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VECTOR LATTICES AND f-ALGEBRAS: THE CLASSICAL INEQUALITIES

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Abstract. We present some of the classical inequalities in analysis in the context of Archimedean (real or complex) vector lattices and f-algebras. In particular, we prove an identity for sesquilinear maps from the Cartesian square of a vector space to a geometric mean closed Archimedean vector lattice, from which a Cauchy–Schwarz inequality follows. A reformulation of this result for sesquilinear maps with a geometric mean closed semiprime Archimedean f-algebra as codomain is also given. In addition, a sufficient and necessary condition for equality is presented. We also prove a Hölder inequality for weighted geometric mean closed Archimedean Φ-algebras, substantially improving results by K. Boulabiar and M. A. Toumi. As a consequence, a Minkowski inequality for weighted geometric mean closed Archimedean Φ-algebras is obtained.

1. Introduction

Rich connections between the theory of Archimedean vector lattices and the classical inequalities in analysis, although hitherto little explored, were implied in [5] and the subsequent developments in [8], [13], [18], and [22]. In particular, [13] presents a relationship between the Cauchy–Schwarz inequality and the theory of multilinear maps on vector lattices, built on the analogy between disjointness in vector lattices and orthogonality in inner-product spaces. The ideas in [13] led to the construction of powers of vector lattices (see [9], [14]), a theory that

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was recently extended to the complex vector lattice environment in [11]. This article follows the complex theme of [11] and in fact contains results that are valid for both real vector lattices and complex vector lattices. We also conjoin the Cauchy–Schwarz, Hölder, and Minkowski inequalities with the theory of geometric mean closed Archimedean vector lattices, as found in [2] and [3]. We first discuss these results more closely.

In [13, Corollary 4], the first author and van Rooij extend the classical Cauchy–Schwarz inequality as follows. If V is a real vector space and A is an Archimedean almost f-algebra, then for every bilinear map $T\colon V\times V\to A$ such that

(1) $T(v,v)\geq 0$ $(v\in V)$ and
(2) $T(u,v)=T(v,u)$ $(u,v\in V)$

we have

$$T(u,v)^2\leq T(u,u)T(v,v)\quad (u,v\in V).$$

This is the classical Cauchy–Schwarz inequality when $A=\mathbb{R}$, which is equivalent to

$$\bigl|T(u,v)\bigr|\leq \bigl(T(u,u)T(v,v)\bigr)^{1/2}=2^{-1}\inf\bigl\{\theta T(u,u)+\theta^{-1}T(v,v):\theta\in(0,\infty)\bigr\}\quad \text{(i)}$$

for $u,v\in V$.

The proof of [13, Corollary 4] easily adapts to a natural complex analogue for sesquilinear maps, although the condition for equality in the classical Cauchy–Schwarz inequality (see, e.g., [15, p. 3]) does not hold in this more general context (see Example 3.3).

In Theorem 3.1 of this article, we extend both the real and complex versions of the classical Cauchy–Schwarz inequality by replacing the codomain of the sesquilinear maps with an Archimedean (real or complex) vector lattice that is closed under the infimum in (i) above. We also prove a convenient formula for the difference between the two sides of the Cauchy–Schwarz inequality and use it to generalize the known condition for equality in the classical case. In Corollary 3.2, we obtain a Cauchy–Schwarz inequality as well as a condition for equality for sesquilinear maps with values in a semiprime Archimedean f-algebra that is closed under the infimum in (i).

Theorem 4.7 of this article proves a Hölder inequality for positive linear maps between Archimedean Φ-algebras that are closed under certain weighted renditions of (i). Our Hölder inequality generalizes [8, Theorem 5, Corollary 6] and [22, Theorem 3.12] by (1) weakening the assumption of uniform completeness, (2) including irrational exponents via explicit formulas without restricting the codomain to the real numbers, (3) providing a result for several variables, and (4) enabling the domain and codomain of the positive linear maps in question to be either both real Φ-algebras or both complex Φ-algebras.

We note that Theorem 4.7 is itself a consequence of Proposition 4.1, which in turn generalizes a reformulation of the classical Hölder inequality by Maligranda [20, (HI_1)]. Indeed, Proposition 4.1 is a reinterpretation of (HI_1) for Archimedean vector lattices that simultaneously extends (HI_1) to several variables. We add that Kusraev [18, Theorem 4.2] independently developed his own version of (HI_1) in the setting of uniformly complete Archimedean vector lattices. Our Proposition 4.1

contains [18, Theorem 4.2]. Noting that Proposition 4.1 relies primarily on the Archimedean vector lattice functional calculus, it (contrary to [18, Theorem 4.2]) depends only on (at most) the countable axiom of choice.

Finally, we employ the Hölder inequality of Theorem 4.7 to prove a Minkowski inequality in Theorem 5.1.

We proceed with some preliminaries.

2. Preliminaries

We refer the reader to [1], [19], and [23] for any unexplained terminology regarding vector lattices and f-algebras. Throughout, $\mathbb{R}$ is used for the real numbers, $\mathbb{C}$ denotes the complex numbers, $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$, and the symbol for the set of strictly positive integers is $\mathbb{N}$.

An Archimedean real vector lattice E is said to be *square mean closed* (see [3, p. 482]) if $\sup\{(\cos\theta)f+(\sin\theta)g : \theta\in[0,2\pi]\}$ exists in E for every $f,g\in E$, and in this case we write

$$f \boxplus g = \sup\bigl\{(\cos\theta)f+(\sin\theta)g : \theta\in[0,2\pi]\bigr\}\quad (f,g\in E).$$

The notion of square mean closedness in vector lattices dates back to a 1973 paper by de Schipper, under the term *property (E)* (see [17, p. 356]). We adopt de Schipper's definition of an Archimedean complex vector lattice but use the terminology found in [3].

Throughout, $V+iV$ denotes the commonly used vector space complexification of a real vector space V. An *Archimedean complex vector lattice* is a complex vector space of the form $E+iE$, where E is a square mean closed Archimedean real vector lattice (see [17, pp. 356–357]).

An Archimedean real vector lattice will also be called an *Archimedean vector lattice over* $\mathbb{R}$, and an Archimedean complex vector lattice will additionally be referred to as an *Archimedean vector lattice over* $\mathbb{C}$. An *Archimedean vector lattice over* $\mathbb{K}$ is a vector space that is either an Archimedean vector lattice over $\mathbb{R}$ or an Archimedean vector lattice over $\mathbb{C}$. Equivalently, an Archimedean vector lattice over $\mathbb{K}$ is a vector space over $\mathbb{K}$ that is equipped with an Archimedean modulus, as defined axiomatically by Mittelmeyer and Wolff in [21, Definition 1.1].

Given an Archimedean vector lattice $E+iE$ over $\mathbb{C}$, we write $\mathrm{Re}(f+ig)=f$ and $\mathrm{Im}(f+ig)=g$ $(f,g\in E)$. For convenience, we write $\mathrm{Re}(f)=f$ and $\mathrm{Im}(f)=0$ $(f\in E)$ when E is an Archimedean vector lattice over $\mathbb{R}$. Lemma 1.2, Korollar 1.4, Proposition 1.5, and Satz 2.2 of [21] together imply that the Archimedean modulus on an Archimedean vector lattice E over $\mathbb{K}$ is given by the formula

$$|f| = \sup\bigl\{\mathrm{Re}(\lambda f) : \lambda\in\mathbb{K}, |\lambda|=1\bigr\}\quad (f\in E).$$

In particular, for an Archimedean vector lattice E over $\mathbb{R}$ we have $|f|=f\vee(-f)$ $(f\in E)$, while $|f+ig|=f\boxplus g$ $(f,g\in E)$ holds in any Archimedean vector lattice $E+iE$ over $\mathbb{C}$.

For an Archimedean vector lattice E over $\mathbb{K}$, we define the *positive cone* E^+ of E by $E^+=\{f\in E : |f|=f\}$, while the real vector lattice $E_\rho=\{f-g : f,g\in E^+\}$ is called the *real part* of E. With this notation, $E=E_\rho$ for every Archimedean vector lattice E over $\mathbb{R}$, whereas $E = E_\rho + iE_\rho$ whenever E is an Archimedean vector lattice over $\mathbb{C}$.

We say that an Archimedean vector lattice E over $\mathbb{K}$ is *uniformly complete* if E_ρ is uniformly complete. Note that every uniformly complete Archimedean vector lattice over $\mathbb{R}$ is square mean closed (see [7, Section 2]).

An Archimedean vector lattice E over $\mathbb{R}$ is said to be *geometric mean closed* (see [3, p. 486]) if $\inf\{\theta f + \theta^{-1} g : \theta \in (0,\infty)\}$ exists in E for every $f, g \in E^+$, and in this case we write

$$f \boxtimes g = 2^{-1} \inf\big\{\theta f + \theta^{-1} g : \theta \in (0,\infty)\big\} \quad (f, g \in E^+).$$

We define an Archimedean vector lattice over $\mathbb{K}$ to be *geometric mean closed* (resp., *square mean closed*) if A_ρ is geometric mean closed (resp., square mean closed). Thus every Archimedean vector lattice over $\mathbb{C}$ is square mean closed.

We next provide some basic information regarding Archimedean f-algebras that will be needed throughout this article.

The multiplication on an Archimedean f-algebra A canonically extends to a multiplication on $A + iA$. We call an Archimedean vector lattice A over $\mathbb{K}$ an *Archimedean f-algebra over* $\mathbb{K}$ if A_ρ is an f-algebra. If in addition A has a multiplicative identity, then we say that A is an *Archimedean Φ-algebra over* $\mathbb{K}$. It was proved in [16, Corollary 10.4] (also see [23, Theorem 142.5]) that every Archimedean Φ-algebra over $\mathbb{R}$ (and therefore every Archimedean Φ-algebra over $\mathbb{K}$) is semiprime.

Given an Archimedean f-algebra A over $\mathbb{K}$, we call a vector sublattice A_0 of A an *f-subalgebra* of A if A_0 is itself an f-algebra under the multiplication inherited from A. If A is an Archimedean Φ-algebra over $\mathbb{K}$ with multiplicative identity e, we call a vector sublattice A_0 of A a *Φ-subalgebra* of A if A_0 is an f-subalgebra of A for which $e \in A_0$. The smallest f-subalgebra (resp., Φ-subalgebra) of an Archimedean f-algebra (resp., Φ-algebra) A over $\mathbb{K}$ that contains $f_1, \dots, f_n \in A$ will be called the *f-subalgebra of A generated by $f_1, \dots, f_n$* (resp., *Φ-subalgebra of A generated by $f_1, \dots, f_n$*).

The multiplication on an Archimedean f-algebra over $\mathbb{K}$ will be denoted by juxtaposition throughout. For Archimedean f-algebras A and B over $\mathbb{K}$, we say that a map $T\colon A \to B$ is *multiplicative* if $T(ab) = T(a)T(b)$ $(a, b \in A)$.

Let A be an Archimedean f-algebra over $\mathbb{K}$, and let $n \in \mathbb{N}$ and $a \in A^+$. If there exists a unique element r of A^+ such that $r^n = a$, we write $r = a^{1/n}$ and say that $a^{1/n}$ exists. If A is an Archimedean semiprime f-algebra and $a, r \in A^+$ satisfy $r^n = a$, then $r = a^{1/n}$ (see [6, Proposition 2(ii)]). Given $m, n \in \mathbb{N}$, we write $a^{m/n} = (a^m)^{1/n}$, provided $(a^m)^{1/n}$ exists.

Every uniformly complete semiprime Archimedean f-algebra A over $\mathbb{R}$ is geometric mean closed (see [2, Theorem 2.21]) and

$$f \boxtimes g = (fg)^{1/2} \quad (f, g \in A^+). \tag{ii}$$

The formula (ii) also holds in the weaker condition when A is geometric mean closed. In fact, the proof of (ii) in [2, Theorem 2.21] does not require uniform completeness, as illustrated in the next proposition. Since [2] is not widely accessible, we reproduce the proof of [2, Theorem 2.21], while (trivially) extending this theorem to include complex f-algebras.

Proposition 2.1. *Let A be a semiprime Archimedean f-algebra over $\mathbb{K}$. If A is geometric mean closed, then $f \boxtimes g = (fg)^{1/2}$ $(f, g \in A^+)$.*

Proof. Evidently, A_ρ is a semiprime Archimedean f-algebra over $\mathbb{R}$. Let $f, g \in A^+$, and let C be the f-subalgebra of A_ρ generated by the elements $f, g, f \boxtimes g$. Suppose that $\phi\colon C \to \mathbb{R}$ is a nonzero multiplicative vector lattice homomorphism. Using [2, Proposition 2.20] or [12, Corollary 3.13] (first equality), we obtain

$$\phi(f \boxtimes g) = \phi(f) \boxtimes \phi(g) = \big(\phi(f)\phi(g)\big)^{1/2} = \big(\phi(fg)\big)^{1/2}.$$

Therefore,

$$\phi\big((f \boxtimes g)^2\big) = \big(\phi(f \boxtimes g)\big)^2 = \phi(fg).$$

Since the set of all nonzero multiplicative vector lattice homomorphisms from C into $\mathbb{R}$ separates the points of C (see [10, Corollary 2.7]), we therefore have $(f \boxtimes g)^2 = fg$. $\square$

We conclude this section with some basic terminology regarding Archimedean vector lattices over $\mathbb{K}$.

Given an Archimedean vector lattice E over $\mathbb{C}$, we define the complex conjugate

$$\overline{f + ig} = f - ig \quad (f, g \in E_\rho).$$

Since every Archimedean vector lattice over $\mathbb{R}$ canonically embeds into an Archimedean vector lattice over $\mathbb{C}$ (see [11, Theorem 3.3]), the previous definition also makes sense in Archimedean vector lattices over $\mathbb{R}$ (via such an embedding). If E is an Archimedean vector lattice over $\mathbb{K}$, then the familiar identities $\mathrm{Re}(f) = 2^{-1}(f + \bar{f})$ and $\mathrm{Im}(f) = (2i)^{-1}(f - \bar{f})$ are valid for every $f \in E$.

Let V be a vector space over $\mathbb{K}$, and suppose that F is an Archimedean vector lattice over $\mathbb{K}$. A map $T\colon V \times V \to F$ is called *positive semidefinite* if $T(v, v) \geq 0$ for every $v \in V$. If $T(u, v) = \overline{T(v, u)}$ for each $u, v \in V$, then T is said to be *conjugate symmetric*. We say that T is *sesquilinear* if

(1) $T(\alpha u_1 + \beta u_2, v) = \alpha T(u_1, v) + \beta T(u_2, v)$ $(\alpha, \beta \in \mathbb{K}, u_1, u_2, v \in V)$, and
(2) $T(u, \alpha v_1 + \beta v_2) = \bar{\alpha} T(u, v_1) + \bar{\beta} T(u, v_2)$ $(\alpha, \beta \in \mathbb{K}, u, v_1, v_2 \in V)$.

3. A Cauchy–Schwarz inequality

We prove a Cauchy–Schwarz inequality for sesquilinear maps with a geometric mean closed Archimedean vector lattice over $\mathbb{K}$ as codomain (see Theorem 3.1) and with a geometric mean closed semiprime Archimedean f-algebra over $\mathbb{K}$ as codomain (see Corollary 3.2). A necessary and sufficient condition for equality in Theorem 3.1 and Corollary 3.2 is given via an explicit formula for the difference between the two sides in the Cauchy–Schwarz inequality of Theorem 3.1 mentioned above. However, Example 3.3 illustrates that the condition for equality in the classical Cauchy–Schwarz inequality fails in Theorem 3.1 and Corollary 3.2.

We proceed to the main result of this section.

Theorem 3.1 (Cauchy–Schwarz inequality)**.** *Let V be a vector space over $\mathbb{K}$, and suppose that F is a geometric mean closed Archimedean vector lattice over $\mathbb{K}$. If a map $T\colon V\times V\to F$ is positive semidefinite, conjugate symmetric, and sesquilinear, then*

(1) $\inf_{z\in\mathbb{K}\setminus\{0\}}\{|z|^{-1}T(zu-v,zu-v)\}$ *exists in F $(u,v\in V)$,*
(2) $|T(u,v)| = T(u,u)\boxtimes T(v,v) - 2^{-1}\inf_{z\in\mathbb{K}\setminus\{0\}}\{|z|^{-1}T(zu-v,zu-v)\}$ *$(u,v\in V)$,*
(3) $|T(u,v)|\leq T(u,u)\boxtimes T(v,v)$ *$(u,v\in V)$, and*
(4) $|T(u,v)| = T(u,u)\boxtimes T(v,v)$ *if and only if* $\inf_{z\in\mathbb{K}\setminus\{0\}}\{|z|^{-1}T(zu-v,zu-v)\}=0$.

Proof. Suppose that $T\colon V\times V\to F$ is a positive semidefinite, conjugate symmetric, sesquilinear map. Let $u,v\in V$, and put $\theta\in(0,\infty)$. Using the sesquilinearity of T and the identity $\mathrm{Re}(f)=2^{-1}(f+\bar{f})$ for $f\in F$, we obtain

$$\begin{aligned} T(\theta u-v,u-\theta^{-1}v) &= T(\theta u,u)+T(v,\theta^{-1}v)-T(\theta u,\theta^{-1}v)-T(v,u)\\ &=\theta T(u,u)+\theta^{-1}T(v,v)-2\,\mathrm{Re}\big(T(u,v)\big). \end{aligned}$$

Therefore,

$$\mathrm{Re}\big(T(u,v)\big)=2^{-1}\big(\theta T(u,u)+\theta^{-1}T(v,v)\big)-(2\theta)^{-1}T(\theta u-v,\theta u-v).$$

In particular, for every $\lambda\in S=\{\lambda\in\mathbb{K}:|\lambda|=1\}$ we have

$$\begin{aligned} \mathrm{Re}\big(\lambda T(u,v)\big) &= \mathrm{Re}\big(T(\lambda u,v)\big)\\ &=2^{-1}\big(\theta T(u,u)+\theta^{-1}T(v,v)\big)-(2\theta)^{-1}T(\theta\lambda u-v,\theta\lambda u-v). \end{aligned}$$

Thus we obtain

$$\begin{aligned} \big|T(u,v)\big| &= \sup_{\lambda\in S}\big\{\mathrm{Re}\big(\lambda T(u,v)\big)\big\}\\ &=2^{-1}\sup_{\lambda\in S}\big\{\theta T(u,u)+\theta^{-1}T(v,v)-\theta^{-1}T(\theta\lambda u-v,\theta\lambda u-v)\big\}\\ &=2^{-1}\big(\theta T(u,u)+\theta^{-1}T(v,v)\big)+2^{-1}\sup_{\lambda\in S}\big\{-\theta^{-1}T(\theta\lambda u-v,\theta\lambda u-v)\big\}\\ &=2^{-1}\big(\theta T(u,u)+\theta^{-1}T(v,v)\big)-2^{-1}\inf_{\lambda\in S}\big\{\theta^{-1}T(\theta\lambda u-v,\theta\lambda u-v)\big\}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} T(u,u)\boxtimes T(v,v) &= \inf_{\theta\in(0,\infty)}\Big\{\big|T(u,v)\big|+2^{-1}\inf_{\lambda\in S}\big\{\theta^{-1}T(\theta\lambda u-v,\theta\lambda u-v)\big\}\Big\}\\ &=\big|T(u,v)\big|+2^{-1}\inf_{\theta\in(0,\infty)}\Big\{\inf_{\lambda\in S}\big\{\theta^{-1}T(\theta\lambda u-v,\theta\lambda u-v)\big\}\Big\}\\ &=\big|T(u,v)\big|+2^{-1}\inf_{\lambda\in S,\theta\in(0,\infty)}\big\{\theta^{-1}T(\theta\lambda u-v,\theta\lambda u-v)\big\}\\ &=\big|T(u,v)\big|+2^{-1}\inf_{z\in\mathbb{K}\setminus\{0\}}\big\{|z|^{-1}T(zu-v,zu-v)\big\}. \end{aligned}$$

This proves statements (1) and (2) of this theorem. Statements (3) and (4) immediately follow from statement (2). $\square$

As a consequence of Theorem 3.1 and Proposition 2.1, we obtain the following.

Corollary 3.2. *Let V be a vector space over $\mathbb{K}$, and suppose that A is a geometric mean closed semiprime Archimedean f-algebra over $\mathbb{K}$. If $T\colon V \times V \to A$ is a positive semidefinite, conjugate symmetric, sesquilinear map, then*

(1) $\inf_{z\in\mathbb{K}\setminus\{0\}}\{|z|^{-1}T(zu-v, zu-v)\}$ *exists in A $(u, v \in V)$,*
(2) $|T(u,v)|^2 = ((T(u,u)T(v,v))^{1/2} - 2^{-1}\inf_{z\in\mathbb{K}\setminus\{0\}}\{|z|^{-1}T(zu-v, zu-v)\})^2$ *$(u, v \in V)$,*
(3) $|T(u,v)|^2 \le T(u,u)T(v,v)$ *$(u, v \in V)$, and*
(4) $|T(u,v)|^2 = T(u,u)T(v,v)$ *if and only if* $\inf_{z\in\mathbb{K}\setminus\{0\}}\{|z|^{-1}T(zu-v, zu-v)\} = 0$.

Proof. Statement (1) follows from Theorem 3.1(1). To prove (2), let $T\colon V\times V \to A$ be a positive semidefinite, conjugate symmetric, sesquilinear map, and also let $u, v \in V$. From Theorem 3.1(2) and Proposition 2.1, we have

$$\big|T(u,v)\big| = \big(T(u,u)T(v,v)\big)^{1/2} - 2^{-1}\inf_{z\in\mathbb{K}\setminus\{0\}}\big\{|z|^{-1}T(zu-v, zu-v)\big\}. \tag{iii}$$

Squaring both sides of (iii) verifies (2) (see [6, Proposition 2(ii)]). As a consequence of (iii), we obtain

$$\big|T(u,v)\big| \le \big(T(u,u)T(v,v)\big)^{1/2},$$

and squaring both sides of the above inequality proves (3) (see [6, Proposition 2(iii)]). To prove (4), note that if $\inf_{z\in\mathbb{K}\setminus\{0\}}\{|z|^{-1}T(zu-v, zu-v)\} = 0$, then $|T(u,v)|^2 = T(u,u)T(v,v)$ by (2). Conversely, if $|T(u,v)|^2 = T(u,u)T(v,v)$, then by [6, Proposition 2(ii)] and Proposition 2.1 we have

$$\big|T(u,v)\big| = \big(T(u,u)T(v,v)\big)^{1/2} = T(u,u) \boxtimes T(v,v).$$

That $\inf_{z\in\mathbb{K}\setminus\{0\}}\{|z|^{-1}T(zu-v, zu-v)\} = 0$ now follows from Theorem 3.1(4). □

For $\mathbb{K} = \mathbb{R}$, Corollary 3.2(3) is contained in [13, Corollary 4]. Similarly, the given statement is contained in the complex analogue of [13, Corollary 4] (mentioned in the Introduction) when $\mathbb{K} = \mathbb{C}$. Corollary 3.2(2), however, depends on the uniqueness of square roots, which implies the semiprime property in Archimedean almost f-algebras. Indeed, let A be an Archimedean almost f-algebra, and suppose that $a^2 = b^2$ implies $a = b(a, b \in A^+)$. Let a be a nilpotent in A. Then $a^3 = 0$ (see [4, Theorem 3.2]), and thus $a^4 = 0$. Using the uniqueness of square roots twice, we obtain $a = 0$. Finally, every semiprime almost f-algebra is automatically an f-algebra (see [4, Theorem 1.11(i)]).

The special case where $A = \mathbb{K}$ in the inequality of Corollary 3.2 is the classical Cauchy–Schwarz inequality. Thus we know in this special case that $|T(u,v)|^2 = T(u,u)T(v,v)$ if and only if there exist $\alpha, \beta \in \mathbb{K}$, not both zero, such that $T(\beta u + \alpha v, \beta u + \alpha v) = 0$ (see, e.g., [15, p. 3]). This criterion no longer holds for Theorem 3.1 nor Corollary 3.2.

Example 3.3. Define $T\colon \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K}^2$ by

$$T\big((z_1, z_2), (w_1, w_2)\big) = (z_1\bar{w}_1, z_2\bar{w}_2) \quad \big((z_1, z_2), (w_1, w_2) \in \mathbb{K}^2\big).$$

Since $\mathbb{C}^2 = \mathbb{R}^2 + i\mathbb{R}^2$, we see that $\mathbb{K}^2$ is a geometric mean closed semiprime Archimedean f-algebra over $\mathbb{K}$ with respect to the coordinatewise vector space operations, coordinatewise ordering, and coordinatewise multiplication. Also, T is a positive semidefinite, conjugate symmetric, sesquilinear map. Note that

$$\left|T\big((1,0),(0,1)\big)\right|^2 = T\big((1,0),(1,0)\big)T\big((0,1),(0,1)\big).$$

Suppose that there exist $\alpha, \beta \in \mathbb{K}$, not both zero, for which

$$T\big(\beta(1,0)+\alpha(0,1), \beta(1,0)+\alpha(0,1)\big) = 0.$$

Then $(|\beta|^2, |\alpha|^2) = (0,0)$, which is a contradiction.

4. A Hölder inequality

We prove a Hölder inequality for positive linear maps between weighted geometric mean closed Archimedean Φ-algebras over $\mathbb{K}$ in this section, extending [8, Theorem 5, Corollary 6] by Boulabiar and [22, Theorem 3.12] by Toumi. We begin with some definitions.

Let A be an Archimedean f-algebra over $\mathbb{K}$, and suppose that $n \in \mathbb{N}$. As usual, we write $\prod_{k=1}^n a_k = a_1 \cdots a_n$ for $a_1, \ldots, a_n \in A$.

For every $r_1, \ldots, r_n \in (0,1)$ such that $\sum_{k=1}^n r_k = 1$, we define a *weighted geometric mean* $\gamma_{r_1,\ldots,r_n} : \mathbb{R}^n \to \mathbb{R}$ by

$$\gamma_{r_1,\ldots,r_n}(x_1,\ldots,x_n) = \prod_{k=1}^n |x_k|^{r_k} \quad (x_1,\ldots,x_n \in \mathbb{R}).$$

The weighted geometric means are concave on $(\mathbb{R}^+)^n$ as well as continuous and positively homogeneous on $\mathbb{R}^n$. Moreover, for each $r_1, \ldots, r_n \in (0,1)$ with $\sum_{k=1}^n r_k = 1$, it follows from [12, Lemma 3.6(iii)] that

$$\begin{aligned}&\gamma_{r_1,\ldots,r_n}(x_1,\ldots,x_n)\\ &\quad = \inf\Big\{\sum_{k=1}^n r_k\theta_k x_k : \theta_k \in (0,\infty), \prod_{k=1}^n \theta_k^{r_k} = 1\Big\} \quad (x_1,\ldots,x_n \in \mathbb{R}^+).\end{aligned}$$

An Archimedean vector lattice E over $\mathbb{K}$ is said to be *weighted geometric mean closed* if $\inf\{\sum_{k=1}^n r_k\theta_k|f_k| : \theta_k \in (0,\infty), \prod_{k=1}^n \theta_k^{r_k} = 1\}$ exists in E for every $f_1, \ldots, f_n \in E$ and every $r_1, \ldots, r_n \in (0,1)$ with $\sum_{k=1}^n r_k = 1$. In this case, we write

$$\mathop{\triangle}_{k=1}^{n}(f_k, r_k) = \inf\Big\{\sum_{k=1}^n r_k\theta_k|f_k| : \theta_k \in (0,\infty), \prod_{k=1}^n \theta_k^{r_k} = 1\Big\} \quad (f_1,\ldots,f_n \in E).$$

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $p \in (1,\infty)$. It follows from [20, (HI_1)] by Maligranda that $|f|^{1/p}|g|^{1-1/p} \in L_1(X,\mu)$ for $f, g \in L_1(X,\mu)$, and

$$\big\||f|^{1/p}|g|^{1-1/p}\big\|_1 \leq \|f\|_1^{1/p}\|g\|_1^{1-1/p}.$$

Following Maligranda's proof, we redevelop and extend [20, (HI_1)] to a multivariate version in the setting of positive operators between vector lattices.

Proposition 4.1. *Let E and F be weighted geometric mean closed Archimedean vector lattices over $\mathbb{K}$, and suppose that $r_1, \dots, r_n \in (0,1)$ satisfy $\sum_{k=1}^n r_k = 1$.*

(1) *For each positive linear map $T\colon E \to F$,*

$$T\Big(\mathop{\triangle}_{k=1}^{n}(f_k, r_k)\Big) \leq \mathop{\triangle}_{k=1}^{n}\big(T\big(|f_k|\big), r_k\big) \quad (f_1, \dots, f_n \in E).$$

(2) *If $T\colon E \to F$ is a linear map, then T is a vector lattice homomorphism if and only if*

$$T\Big(\mathop{\triangle}_{k=1}^{n}(f_k, r_k)\Big) = \mathop{\triangle}_{k=1}^{n}\big(T(f_k), r_k\big) \quad (f_1, \dots, f_n \in E).$$

(3) *If G is a (not necessarily weighted geometric mean closed) vector sublattice of E, if $T\colon G \to F$ is a vector lattice homomorphism, and if also $\{f_1, \dots, f_n, \triangle_{k=1}^n(f_k, r_k)\} \subseteq G$, then*

$$T\Big(\mathop{\triangle}_{k=1}^{n}(f_k, r_k)\Big) = \mathop{\triangle}_{k=1}^{n}\big(T(f_k), r_k\big).$$

Proof. (1) Assume that $T\colon E \to F$ is a positive linear map. Let $f_1, \dots, f_n \in E$, and suppose that $\theta_1, \dots, \theta_n \in (0,\infty)$ are such that $\prod_{k=1}^n \theta_k^{r_k} = 1$. From the positivity and linearity of T we have

$$T\Big(\mathop{\triangle}_{k=1}^{n}(f_k, r_k)\Big) \leq T\Big(\sum_{k=1}^{n} r_k\theta_k|f_k|\Big) = \sum_{k=1}^{n} r_k\theta_k T\big(|f_k|\big).$$

Then

$$T\Big(\mathop{\triangle}_{k=1}^{n}(f_k, r_k)\Big) \leq \inf\Big\{\sum_{k=1}^{n} r_k\theta_k T\big(|f_k|\big) : \theta_k \in (0,\infty), \prod_{k=1}^{n}\theta_k^{r_k} = 1\Big\}.$$

(2) Suppose that $T\colon E \to F$ is a vector lattice homomorphism. It follows from [12, Theorem 3.7(2)] that $\gamma_{r_1,\dots,r_n}(f_1, \dots, f_n)$, which is defined via functional calculus (see [10, Definition 3.1]), exists in E for every $f_1, \dots, f_n \in E^+$ and

$$\gamma_{r_1,\dots,r_n}(f_1, \dots, f_n) = \mathop{\triangle}_{k=1}^{n}(f_k, r_k) \quad (f_1, \dots, f_n \in E^+).$$

It is readily checked using [10, Definition 3.1] and the identity

$$\gamma_{r_1,\dots,r_n}(x_1, \dots, x_n) = \gamma_{r_1,\dots,r_n}\big(|x_1|, \dots, |x_n|\big) \quad (x_1, \dots, x_n \in \mathbb{R})$$

that $\gamma_{r_1,\dots,r_n}(f_1, \dots, f_n)$ exists in E for all $f_1, \dots, f_n \in E$ and

$$\gamma_{r_1,\dots,r_n}(f_1, \dots, f_n) = \gamma_{r_1,\dots,r_n}\big(|f_1|, \dots, |f_n|\big) \quad (f_1, \dots, f_n \in E).$$

It follows that

$$\gamma_{r_1,\dots,r_n}(f_1, \dots, f_n) = \mathop{\triangle}_{k=1}^{n}(f_k, r_k) \quad (f_1, \dots, f_n \in E).$$

By [12, Theorem 3.11] (second equality), we have for all $f_1, \dots, f_n \in E$,

$$\begin{aligned} T\Big(\mathop{\triangle}_{k=1}^{n}(f_k, r_k)\Big) &= T\big(\gamma_{r_1,\dots,r_n}(f_1, \dots, f_n)\big) \\ &= \gamma_{r_1,\dots,r_n}\big(T(f_1), \dots, T(f_n)\big) \\ &= \mathop{\triangle}_{k=1}^{n}\big(T(f_k), r_k\big). \end{aligned}$$

On the other hand, assume that $T\colon E \to F$ is a linear map and that

$$T\Big(\mathop{\triangle}_{k=1}^{n}(f_k, r_k)\Big) = \mathop{\triangle}_{k=1}^{n}\big(T(f_k), r_k\big) \quad (f_1, \dots, f_n \in E).$$

From [12, Theorem 3.11], we conclude that

$$T\big(\gamma_{r_1,\dots,r_n}(f_1, \dots, f_n)\big) = \gamma_{r_1,\dots,r_n}\big(T(f_1), \dots, T(f_n)\big) \quad (f_1, \dots, f_n \in E),$$

and (since $\gamma_{r_1,\dots,r_n}(x, \dots, x) = |x|$ $(x \in \mathbb{R})$) that T is a vector lattice homomorphism.

(3) Let G be a vector sublattice of E, and let $T : G \to F$ be a vector lattice homomorphism. Let $\mathcal{D}$ be the collection of all weighted geometric means. That is, let

$$\mathcal{D} = \Big\{\gamma_{r_1,\dots,r_n} : r_1, \dots, r_n \in (0,1) \text{ and } \sum_{k=1}^{n} r_k = 1\Big\}.$$

It follows from [12, Theorem 3.7(2)] that F is $\mathcal{D}$-complete (see [12, Definition 3.2]). By [12, Theorem 3.17], it holds that T uniquely extends to a vector lattice homomorphism $T^{\mathcal{D}} : G^{\mathcal{D}} \to F$, where $G^{\mathcal{D}}$ denotes the $\mathcal{D}$-completion of G (see [12, Definition 3.10]). Note that $G^{\mathcal{D}}$ is $\mathcal{D}$-complete by definition. Thus [12, Theorem 3.7(2)] implies that $G^{\mathcal{D}}$ is weighted geometric mean closed. An appeal to (2) now verifies (3). □

Let A be an Archimedean Φ-algebra over $\mathbb{K}$. If A is uniformly complete, then $a^{1/n}$ exists in A for every $a \in A^+$ and every $n \in \mathbb{N}$ (see [6, Corollary 6]). It follows that a^q exists in A for every $a \in A^+$ and every $q \in \mathbb{Q} \cap (0, \infty)$. The assumption of uniform completeness in [6, Corollary 6] can be weakened, which is the content of our next lemma.

Lemma 4.2. *Let A be a weighted geometric mean closed Archimedean Φ-algebra over $\mathbb{K}$. Then a^q exists in A for all $a \in A^+$ and $q \in \mathbb{Q} \cap (0, \infty)$. Furthermore, if $q_1, \dots, q_n \in \mathbb{Q} \cap (0,1)$ are such that $\sum_{k=1}^{n} q_k = 1$, then for every $a_1, \dots, a_n \in A^+$,*

$$\prod_{k=1}^{n} a_k^{q_k} = \mathop{\triangle}_{k=1}^{n}(a_k, q_k) \in A.$$

Proof. Denote the multiplicative identity of A by e, and let $a \in A^+$. In order to prove that a^q is defined in A for every $q \in \mathbb{Q} \cap (0, \infty)$, it suffices to verify that $a^{1/n}$ exists in A for all $n \in \mathbb{N}$. To this end, let $n \in \mathbb{N} \setminus \{1\}$. Let C be the Archimedean Φ-subalgebra of A_ρ generated by the elements $a \in A^+$ and

$$b = \inf\big\{n^{-1}\theta_1 a + (1 - n^{-1})\theta_2 e : \theta_1, \theta_2 \in (0, \infty), \theta_1^{1/n}\theta_2^{1-1/n} = 1\big\} \in A^+.$$

Suppose that $\omega\colon C\to\mathbb{R}$ is a nonzero multiplicative vector lattice homomorphism. It follows that $\omega(e)=1$. Using Proposition 4.1(3) (third equality), we obtain

$$\begin{aligned}\omega(b^n)&=\omega(b)^n\\&=\Big(\omega\big(\inf\big\{n^{-1}\theta_1 a+(1-n^{-1})\theta_2 e:\theta_1,\theta_2\in(0,\infty),\theta_1^{1/n}\theta_2^{1-1/n}=1\big\}\big)\Big)^n\\&=\big(\inf\big\{n^{-1}\theta_1\omega(a)+(1-n^{-1})\theta_2:\theta_1,\theta_2\in(0,\infty),\theta_1^{1/n}\theta_2^{1-1/n}=1\big\}\big)^n\\&=\big(\gamma_{\frac{1}{n},\frac{n-1}{n}}\big(\omega(a),1\big)\big)^n=\big(\big(\omega(a)\big)^{1/n}\big)^n=\omega(a).\end{aligned}$$

Since the set of all nonzero multiplicative vector lattice homomorphisms $\omega\colon C\to\mathbb{R}$ separates the points of C (see [10, Corollary 2.7]), we have in A_ρ that $b^n=a$. But then $a^{1/n}=b$.

Finally, a similar proof verifies that

$$\prod_{k=1}^{n}a_k^{q_k}=\mathop{\triangle}_{k=1}^{n}(a_k,q_k)$$

for every $a_1,\dots,a_n\in A^+$ and all $q_1,\dots,q_n\in\mathbb{Q}\cap(0,1)$ such that $\sum_{k=1}^n q_k=1$. $\square$

We next use the proof of Lemma 4.2 as a guide to define strictly positive irrational powers of positive elements in weighted geometric mean closed Archimedean Φ-algebras in an intrinsic manner that does not require representation theory dependent on more than the countable axiom of choice. For $r\in(0,\infty)$, define

$$\lfloor r\rfloor=\max\big\{n\in\mathbb{N}\cup\{0\}:n\le r\big\}\qquad\text{and}\qquad\tilde r=r-\lfloor r\rfloor.$$

Definition 4.3. Suppose that A is a weighted geometric mean closed Archimedean Φ-algebra over $\mathbb{K}$. Let e be the multiplicative identity of A. For $a\in A^+$ and $r\in(0,\infty)$, define

$$a^r=a^{\lfloor r\rfloor}\inf\big\{\tilde r\theta_1 a+(1-\tilde r)\theta_2 e:\theta_1,\theta_2\in(0,\infty),\theta_1^{\tilde r}\theta_2^{1-\tilde r}=1\big\},$$

where $a^{\lfloor r\rfloor}$ is taken to equal e in the case where $\lfloor r\rfloor=0$.

By Lemma 4.2, the above definition of strictly positive real exponents extends the natural definition of strictly positive rational exponents previously discussed. We next give an easy corollary of Proposition 4.1(3).

Corollary 4.4. *Let A and B be weighted geometric mean closed Archimedean Φ-algebras over $\mathbb{K}$ with multiplicative identities e and e', respectively. Suppose that C is a Φ-subalgebra of A and that $T\colon C\to B$ is a multiplicative vector lattice homomorphism such that $T(e)=e'$. Let $a\in A^+$ and $r\in(0,\infty)$. If $a,a^r\in C$, then $T(a^r)=(T(a))^r$.*

The following lemma verifies some familiar (and needed for Theorems 4.7 and 5.1) arithmetical rules for positive real exponents in weighted geometric mean closed Archimedean Φ-algebras over $\mathbb{K}$.

Lemma 4.5. *Let $p, q \in (0, \infty)$, and let A be a weighted geometric mean closed Archimedean Φ-algebra over $\mathbb{K}$. For each $a \in A^+$, the following hold:*

(1) $(a^p)^q = a^{pq}$,
(2) $a^p a^q = a^{p+q}$.

Proof. We prove (1), leaving the similar proof of (2) to the reader. To this end, let $a \in A^+$. Let C be the real Φ-subalgebra of A_ρ generated by $a, a^{\tilde{p}}, (a^p)^{\tilde{q}}$, and $a^{\widetilde{pq}}$, and note that $a, a^p, (a^p)^q, a^{pq} \in C$. If $\omega\colon C \to \mathbb{R}$ is a nonzero multiplicative vector lattice homomorphism, then $\omega(e) = 1$, where e is the multiplicative identity of A. Using Corollary 4.4, we obtain

$$\omega\big((a^p)^q\big) = \big(\omega(a^p)\big)^q = \big(\omega(a)\big)^{pq} = \omega(a^{pq}).$$

Since the nonzero multiplicative vector lattice homomorphisms separate the points of C (see [10, Corollary 2.7]), we conclude that $(a^p)^q = a^{pq}$. □

In light of Definition 4.3, the second part of Lemma 4.2 can now be improved to include irrational exponents. The proof of Lemma 4.6 uses real-valued multiplicative vector lattice homomorphisms, similar to what is found in the proofs of Lemmas 4.2 and 4.5. Therefore, the proof is omitted.

Lemma 4.6. *Let A be a weighted geometric mean closed Archimedean Φ-algebra over $\mathbb{K}$. If $r_1, \dots, r_n \in (0,1)$ are such that $\sum_{k=1}^n r_k = 1$, then we have, for every $a_1, \dots, a_n \in A^+$,*

$$\prod_{k=1}^n a_k^{r_k} = \mathop{\triangle}_{k=1}^n (a_k, r_k).$$

We proceed with the main theorem of this section.

Theorem 4.7 (Hölder inequality)**.** *Suppose that $p_1, \dots, p_n \in (1, \infty)$ with $\sum_{k=1}^n p_k^{-1} = 1$. Assume that A is a weighted geometric mean closed Archimedean Φ-algebra over $\mathbb{K}$.*

(1) *If B is also a weighted geometric mean closed Archimedean Φ-algebra over $\mathbb{K}$ and if $T\colon A \to B$ is a positive linear map, then*

$$T\Big(\prod_{k=1}^n |a_k|\Big) \le \prod_{k=1}^n \big(T\big(|a_k|^{p_k}\big)\big)^{1/p_k} \quad (a_1, \dots, a_n \in A).$$

(2) *If B is a weighted geometric mean closed Archimedean vector lattice over $\mathbb{K}$ and if $T\colon A \to B$ is a positive linear map, then*

$$T\Big(\prod_{k=1}^n |a_k|\Big) \le \mathop{\triangle}_{k=1}^n \big(T\big(|a_k|^{p_k}\big), 1/p_k\big) \quad (a_1, \dots, a_n \in A).$$

Proof. We only prove (1) since the proof of (2) is similar. To this end, let B be a weighted geometric mean closed Archimedean Φ-algebra over $\mathbb{K}$, and suppose that $T\colon A \to B$ is a positive linear map. Using Lemma 4.5(1) (first equality), Lemma 4.6 (second equality and last equality), and Proposition 4.1(1) (for the inequality), we have

$$\begin{aligned} T\Big(\prod_{k=1}^{n}|a_k|\Big) &= T\Big(\prod_{k=1}^{n}\big(|a_k|^{p_k}\big)^{1/p_k}\Big) \\ &= T\Big(\mathop{\triangle}_{k=1}^{n}\big(|a_k|^{p_k}, 1/p_k\big)\Big) \\ &\leq \mathop{\triangle}_{k=1}^{n}\big(T\big(|a_k|^{p_k}\big), 1/p_k\big) \\ &= \prod_{k=1}^{n}\big(T\big(|a_k|^{p_k}\big)\big)^{1/p_k}. \end{aligned}$$

□

5. A Minkowski inequality

We employ the Hölder inequality in Theorem 4.7(1) to prove a Minkowski inequality in the setting of Archimedean Φ-algebras over $\mathbb{K}$ in this section.

Theorem 5.1 (Minkowski inequality). *Let $p \in (1,\infty)$. Suppose that A and B are both weighted geometric mean closed Archimedean Φ-algebras over $\mathbb{K}$. For every positive linear map $T\colon A \to B$, we have*

$$\Big(T\Big(\Big|\sum_{k=1}^{n} a_k\Big|^p\Big)\Big)^{1/p} \leq \sum_{k=1}^{n}\big(T\big(|a_k|^p\big)\big)^{1/p} \quad (a_1,\dots,a_n \in A).$$

Proof. We prove the result for $n = 2$ and we note that the rest of the proof follows from a standard induction argument. To this end, let $T\colon A \to B$ be a positive linear map, and assume that $a, b \in A$. Let $q \in (1,\infty)$ satisfy $q^{-1} + p^{-1} = 1$. By Lemma 4.5(2) (first equality), Theorem 4.7(1) (second inequality), and Lemma 4.5(1) (third equality),

$$\begin{aligned} T\big(|a+b|^p\big) &= T\big(|a+b|^{p-1}|a+b|\big) \\ &\leq T\big(|a+b|^{p-1}\big(|a|+|b|\big)\big) \\ &= T\big(|a+b|^{p-1}|a|\big) + T\big(|a+b|^{p-1}|b|\big) \\ &\leq T\big(\big(|a+b|^{p-1}\big)^q\big)^{1/q}T\big(|a|^p\big)^{1/p} + T\big(\big(|a+b|^{p-1}\big)^q\big)^{1/q}T\big(|b|^p\big)^{1/p} \\ &= T\big(|a+b|^p\big)^{1/q}T\big(|a|^p\big)^{1/p} + T\big(|a+b|^p\big)^{1/q}T\big(|b|^p\big)^{1/p} \\ &= T\big(|a+b|^p\big)^{1/q}\big(T\big(|a|^p\big)^{1/p} + T\big(|b|^p\big)^{1/p}\big). \end{aligned}$$

Setting $f = T(|a+b|^p)$ and $g = T(|a|^p)^{1/p} + T(|b|^p)^{1/p}$, we have $f \leq f^{1/q}g$. Next let C be the Φ-subalgebra of B_ρ generated by $f, f^{1/p}, f^{1/q}$, and g. Let $\omega\colon C \to \mathbb{R}$ be a nonzero multiplicative vector lattice homomorphism. It follows that $\omega(e) = 1$, where e is the multiplicative identity of B. Using Corollary 4.4, we have

$$\omega(f) \leq \omega(f^{1/q}g) = \omega(f^{1/q})\omega(g) = \omega(f)^{1/q}\omega(g).$$

Thus if $\omega(f) \neq 0$, then $\omega(f)^{1/p} \leq \omega(g)$. Of course, $\omega(f)^{1/p} \leq \omega(g)$ also holds in the case that $\omega(f) = 0$, since $g \in B^+$. By Corollary 4.4 again, $\omega(f^{1/p}) \leq \omega(g)$.

Since the collection of all nonzero multiplicative vector lattice homomorphisms separate the points of C (see [10, Corollary 2.7]), we conclude that $f^{1/p} \leq g$. Therefore, we obtain

$$T\big(|a+b|^p\big)^{1/p} \leq T\big(|a|^p\big)^{1/p} + T\big(|b|^p\big)^{1/p}. \qquad \square$$

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