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NORM CONVERGENCE OF LOGARITHMIC MEANS ON UNBOUNDED VILENKIN GROUPS

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Abstract. In this paper we prove that, in the case of some unbounded Vilenkin groups, the Riesz logarithmic means converges in the norm of the spaces $X(G)$ for every $f \in X(G)$, where by $X(G)$ we denote either the class of continuous functions with supremum norm or the class of integrable functions.

Let $\mathbb{P}$ denote the set of positive integers, and let $\mathbb{N} := \mathbb{P} \cup \{0\}$ be the set of nonnegative integers. Let $m := (m_0, m_1, \ldots)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$ the additive group of integers modulo m_k. Define the group G as the complete direct product of the groups Z_{m_j}, with the product of the discrete topologies of Z_{m_j}'s. The direct product μ of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in Z_{m_k})$$

is the Haar measure on G with $\mu(G) = 1$. The elements of G can be represented by sequences $x := (x_0, x_1, \ldots, x_j, \ldots)$, $(x_j \in Z_{m_j})$. The group operation $+$ in G is given by $x + y = (x_0 + y_0 (\mathrm{mod}\, m_0), \ldots, x_k + y_k (\mathrm{mod}\, m_k), \ldots)$, where $x = (x_0, \ldots, x_k, \ldots)$ and $y = (y_0, \ldots, y_k, \ldots) \in G$. The inverse of $+$ will be denoted by $-$.

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It is easy to give a base for the neighborhoods of G:

$$I_0(x) := G,$$

$$I_n(x) := \{y \in G_m | y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$$

for $x \in G, n \in \mathbb{N}$. Define $I_n := I_n(0)$ for $n \in \mathbb{N}$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$ the nth coordinate of which is 1 and the rest are zeros ($n \in \mathbb{N}$). If we define the so-called *generalized number system* based on m as $M_0 := 1, M_{k+1} := m_k M_k$ ($k \in \mathbb{N}$), then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ ($j \in \mathbb{N}$) and only a finite number of n_j's differ from zero.

Next, we introduce on G an orthonormal system which is called the *Vilenkin system* (see [1]). First, define the complex valued functions $r_k(x) : G \to \mathbb{C}$, the generalized Rademacher functions, in this way:

$$r_k(x) := \exp \frac{2\pi \imath x_k}{m_k} \quad (\imath^2 = -1, x \in G, k \in \mathbb{N}).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G as follows:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system the *Walsh–Paley* one if $m_i \equiv 2, i \in \mathbb{N}$. Dirichlet kernels are defined as follows:

$$D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbb{N}).$$

Recall that (see [17])

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \in G \backslash I_n. \end{cases} \tag{1}$$

It is well know that (see [17])

$$D_n = \psi_n \sum_{j=0}^{\infty} D_{M_j} \sum_{a=m_j-n_j}^{m_j-1} r_j^a. \tag{2}$$

The norm of the space $L_p(G)$ is defined by

$$\|f\|_p^p := \int_G \left|f(x)\right|^p d\mu(x), \quad 1 \le p < \infty.$$

By $C(G)$ we denote the space of continuous functions on G, with the supremum norm

$$\|f\|_C = \sup_{x \in G} \left|f(x)\right| \quad \big(f \in C(G)\big).$$

For the sake of simplicity in notation, in the case of $p = \infty$ we sometimes write $L_\infty(G)$, which has the meaning $C(G)$.

Let X be $C(G)$ or $L_1(G)$. The partial sums of the Vilenkin–Fourier series are defined as

$$S_n(f;x) := \sum_{i=0}^{n-1} \widehat{f}(i)\psi_i(x),$$

where the number

$$\widehat{f}(i) = \int_G f(x)\overline{\psi}_i(x)\,d\mu(x)$$

is said to be the ith Vilenkin–Fourier coefficient of the function f. The Fejér means is defined as follows:

$$\sigma_n(x;f) := \frac{1}{n}\sum_{j=1}^{n} S_j(x;f).$$

Then

$$\sigma_n(f) = f * K_n,$$

where

$$K_n := \frac{1}{n}\sum_{j=1}^{n} D_j.$$

The notion of the Riesz logarithmic means of a Fourier series was previously introduced in the literature (see [15], [20]). The nth Riesz logarithmic means of the Fourier series of the integrable function f is defined by

$$R_n(f) := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{k}, \qquad l_n := \sum_{k=1}^{n-1}\frac{1}{k},$$

where $S_k(f)$ is the kth partial sum of its Fourier series. This Riesz logarithmic means with respect to the trigonometric system has been studied by a considerable number of authors. We mention for instance the work of Szász [19] and Yabuta [21]; the means with respect to the Walsh and Vilenkin system is discussed by Simon in [18] and by Gát in [9].

Let $\{q_k : k \ge 0\}$ be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of f is defined by

$$\frac{1}{\sum_{k=1}^{n-1} q_k}\sum_{k=1}^{n-1} q_k S_{n-k}(f).$$

If $q_k = \frac{1}{k}$, then we get the (Nörlund) logarithmic means,

$$t_n(f;x) := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_{n-k}(f)}{k}, \tag{3}$$

even though it is a kind of "reverse" Riesz logarithmic means. In [10] the present authors proved some convergence and divergence properties of the logarithmic means of Walsh–Fourier series of functions in the class of continuous functions, and in the Lebesgue space L_1. In particular, in [10] it was proved (for the Walsh–Paley system) that there exists a function $f \in X(G)$ for which

$$\left\|t_n(f) - f\right\|_X \nrightarrow 0.$$

We now review some approximation results with respect to the Vilenkin system. It is well known that the partial sums converge to the function in norm. That is,

$$\left\|S_n(f) - f\right\|_p \to 0 \quad \text{as } n \to \infty$$

for all $f \in L_p(G)$, where $1 < p < \infty$ (see [16], [18], [22]). This does not hold for the cases $p = 1$ or $p = \infty$.

Moreover, if we use the partial sequence M_n, then from [1] the convergence in the supremum norm for functions, and in the Lebesgue norm L_1 for functions $f \in L_1(G)$,

$$\left\|S_{M_n}(f) - f\right\|_X \to 0 \quad \text{as } n \to \infty$$

is well known. The properties of the convergence are better using the Fejér means on the bounded Vilenkin system (see [13]):

$$\left\|\sigma_n(f) - f\right\|_p \to 0 \quad \text{as } n \to \infty, f \in L_p(G), 1 \leq p \leq \infty.$$

On the arbitrary Vilenkin system,

$$\left\|\sigma_n(f) - f\right\|_p \to 0 \quad \text{as } n \to \infty, f \in L_p(G), 1 < p < \infty.$$

On the other hand, in the case of unbounded Vilenkin systems and $p = 1$ or $p = \infty$, the preceding math does not hold (for this fact, see Price [14]).

What is the situation regarding the Fejér means in the case of special subsequence M_n? It is interesting that for any unbounded sequence m there exists $f \in X(G)$ (see [14]) such that

$$\left\|\sigma_{M_n}(f) - f\right\|_X \nrightarrow 0.$$

On the other hand, it is proved by the first author in [9] that for every Vilenkin system,

$$\sigma_{M_n}(f) \to f \quad \text{a.e. as } n \to \infty, \forall f \in L_1(G).$$

The first author and Blahota in [6] proved that there exists a class of unbounded Vilenkin groups such that

$$\left\|t_{M_n}(f) - f\right\|_X \to 0 \quad \text{as } n \to \infty, f \in L_1(G).$$

So, for special subsequence M_n, approximate properties of Nörlund logarithmic means are better than approximate properties of Fejér means. In this article we prove that, in the case of some unbounded Vilenkin group, the Riesz logarithmic means converges in the norm of the spaces $X(G)$ for every $f \in X(G)$. In particular, the following theorem is true.

Theorem 1. *Let $f \in X(G)$ and let*

$$\overline{\lim_{n\to\infty}} \frac{\sum_{k=0}^{n-1} \log^2 m_k}{\log M_n} < \infty.$$

Then

$$\lim_{n\to\infty}\left\|R_n(f)-f\right\|_X=0. \tag{4}$$

We mention that the condition of Theorem 1 is the same one which was given for the L_1-norm convergence of Nörlund logarithmic means in [6]. We note that the problems of norm convergence and approximation of Fourier series with respect to Walsh and Vilenkin systems have been investigated by Fine [8]; Efimov [7]; Avdispahić and Memić [3]; Avdispahić [2]; Avdispahić and Pepić [5], [4]; the present authors [10]; and the second author [11], [12]. For some methods and more details with respect to unbounded Vilenkin systems, see [9].

In order to prove Theorem 1, we need several lemmas. Set

$$\beta_A:=\frac{1}{M_A}\sum_{t=0}^{A-1}M_{t+1}\log m_t. \tag{5}$$

The notation $a\lesssim b$ in the whole paper stands for $a\le cb$, where c is a positive constant. The following lemma was proved by Blahota and the first author in [6, Lemma 3].

Lemma 2 (Blahota, Gát)**.** *Let $M_A\le n<M_{A+1}$. Then*

$$\|K_n\|_1\lesssim\sum_{t=0}^{A+1}\frac{\beta_{A+1-t}}{2^t}.$$

Lemma 3. *Let $M_k\le n<M_{k+1}$. Then*

$$\int_{I_k\setminus I_{k+1}}\Big|\sum_{j=M_k}^{n-1}\frac{D_j}{j}\Big|\lesssim\log^2(1+n_k).$$

Proof. Let $x\in I_k\setminus I_{k+1}$. Then from (2), we have

$$D_j(x)=\psi_j(x)\Big(\sum_{a=0}^{k-1}j_aM_a+M_k\sum_{b=m_k-j_k}^{m_k-1}r_k^b(x)\Big).$$

Thus, we have

$$\begin{aligned}&\int_{I_k\setminus I_{k+1}}\Big|\sum_{j=M_k}^{n-1}\frac{D_j}{j}\Big|\\&\lesssim\log\frac{n}{M_k}+M_k\int_{I_k\setminus I_{k+1}}\sum_{j_k=1}^{n_k-1}\sum_{u=0}^{M_k-1}\frac{1}{j_kM_k+u}\Big|\sum_{b=0}^{j_k-1}r_k^b(x)\Big|\,d\mu(x)\\&\quad+M_k\int_{I_k\setminus I_{k+1}}\sum_{u=0}^{n_{(k)}-1}\frac{1}{n_kM_k+u}\Big|\sum_{b=0}^{n_k-1}r_k^b(x)\Big|\,d\mu(x)\\&=:A_1+A_2+A_3,\end{aligned}$$

where $n_{(k)} = \sum_{i=0}^{k-1} n_i M_i$. Note that

$$\begin{aligned}\sum_{z_s=1}^{m_s-1} \frac{|\sin(\frac{\pi n_s z_s}{m_s})|}{|\sin(\frac{\pi z_s}{m_s})|} &\leq \sum_{0\leq k\leq n_s} \sum_{[k\frac{m_s}{n_s}]+1\leq z_s<[(k+1)\frac{m_s}{n_s}]} \frac{|\sin(\frac{\pi n_s z_s}{m_s})|}{|\sin(\frac{\pi z_s}{m_s})|} \\ &\lesssim \sum_{0\leq k\leq n_s/2} \sum_{1\leq z_s\leq \frac{m_s}{n_s}} \frac{(n_s/m_s)z_s}{z_s/m_s+k/n_s} \\ &\lesssim \frac{cn_s^2}{m_s}\frac{m_s^2}{n_s^2}\log(1+n_s) = cm_s\log(1+n_s). \qquad (6)\end{aligned}$$

For A_3, we have

$$\begin{aligned}A_3 &\lesssim M_k \int_{I_k\setminus I_{k+1}} \frac{1}{n_k}\Big|\frac{\sin(\pi n_k x_k/m_k)}{\sin(\pi x_k/m_k)}\Big|\,d\mu(x) \\ &= cM_k \sum_{x_k=1}^{m_k-1}\int_{I_{k+1}(x_k e_k)} \frac{1}{n_k}\Big|\frac{\sin(\pi n_k x_k/m_k)}{\sin(\pi x_k/m_k)}\Big|\,d\mu(x) \\ &\lesssim \frac{1}{m_k n_k}\sum_{x_k=1}^{m_k-1}\Big|\frac{\sin(\pi n_k x_k/m_k)}{\sin(\pi x_k/m_k)}\Big| \\ &\lesssim \frac{\log(1+n_k)}{n_k} \lesssim 1.\end{aligned}$$

For A_2, we have

$$\begin{aligned}A_2 &\lesssim M_k \sum_{j_k=1}^{n_k-1}\frac{1}{j_k}\int_{I_k\setminus I_{k+1}}\Big|\sum_{b=0}^{j_k-1} r_k^b(x)\Big|\,d\mu(x) \\ &\lesssim \sum_{j_k=1}^{n_k-1}\frac{\log(1+j_k)}{j_k} \lesssim \log^2(1+n_k).\end{aligned}$$

That is,

$$\int_{I_k\setminus I_{k+1}}\Big|\sum_{j=M_k}^{n-1}\frac{D_j}{j}\Big| \lesssim \log\frac{n}{M_k} + 1 + \log^2(1+n_k) \lesssim \log^2(1+n_k).$$

This completes the proof of Lemma 3. □

We now investigate the integrals of the function $|\sum_{j=M_k}^{n-1}\frac{D_j}{j}|$ on the set $I_t\setminus I_{t+1}$ for $t<k$.

Lemma 4. *Let $M_k \leq n < M_{k+1}, t<k$. Then*

$$\int_{I_t\setminus I_{t+1}}\Big|\sum_{j=M_k}^{n-1}\frac{D_j}{j}\Big| \lesssim \frac{M_{t+1}\log m_t}{M_k}\Big(\log(1+n_k)+\log\Big(\frac{M_k}{M_{t+1}}\Big)\Big).$$

Proof. For $x \in I_t \setminus I_{t+1}$, we have

$$\sum_{j=M_k}^{n-1} \frac{D_j}{j} = \sum_{j=M_k}^{n-1} \frac{\psi_j(x)}{j}\Big(\sum_{a=0}^{t-1} j_a M_a + M_t \sum_{b=m_t-j_t}^{m_t-1} r_t^b(x)\Big) =: B_1 + B_2.$$

First check B_1. It will be divided into two addends,

$$B_1 = \sum_{j=M_k}^{n^{(t+1)}-1} \psi_j(x) j_{(t)} \frac{1}{j} + \sum_{j=n^{(t+1)}}^{n-1} \psi_j(x) j_{(t)} \frac{1}{j} =: B_{1,1} + B_{1,2},$$

where $n^{(t+1)} = \sum_{i=t+1}^{\infty} n_i M_i$. It is easy to have an upper estimate for the integral of $|B_{1,2}|$:

$$\begin{aligned}\int_{I_t \setminus I_{t+1}} |B_{1,2}| &\le \frac{1}{M_t} \sum_{j=n^{(t+1)}}^{n-1} \frac{M_t}{j} \\ &\le \frac{1}{n_k M_k} \sum_{j=n^{(t+1)}}^{n-1} 1 \\ &= \frac{1}{n_k M_k}(n - n^{(t+1)}) = \frac{n_{(t+1)}}{n_k M_k} \le \frac{(1+n_t)M_t}{n_k M_k}.\end{aligned}$$

Let function $\alpha_i : \mathbb{N}^{k-i} \to \mathbb{N}$ for $i = k-1, k-2, \dots, t+1$ be introduced in the following way:

$$\alpha_i(j_{i+1}, j_{i+2}, \dots, j_k) = \begin{cases} n_i & \text{if } j_{i+1} = n_{i+1}, \dots, j_k = n_k \text{ and } i > t+1, \\ n_i - 1 & \text{if } j_{i+1} = n_{i+1}, \dots, j_k = n_k \text{ and } i = t+1, \\ m_i - 1 & \text{otherwise.} \end{cases}$$

If it does not cause misunderstanding, then $\alpha_i(j_{i+1}, j_{i+2}, \dots, j_k)$ is simply denoted by α_i.

For $B_{1,1}$, we can write

$$B_{1,1} = \sum_{j_k=1}^{n_k} \sum_{j_{k-1}=0}^{\alpha_{k-1}} \cdots \sum_{j_{t+1}=0}^{\alpha_{t+1}} \psi_{j^{(t+1)}}(x) \sum_{j_{t-1}=0}^{m_{t-1}-1} \cdots \sum_{j_0=0}^{m_0-1} j_{(t)} \sum_{j_t=0}^{m_t-1} \frac{r_t^{j_t}(x)}{j}.$$

We discuss the innermost sum by the help of Abel transform

$$\begin{aligned}\sum_{j_t=0}^{m_t-1} \frac{r_t^{j_t}(x)}{j} &= \sum_{j_t=0}^{m_t-2} \Big(\frac{1}{j} - \frac{1}{j+M_t}\Big) \sum_{i=0}^{j_t} r_t^i(x) \\ &\quad + \frac{1}{j^{(t+1)} + (m_t-1)M_t + j_{(t)}} \sum_{i=0}^{m_t-1} r_t^i(x)\end{aligned}$$

since

$$\sum_{i=0}^{m_t-1} r_t^i(x) = 0, \quad x \in I_t \setminus I_{t+1}$$

and

$$\Big|\frac{1}{j}-\frac{1}{j+M_t}\Big|\leq\frac{M_t}{j_k^2M_k^2};$$

then we have

$$\begin{aligned}&\int_{I_t\backslash I_{t+1}}|B_{1,1}|\\&\leq\sum_{j_k=1}^{n_k}\sum_{j_{k-1}=0}^{\alpha_{k-1}}\cdots\sum_{j_{t+1}=0}^{\alpha_{t+1}}\sum_{a=0}^{M_t-1}a\frac{M_t}{j_k^2M_k^2}\sum_{j_t=0}^{m_t-1}\int_{I_t\backslash I_{t+1}}\Big|\sum_{i=0}^{j_t}r_t^i(x)\Big|\,d\mu(x)\\&\lesssim\sum_{j_k=1}^{n_k}m_{k-1}\cdots m_{t+1}\frac{M_t^2}{j_k^2M_k^2}\sum_{j_t=0}^{m_t-1}\log(1+j_t)\\&\lesssim\frac{M_t}{M_k}\log(1+m_t).\end{aligned}$$

That is,

$$\int_{I_t\backslash I_{t+1}}|B_1|\leq\frac{M_{t+1}}{M_k}.\tag{7}$$

Next, we check ($x\in I_t\backslash I_{t+1}$),

$$\begin{aligned}B_2&=M_t\sum_{j=M_k}^{n-1}\frac{\psi_{j^{(t+1)}}(x)}{j}\sum_{b=0}^{j_t-1}r_t^b(x)\\&=M_t\sum_{j=M_k}^{n^{(t+1)}-1}\frac{\psi_{j^{(t+1)}}(x)}{j}\sum_{b=0}^{j_t-1}r_t^b(x)+M_t\sum_{j=n^{(t+1)}}^{n-1}\frac{\psi_{j^{(t+1)}}(x)}{j}\sum_{b=0}^{j_t-1}r_t^b(x)\\&=:B_{2,1}+B_{2,2}.\end{aligned}$$

From (6), we can write

$$M_t\int_{I_t\backslash I_{t+1}}\Big|\sum_{b=0}^{j_t-1}r_t^b(x)\Big|\,d\mu(x)\lesssim\log(1+j_t).$$

Since $j_t\leq n_t$, we get

$$\begin{aligned}\int_{I_t\backslash I_{t+1}}|B_{2,2}|&\lesssim\frac{1}{n_kM_k}n_{(t+1)}\log(1+n_t)\\&\leq\frac{(1+n_t)M_t}{n_kM_k}\log(1+n_t).\end{aligned}\tag{8}$$

Finally, we have to investigate $B_{2,1}$:

$$B_{2,1}=\sum_{j_k=1}^{n_k}\sum_{j_{k-1}=0}^{\alpha_{k-1}}\cdots\sum_{j_{t+1}=0}^{\alpha_{t+1}}\psi_{j^{(t+1)}}(x)\sum_{j_{t-1}=0}^{m_{t-1}-1}\cdots\sum_{j_0=0}^{m_0-1}M_t\sum_{j_t=0}^{m_t-1}\frac{1}{j}\sum_{b=0}^{j_t-1}r_t^b(x).$$

Apply the Abel transform for the innermost sum in term $B_{2,1}$. It equals with

$$\sum_{j_t=0}^{m_t-2}\Big(\frac{1}{j}-\frac{1}{j+M_t}\Big)\sum_{i=0}^{j_t}\sum_{b=0}^{i-1} r_t^b(x) \\ +\frac{1}{j^{(t+1)}+(m_t-1)M_t+j_{(t)}}\sum_{i=0}^{m_t-1}\sum_{b=0}^{i-1} r_t^b(x).$$

Then let $B_{2,1}=B_{2,1,1}+B_{2,1,2}$, where

$$B_{2,1,1}:=M_t\sum_{j_k=1}^{n_k}\sum_{j_{k-1}=0}^{\alpha_{k-1}}\cdots\sum_{j_{t+1}=0}^{\alpha_{t+1}}\psi_{j^{(t+1)}}(x) \\ \times\sum_{j_{t-1}=0}^{m_{t-1}-1}\cdots\sum_{j_0=0}^{m_0-1}\sum_{j_t=0}^{m_t-2}\Big(\frac{1}{j}-\frac{1}{j+M_t}\Big)\sum_{i=0}^{j_t}\sum_{b=0}^{i-1} r_t^b(x) \tag{9}$$

and

$$B_{2,1,2}:=M_t\sum_{j_k=1}^{n_k}\sum_{j_{k-1}=0}^{\alpha_{k-1}}\cdots\sum_{j_{t+1}=0}^{\alpha_{t+1}}\psi_{j^{(t+1)}}(x) \\ \times\sum_{j_{t-1}=0}^{m_{t-1}-1}\cdots\sum_{j_0=0}^{m_0-1}\frac{1}{j^{(t+1)}+(m_t-1)M_t+j_{(t)}}\sum_{i=0}^{m_t-1}\sum_{b=0}^{i-1} r_t^b(x). \tag{10}$$

We investigate $B_{2,1,1}$ first. We have that

$$\Big|\frac{1}{j}-\frac{1}{j+M_t}\Big|\le\frac{M_t}{j_k^2M_k^2}$$

and that (see (6))

$$M_t\int_{I_t\setminus I_{t+1}}\Big|\sum_{i=0}^{j_t}\sum_{b=0}^{i-1} r_t^b(x)\Big|\,d\mu(x)\lesssim j_t\log(1+j_t)$$

give

$$\int_{I_t\setminus I_{t+1}}|B_{2,1,1}| \\ \lesssim M_t\sum_{j_k=1}^{n_k}\sum_{j_{k-1}=0}^{\alpha_{k-1}}\cdots\sum_{j_{t+1}=0}^{\alpha_{t+1}}\sum_{j_{t-1}=0}^{m_{t-1}-1}\cdots\sum_{j_0=0}^{m_0-1}\sum_{j_t=0}^{m_t-1}\frac{j_t\log(1+j_t)}{j_k^2M_k^2} \\ \lesssim M_t\frac{M_k}{m_t}\frac{1}{M_k^2}\sum_{j_t=0}^{m_t-1} j_t\log(1+j_t) \\ \lesssim\frac{M_t}{m_tM_k}m_t^2\log m_t=\frac{M_{t+1}\log m_t}{M_k}. \tag{11}$$

Now, we discuss $B_{2,1,2}$. We can write

$$\sum_{i=0}^{m_t-1}\sum_{b=0}^{i-1} r_t^b(x) = \sum_{i=0}^{m_t-1}\frac{r_t^i(x)-1}{r_t(x)-1} = \frac{m_t}{1-r_t(x)}.$$

That is,

$$B_{2,1,2} = \frac{M_{t+1}}{1-r_t}\sum_{j_k=1}^{n_k}\sum_{j_{k-1}=0}^{\alpha_{k-1}}\cdots\sum_{j_{t+1}=0}^{\alpha_{t+1}}\psi_{j^{(t+1)}}\sum_{u=0}^{M_t-1}\frac{1}{j^{(t+1)}+M_{t+1}-M_t+u}.$$

In order to give an upper bound for term $B_{2,1,2}$, we investigate two cases: $x-x_te_t \in I_k$ and $x-x_te_t \notin I_k$. If $x-x_te_t\in I_k$, then

$$\begin{aligned}\sum_{x_t=1}^{m_t-1}&\int_{I_k(x_te_t)}\left|B_{2,1,2}(x)\right| d\mu(x)\\ &\leq \frac{1}{M_k}\sum_{x_t=1}^{m_t-1}\frac{M_{t+1}}{|1-r_t(x)|}\sum_{j_k=1}^{n_k}\frac{M_k}{M_{t+1}}\frac{M_t}{j_kM_k}\\ &\lesssim \frac{M_{t+1}\log m_t\log(1+n_k)}{M_k}. \end{aligned} \tag{12}$$

If $x-x_te_t\notin I_k$, then there exits an integer s, such that $t+1\leq s\leq k-1$ and $x-x_te_t\in I_s\setminus I_{s+1}$. Then, to investigate $B_{2,1,2}$, we divide the sum $\sum_{j_{k-1},\dots,j_{t+1}}$—that is, $B_{2,1,2}$—into two parts. Let $n_s^*=n_s$ for $s=t+2,\dots,k-1$ and $n_{t+1}^*=n_{t+1}-1$.

In part one, $n_s^*<\alpha_s$, which also indicates that $\alpha_s=m_s-1$, $\alpha_{s-1}=m_{s-1}-1$, $\dots$, $\alpha_{t+1}=m_{t+1}-1$, and consequently the order of sums $\sum_{j_s},\sum_{j_{s-1}},\dots,\sum_{j_{t+1}}$ can be changed. So now let

$$\begin{aligned}B_{2,1,2}^{\dagger} &:= \frac{M_{t+1}}{1-r_t}\\ &\quad\times\sum_{j_k=1}^{n_k}\sum_{\substack{j_i\leq\alpha_i\\ i\in\{t+1,\dots,k-1\}\setminus\{s\}\\ n_s^*<\alpha_s}}\prod_{\substack{i=t+1\\ i\neq s}}^{k-1} r_i^{j_i}\sum_{j_s=0}^{m_s-1} r_s^{j_s}\sum_{u=0}^{M_t-1}\frac{1}{j^{(t+1)}+M_{t+1}-M_t+u}.\end{aligned}$$

In part two, we have $n_s^*=\alpha_s$, which also indicates that $j_{s+1}=n_{s+1},\dots,j_k=n_k$. That is, let

$$\begin{aligned}&B_{2,1,2}^{\ddagger}\\ &\quad:= \frac{M_{t+1}}{1-r_t}\sum_{j_s=0}^{n_s^*}\sum_{j_{s-1}=0}^{\alpha_{s-1}}\cdots\sum_{j_{t+1}=0}^{\alpha_{t+1}} r_k^{n_k}\cdots r_{s+1}^{n_{s+1}}\\ &\qquad\times r_s^{j_s}\cdots r_{t+1}^{j_{t+1}}\sum_{u=0}^{M_t-1}\frac{1}{j^{(t+1)}+M_{t+1}-M_t+u}.\end{aligned} \tag{13}$$

That is,

$$B_{2,1,2} = B_{2,1,2}^{\dagger} + B_{2,1,2}^{\ddagger}. \tag{14}$$

To discuss $B^{\dagger}_{2,1,2}$, we use the Abel transform again:

$$\sum_{j_s=0}^{m_s-1} r_s^{j_s} \sum_{u=0}^{M_t-1} \frac{1}{j^{(t+1)}+M_{t+1}-M_t+u}$$
$$= \sum_{j_s=0}^{m_s-2}\Big(\sum_{u=0}^{M_t-1}\Big(\frac{1}{j^{(t+1)}+M_{t+1}-M_t+u}$$
$$-\frac{1}{j^{(t+1)}+M_s+M_{t+1}-M_t+u}\Big)\Big)\sum_{a=0}^{j_s} r_s^a$$
$$+\sum_{u=0}^{M_t-1}\frac{1}{j_{(s)}-j_{(t+1)}+j^{(s+1)}+M_{s+1}-M_s+M_{t+1}-M_t+u}\sum_{a=0}^{m_s-1} r_s^a$$
$$= \sum_{j_s=0}^{m_s-1}\Big(\sum_{u=0}^{M_t-1}\Big(\frac{1}{j^{(t+1)}+M_{t+1}-M_t+u}$$
$$-\frac{1}{j^{(t+1)}+M_s+M_{t+1}-M_t+u}\Big)\Big)\sum_{a=0}^{j_s} r_s^a.$$

We have that

$$\sum_{u=0}^{M_t-1}\Big|\frac{1}{j^{(t+1)}+M_{t+1}-M_t+u}-\frac{1}{j^{(t+1)}+M_s+M_{t+1}-M_t+u}\Big| \le \frac{M_tM_s}{j_k^2M_k^2}$$

gives

$$\sum_{s=t+1}^{k-1}\sum_{x_t=1}^{m_t-1}\int_{I_s(x_te_t)\setminus I_{s+1}(x_te_t)}\big|B^{\dagger}_{2,1,2}(x)\big|\,d\mu(x)$$
$$\le \sum_{j_k=1}^{n_k}\sum_{s=t+1}^{k-1}\sum_{x_t=1}^{m_t-1} M_{t+1}\frac{m_{t+1}\cdots m_{k-1}}{m_s}\frac{M_tM_s}{j_k^2M_k^2}$$
$$\times\sum_{x_s=1}^{m_s-1}\int_{I_{s+1}(x_te_t+x_se_s)}\sum_{j_s=0}^{m_s-1}\Big|\frac{r_s^{j_s+1}(x)-1}{r_s(x)-1}\frac{1}{1-r_t(x)}\Big|\,d\mu(x)$$
$$\lesssim \sum_{s=t+1}^{k-1} M_{t+1}\frac{M_k}{M_{t+1}}\frac{M_t}{m_sM_k^2}m_sm_t\log m_s\log m_t$$
$$=\frac{M_{t+1}}{M_k}\log m_t\sum_{s=t+1}^{k-1}\log m_s=\frac{M_{t+1}}{M_k}\log m_t\log\Big(\frac{M_k}{M_{t+1}}\Big).$$

On the other hand, in the term $B^{\ddagger}_{2,1,2}$, we have $\alpha_s = n_s^*$ and then $j_{s+1} = n_{s+1},\dots,j_k=n_k$. Then, by (13), it is easy to have

$$|B^{\ddagger}_{2,1,2}| \le \frac{M_{t+1}}{|1-r_t|}m_{s-1}\cdots m_{t+1}\Big|\frac{r_s^{n_s+1}-1}{r_s(x)-1}\Big|\frac{M_t}{n_kM_k}.$$

Moreover,

$$\begin{aligned}
&\sum_{s=t+1}^{k-1}\sum_{x_t=1}^{m_t-1}\int_{I_s(x_te_t)\setminus I_{s+1}(x_te_t)}\left|B^{\ddagger}_{2,1,2}(x)\right|d\mu(x)\\
&\quad\le\sum_{s=t+1}^{k-1}\sum_{x_t=1}^{m_t-1}\frac{M_{t+1}M_t}{n_kM_k}\frac{M_s}{M_{t+1}}\int_{I_s(x_te_t)\setminus I_{s+1}(x_te_t)}\left|\frac{r_s^{n_s+1}(x)-1}{r_s(x)-1}\frac{1}{1-r_t(x)}\right|dx\\
&\quad\lesssim\sum_{s=t+1}^{k-1}\frac{M_t}{n_kM_k}\log(1+n_s)m_t\log m_t\\
&\quad\le\frac{M_{t+1}\log m_t}{M_k}\log\Big(\frac{M_k}{M_{t+1}}\Big), \qquad (15)
\end{aligned}$$

which is nothing else but the same bound in the case of $B^{\dagger}_{2,1,2}$. That is, we have the same upper bound for the integrals of the two addends of $B_{2,1,2}$.

Finally, inequalities (7)–(15) give

$$\begin{aligned}
&\int_{I_t\setminus I_{t+1}}\Big|\sum_{j=M_k}^{n-1}\frac{D_j}{j}\Big|\\
&\quad\lesssim\frac{M_{t+1}}{M_k}+\frac{(1+n_t)M_t}{n_kM_k}\log(1+n_t)+\frac{M_{t+1}\log m_t}{M_k}\\
&\qquad+\frac{M_{t+1}\log m_t\log(1+n_k)}{M_k}+\frac{M_{t+1}}{M_k}\log m_t\log\Big(\frac{M_k}{M_{t+1}}\Big)\\
&\quad\lesssim\frac{M_{t+1}\log m_t}{M_k}\Big(\log(1+n_k)+\log\Big(\frac{M_k}{M_{t+1}}\Big)\Big).
\end{aligned}$$

This completes the proof of Lemma 4. □

Lemma 5. *Let $M_k\le n<M_{k+1}$. Then*

$$\begin{aligned}
&\int_G\Big|\sum_{j=M_k}^{n-1}\frac{D_j}{j}\Big|\\
&\quad\lesssim\log^2(1+n_k)+\sum_{t=0}^{k-1}\frac{M_{t+1}\log m_t}{M_k}\Big(\log(1+n_k)+\log\Big(\frac{M_k}{M_{t+1}}\Big)\Big).
\end{aligned}$$

Proof. For $x\in I_{k+1}$, we have $\sum_{j=M_k}^{n-1}\frac{D_j}{j}(x)=\sum_{j=M_k}^{n-1}1=n-M_k$, and then

$$\int_{I_{k+1}}\Big|\sum_{j=M_k}^{n-1}\frac{D_j}{j}\Big|\le 1.$$

The rest of the proof of Lemma 5 is given by Lemma 3 and Lemma 4. □

Lemma 6. *Let $M_k\le n<M_{k+1}$. Then*

$$\Big\|\frac{1}{l_n}\sum_{j=1}^{n}\frac{D_j}{j}\Big\|_1\lesssim\frac{\sum_{j=0}^{k-1}\log^2 m_j}{\log M_k}+\frac{\sum_{j=0}^{k}\log^2 m_j}{\log M_{k+1}}.$$

Proof. We can write

$$\int_G \frac{1}{l_n}\Big|\sum_{j=1}^{n-1}\frac{D_j(x)}{j}\Big|\,d\mu(x)$$
$$\leq \int_G \frac{1}{l_n}\Big|\sum_{j=1}^{M_k-1}\frac{D_j(x)}{j}\Big|\,d\mu(x)+\int_G \frac{1}{l_n}\Big|\sum_{j=M_k}^{n-1}\frac{D_j(x)}{j}\Big|\,d\mu(x)$$
$$:= I+II. \tag{16}$$

Using Lemma 5 for II, we have

$$II \lesssim \frac{1}{\log(n_kM_k)}\Big\{\log^2(1+n_k)+\sum_{t=0}^{k-1}\frac{M_{t+1}\log m_t}{M_k}$$
$$\times\Big(\log(1+n_k)+\log\Big(\frac{M_k}{M_{t+1}}\Big)\Big)\Big\}$$
$$\lesssim \frac{\log^2(1+n_k)}{\log(n_kM_k)}+\frac{1}{\log(n_kM_k)}\sum_{t=0}^{k-1}\frac{M_{t+1}\log m_t}{M_k}\log(1+n_k)$$
$$+\frac{1}{\log(n_kM_k)}\sum_{t=0}^{k-1}\frac{M_{t+1}\log m_t}{M_k}\log\Big(\frac{M_k}{M_{t+1}}\Big)$$
$$\lesssim \frac{\log^2 m_k}{\log M_{k+1}}+\frac{\log m_k}{\log M_{k+1}}\max_{0\leq t<k}\log m_t$$
$$+\frac{1}{\log M_k}\sum_{t=0}^{k-1}\log m_t \lesssim \frac{\sum_{j=0}^{k}\log^2 m_j}{\log M_{k+1}}. \tag{17}$$

Using Abel's transformation for I, we get

$$I\leq \int_G \frac{1}{l_n}\Big|\sum_{j=1}^{M_k-2}\Big(\frac{1}{j}-\frac{1}{j+1}\Big)jK_j(x)\Big|\,d\mu(x)$$
$$+\int_G\frac{1}{l_n}\big|K_{M_k-1}(x)\big|\,d\mu(x)=I_1+I_2. \tag{18}$$

By Lemma 2 and taking account the above, we can write

$$I_1\leq \frac{1}{l_n}\sum_{j=1}^{M_k-1}\frac{\|K_j\|_1}{j}=\frac{1}{l_n}\sum_{r=0}^{k-1}\sum_{j=M_r}^{M_{r+1}-1}\frac{\|K_j\|_1}{j}$$
$$=\frac{1}{l_n}\sum_{r=0}^{k-1}\sum_{a=1}^{m_r-1}\sum_{j=aM_r}^{(a+1)M_r-1}\frac{\|K_j\|_1}{j}$$
$$\lesssim \frac{1}{\log n}\sum_{r=0}^{k-1}\log m_r\sum_{i=0}^{r+1}\frac{\beta_{r+1-i}}{2^i}.$$

Since by (5)

$$\log m_r \leq \beta_{r+1} \leq \sum_{t=0}^{r} \frac{\log m_t}{2^{r-t}} \tag{19}$$

we have

$$\begin{aligned}
I_1 &\lesssim \frac{1}{\log n} \sum_{r=0}^{k-1} \log m_r \sum_{i=0}^{r+1} \frac{1}{2^{i+1}} \sum_{t=0}^{r-i} \frac{\log m_t}{2^{r-i-t}} \\
&= \frac{c}{\log n} \sum_{r=0}^{k-1} \log m_r \sum_{t=1}^{r} \sum_{i=0}^{r-t} \frac{\log m_t}{2^{r-t+1}} \\
&\lesssim \frac{1}{\log n} \sum_{r=0}^{k-1} \sum_{t=1}^{r} \frac{r-t+1}{2^{r-t+1}} \log m_t \log m_r \\
&\lesssim \frac{1}{\log n} \sum_{r=0}^{k-1} \sum_{t=1}^{r} \frac{r-t+1}{2^{r-t+1}} (\log^2 m_t + \log^2 m_r) \\
&= \frac{c}{\log n} \sum_{r=0}^{k-1} \sum_{t=1}^{r} \frac{r-t+1}{2^{r-t+1}} \log^2 m_t \\
&\quad + \frac{c}{\log n} \sum_{r=0}^{k-1} \sum_{t=1}^{r} \frac{r-t+1}{2^{r-t+1}} \log^2 m_r \\
&\lesssim \frac{1}{\log M_k} \sum_{r=0}^{k-1} \log^2 m_r.
\end{aligned} \tag{20}$$

Using Lemma 2 and inequality (19), we have

$$\begin{aligned}
I_2 &\lesssim \frac{1}{\log n} \sum_{i=0}^{k} \frac{\beta_{k-i}}{2^i} \\
&\lesssim \frac{1}{\log n} \sum_{t=0}^{k-1} \sum_{i=0}^{k-t-1} \frac{\log m_t}{2^{k-t}} \\
&\lesssim \frac{1}{\log n} \sum_{t=0}^{k-1} \log m_t \lesssim 1.
\end{aligned} \tag{21}$$

From (18)–(21) we obtain

$$I \lesssim \frac{1}{\log M_k} \sum_{r=0}^{k-1} \log^2 m_r. \tag{22}$$

Combining (16), (17), and (22), we complete the proof of Lemma 6. □

Proof of Theorem 1. Using the theorem of Fubini and Lemma 6, we have

$$\begin{aligned}\left\|R_n(f)\right\|_X &= \left\|f * \left(\frac{1}{l_n}\sum_{j=1}^{n-1}\frac{D_j}{j}\right)\right\|_X \\ &\lesssim \left\|\frac{1}{l_n}\sum_{j=1}^{n}\frac{D_j}{j}\right\|_1 \|f\|_X \lesssim \|f\|_X. \end{aligned} \tag{23}$$

Observe that (4) trivially holds for the Vilenkin polynomials. Indeed, let T be a Vilenkin polynomial of degree s. Then we have

$$\begin{aligned}R_n(T) &= \frac{1}{l_n}\sum_{j=1}^{s}\frac{S_j(T)}{j} + \frac{1}{l_n}\sum_{j=s+1}^{n}\frac{S_j(T)}{j} \\ &= \frac{1}{l_n}\sum_{j=1}^{s}\frac{S_j(T)}{j} + \frac{1}{l_n}\sum_{j=s+1}^{n}\frac{T}{j}.\end{aligned}$$

Hence

$$\left\|R_n(T) - T\right\|_X \lesssim \frac{s}{l_n} \to 0 \quad \text{as } n \to \infty.$$

Then from (23), we conclude that

$$\begin{aligned}\left\|R_n(f) - f\right\|_X &\le \left\|R_n(f-T)\right\|_X + \left\|R_n(T) - T\right\|_X + \|f-T\|_X \\ &\lesssim \|f-T\|_X + \left\|R_n(T) - T\right\|_X.\end{aligned}$$

Hence

$$\left\|R_n(f) - f\right\|_X \to 0 \quad \text{as } n \to \infty.$$

Theorem 1 is proved. □

Example 7. We give an example for an unbounded sequence m satisfying the condition of Theorem 1. Basically, this condition shows that the number of "big" elements of the generating sequence m is relatively "small" with respect to all the elements. That is, for example, let

$$m_i \le \begin{cases} 2^{\sqrt[3]{i}} & \text{if } i = s^3, \\ C & \text{otherwise,}\end{cases}$$

for some natural number s. Then by $M_n \ge 2^n$, we have

$$\overline{\lim_{n\to\infty}} \frac{\sum_{k=0}^{n-1} \log^2 m_k}{\log M_n} \le \overline{\lim_{n\to\infty}} \frac{Cn + \sum_{s=0}^{\lfloor\sqrt[3]{n}\rfloor} s^2}{n} < \infty.$$

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References

1. G. N. Agaev, N. Ya. Vilenkin, G. M. Dzhafarli, and A. I. Rubinshteĭn, *Multiplicative Systems of Functions and Harmonic Analysis on Zero-dimensional Groups*, Ehlm, Baku, 1981. MR0679132. 423, 425
2. M. Avdispahić, *Concepts of generalized bounded variation and the theory of Fourier series*, Int. J. Math. Math. Sci. **9** (1986), no. 2, 223–244. Zbl 0595.42012. MR0848219. DOI 10.1155/S0161171286000285. 426
3. M. Avdispahić and N. Memić, *On the Lebesgue test for convergence of Fourier series on unbounded Vilenkin groups*, Acta Math. Hungar. **129** (2010), no. 4, 381–392. Zbl 1274.43006. MR2739772. DOI 10.1007/s10474-010-0023-9. 426
4. M. Avdispahić and M. Pepić, *On summability in L_p-norm on general Vilenkin groups*, Taiwanese J. Math. **4** (2000), no. 2, 285–296. Zbl 0969.42013. MR1757407. 426
5. M. Avdispahić and M. Pepić, *Summability and integrability of Vilenkin series*, Collect. Math. **51** (2000), no. 3, 237–254. Zbl 0976.42013. MR1814327. 426
6. I. Blahota and G. Gát, *Norm summability of Nörlund logarithmic means on unbounded Vilenkin groups*, Anal. Theory Appl. **24** (2008), no. 1, 1–17. Zbl 1164.42022. MR2422455. DOI 10.1007/s10496-008-0001-z. 425, 426
7. A. V. Efimov, *On certain approximation properties of periodic multiplicative orthonormal systems*, Mat. Sb. (N.S.) **69** (1966), 354–370. Zbl 0171.31901. MR0196419. 426
8. N. J. Fine, *On the Walsh functions*, Trans. Amer. Math. Soc. **65** (1949), 372–414. Zbl 0036.03604. MR0032833. DOI 10.2307/1990619. 426
9. G. Gát, *Cesàro means of integrable functions with respect to unbounded Vilenkin systems*, J. Approx. Theory **124** (2003), no. 1, 25–43. Zbl 1032.43003. MR2010779. DOI 10.1016/S0021-9045(03)00075-3. 424, 425, 426
10. G. Gát and U. Goginava, *Uniform and L-convergence of logarithmic means of Walsh-Fourier series*, Acta Math. Sin. (Engl. Ser.) **22** (2006), no. 2, 497–506. Zbl 1129.42411. MR2214371. DOI 10.1007/s10114-005-0648-8. 424, 425, 426
11. U. Goginava, *On the uniform convergence of Walsh-Fourier series*, Acta Math. Hungar. **93** (2001), no. 1–2, 59–70. Zbl 0992.42012. MR1924667. DOI 10.1023/A:1013865315680. 426
12. U. Goginava, *Uniform convergence of Cesàro means of negative order of double Walsh-Fourier series*, J. Approx. Theory **124** (2003), no. 1, 96–108. Zbl 1029.42022. MR1924667. DOI 10.1016/S0021-9045(03)00134-5. 426
13. J. Pál and P. Simon, *On a generalization of the concept of derivative*, Acta Math. Acad. Sci. Hungar. **29** (1977), no. 1–2, 155–164. Zbl 0345.42011. MR0450884. 425
14. J. Price, *Certain groups of orthonormal step functions*, Canad. J. Math. **9** (1957), 413–425. Zbl 0079.09204. MR0087792. DOI 10.4153/CJM-1957-049-x. 425
15. M. Riesz, *Sur un théorème de la moyenne et ses applications*, Acta Litt. ac Scient. Univ. Hung. **1** (1923), 114–126. Acta. Sci. Math. (Szeged), **1** (1922), 114-126. JFM 49.0707.01. 424
16. F. Schipp, *On L_p-norm convergence of series with respect to product systems*, Anal. Math. **2** (1976), no. 1, 49–64. Zbl 0364.40009. MR0428430. 425
17. F. Schipp, W. R. Wade, P. Simon, *Walsh Series: An Introduction to Dyadic Harmonic Analysis*, Adam Hilger, Bristol, 1990. Zbl 0727.42017. MR1117682. 423
18. P. Simon, *Verallgemeinerte Walsh-Fourierreihen, II.* Acta Math. Acad. Sci. Hungar. **27** (1976), no. 3–4, 329–341. Zbl 0335.42009. MR0470601. DOI 10.1007/BF01902112. 424, 425
19. O. Szász, *On the logarithmic means of rearranged partial sums of Fourier series*, Bull. Amer. Math. Soc. **48** (1942), 705–711. Zbl 0060.18101. MR0006756. DOI 10.1090/S0002-9904-1942-07763-9. 424
20. F. T. Wang, *On the summability of Fourier series by Riesz's logarithmic means, II*, Tohoku Math. J. **40** (1935), 392–397. JFM 61.0282.03. 424

21. K. Yabuta, *Quasi-Tauberian theorems, applied to the summability of Fourier series by Riesz's logarithmic means*, Tohoku Math. J. (2) **22** (1970), 117–129. Zbl 0192.42601. MR0270063. DOI 10.2748/tmj/1178242866. 424
22. W.-S. Young, *Mean convergence of generalized Walsh-Fourier series*, Trans. Amer. Math. Soc. **218** (1976), 311–320. Zbl 0327.43009. MR0394022. DOI 10.2307/1997441. 425

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