\mathcal{C})$ denotes the *total K-theory* of $\mathcal{C}$ (e.g., see [13, Section 5.8]). For all $n \geq 0$, let C_n be the 2-dimensional CW-complex obtained by attaching a 2-cell to the S^1 via the degree n map from S^1 to S^1. We let $\mathbb{P}(\mathcal{C})$ denote the image, in $\underline{K}(\mathcal{C})$, of the set of projections in $\bigcup_{n \geq 0} \mathbb{M}_\infty(\mathcal{C} \otimes C(C_n) \otimes C(S^1))$ (see [13, Chapter 5] and [10, Section 2.1]).

The next result is from [14, Theorem 4.6] (see also [6]).

Theorem 2.5. *Let X be a compact metric space, $\epsilon > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. Let $\nu > 0$ be such that $|f(x) - f(y)| < \epsilon/8$ if $\mathrm{dist}(x, y) < \nu$ for all $f \in \mathcal{F}$ and all $x, y \in X$. Then for any $s \geq 1$, any finite $\nu/2$-dense subset $\{x_1, x_2, \ldots, x_m\}$ of X for which $O_j \cap O_k = \emptyset$ for $j \neq k$, where*

$$O_k =_{df} \big\{x \in X : \mathrm{dist}(x, x_k) < \nu/(2s)\big\},$$

and for any $1/(2s) > \sigma > 0$, there exist $\gamma > 0$, a finite subset $\mathcal{G} \subset C(X)$, $\delta > 0$, and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ satisfying the following.

For any unital separable simple C^*-algebra $\mathcal{A}$ with tracial rank 0, and $\mathcal{G}$-δ-multiplicative unital c.p.c. maps $\phi, \psi : C(X) \to \mathcal{A}$ with $\tau \circ \phi(g)$ within γ of $\tau \circ \psi(g)$ for all $g \in \mathcal{G}$ and for all $\tau \in T(\mathcal{A})$, if

(i) $\mu_{\tau\circ\phi}(O_k), \mu_{\tau\circ\psi}(O_k) > \sigma\nu$ *for all k and for all $\tau \in T(\mathcal{A})$, and*

(ii) $[\phi]|\mathcal{P} = [\psi]|\mathcal{P}$,

then there exists a unitary $u \in \mathcal{A}$ such that $u\phi(f)u^*$ is within ϵ of $\psi(f)$ for all $f \in \mathcal{F}$. Finally, if, in the above, the elements of $\mathcal{F}$ all have norm less than or equal to 1, then we can choose $\mathcal{G}$ so that its elements all have norm less than or equal to 1.

The next stable uniqueness theorem can be found in [4, Theorem 3.1] (see also [4, Remark 1.1], [5], and [14]).

Theorem 2.6. *Let X be a compact metric space. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$, $\eta > 0$, and integer $N \geq 1$, a finite subset $\mathcal{G} \subset C(X)$, and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ satisfying the following.*

For any unital separable simple C^*-algebra $\mathcal{A}$ with real rank 0, stable rank 1, and weakly unperforated K_0 group, for any η-dense subset $\{x_1, x_2, \ldots, x_m\}$ in X, and any $\mathcal{G}$-δ-multiplicative c.p.c. maps $\phi, \psi : C(X) \to \mathcal{A}$, if

$$[\phi]|\mathcal{P} = [\psi]|\mathcal{P},$$

then there exists a unitary $u \in \mathbb{M}_{Nm+1}(\mathcal{A})$ such that

$$\begin{aligned} &u\big(\phi(f) \oplus f(x_1)1_N \oplus f(x_2)1_N \oplus \cdots \oplus f(x_m)1_N\big)u^* \\ &\quad \approx_\epsilon \psi(f) \oplus f(x_1)1_N \oplus f(x_2)1_N \oplus \cdots \oplus f(x_m)1_N \end{aligned}$$

for all $f \in \mathcal{F}$.

The next result follows from [13, Theorem 6.2.9].

Theorem 2.7. *Let X be a finite CW-complex, and let $\mathcal{A}$ be a unital simple AH-algebra with bounded dimension growth and real rank 0. Let $\mathcal{P} \subset \mathbb{P}(C(X))$ be a finite subset, and let $\alpha \in KL(C(X), \mathcal{A})$ be such that*

$$\alpha\big(\underline{K}\big(C(X)\big)_+\big) \subseteq \underline{K}(\mathcal{A})_+$$

and $\alpha([1_{C(X)}]) = [1_\mathcal{A}]$ (with equivalence classes in K_0). Then there exists a sequence of unital c.p.c. maps $L_n : C(X) \to \mathcal{A}$ such that

$$[L_n]|_\mathcal{P} = \alpha|_\mathcal{P}$$

and

$$\big\|L_n(f)L_n(g) - L_n(fg)\big\| \to 0$$

as $n \to \infty$, for all $f, g \in C(X)$.

Lemma 2.8. *Let X be a finite CW-complex, let $\epsilon > 0$, and let $\mathcal{F} \subset C(X)$ be a finite subset. Then there exist $\nu_1, \nu_2 > 0$ such that for all finite $\nu_1/2$-dense subsets $\{x_1, x_2, \ldots, x_m\} \subset X$, for all finite $\nu_2/2$-dense subsets $\{y_1, y_2, \ldots, y_n\} \subset X$, for all integers $s, s' \geq 3$ for which*

$$\overline{O^0_j} \cap \overline{O^0_k} = \emptyset$$

for all $j \neq k$ and

$$\overline{O_j^1} \cap \overline{O_k^1} = \emptyset$$

for all $j \neq k$ (here, $O_j^0 =_{df} B(x_j, \nu_1/s)$ and $O_k^1 =_{df} B(y_k, \nu_2/s')$), for all $\sigma, \sigma' > 0$ with

$$\frac{1}{4s} > \sigma > 0$$

and

$$\frac{1}{2s'} > \sigma' > 0,$$

there exist $\delta > 0$, $\gamma > 0$, a finite subset $\mathcal{G} \subset C(X)$, a finite subset $\mathcal{G}' \subset C(X)$, and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ such that the following holds.

For every unital separable simple finite real rank 0 $\mathcal{Z}$-stable C-algebra $\mathcal{A}$, for all unital c.p.c. $\mathcal{G}$-δ-multiplicative maps $\phi, \psi : C(X) \to \mathcal{A}$,*

if

$$[\phi]|\mathcal{P} = [\psi]|\mathcal{P},$$

$$\big|\tau \circ \phi(g) - \tau \circ \psi(g)\big| < \gamma$$

for all $g \in \mathcal{G}'$ and for all $\tau \in T(\mathcal{A})$,

$$\mu_{\tau\circ\phi}(O_j^2), \mu_{\tau\circ\psi}(O_j^2) > 2\sigma\nu_1$$

and

$$\mu_{\tau\circ\phi}(O_k^3), \mu_{\tau\circ\psi}(O_k^3) > \sigma'\nu_2$$

for all j, k, for all $\tau \in T(\mathcal{A})$ (here, $O_j^2 =_{df} B(x_j, \frac{\nu_1}{2(s+2)})$ and $O_k^3 =_{df} B(y_k, \frac{\nu_2}{2s'})$), then there exists a unitary $u \in \mathcal{A}$ such that

$$u\phi(f)u^* \approx_\epsilon \psi(f)$$

for all $f \in \mathcal{F}$.

Proof. Let X be a finite CW-complex, let $\mathcal{F} \subset C(X)$ be a finite subset, and let $\epsilon > 0$ be given. Let d be the metric for X. We may assume that all the elements of $\mathcal{F}$ have norm less than or equal to 1. We may also assume that $1_{C(X)} \in \mathcal{F}$. Let $\nu_1 > 0$ be such that for all $x, y \in X$, if $d(x, y) < 2\nu_1$, then for all $f \in \mathcal{F}$, $|f(x) - f(y)| < \epsilon/80$. Let $s \geq 3$ be an arbitrary integer. Let $\{x_1, x_2, \ldots, x_m\} \subset X$ be a $\nu_1/2$-dense subset of X for which

$$\overline{O_j^a} \cap \overline{O_k^a} = \emptyset$$

for all $j \neq k$, and let $\frac{1}{4s} > \sigma > 0$ be arbitrary. (In the above, $O_j^a =_{df} B(x_j, \nu_1/s) = \{x \in X : d(x, x_j) < \nu_1/s\}$ for all j.) Plug X, $\mathcal{F}$, $\epsilon/10$, ν_1, $\{x_1, x_2, \ldots, x_m\}$, σ and s into Theorem 2.5 to get $\gamma_1 > 0$, a finite subset $\mathcal{G}_1 \subset C(X)$, $\delta_1 > 0$, and a finite subset $\mathcal{P}_1 \subset \mathbb{P}(C(X))$.

We may assume that all the elements of $\mathcal{G}_1$ have norm less than or equal to 1. We may also assume that $\delta_1 < \epsilon/10$, $\gamma_1 < \epsilon/10$, and $\mathcal{F} \subset \mathcal{G}_1$. We denote the above statements by "(*)".

Let $\delta_2 > 0$ and a finite subset $\mathcal{G}_2 \subset C(X)$ be such that for every unital C*-algebra $\mathcal{C}$, for all unital c.p.c. $\mathcal{G}_2$-δ_2-multiplicative maps $\alpha_1, \alpha_2 : C(X) \to \mathcal{C}$, if

$\|\alpha_1(f) - \alpha_2(f)\| < \delta_2$ for all $f \in \mathcal{G}_2$ and $\mu_{\tau\circ\alpha_1}(O_j^b) > \frac{5}{4}\sigma\nu_1$ for all $\tau \in T(\mathcal{C})$ and for all j, then α_1, α_2 are well defined on $\mathcal{P}_1$,

$$[\alpha_1]|\mathcal{P}_1 = [\alpha_2]|\mathcal{P}_1$$

$$\left|\tau \circ \alpha_1(g) - \tau \circ \alpha_2(g)\right| < \frac{\gamma_1}{10}$$

for all $g \in \mathcal{G}_1$ and

$$\mu_{\tau\circ\alpha_1}(O_j^a), \mu_{\tau\circ\alpha_2}(O_j^a) > \sigma\nu_1$$

for all $\tau \in T(\mathcal{C})$ and for all j. (Here, $O_j^b =_{df} B(x_j, \frac{\nu_1}{2(s+1)}) = \{x \in X : d(x, x_j) < \frac{\nu_1}{2(s+1)}\}$.) We may assume that $\delta_2 < \frac{\delta_1}{10} < \frac{\epsilon}{100}$, $(\mathcal{F} \subseteq)\mathcal{G}_1 \subseteq \mathcal{G}_2$, and all the elements of $\mathcal{G}_2$ have norm less than or equal to 1. We denote the above statements by "(**)".

Plug X, $\delta_2/10$, and $\mathcal{G}_2$ into Theorem 2.6 to get $\delta_3 > 0$, $\eta_3 > 0$, an integer $N_1 \geq 1$, a finite subset $\mathcal{G}_3 \subset C(X)$, and a finite subset $\mathcal{P}_3 \subset \mathbb{P}(C(X))$. Expanding $\mathcal{P}_3$ if necessary, we may assume that for all unital separable simple real rank 0 C*-algebra $\mathcal{D}$ with weak unperforation and stable rank 1, if a c.p.c. map $\alpha_1 : C(X) \to \mathcal{D}$ is multiplicative enough to induce a well-defined map from $\mathcal{P}_3$ into $\underline{K}(\mathcal{D})$, then α_1 induces a well-defined element $[\alpha_1] \in KL(C(X), \mathcal{D})$ and $[\alpha_1]|_{K_0(C(X))}$ is positive (see the discussion in [10, p. 1274]). We may also assume that $\delta_3 < \frac{\delta_2}{10} < \frac{\epsilon}{1000}$, $\eta_3 < \frac{\epsilon}{10}$, $\mathcal{G}_2 \subseteq \mathcal{G}_3$, $\mathcal{P}_1 \subseteq \mathcal{P}_3$, and all the elements of $\mathcal{G}_3$ have norm less than or equal to 1. We denote the above statements by "(***)".

Let $\delta_4 > 0$ and $\mathcal{G}_4 \subset C(X)$ a finite subset be such that for every unital C*-algebra $\mathcal{C}$, if $\beta_1, \beta_2 : C(X) \to \mathcal{C}$ are unital c.p.c. $\mathcal{G}_4$-δ_4-multiplicative maps for which

$$\left\|\beta_1(f) - \beta_2(f)\right\| < \delta_4$$

for all $f \in \mathcal{G}_4$ and

$$\mu_{\tau\circ\beta_1}(O_j^c), \mu_{\tau\circ\beta_2}(O_j^c) > 2\sigma\nu_1$$

and for all j (where $O_j^c =_{df} B(x_j, \frac{\nu_1}{2(s+2)}) = \{x \in X : d(x, x_j) < \frac{\nu_1}{2(s+2)}\}$), then β_1 and β_2 are well defined on $\mathcal{P}_3$,

$$[\beta_1]|\mathcal{P}_3 = [\beta_2]|\mathcal{P}_3,$$

$$\left|\tau \circ \beta_1(f) - \tau \circ \beta_2(f)\right| < \frac{\gamma_1}{10}$$

for all $f \in \mathcal{G}_3$, and

$$\mu_{\tau\circ\beta_1}(O_j^b), \mu_{\tau\circ\beta_2}(O_j^b) > \frac{3}{2}\sigma\nu_1$$

for all $\tau \in T(\mathcal{C})$ and for all j. We may assume that $\delta_4 < \frac{\delta_3}{10} < \frac{\epsilon}{10000}$, $\mathcal{F} \subseteq \mathcal{G}_3 \subseteq \mathcal{G}_4$, and the elements of $\mathcal{G}_4$ all have norm less than or equal to 1. We denote the above statements by "(****)".

Let $\nu_2 > 0$ be such that $\nu_2 < \eta_3$ and for all $f \in \mathcal{G}_4$, for all $x, y \in X$, if $d(x, y) < 2\nu_2$, then $|f(x) - f(y)| < \epsilon/80$. Let $s' \geq 3$, and let $\{y_1, y_2, \ldots, y_n\} \subset X$ be a $\frac{\nu_2}{2}$-dense subset such that $\overline{O_j^d} \cap \overline{O_k^d} = \emptyset$ for all $j \neq k$ where $O_j^d =_{df} B(y_j, \frac{\nu_2}{s'})$. Let $\frac{1}{2s'} > \sigma' > 0$ be arbitrary. Plug $X, \delta_4, \mathcal{G}_4, N_1, \nu_2, \{y_1, y_2, \ldots, y_n\}, s'$, and σ' into Proposition 2.4 to get $\mathcal{G}_5$ and δ_5. We may assume that $\delta_5 < \frac{\delta_4}{10} < \frac{\epsilon}{100000}$, $\mathcal{G}_4 \subseteq \mathcal{G}_5$, and all the elements of $\mathcal{G}_5$ have norm less than or equal to 1. We denote the above statements by "(+)".

In the notation of the statement of Lemma 2.8, we take $\delta = \delta_5$, $\gamma = \frac{\gamma_1}{10}$, $\mathcal{G} = \mathcal{G}_5$, $\mathcal{G}' = \mathcal{G}_1$, and $\mathcal{P} = \mathcal{P}_3$. Now suppose that $\mathcal{A}$ is an arbitrary unital simple separable finite real rank 0 $\mathcal{Z}$-stable C*-algebra, and suppose that

$$\phi, \psi : C(X) \to \mathcal{A}$$

are both unital c.p.c., $\mathcal{G}_5$-δ_5-multiplicative maps such that

$$[\phi]|\mathcal{P}_3 = [\psi]|\mathcal{P}_3,$$
$$\left|\tau \circ \phi(f) - \tau \circ \psi(f)\right| < \frac{\gamma_1}{10}$$

for all $f \in \mathcal{G}_1$,

$$\mu_{\tau\circ\phi}(O^e_j), \mu_{\tau\circ\psi}(O^e_j) > \sigma'\nu_2$$

for all $\tau \in T(\mathcal{A})$ and for all j (where $O^e_j =_{df} B(y_j, \frac{\nu_2}{2s'})$), and

$$\mu_{\tau\circ\phi}(O^c_j), \mu_{\tau\circ\psi}(O^c_j) > 2\sigma\nu_1$$

for all j and for all $\tau \in T(\mathcal{A})$. Let us denote the above statements by "(++)".

By (++) and by Proposition 2.4, there exist projections $p, q \in \mathcal{A}$ and $\mathcal{G}_4$-δ_4-multiplicative c.p.c. maps

$$L_\phi : C(X) \to p\mathcal{A}p$$

and

$$L_\psi : C(X) \to q\mathcal{A}q$$

such that the following statements are true.

(1) There exist pairwise orthogonal projections $p_0, p_1, \ldots, p_n, t \in \mathcal{A}$ and pairwise orthogonal projections $q_0, q_1, \ldots, q_n, t_1 \in \mathcal{A}$ with $p_0 = p$, $q_0 = q$, $\sum_{j=0}^n p_j + t = 1_{\mathcal{A}} = \sum_{j=0}^n q_j + t_1$, $N_1[p] \leq p_j$ and $N_1[q] \leq q_j$ for all $j \neq 0$.

(2) There exist finite-dimensional *-homomorphisms $h_1 : C(X) \to t\mathcal{A}t$ and $h_2 : C(X) \to t_1\mathcal{A}t_1$ such that

$$\phi(f) \approx_{\delta_4} \phi_1(f)$$

and

$$\psi(f) \approx_{\delta_4} \psi_1(f)$$

for all $f \in \mathcal{G}_4$, where

$$\phi_1(f) =_{df} L_\phi(f) + \sum_{j=1}^n f(y_j)p_j + h_1(f)$$

and

$$\psi_1(f) =_{df} L_\psi(f) + \sum_{j=1}^n f(y_j)q_j + h_2(f).$$

Note that $\delta_4 < \epsilon/10000$, and that both ϕ_1 and ψ_1 are c.p.c. $\mathcal{G}_4$-δ_4-multiplicative maps. We denote the above statements by "(+++)".

By (****), (++), and (+++), we also have that ϕ_1 and ψ_1 are well defined on $\mathcal{P}_3$,

$$[\phi_1]|\mathcal{P}_3 = [\psi_1]|\mathcal{P}_3,$$

$$\left|\tau \circ \phi_1(f) - \tau \circ \psi_1(f)\right| < \frac{3\gamma_1}{10}$$

for $f \in \mathcal{G}_3$, and

$$\mu_{\tau\circ\phi_1}(O_j^b), \mu_{\tau\circ\psi_1}(O_j^b) > \frac{3}{2}\sigma\nu_1$$

for all $\tau \in T(\mathcal{A})$ and for all j. We denote the above statements by "(++++)".

By [12, Theorem 4.5], there exists a unital simple AH-algebra $\mathcal{A}_0$ with bounded dimension growth and real rank 0, and there exists a unital *-homomorphism $\Phi : \mathcal{A}_0 \to \mathcal{A}$ which induces an order isomorphism (which, of course, respects the Bockstein operations) between the full K groups $\underline{K}(\mathcal{A}_0)$ and $\underline{K}(\mathcal{A})$. And since $\mathcal{A}_0$ and $\mathcal{A}$ are both real rank 0, this induces an isomorphism between the tracial simplexes $T(\mathcal{A})$ and $T(\mathcal{A}_0)$. Henceforth, to simplify notation, we view $\mathcal{A}_0$ as a *-subalgebra of $\mathcal{A}$, we view Φ as the inclusion map, we identify $\underline{K}(\mathcal{A}_0)$, $T(\mathcal{A}_0)$ with $\underline{K}(\mathcal{A})$, $T(\mathcal{A})$, and we take the induced maps $\underline{K}(\Phi)$, $T(\Phi)$ to be identity maps.

Let $\{p'_j\}_{j=0}^n \cup \{t'\}$ be pairwise orthogonal projections in $\mathcal{A}_0$, and let $\{q'_j\}_{j=0}^n \cup \{t'_1\}$ be pairwise orthogonal projections in $\mathcal{A}_0$ such that $p'_j \sim p_j$, $t' \sim t$, $q'_j \sim q_j$, and $t'_1 \sim t_1$ for all j, and

$$\sum_{j=0}^n p'_j + t' = 1_{\mathcal{A}_0} = 1_{\mathcal{A}} = \sum_{j=0}^n q'_j + t'_1.$$

Let $h_{0,1} : C(X) \to t'\mathcal{A}_0 t'$ and $h_{0,2} : C(X) \to t'_1\mathcal{A}_0 t'_1$ be finite-dimensional *-homomorphisms which are unitarily equivalent (in $\mathcal{A}$) to h_1, h_2, respectively.

By Theorem 2.7 and by (***), let $L_{0,\phi} : C(X) \to p'_0\mathcal{A}_0 p'_0$ and $L_{0,\psi} : C(X) \to q'_0\mathcal{A}_0 q'_0$ be unital c.p.c. $\mathcal{G}_3$-δ_3-multiplicative maps such that

$$[L_{0,\phi}]|\mathcal{P}_3 = [L_\phi]|\mathcal{P}_3$$

and

$$[L_{0,\psi}]|\mathcal{P}_3 = [L_\psi]|\mathcal{P}_3.$$

Let $\phi_2, \psi_2 : C(X) \to \mathcal{A}_0$ be the c.p.c. $\mathcal{G}_3$-δ_3-multiplicative maps given by

$$\phi_2(f) =_{df} L_{0,\phi}(f) + \sum_{j=1}^n f(y_j)p'_j + h_{0,1}(f)$$

and

$$\psi_2(f) =_{df} L_{0,\psi}(f) + \sum_{j=1}^n f(y_j)q'_j + h_{0,2}(f)$$

for all $f \in C(X)$. Therefore, by (***) and Theorem 2.6, let $u, v \in \mathcal{A}$ be unitaries such that

$$u\phi(f)u^* \approx_{\delta_4} u\phi_1(f)u^* \approx_{\delta_2/10} \phi_2(f)$$

and

$$v\psi(f)v^* \approx_{\delta_4} v\psi_1(f)v^* \approx_{\delta_2/10} \psi_2(f)$$

for all $f \in \mathcal{G}_2$. So

$$u\phi(f)u^* \approx_{\delta_4+\delta_2/10} \phi_2(f)$$

and

$$v\psi(f)v^* \approx_{\delta_4+\delta_2/10} \psi_2(f)$$

for all $f \in \mathcal{G}_2$. Note that $\delta_4 < \delta_3/10 < \delta_2/100$. So $\delta_4 + \delta_2/10 < 11\delta_2/100$. Hence, by (**),

$$[\phi_2]|\mathcal{P}_1 = \big[\mathrm{Ad}(u)\phi\big]|\mathcal{P}_1 = [\phi]|\mathcal{P}_1 = [\psi]|\mathcal{P}_1 = \big[\mathrm{Ad}(v)\psi\big]|\mathcal{P}_1 = [\psi_2]|\mathcal{P}_1,$$
$$\big|\tau \circ \phi_2(g) - \tau \circ \phi(g)\big| < \gamma_1/10$$

and

$$\big|\tau \circ \psi_2(g) - \tau \circ \psi(g)\big| < \gamma_1/10$$

for all $\tau \in T(\mathcal{A})$ and for all $g \in \mathcal{G}_1$. Hence, by (++),

$$\big|\tau \circ \phi_2(g) - \tau \circ \psi_2(g)\big| < 3\gamma_1/10$$

for all $\tau \in T(\mathcal{A}_0)$ and for all $g \in \mathcal{G}_1$.

Again, by (++++) and (**),

$$\mu_{\tau\circ\phi_2}(O_j^a), \mu_{\tau\circ\psi_2}(O_j^a) > \sigma\nu_1$$

for all $\tau \in T(\mathcal{A}_0)$ and for all j. Hence, by (*) and Theorem 2.5, let $w \in \mathcal{A}_0$ be a unitary such that

$$w\phi_2(f)w^* \approx_{\epsilon/10} \psi_2(f)$$

for all $f \in \mathcal{F}$. Hence,

$$wu\phi(f)u^*w^* \approx_{\epsilon/10+2\delta_4+\delta_2/5} v\psi(f)v^*$$

for all $f \in \mathcal{F}$. Now $2\delta_4 + \delta_2/5 < 11\delta_2/50 < 11\epsilon/500$. Hence,

$$v^*wu\phi(f)u^*w^*v \approx_{\epsilon} \psi(f)$$

for all $f \in \mathcal{F}$. □

Recall our standing hypotheses for this paper in (i). For all unital simple separable stably finite C*-algebras appearing in this paper, all quasitraces are assumed to be traces (ii). All c.p.c. maps between C*-algebras in this paper are assumed to be linear. Also, for a unital C*-algebra $\mathcal{C}$ and for $\tau \in T(\mathcal{C})$, recall that $d_\tau : \mathcal{C}_+ \to [0, \infty)$ is the map given by $d_\tau(c) =_{df} \limsup_{n\to\infty} \tau(c^{1/n})$ for all $c \in \mathcal{C}_+$.

Theorem 2.9. *Let X be a finite CW-complex, $\epsilon > 0$, and $\mathcal{F} \subset C(X)$ a finite subset. Then there exists a nonempty finite subset $\mathcal{E} \subset C(X)_+ - \{0\}$ such that for all $\lambda > 0$, there exist a finite subset $\mathcal{G} \subset C(X)$, $\delta > 0$, a finite subset $\mathcal{E}' \subset C(X)$, $\gamma > 0$, and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ such that the following hold.*

For all unital separable simple finite real rank 0 $\mathcal{Z}$-stable C-algebras $\mathcal{A}$, for all unital c.p.c. $\mathcal{G}$-δ-multiplicative maps $\phi, \psi : C(X) \to \mathcal{A}$, if*

$$[\phi]|\mathcal{P} = [\psi]|\mathcal{P},$$
$$\big|\tau \circ \phi(f) - \tau \circ \psi(f)\big| < \gamma$$

for all $f \in \mathcal{E}'$, for all $\tau \in T(\mathcal{A})$, and

$$d_\tau\big(\phi(g)\big), d_\tau\big(\psi(g)\big) > \lambda$$

for all $\tau \in T(\mathcal{A})$ and all $g \in \mathcal{E}$, then there exists a unitary $u \in \mathcal{A}$ such that

$$u\phi(f)u^* \approx_\epsilon \psi(f)$$

for all $f \in \mathcal{F}$.

Proof. This follows immediately from Lemma 2.8. □

Let X be a compact metric space, and let $\mathcal{A}$ be a C*-algebra. Recall that a *-homomorphism $\phi : C(X) \to \mathcal{A}$ is regarded as *finite-dimensional* if there exist $x_1, x_2, \ldots, x_n \in X$ and pairwise orthogonal projections $p_1, p_2, \ldots, p_n \in \mathcal{A}$ such that $\phi(f) = \sum_{j=1}^n f(x_j)p_j$ for all $f \in C(X)$.

Definition 2.10. Let X be a compact metric space, and let $\mathcal{A}$ be a C*-algebra. Let $\mathcal{P} \subseteq \mathbb{P}(C(X))$. Then $\mathfrak{N}_\mathcal{P}$ denotes the set of all maps $\alpha : \mathcal{P} \to \underline{K}(\mathcal{A})$ such that there exists a finite-dimensional *-homomorphism $\phi : C(X) \to \mathbb{M}_k \otimes \mathcal{A}$ for which $[\phi]|_\mathcal{P} = \alpha$. Let $\mathfrak{N}$ denote the set of $\alpha \in KL(C(X), \mathcal{A})$ such that there exists a finite-dimensional *-homomorphism $\phi : C(X) \to \mathbb{M}_k \otimes \mathcal{A}$ for which $[\phi] = \alpha$.

Theorem 2.11. *Let X be a finite CW-complex, $\epsilon > 0$, and $\mathcal{F} \subset C(X)$ be a finite subset. Then there exists a nonempty finite subset $\mathcal{E} \subset C(X)_+ - \{0\}$ such that for all $\lambda > 0$, there exist a finite subset $\mathcal{G} \subset C(X)$, $\delta > 0$, and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ such that the following hold.*

For all unital separable simple finite real rank 0 $\mathcal{Z}$-stable C-algebras $\mathcal{A}$, for every unital c.p.c. $\mathcal{G}$-δ-multiplicative map $\phi : C(X) \to \mathcal{A}$, if $[\phi]|_\mathcal{P} : \mathcal{P} \to \underline{K}(\mathcal{A})$ lies in $\mathfrak{N}_\mathcal{P}$, and*

$$d_\tau\big(\phi(g)\big) > \lambda$$

*for all $\tau \in T(\mathcal{A})$ and all $g \in \mathcal{E}$, then there exists a unital *-homomorphism $\psi : C(X) \to \mathcal{A}$ with finite-dimensional range such that*

$$\big\|\phi(f) - \psi(f)\big\| < \epsilon$$

for all $f \in \mathcal{F}$.

Proof. By considering each path-connected component, we may assume that X is path-connected. Let $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$ be given. Plug $\epsilon/10$, $\mathcal{F}$ into Theorem 2.6 to get δ_1, η_1, $N \geq 1$, $\mathcal{G}_1 \subset C(X)$ (finite subset), and $\mathcal{P} \subset \mathbb{P}(C(X))$ (finite subset). Choose $\delta_2 > 0$ and a finite subset $\mathcal{G}_2 \subset C(X)$

such that for all unital C*-algebras $\mathcal{C}$, if $\psi_1, \psi_2 : C(X) \to \mathcal{C}$ are unital c.p.c. $\mathcal{G}_2$-δ_2-multiplicative maps such that

$$\psi_1(f) \approx_{\delta_2} \psi_2(f)$$

for all $f \in \mathcal{G}_2$, then

$$[\psi_1]|_{\mathcal{P}} = [\psi_2]|_{\mathcal{P}}.$$

We may assume that $\delta_2 < \frac{1}{100}\min\{\epsilon, \delta_1\}$ and $\mathcal{G}_1 \cup \mathcal{F} \subset \mathcal{G}_2$. Choose $\eta_2 > 0$ with $\eta_2 < \eta_1$ such that for all $f \in \mathcal{G}_2$, $|f(x) - f(x')| < \delta_2/10$ if $\mathrm{dist}(x, x') < 2\eta_2$. Let $\{x_1, x_2, \ldots, x_n\} \subset X$ be an $\eta_2/2$-dense subset such that $\overline{B(x_j, \eta_2/4)} \cap \overline{B(x_k, \eta_2/4)} = \emptyset$ for all $j \neq k$. Let $\mathcal{E} =_{df} \{f_j : 1 \leq j \leq n\}$ where for all j, $f_j \in C(X)$ is such that

$$f_j(x) \begin{cases} > 0 & x \in B(x_j, \eta_2/8), \\ = 0 & x \notin B(x_j, \eta_2/8). \end{cases}$$

Let $\lambda > 0$ be arbitrary. Choose σ with $1/8 > \sigma > 0$ so that

$$\lambda/2 > \sigma\eta_2.$$

Plug $X, \delta_2, \mathcal{G}_2, N, \eta_2$, and σ into Proposition 2.4 to get δ and $\mathcal{G}$.

Now suppose that $\mathcal{A}$ is a unital separable simple finite real rank 0 $\mathcal{Z}$-stable C*-algebra, and suppose that $\phi : C(X) \to \mathcal{A}$ is a c.p.c. $\mathcal{G}$-δ-multiplicative map such that

$$[\phi] \in \mathfrak{N}_{\mathcal{P}}$$

and

$$d_\tau\big(\phi(f)\big) > \lambda$$

for all $f \in \mathcal{E}$ and for all $\tau \in T(\mathcal{A})$. By Proposition 2.4, there exist a projection $p \in \mathcal{A}$ and a unital c.p.c. $\mathcal{G}_2$-δ_2-multiplicative map $L_1 : C(X) \to p\mathcal{A}p$ such that the following statements are true.

(1) There exist pairwise orthogonal projections $p_0, p_1, \ldots, p_n, t \in \mathcal{A}$ with $p_0 = p$, $\sum_{j=0}^n p_j + t = 1_{\mathcal{A}}$ and $Np \preceq p_j$ for all $j \neq 0$.

(2) There exists a finite-dimensional *-homomorphism $h_1 : C(X) \to t\mathcal{A}t$ such that for all $f \in \mathcal{G}_2$,

$$\phi(f) \approx_{\delta_2} L_1(f) + \sum_{j=1}^n f(x_j)p_j + h_1(f).$$

Let $h_2 : C(X) \to (1-p)\mathcal{A}(1-p)$ be the finite-dimensional *-homomorphism given by

$$h_2(f) =_{df} \sum_{j=1}^n f(x_j)p_j + h_1(f)$$

for all $f \in C(X)$. By the definition of δ_2 and $\mathcal{G}_2$,

$$[\phi]|_{\mathcal{P}} = [L_1 + h_2]|_{\mathcal{P}}.$$

Therefore, since $[\phi]|_{\mathcal{P}} \in \mathfrak{N}_{\mathcal{P}}$, $[L_1+h_2]|_{\mathcal{P}} \in \mathfrak{N}_{\mathcal{P}}$. Let $x_0 \in X$, and let $h_3 : C(X) \to \mathcal{A}$ be the finite-dimensional *-homomorphism given by $h_3(f) =_{df} f(x_0)1_{\mathcal{A}}$ for all $f \in C(X)$. Since X is path-connected, $[L_1 + h_2]|_{\mathcal{P}} = [h_3]|_{\mathcal{P}}$. Hence,

$$[L_1]|_{\mathcal{P}} + [h_2]|_{\mathcal{P}} = [ph_3]|_{\mathcal{P}} + \big[(1-p)h_3\big]|_{\mathcal{P}}.$$

(Note that $ph_3(f) = f(x_0)p$ and $(1-p)h_3(f) = f(x_0)(1-p)$ for all $f \in C(X)$.)

Clearly, $[h_2] = [(1-p)h_3]$. Hence,

$$[L_1]|_{\mathcal{P}} = [ph_3]|_{\mathcal{P}}.$$

Let $h_4 : C(X) \to \mathcal{A}$ be the finite-dimensional unital *-homomorphism given by

$$h_4(f) =_{df} ph_3(f) + h_2(f)$$

for all $f \in C(X)$. Therefore, by Theorem 2.6, there exists a unitary $u \in \mathcal{A}$ such that

$$L_1(f) + h_2(f) \approx_{\epsilon/10} uh_4(f)u^*$$

for all $f \in \mathcal{F}$. Hence,

$$\phi(f) \approx_{\epsilon} uh_4(f)u^*$$

for all $f \in C(X)$. □

We note that in our context (finite CW-complexes), our K-theoretic condition is the same as that of [6] and a special case of that of [16] (see, e.g., the discussion in [10, Section 1.2]). We also note that since X is a finite CW-complex, $KK(C(X), \mathcal{A}) = KL(C(X), \mathcal{A})$.

Theorem 2.12. *Let X be a finite CW-complex, and let $\mathcal{A}$ be a unital simple separable finite $\mathcal{Z}$-stable C^*-algebra with real rank 0. Let $\phi : C(X) \to \mathcal{A}$ be a unital *-monomorphism. Then ϕ is the pointwise-norm limit of finite-dimensional *-homomorphisms if and only if $[\phi] \in \mathfrak{N}$.*

Proof. This follows immediately from Theorem 2.11. □

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