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CHARACTERIZATIONS OF ASYMMETRIC TRUNCATED TOEPLITZ OPERATORS

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Abstract. The aim of this paper is to investigate asymmetric truncated Toeplitz operators with L^2-symbols between two different model spaces given by inner functions such that one divides the other. The class of symbols corresponding to the zero operator is described. Asymmetric truncated Toeplitz operators are characterized in terms of operators of rank at most 2, and the relations with the corresponding symbols are studied.

1. Introduction

Toeplitz operators on the Hardy space H^2, which are compositions of a multiplication operator and the orthogonal projection from L^2 onto H^2, constitute a classical topic in operator theory. In his important paper [19], Sarason explored truncated Toeplitz operators, thus generating huge interest in this class of operators (see, e.g., [4], [6], [7], [9]–[12], and [15]). Instead of the Hardy space H^2, they act on a model space $K^2_\theta = H^2 \ominus \theta H^2$ associated with a given nonconstant inner function θ, and a multiplication operator is composed with the orthogonal projection from H^2 onto K^2_θ.

This work was inspired, on the one hand, by the work of Sarason, and, on the other hand, by [8], where asymmetric truncated Toeplitz operators were introduced (in the context of the Hardy space H^p of the half-plane, with $1 < p < \infty$)

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and studied in the case of bounded symbols. Asymmetric truncated Toeplitz operators involve the composition of a multiplication operator with two projections from H^2 onto a model space, associated with (possibly different) nonconstant inner functions α and θ. They are a natural generalization of rectangular Toeplitz matrices, which appear in various contexts, such as in the study of finite-time convolution equations, signal processing, control theory, probability, approximation theory, and diffraction problems (see, e.g., [2]–[4], [16], [17], [1], and [20]).

Here we consider bounded asymmetric truncated Toeplitz operators with L^2-symbols, defined between two model spaces K_θ^2 and K_α^2, where α divides θ ($\alpha \leq \theta$). We study various properties of these operators and their relations with the corresponding symbols, and we obtain a necessary and sufficient condition for a bounded operator between two model spaces to be an asymmetric truncated Toeplitz operator in terms of rank 2 operators, thus generalizing a corresponding result by Sarason for the case where $\alpha = \theta$. In the asymmetric case, however, a more complex connection between the operators and their symbols is revealed, which is not apparent when the two model spaces involved are the same.

Our article is organized as follows. In Section 2, we present some auxiliary results on model spaces, their direct sum decompositions, and the associated projections and conjugations. In Section 3, asymmetric truncated Toeplitz operators with L^2-symbols are introduced and some of their basic properties are presented. In Section 4, the class of all possible symbols for a given asymmetric truncated Toeplitz operator is described, and a necessary and sufficient condition for the operator to be zero is obtained. In Sections 5 and 6, we generalize Sarason's characterization of truncated Toeplitz operators in terms of operators of rank 2 at most, highlighting the similarities and differences between the symmetric and the asymmetric cases. In Section 7, we obtain conditions under which that characterization can be done in terms of a rank 1 operator. In Section 8, we address the inverse problem of determining a symbol for a given asymmetric truncated Toeplitz operator in terms of the action of A and its adjoint on certain reproducing kernel functions. Moreover, we give the formulas for a symbol of the operator satisfying the characterizations in terms of rank 2 operators presented in Sections 5 and 6.

2. Model spaces, conjugations, and decompositions

Let L^2 denote the space $L^2(\mathbb{T}, m)$, where $\mathbb{T}$ is the unit circle and m is the normalized Lebesgue measure on $\mathbb{T}$, and let H^2 be the Hardy space on the unit disk $\mathbb{D}$, identified as usual with a subspace of L^2. Similarly, $L^\infty = L^\infty(\mathbb{T}, m)$, and we denote by H^∞ the space of all analytic and bounded functions on $\mathbb{D}$. Denoting by H_0^2 the subspace consisting of all functions in H^2 which vanish at 0, we have $L^2 \ominus H^2 = \overline{H_0^2}$, and we denote by P and P^- the orthogonal projections from L^2 onto H^2 and $\overline{H_0^2}$, respectively.

With any given inner function θ we associate the so-called *model space* K_θ^2, defined by $K_\theta^2 = H^2 \ominus \theta H^2$. We also have $K_\theta^2 = H^2 \cap \theta \overline{H_0^2}$; thus,

$$f \in K_\theta^2 \quad \text{if and only if} \quad \bar{\theta} f \in \overline{H_0^2} \quad \text{and} \quad f \in H^2.$$

In particular, if $f \in K_\theta^2$, then $\theta\bar{f} \in H_0^2$. Let P_θ be the orthogonal projection $P_\theta\colon L^2 \to K_\theta^2$. We then have the following.

Proposition 2.1. *Let θ be a nonconstant inner function, and let $K_\theta^2 = H^2 \ominus \theta H^2$ be the associated model space. Then*

(1) $P_\theta = \theta P^- \bar{\theta} P = \theta P^- \bar{\theta} - P^-$,
(2) $P_\theta f = \theta P^- \bar{\theta} f = f - \theta P \bar{\theta} f$ *for all* $f \in H^2$, *and*
(3) $P_\theta \bar{f} = P_\theta P \bar{f} = \overline{f(0)} P_\theta 1 = \overline{f(0)}(1 - \overline{\theta(0)}\theta)$ *for all* $f \in H^2$.

Model spaces are also equipped with *conjugations* (antilinear isometric involutions), which are important tools in the study of model spaces and truncated Toeplitz operators (see, e.g., [13], [14], and [18]). For a given inner function θ, the conjugation C_θ is defined by $C_\theta\colon L^2 \to L^2$,

$$C_\theta f(z) = \theta \overline{z f(z)}.$$

It is worth noting that C_θ preserves the space K_θ^2 and maps θH^2 onto $L^2 \ominus H^2$.

Recall that, for $\lambda \in \mathbb{D}$, the *kernel function* in H^2 denoted by k_λ is given by $k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$. Similarly, for an inner function θ, in K_θ^2 the kernel function k_λ^θ is given by $k_\lambda^\theta = P_\theta k_\lambda$; that is, $k_\lambda^\theta = k_\lambda(1 - \overline{\theta(\lambda)}\theta)$. The set $\{k_\lambda^\theta : \lambda \in \mathbb{D}\}$ is linearly dense in K_θ^2. Since $k_\lambda^\theta \in K_\theta^\infty$, where K_θ^∞ denotes the subspace $H^\infty \cap K_\theta^2$, the space K_θ^∞ is dense in K_θ^2 (see [19, p. 494]).

Defining $\tilde{k}_\lambda^\theta = C_\theta k_\lambda^\theta$, we have in particular

$$k_0^\theta(z) = 1 - \overline{\theta(0)}\theta(z), \qquad \tilde{k}_0^\theta(z) = \bar{z}\big(\theta(z) - \theta(0)\big).$$

It is easy to see that, for all $f \in K_\theta^2$,

$$\langle f, k_0^\theta \rangle = f(0), \qquad \langle f, \tilde{k}_0^\theta \rangle = \overline{(C_\theta f)(0)}.$$

Now let us consider two nonconstant inner functions α and θ. If $\bar{\alpha}\theta$ is an inner function, then we say that α *divides* θ, and we write $\alpha \leq \theta$.

Proposition 2.2. *Let α, θ be nonconstant inner functions such that $\alpha \leq \theta$. Then the following hold:*

(1) $K_\theta^2 = K_\alpha^2 \oplus \alpha K_{\frac{\theta}{\alpha}}^2$,
(2) $P_\theta = P_\alpha + \alpha P_{\frac{\theta}{\alpha}} \bar{\alpha}$,
(3) $k_0^\theta = k_0^\alpha + \overline{\alpha(0)} \alpha k_0^{\frac{\theta}{\alpha}}$,
(4) $\tilde{k}_0^\theta = \frac{\theta}{\alpha}(0) \tilde{k}_0^\alpha + \alpha \tilde{k}_0^{\frac{\theta}{\alpha}}$, *and*
(5) $P_\alpha k_0^\theta = k_0^\alpha, P_\alpha \tilde{k}_0^\theta = \frac{\theta}{\alpha}(0)\tilde{k}_0^\alpha$.

Proof. Items (1), (2), and (3) were proved in [12, p. 97]. We need to prove only (4), and (5) follows immediately from (3) and (4). In fact, we have

$$\tilde{k}_0^\theta = \bar{z}\big(\theta - \theta(0)\big) = \alpha \bar{z} \tfrac{\theta}{\alpha} - \bar{z}\alpha(0)\tfrac{\theta}{\alpha}(0) = \alpha\bar{z}\big(\tfrac{\theta}{\alpha} - \tfrac{\theta}{\alpha}(0)\big) + \tfrac{\theta}{\alpha}(0)\bar{z}\big(\alpha - \alpha(0)\big),$$

where $\bar{z}(\frac{\theta}{\alpha} - \frac{\theta}{\alpha}(0)) = \tilde{k}_0^{\frac{\theta}{\alpha}}, \bar{z}(\alpha - \alpha(0)) = \tilde{k}_0^\alpha$. □

The following proposition describes some relations between decompositions and conjugations. Note that, if $\alpha \leq \theta$, then any $f \in K_\theta^2$ can be uniquely decomposed as $f = f_1 + \alpha f_2$ for some $f_1 \in K_\alpha^2$ and some $f_2 \in K_{\frac{\theta}{\alpha}}^2$, or as $f = f_2 + \frac{\theta}{\alpha} f_1$ for some $f_1 \in K_\alpha^2$ and some $f_2 \in K_{\frac{\theta}{\alpha}}^2$. Then the conjugation C_θ can be seen as $C_\theta \colon K_\theta^2 = K_\alpha^2 \oplus \alpha K_{\frac{\theta}{\alpha}}^2 \to K_\theta^2 = K_{\frac{\theta}{\alpha}}^2 \oplus \frac{\theta}{\alpha} K_\alpha^2$, or as $C_\theta \colon K_\theta^2 = K_{\frac{\theta}{\alpha}}^2 \oplus \frac{\theta}{\alpha} K_\alpha^2 \to K_\theta^2 = K_\alpha^2 \oplus \alpha K_{\frac{\theta}{\alpha}}^2$. Now we have the following.

Proposition 2.3. *Let α, θ be nonconstant inner functions such that $\alpha \leq \theta$. If $f_1 \in K_\alpha^2$ and $f_2 \in K_{\frac{\theta}{\alpha}}^2$, then*

(1) $C_\theta(f_1 + \alpha f_2) = C_{\frac{\theta}{\alpha}} f_2 + \frac{\theta}{\alpha} C_\alpha f_1$, *and*
(2) $C_\theta(f_2 + \frac{\theta}{\alpha} f_1) = C_\alpha f_1 + \alpha C_{\frac{\theta}{\alpha}} f_2$.

Proof. If we let $f_1 \in K_\alpha^2$, $f_2 \in K_{\frac{\theta}{\alpha}}^2$, then $C_\theta(f_1 + \alpha f_2) = \theta \bar{z} \bar{f}_1 + \theta \bar{z} \bar{\alpha} \bar{f}_2 = \frac{\theta}{\alpha} \alpha \bar{z} \bar{f}_1 + \frac{\theta}{\alpha} \bar{z} \bar{f}_2 = C_{\frac{\theta}{\alpha}}(f_2) + \frac{\theta}{\alpha} C_\alpha(f_1)$, and $C_\theta(f_2 + \frac{\theta}{\alpha} f_1) = \theta \bar{z} \bar{f}_2 + \theta \bar{z} \frac{\bar{\theta}}{\bar{\alpha}} \bar{f}_1 = \alpha \bar{z} \bar{f}_1 + \alpha \frac{\theta}{\alpha} \bar{z} \bar{f}_2 = C_\alpha(f_1) + \alpha C_{\frac{\theta}{\alpha}}(f_2)$. □

Corollary 2.4. *Let α, θ be nonconstant inner functions such that $\alpha \leq \theta$. We have, for $f \in K_\theta^2$,*

(1) $P_\alpha C_\theta f = C_\alpha P_\alpha (\frac{\bar{\theta}}{\bar{\alpha}} f)$, *and*
(2) $P_{\frac{\theta}{\alpha}} C_\theta f = C_{\frac{\theta}{\alpha}} P_{\frac{\theta}{\alpha}} \bar{\alpha} f$.

Let S be the unilateral shift on the Hardy space H^2, and, for a nonconstant inner function θ, let $S_\theta = P_\theta S_{|K_\theta^2}$ be the compression of S to K_θ^2. The space K_θ^2 is invariant for S^*; thus, $(S_\theta)^* = S^*{}_{|K_\theta^2}$. Note that, for any $f \in K_\theta^2$,

$$S_\theta f = zf - \overline{(C_\theta f)(0)} \theta = Sf - \langle f, \tilde{k}_0^\theta \rangle \theta, \tag{2.1}$$

$$S_\theta^* f = \bar{z}\big(f - f(0)\big). \tag{2.2}$$

In particular,

$$S_\theta^* k_0^\theta = -\overline{\theta(0)} \tilde{k}_0^\theta, \qquad S_\theta \tilde{k}_0^\theta = -\theta(0) k_0^\theta. \tag{2.3}$$

The function k_0^θ is a cyclic vector for S_θ, and $\tilde{k}_0^\theta$ is a cyclic vector for S_θ^* (see [19, Lemma 2.3]). In what follows we will use the *defect operators* $I_{K_\theta^2} - S_\theta S_\theta^* = k_0^\theta \otimes k_0^\theta$ and $I_{K_\theta^2} - S_\theta^* S_\theta = \tilde{k}_0^\theta \otimes \tilde{k}_0^\theta$, using the notation $(x \otimes y)z = \langle z, y \rangle x$ for any x, y, z in a Hilbert space H (see [19, Lemma 2.4]). We list below some simple properties that will be used later.

Proposition 2.5. *Let α and θ be nonconstant inner functions such that $\alpha \leq \theta$. Then*

(1) $P_\alpha S_\theta = S_\alpha P_\alpha$ *on* K_θ^2,
(2) $S_\theta^* P_\alpha = S_\alpha^*$ *on* K_α^2,
(3) $S_\theta^n k_0^\theta \in K_\theta^\infty$ *for* $n \geq 0$, *and*
(4) $(S_\theta^n)^* \tilde{k}_0^\theta \in K_\theta^\infty$ *for* $n \geq 0$.

Proof. The proof of (1) and (2) follows immediately. By [12, Theorem 9.2.2] we know that $S_\theta^n f = P_\theta S^n f$ for any $f \in K_\theta^2$, $n \geq 0$; hence,

$$S_\theta^n k_0^\theta = P_\theta S^n k_0^\theta = P_\theta\big(z^n\big(1 - \overline{\theta(0)}\theta\big)\big) = P_\theta z^n = z^n - \theta P(\bar\theta z^n).$$

Since $P(\bar\theta z^n)$ is a polynomial, then we get $S_\theta^n k_0^\theta \in K_\theta^\infty$, which proves (3). On the other hand,

$$(S_\theta^*)^n \tilde{k}_0^\theta = P_\theta \bar z^{n+1}\big(\theta - \theta(0)\big) = \theta \bar z^{n+1} - P^- \theta \bar z^{n+1};$$

since $P^-(\theta \bar z^n)$ is a polynomial in $\bar z$, we get (4). □

3. Asymmetric truncated Toeplitz operators

Let α, θ be nonconstant inner functions. For $\varphi \in L^2$, we define an operator $A_\varphi^{\theta,\alpha} \colon \mathcal{D} \subset K_\theta^2 \to K_\alpha^2$ as $A_\varphi^{\theta,\alpha} f = P_\alpha(\varphi f)$ having domain $\mathcal{D} = \mathcal{D}(A_\varphi^{\theta,\alpha}) = \{f \in K_\theta^2 \colon \varphi f \in L^2\}$. The operator $A_\varphi^{\theta,\alpha}$ is closed and densely defined in K_θ^2. Note that $K_\theta^\infty \subset \mathcal{D}(A_\varphi^{\theta,\alpha})$. The operator $A_\varphi^{\theta,\alpha}$ will be called an *asymmetric truncated Toeplitz operator*. If this operator is bounded, then it admits a unique bounded extension to K_θ^2, $A_\varphi^{\theta,\alpha} \colon K_\theta^2 \to K_\alpha^2$. By $\mathcal{T}(\theta,\alpha)$ we denote the space of all bounded asymmetric truncated Toeplitz operators. For $\alpha = \theta$ we will write A_φ^θ instead of $A_\varphi^{\theta,\theta}$. (Such operators are called *truncated Toeplitz operators*; they were studied by Sarason in [19].) In addition we will write $\mathcal{T}(\theta)$ instead of $\mathcal{T}(\theta,\theta)$. It is easy to see that the following holds.

Proposition 3.1. *Let α, θ be nonconstant inner functions such that $\alpha \leq \theta$. Then*

$$A_{k_0^\alpha}^{\theta,\alpha} = P_\alpha = A_{\overline{k_0^\theta}}^{\theta,\alpha}.$$

If $\varphi \in L^2$, then we have, for all $f \in K_\alpha^\infty$,

$$A_\varphi^{\theta,\alpha} f = A_\varphi^\alpha f.$$

Proposition 3.2. *Let α, θ be any inner functions, and let $\varphi \in L^2$. Then*

$$\langle A_\varphi^{\theta,\alpha} f, g\rangle = \langle f, A_{\bar\varphi}^{\alpha,\theta} g\rangle \quad \text{for all } f \in \mathcal{D}(A_\varphi^{\theta,\alpha}), g \in \mathcal{D}(A_{\bar\varphi}^{\alpha,\theta}).$$

Moreover, $\mathcal{D}(A_{\bar\varphi}^{\alpha,\theta}) = \mathcal{D}((A_\varphi^{\theta,\alpha})^*)$, and $(A_\varphi^{\theta,\alpha})^* = A_{\bar\varphi}^{\alpha,\theta}$.

Proof. A straightforward calculation shows that

$$\begin{aligned}\langle A_\varphi^{\theta,\alpha} f, g\rangle &= \big\langle P_\alpha(\varphi f), g\big\rangle = \langle \varphi f, g\rangle = \int f\varphi\bar g\, dm \\ &= \langle f, \bar\varphi g\rangle = \big\langle f, P_\theta(\bar\varphi g)\big\rangle = \langle f, A_{\bar\varphi}^{\alpha,\theta} g\rangle.\end{aligned}$$

Note also that $\mathcal{D}(A_{\bar\varphi}^{\alpha,\theta}) = \{g \in K_\alpha^2 \colon \bar\varphi g \in L^2\}$. Hence, if $g \in \mathcal{D}(A_{\bar\varphi}^{\alpha,\theta})$, then there is $h \in K_\theta^2$ such that

$$\langle A_\varphi^{\theta,\alpha} f, g\rangle = \langle f, h\rangle$$

for all $f \in \mathcal{D}(A_\varphi^{\theta,\alpha})$, that is, such that $\varphi f \in L^2$; therefore, $g \in \mathcal{D}((A_\varphi^{\theta,\alpha})^*)$. In fact, taking $h = P_\theta(\bar\varphi f)$, we have $\langle A_\varphi^{\theta,\alpha} f, g\rangle = \langle P_\alpha(\varphi f), g\rangle = \langle \varphi f, g\rangle = \langle f, \bar\varphi g\rangle = \langle f, P_\theta(\bar\varphi g)\rangle = \langle f, h\rangle$. The converse is similarly true. □

Proposition 3.3. *Let α, θ be nonconstant inner functions such that $\alpha \leq \theta$. Let $A_\psi^{\theta,\alpha}$ be an asymmetric truncated Toeplitz operator with $\psi \in H^2$. Then*

$$S_\alpha A_\psi^{\theta,\alpha} f = A_\psi^{\theta,\alpha} S_\theta f \quad \text{for all } f \in K_\theta^\infty.$$

Proof. Let $f \in K_\theta^\infty$. Then by Proposition 2.1,

$$S_\alpha A_\psi^{\theta,\alpha} f = S_\alpha P_\alpha(\psi f) = P_\alpha z\big(\psi f - \alpha P(\bar{\alpha}\psi f)\big) = P_\alpha(z\psi f)$$

since $P_\alpha(\alpha z P(\bar{\alpha}\psi f)) = 0$. On the other hand, taking (2.1) into account, for all $f \in K_\theta^\infty$, we have

$$A_\psi^{\theta,\alpha} S_\theta f = P_\alpha \psi S_\theta f = P_\alpha \psi\big(zf - \langle f, \tilde{k}_0^\theta\rangle\theta\big) = P_\alpha(z\psi f)$$

since $P_\alpha(\psi\theta) = 0$. □

Remark 3.4. Theorem 3.1.16 of [5] implies that, for nonconstant inner functions α, θ such that $\alpha \leq \theta$, if a bounded operator $A\colon K_\theta^2 \to K_\alpha^2$ intertwines S_α, S_θ (i.e., $S_\alpha A = AS_\theta$), then $A = A_\psi^{\theta,\alpha}$ for some $\psi \in H^\infty$.

Example 3.5. One can ask whether a similar result as in Proposition 3.3 can be obtained for $A_\psi^{\alpha,\theta}$ with $\alpha \leq \theta$ and $\psi \in H^2$; the answer is no. For example, let $\alpha = z^2$, $\theta = z^n$, let $n > 5$, $\psi = z^3$, and let $f = z$. Then $S_\theta A_\psi^{\alpha,\theta} f = z^5$, but $A_\psi^{\alpha,\theta} S_\alpha f = 0$.

In the next proposition, we describe the action of some (not necessarily bounded) asymmetric truncated Toeplitz operators on some particular functions. These properties will be used later on.

Proposition 3.6. *Let α, θ be nonconstant inner functions such that $\alpha \leq \theta$, and let $\psi \in K_\alpha^2, \chi \in K_\theta^2$ with $\chi = \chi_{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha}\chi_\alpha$, where $\chi_{\frac{\theta}{\alpha}} \in K_{\frac{\theta}{\alpha}}^2$ and $\chi_\alpha \in K_\alpha^2$. Then*

1. $A_\psi^{\theta,\alpha} k_0^\theta = \psi$,
2. $A_{\bar{\chi}}^{\theta,\alpha} k_0^\theta = \overline{\chi(0)} k_0^\alpha - \overline{\theta(0)}\alpha(\bar{\chi}_\alpha - \overline{\chi_\alpha(0)})$,
3. $A_\psi^{\theta,\alpha} \tilde{k}_0^\theta = \frac{\theta}{\alpha}(0)\psi(0)\tilde{k}_0^\alpha - \theta(0)S_\alpha^*\psi = \bar{z}(\frac{\theta}{\alpha}(0)\psi(0)\alpha - \theta(0)\psi)$,
4. $A_{\bar{\chi}}^{\theta,\alpha} \tilde{k}_0^\theta = P_\alpha C_\theta \chi = C_\alpha P_\alpha(\frac{\bar{\theta}}{\bar{\alpha}}\chi) = C_\alpha \chi_\alpha$,
5. $A_{\bar{\psi}}^{\alpha,\theta} k_0^\alpha = \overline{\psi(0)} k_0^\theta - \overline{\alpha(0)}(\alpha\bar{\psi} - \overline{\frac{\theta}{\alpha}(0)\psi(0)}\theta)$,
6. $A_\chi^{\alpha,\theta} k_0^\alpha = \chi - \overline{\alpha(0)}\alpha\chi_{\frac{\theta}{\alpha}}$,
7. $A_{\bar{\psi}}^{\alpha,\theta} \tilde{k}_0^\alpha = C_\alpha \psi$, *and*
8. $A_\chi^{\alpha,\theta} \tilde{k}_0^\alpha = P_\theta C_\theta(\bar{\chi}\frac{\theta}{\alpha}) - \alpha(0)S_\theta^*\chi = S^*(\alpha\chi_{\frac{\theta}{\alpha}}) + \langle\chi_\alpha, k_0^\alpha\rangle\tilde{k}_0^\theta - \alpha(0)S_\theta^*\chi$.

Proof. Item (1) follows from

$$A_\psi^{\theta,\alpha} k_0^\theta = P_\alpha\big(\psi - \overline{\theta(0)}\theta\psi\big) = \psi.$$

To prove item (2), we first calculate

$$P_\alpha(\theta\bar{\chi}) = P_\alpha\left(\alpha\frac{\theta}{\alpha}\bar{\chi}_{\frac{\theta}{\alpha}}\right) + P_\alpha(\alpha\bar{\chi}_\alpha) = P_\alpha(\alpha\bar{\chi}_\alpha) = \alpha\big(\bar{\chi}_\alpha - \overline{\chi_\alpha(0)}\big).$$

Hence, by Proposition 2.1(3), we have

$$A_{\bar{\chi}}^{\theta,\alpha} k_0^\theta = P_\alpha \bar{\chi} - \overline{\theta(0)} P_\alpha(\theta\bar{\chi}) = \overline{\chi(0)} k_0^\alpha - \overline{\theta(0)} \alpha\big(\bar{\chi}_\alpha - \overline{\chi_\alpha(0)}\big).$$

To prove item (3), note that, by Proposition 2.2,

$$\begin{aligned} A_\psi^{\theta,\alpha} \tilde{k}_0^\theta &= P_\alpha(\psi P_\theta \tilde{k}_0^\theta) = P_\alpha(\psi P_\alpha \tilde{k}_0^\theta) + P_\alpha(\psi \alpha P_{\frac{\theta}{\alpha}} \bar{\alpha} \tilde{k}_0^\theta) = P_\alpha(\psi P_\alpha \tilde{k}_0^\theta) \\ &= P_\alpha\left(\psi \frac{\theta}{\alpha}(0) \tilde{k}_0^\alpha\right) = \frac{\theta}{\alpha}(0) P_\alpha(\psi \bar{z} \alpha) - \theta(0) P_\alpha(\psi \bar{z}) \\ &= \frac{\theta}{\alpha}(0)\psi(0)\tilde{k}_0^\alpha - \theta(0) S_\alpha^* \psi \end{aligned}$$

since

$$\begin{aligned} P_\alpha \psi \bar{z} \alpha &= (\alpha P^- \bar{\alpha} - P^-)(\psi \alpha \bar{z}) = \alpha P^-(\psi \bar{z}) - \psi(0)\alpha(0)\bar{z} \\ &= \alpha \psi(0) \bar{z} - \psi(0)\alpha(0)\bar{z} = \psi(0)\tilde{k}_0^\alpha. \end{aligned}$$

To show item (4), we calculate

$$A_{\bar{\chi}}^{\theta,\alpha} \tilde{k}_0^\theta = P_\alpha\big(\bar{\chi}\bar{z}\big(\theta - \theta(0)\big)\big) = P_\alpha(\theta \bar{\chi} \bar{z}) - \theta(0) P_\alpha(\bar{\chi}\bar{z}) = P_\alpha C_\theta \chi,$$

and by Corollary 2.4,

$$P_\alpha(C_\theta \chi) = C_\alpha P_\alpha \frac{\bar{\theta}}{\bar{\alpha}} \chi = C_\alpha \chi_\alpha.$$

The equality in item (5) follows from

$$A_{\bar{\psi}}^{\alpha,\theta} k_0^\alpha = P_\theta\big(\bar{\psi}\big(1 - \overline{\alpha(0)}\alpha\big)\big) = \overline{\psi(0)} k_0^\theta - \overline{\alpha(0)} P_\theta(\alpha \bar{\psi})$$

and

$$P_\theta(\alpha\bar{\psi}) = \alpha\bar{\psi} - \theta P(\bar{\theta}\alpha\bar{\psi}) = \alpha\bar{\psi} - \theta P \frac{\bar{\theta}}{\bar{\alpha}} \bar{\psi} = \alpha\bar{\psi} - \theta \overline{\frac{\theta}{\alpha}(0)\psi(0)}.$$

The equality in item (6) follows from

$$A_\chi^{\alpha,\theta} k_0^\alpha = P_\theta\big(\chi\big(1 - \overline{\alpha(0)}\alpha\big)\big) = \chi - \overline{\alpha(0)} P_\theta(\alpha\chi) = \chi - \overline{\alpha(0)} \alpha \chi_{\frac{\theta}{\alpha}},$$

and item (7) follows from

$$A_{\bar{\psi}}^{\alpha,\theta} \tilde{k}_0^\alpha = P_\theta\big(\bar{\psi}\bar{z}\big(\alpha - \alpha(0)\big)\big) = P_\theta(\alpha \bar{z} \bar{\psi}) - \alpha(0) P_\theta(\bar{z}\bar{\psi}) = P_\theta C_\alpha \psi = C_\alpha \psi.$$

Finally, item (8) follows from

$$A_\chi^{\alpha,\theta} \tilde{k}_0^\alpha = P_\theta(\chi \tilde{k}_0^\alpha) = P_\theta(\alpha \chi \bar{z}) - \alpha(0) P_\theta(\bar{z}\chi) = P_\theta C_\theta\left(\bar{\chi}\frac{\theta}{\alpha}\right) - \alpha(0) S_\theta^* \chi.$$

Since $\chi = \chi_{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha}\chi_\alpha$, we have $\frac{\theta}{\alpha}\bar{\chi} = \frac{\theta}{\alpha}\bar{\chi}_{\frac{\theta}{\alpha}} + \bar{\chi}_\alpha$; hence

$$\begin{aligned} P_\theta\left(\frac{\theta}{\alpha}\bar{\chi}\right) &= P_\theta\left(\frac{\theta}{\alpha}\bar{\chi}_{\frac{\theta}{\alpha}}\right) + P_\theta \bar{\chi}_\alpha = \theta P^- \bar{\theta} \frac{\theta}{\alpha} \bar{\chi}_{\frac{\theta}{\alpha}} + P_\theta \overline{\chi_\alpha(0)} \\ &= \theta\big(\bar{\alpha}\bar{\chi}_{\frac{\theta}{\alpha}} - \bar{\alpha}(0)\bar{\chi}_{\frac{\theta}{\alpha}}(0)\big) + \overline{\chi_\alpha(0)} k_0^\theta. \end{aligned}$$

Thus

$$P_\theta C_\theta\left(\frac{\theta}{\alpha}\bar{\chi}\right) = C_\theta P_\theta\left(\frac{\theta}{\alpha}\bar{\chi}\right) = \bar{z}\big(\alpha\chi_{\frac{\theta}{\alpha}} - \alpha(0)\chi_{\frac{\theta}{\alpha}}(0)\big) + \chi_\alpha(0)C_\theta k_0^\theta$$
$$= S^*(\alpha\chi_{\frac{\theta}{\alpha}}) + \langle\chi_\alpha, k_0^\alpha\rangle\tilde{k}_0^\theta. \qquad \square$$

4. The symbols of zero in $\mathcal{T}(\theta,\alpha)$

In what follows, for $f \in H^2$, let f_i, f_o denote the inner and outer factors, respectively, in an inner-outer factorization $f = f_i f_o$. Recall that f_i and f_o are defined up to multiplication by a constant. For α and θ nonconstant inner functions, let $GCD(\theta,\alpha)$ denote the greatest common divisor of θ and α, which is also defined up to a constant.

In this section, we study the symbols for which the corresponding asymmetric truncated Toeplitz operator is the zero operator. This is equivalent to the kernel of the operator being equal to K_θ^2. We start by characterizing the kernels of some asymmetric truncated Toeplitz operators. The following result generalizes the disk versions of Theorem 7.2 in [8] and Theorem 3.2 in [7]. Note also that in Theorem 4.1, Lemma 4.2, and Corollary 4.3, we do not assume that α divides θ.

Theorem 4.1. *Let α and θ be nonconstant inner functions, and let $\varphi \in H^\infty$, $\varphi \neq 0$ with inner factor φ_i. Then*

(1) $\ker A_\varphi^{\theta,\alpha} = K_\theta^2 \cap \frac{\alpha}{\psi}H^2$ *with $\psi = GCD(\alpha,\varphi_i)$; in particular, if $\frac{\alpha}{\psi} \leq \theta$, then* $\ker A_\varphi^{\theta,\alpha} = \frac{\alpha}{\psi}K^2_{\frac{\theta\psi}{\alpha}}$*; and*

(2) *if $\theta \leq \alpha$, then* $\ker A_{\bar{\varphi}}^{\theta,\alpha} = \ker(A_\varphi^{\alpha,\theta})^* = K^2_{GCD(\theta,\psi)}$ *with $\psi = GCD(\alpha,\varphi_i)$.*

Proof. (1) Let $f \in K_\theta^2$. Note that $f \in \ker A_\varphi^{\theta,\alpha}$ if and only if $\varphi f \in \alpha H^2$, which is equivalent to α dividing $\varphi_i f_i$, where f_i is the inner factor of f. Since $\frac{\alpha}{\psi}$ and $\frac{\varphi_i}{\psi}$ are relatively prime, then f_i is divisible by $\frac{\alpha}{\psi}$; hence $\ker A_\varphi^{\theta,\alpha} = \{f \in K_\theta^2 : f \in \frac{\alpha}{\psi}H^2\} = K_\theta^2 \cap \frac{\alpha}{\psi}H^2$. If $\frac{\alpha}{\psi} \leq \theta$, then the result follows from the decomposition $K_\theta^2 = K^2_{\frac{\alpha}{\psi}} \oplus \frac{\alpha}{\psi}K^2_{\frac{\theta\psi}{\alpha}}$.

(2) If $\theta \leq \alpha$, in which case $K_\theta^2 \subset K_\alpha^2$, then we have

$$\begin{aligned}\ker A_{\bar{\varphi}}^{\theta,\alpha} &= \big\{f \in K_\theta^2\colon P_\alpha(\bar{\varphi}f) = 0\big\} = \big\{f \in K_\theta^2\colon P(\bar{\varphi}f) = 0\big\}\\ &= \{f \in K_\theta^2\colon \bar{\varphi}f \in \overline{H_0^2}\} = \{f \in K_\theta^2\colon \varphi\alpha\bar{z}\bar{f} \in \alpha H^2\}\\ &= \{f \in K_\theta^2\colon \varphi C_\alpha f \in \alpha H^2\} = \left\{f \in K_\theta^2\colon C_\alpha f \in \ker A_\varphi^\alpha = K_\alpha^2 \cap \frac{\alpha}{\psi}H^2\right\}\end{aligned}$$

by (1). Now $C_\alpha f \in K_\alpha^2 \cap \frac{\alpha}{\psi}H^2$ if and only if $f \in K_\alpha^2 \cap \psi\overline{H_0^2}$, which is equivalent to $f \in K_\alpha^2 \cap K_\psi^2 = K_\psi^2$; hence $f \in K_\theta^2 \cap K_\psi^2 = K^2_{GCD(\theta,\psi)}$. $\square$

Lemma 4.2. *Let α and θ be nonconstant inner functions, and let $\varphi \in H^2$. If we assume that $A_\varphi^{\alpha,\theta}$ is an asymmetric truncated Toeplitz operator, then $A_\varphi^{\alpha,\theta} = 0$ if and only if $\varphi \in \theta H^2$.*

Proof. If $\varphi \in \theta H^2$ and $f \in K_\alpha^\infty$, then $A_\varphi^{\alpha,\theta} f = P_\theta(\varphi f) = P_\theta(\theta\varphi_1 f) = 0$, where $\varphi = \theta\varphi_1$ and $\varphi_1 \in H^2$. Since K_α^∞ is dense in K_α^2, we have $A_\varphi^{\alpha,\theta} = 0$.

For the converse implication, let $\psi = GCD(\theta, \varphi_i)$. If $A_\varphi^{\alpha,\theta} = 0$ and $f \in K_\alpha^\infty$, then $\varphi f \in \theta H^2$; hence, θ divides $\varphi_i f_i$, and as a result $\frac{\varphi_i}{\psi} f_i \in \frac{\theta}{\psi} H^2$. Since $\frac{\varphi_i}{\psi}$ and $\frac{\theta}{\psi}$ are relatively prime, we have $f_i \in \frac{\theta}{\psi} H^2$. By the density of K_α^∞ in K_α^2, we have $K_\alpha^2 \subset \frac{\theta}{\psi} H^2$. Since K_α^2 is the kernel of the Toeplitz operator $T_{\bar\alpha}$, then it is near invariant with respect to functions from $\overline{H^\infty}$; that is, if $g \in \ker T_{\bar\alpha}$ and $h \in H^\infty$, then $\bar h g \in H^2$ implies $\bar h g \in \ker T_{\bar\alpha}$. Following the proof of [6, Proposition 2.4], $K_\alpha^2 \subset \frac{\theta}{\psi} H^2$ implies that, for every $g_0 \in K_\alpha^2$, we have $\overline{\frac{\theta}{\psi}} g_0 = g_1 \in H^2$ so that $g_1 \in K_\alpha^2$. Repeating this reasoning for g_1 we conclude that g_0 can be indefinitely divided by $\frac{\theta}{\psi}$, which is possible only if $\frac{\theta}{\psi}$ is a constant. □

Corollary 4.3. *If $\varphi \in H^\infty$, then $A_\varphi^{\theta,\alpha} = 0$ if and only if $\varphi \in \alpha H^\infty$, and $A_{\bar\varphi}^{\theta,\alpha} = 0$ if and only if $\varphi \in \theta H^\infty$.*

Proof. If $A_\varphi^{\theta,\alpha} = 0$, then, by the previous lemma, $\bar\alpha\varphi \in H^2 \cap L^\infty = H^\infty$; thus $\varphi \in \alpha H^\infty$. The converse is obvious. □

The next theorem establishes a necessary and sufficient condition for a bounded asymmetric truncated Toeplitz operator to be the zero operator in terms of its symbol.

Theorem 4.4. *Let α, θ be nonconstant inner functions such that $\alpha \le \theta$. If we let $A_\varphi^{\theta,\alpha}\colon K_\theta^2 \to K_\alpha^2$ be a bounded asymmetric truncated Toeplitz operator with $\varphi \in L^2$, then $A_\varphi^{\theta,\alpha} = 0$ if and only if $\varphi \in \alpha H^2 + \overline{\theta H^2}$.*

Proof. If we assume that $\varphi = \alpha h_1 + \bar\theta\bar h_2$ for some $h_1, h_2 \in H^2$, and we take $f \in K_\theta^\infty$, then $\varphi f = \alpha h_1 f + \bar\theta \bar h_2 f$ and $P_\alpha(\alpha h_1 f) = 0$. Since $\bar\theta f \in \bar\theta K_\theta^2 \subset \overline{zH^2}$, we have $\bar\theta f \bar h_2 \in \overline{zH^2}$ and $P_\alpha(\bar\theta f \bar h_2) = 0$; hence $A_\varphi^{\theta,\alpha} = 0$ on a dense subset of K_θ^2, which implies that $A_\varphi^{\theta,\alpha} = 0$.

For the converse implication, let us take $\varphi = \psi + \bar\chi \in L^2$ with $\psi, \chi \in H^2$ such that $A_\varphi^{\theta,\alpha} = 0$. Note that $A_\psi^{\theta,\alpha} = -A_{\bar\chi}^{\theta,\alpha}$; thus $A_\psi^\alpha f = A_\psi^{\theta,\alpha} f = -A_{\bar\chi}^{\theta,\alpha} f = -A_{\bar\chi}^\alpha f$ for all $f \in K_\alpha^\infty$ by Proposition 3.1. Hence A_ψ^α commutes on K_α^∞ not only with S_α but also with S_α^* (see [19] and Proposition 3.3). Consequently, since $k_0^\alpha \in K_\alpha^\infty$, $S_\alpha^* k_0^\alpha = -\overline{\alpha(0)}\tilde k_0^\alpha \in K_\alpha^\infty$ (cf. (2.3)), and $(I_{K_\alpha^2} - S_\alpha S_\alpha^*)k_0^\alpha \in K_\alpha^\infty$, it follows that

$$\begin{aligned} A_\psi^{\theta,\alpha}(I_{K_\alpha^2} - S_\alpha S_\alpha^*)k_0^\alpha &= (A_\psi^\alpha - S_\alpha A_\psi^\alpha S_\alpha^*)k_0^\alpha = (A_\psi^\alpha - S_\alpha S_\alpha^* A_\psi^\alpha)k_0^\alpha \\ &= (I_{K_\alpha^2} - S_\alpha S_\alpha^*)A_\psi^\alpha k_0^\alpha = (k_0^\alpha \otimes k_0^\alpha)A_\psi^\alpha k_0^\alpha = \langle A_\psi^\alpha k_0^\alpha, k_0^\alpha\rangle k_0^\alpha. \end{aligned}$$

On the other hand, by Lemma 2.4 in [19],

$$A_\psi^{\theta,\alpha}(I_{K_\alpha^2} - S_\alpha S_\alpha^*)k_0^\alpha = A_\psi^\alpha(k_0^\alpha \otimes k_0^\alpha)k_0^\alpha = \big((A_\psi^\alpha k_0^\alpha) \otimes k_0^\alpha\big)k_0^\alpha = \langle k_0^\alpha, k_0^\alpha\rangle A_\psi^\alpha k_0^\alpha;$$

hence,

$$\langle k_0^\alpha, k_0^\alpha\rangle A_\psi^\alpha k_0^\alpha = \langle A_\psi^\alpha k_0^\alpha, k_0^\alpha\rangle k_0^\alpha,$$

and it follows that there is $c \in \mathbb{C}$ such that

$$A_\psi^\alpha k_0^\alpha = c k_0^\alpha. \tag{4.1}$$

Accordingly,

$$0 = (A^{\alpha}_{\psi} - cI_{K^2_\alpha})k^{\alpha}_0 = P_\alpha\big((\psi - c)\big(1 - \overline{\alpha(0)}\alpha\big)\big) = P_\alpha(\psi - c),$$

which implies that $\psi - c \in \alpha H^2$.

Let us now consider $A^{\theta,\alpha}_{\bar\chi}$. We have $A^{\theta,\alpha}_{\bar\chi} = -A^{\theta,\alpha}_{\psi}$, and $A^{\theta,\alpha}_{\psi}$ intertwines S_α and S_θ on K^∞_θ by Proposition 3.3. Thus $A^{\theta,\alpha}_{\bar\chi}S_\theta f = S_\alpha A^{\theta,\alpha}_{\bar\chi} f$ for $f \in K^\infty_\theta$. In addition, $A^{\theta,\alpha}_{\bar\chi}S^*_\theta P_\alpha = A^{\alpha}_{\bar\chi}S^*_\alpha P_\alpha$ since K^2_α is invariant for both S^*_θ and $A^{\theta,\alpha}_{\bar\chi}$. Since $P_\alpha k^\theta_0 = k^\alpha_0 \in K^\infty_\alpha \subset K^\infty_\theta$, and, as above, $S^*_\theta P_\alpha k^\theta_0 \in K^\infty_\theta$, we have

$$\begin{aligned}
A^{\theta,\alpha}_{\bar\chi}(I_{K^2_\theta} - S_\theta S^*_\theta)P_\alpha k^\theta_0 &= (A^{\theta,\alpha}_{\bar\chi}P_\alpha - S_\alpha A^{\theta,\alpha}_{\bar\chi}S^*_\theta P_\alpha)k^\theta_0 \\
&= (A^{\theta,\alpha}_{\bar\chi}P_\alpha - S_\alpha A^{\alpha}_{\bar\chi}S^*_\alpha P_\alpha)k^\theta_0 = (A^{\theta,\alpha}_{\bar\chi}P_\alpha - S_\alpha S^*_\alpha A^{\alpha}_{\bar\chi}P_\alpha)k^\theta_0 \\
&= (I_{K^2_\alpha} - S_\alpha S^*_\alpha)(A^{\theta,\alpha}_{\bar\chi})_{|K^2_\alpha}P_\alpha k^\theta_0 = (k^\alpha_0 \otimes k^\alpha_0)A^{\theta,\alpha}_{\bar\chi}k^\alpha_0 \\
&= \langle A^{\theta,\alpha}_{\bar\chi}k^\alpha_0, k^\alpha_0\rangle k^\alpha_0
\end{aligned}$$

and

$$\begin{aligned}
A^{\theta,\alpha}_{\bar\chi}(I_{K^2_\theta} - S_\theta S^*_\theta)P_\alpha k^\theta_0 &= A^{\theta,\alpha}_{\bar\chi}(k^\theta_0 \otimes k^\theta_0)P_\alpha k^\theta_0 \\
&= (A^{\theta,\alpha}_{\bar\chi}k^\theta_0 \otimes k^\theta_0)k^\alpha_0 = \langle k^\alpha_0, k^\theta_0\rangle A^{\theta,\alpha}_{\bar\chi}k^\theta_0.
\end{aligned}$$

It then follows that $A^{\theta,\alpha}_{\bar\chi}k^\theta_0 = c_1 k^\alpha_0$.

Since $A^{\theta,\alpha}_{\bar\chi} = -A^{\theta,\alpha}_{\psi}$, by Proposition 3.3 the operator $A^{\theta,\alpha}_{\bar\chi}$ intertwines S_θ and S_α on K^∞_θ. By Proposition 2.5, we know that $S^n_\theta k^\theta_0 \in K^\infty_\theta$ for $n \geq 0$; thus we have

$$A^{\theta,\alpha}_{\bar\chi}S^n_\theta k^\theta_0 = S^n_\alpha A^{\theta,\alpha}_{\bar\chi}k^\theta_0 = c_1 S^n_\alpha P_\alpha k^\theta_0 = c_1 P_\alpha S^n_\theta k^\theta_0.$$

From the linear density of $S^n_\theta k^\theta_0$, $n \geq 0$ in K^2_θ, we have

$$A^{\theta,\alpha}_{\bar\chi} = c_1 P_\alpha \in \mathcal{T}(\theta,\alpha). \tag{4.2}$$

Since $A^{\theta,\alpha}_{\bar\chi} = -A^{\theta,\alpha}_{\psi}$, by (4.1) and (4.2) we then have that $c_1 = -c$. Note that $(A^{\theta,\alpha}_{\bar\chi+c})^* = A^{\alpha,\theta}_{\chi+\bar c}$, by Proposition 3.2. Using Lemma 4.2, we obtain that $A^{\alpha,\theta}_{\chi+\bar c} = 0$ if and only if $\bar\chi + c \in \overline{\theta H^2}$; therefore $\varphi = (\psi - c) + (\bar\chi + c)$ is in $\alpha H^2 + \overline{\theta H^2}$. $\square$

Corollary 4.5. *Let $\alpha \leq \theta$ be nonconstant inner functions, and let $A^{\theta,\alpha}_{\varphi} \in \mathcal{T}(\theta,\alpha)$. For $\varphi \in L^2$ there are functions $\psi \in K^2_\alpha$ and $\chi \in K^2_\theta$ such that $A^{\theta,\alpha}_{\varphi} = A^{\theta,\alpha}_{\psi+\bar\chi}$. Moreover, $A^{\theta,\alpha}_{\psi+\bar\chi} = A^{\theta,\alpha}_{\psi_1+\bar\chi_1}$ if and only if $\psi_1 = \psi + ck^\alpha_0$, $\chi_1 = \chi - \bar c k^\theta_0$ for some constant c.*

Proof. Let $\varphi = \varphi_+ + \varphi_-$ with $\varphi_+ \in H^2$ and $\varphi_- \in \overline{H^2}$. If we put $\psi = P_\alpha\varphi_+$ and $\chi = P_\theta\bar\varphi_-$, then $\varphi - \psi - \bar\chi \in \alpha H^2 + \overline{\theta H^2}$; hence $A^{\theta,\alpha}_{\varphi} = A^{\theta,\alpha}_{\psi+\bar\chi}$ by Theorem 4.4.

The proof of the second statement is similar to the proof of the corollary on page 499 in [19]. Note first that, by Proposition 3.1, $A^{\theta,\alpha}_{\varphi} = A^{\theta,\alpha}_{\psi+\bar\chi} = A^{\theta,\alpha}_{\psi_1+\bar\chi_1}$. On the other hand, if $A^{\theta,\alpha}_{\psi_1+\bar\chi_1} = A^{\theta,\alpha}_{\varphi}$, then $\psi - \psi_1 + \bar\chi - \bar\chi_1 \in \alpha H^2 + \overline{\theta H^2}$; hence $\psi - \psi_1 + \bar\chi - \bar\chi_1 = \alpha h_1 + \bar\theta\bar h_2$ for some $h_1, h_2 \in H^2$. Applying P_α and taking into account that $P_\alpha 1 = k^\alpha_0$, we get $\psi - \psi_1 = -P_\alpha(\bar\chi - \bar\chi_1) + P_\alpha\bar\theta\bar h_2 = -c_1 P_\alpha 1 = -c_1 k^\alpha_0$ for some constant c_1 since $P_\alpha\overline{H^2}$ contains only constant functions. Similarly, we also have that $\bar\psi-\bar\psi_1+\chi-\chi_1=\bar\alpha\bar h_1+\theta h_2$, and, applying P_θ, we obtain analogously that $\chi-\chi_1=c_2k_0^\theta$. Since we must have $A^{\theta,\alpha}_{\psi+\bar\chi}-A^{\theta,\alpha}_{\psi_1+\bar\chi_1}=A^{\theta,\alpha}_{-c_1k_0^\alpha+\bar c_2\bar k_0^\theta}=0$, by Proposition 3.1, $-c_1P_\alpha+\bar c_2P_\alpha=0$, which implies that $\bar c_2=c_1$. □

The following properties can be immediately obtained from the previous results by taking adjoints.

Corollary 4.6. *If we let $A^{\alpha,\theta}_\varphi\colon K^2_\alpha\to K^2_\theta$, $A^{\alpha,\theta}_\varphi\in\mathcal{T}(\alpha,\theta)$, $\alpha\le\theta$, $\varphi\in L^2$, then $A^{\alpha,\theta}_\varphi=0$ if and only if $\varphi\in\theta H^2+\overline{\alpha H^2}$.*

Corollary 4.7. *If we let $A^{\alpha,\theta}_\varphi\colon K^2_\alpha\to K^2_\theta$, $A^{\alpha,\theta}_\varphi\in\mathcal{T}(\alpha,\theta)$, $\alpha\le\theta$, $\varphi\in L^2$, then there are functions $\psi\in K^2_\alpha$, $\chi\in K^2_\theta$ such that $A^{\alpha,\theta}_\varphi=A^{\alpha,\theta}_{\bar\psi+\chi}$.*

5. First characterization in terms of rank 2 operators

In [19, Theorem 4.1], a characterization of truncated Toeplitz operators in $\mathcal{T}(\theta)$ was given by using certain rank 2 operators defined in terms of the kernel function k_0^θ. Here we obtain an analogous result for asymmetric truncated Toeplitz operators $\mathcal{T}(\theta,\alpha)$ using the kernel functions k_0^α and k_0^θ.

Theorem 5.1. *Let α,θ be nonconstant inner functions such that $\alpha\le\theta$, and let $A:K^2_\theta\to K^2_\alpha$ be a bounded operator. Then $A\in\mathcal{T}(\theta,\alpha)$ if and only if there are $\psi\in K^2_\alpha$, $\chi\in K^2_\theta$ such that*

$$A-S_\alpha AS_\theta^*=\psi\otimes k_0^\theta+k_0^\alpha\otimes\chi. \tag{5.1}$$

Proof. If we assume that $A\in\mathcal{T}(\theta,\alpha)$, then $A=A^{\theta,\alpha}_{\psi+\bar\chi}$ for some $\psi\in K^2_\alpha$, $\chi\in K^2_\theta$. Note that if $f\in K^\infty_\theta$, then $S^*_\theta f\in K^\infty_\theta$ (cf. (2.2)). Hence by Proposition 3.3,

$$\begin{aligned}(A^{\theta,\alpha}_\psi-S_\alpha A^{\theta,\alpha}_\psi S^*_\theta)f&=(A^{\theta,\alpha}_\psi-A^{\theta,\alpha}_\psi S_\theta S^*_\theta)f\\&=\big(A^{\theta,\alpha}_\psi(I_{K^2_\theta}-S_\theta S^*_\theta)\big)f=(A^{\theta,\alpha}_\psi k_0^\theta\otimes k_0^\theta)f\\&=\big(P_\alpha\big(\psi\big(1-\overline{\theta(0)}\theta\big)\big)\otimes k_0^\theta\big)f=(\psi\otimes k_0^\theta)f.\end{aligned}$$

Moreover, since $A^\theta_{\bar\chi}$ commutes with S^*_θ on K^∞_θ (see [19, p. 498]) and by Proposition 2.5, we have

$$\begin{aligned}(A^{\theta,\alpha}_{\bar\chi}-S_\alpha A^{\theta,\alpha}_{\bar\chi}S^*_\theta)f&=(A^{\theta,\alpha}_{\bar\chi}-S_\alpha P_\alpha A^\theta_{\bar\chi}S^*_\theta)f\\&=(A^{\theta,\alpha}_{\bar\chi}-S_\alpha P_\alpha S^*_\theta A^\theta_{\bar\chi})f=(P_\alpha A^\theta_{\bar\chi}-P_\alpha S_\theta S^*_\theta A^\theta_{\bar\chi})f\\&=P_\alpha(I_{K^2_\theta}-S_\theta S^*_\theta)A^\theta_{\bar\chi}f=P_\alpha(k_0^\theta\otimes k_0^\theta)A^\theta_{\bar\chi}f\\&=\big((P_\alpha k_0^\theta)\otimes(A^\theta_\chi k_0^\theta)\big)f=(k_0^\alpha\otimes\chi)f.\end{aligned}$$

Since K^∞_θ is dense in K^2_θ, we obtain (5.1).

For the converse implication, note that, for $\psi\in K^2_\alpha$, $\chi\in K^2_\theta$ and for all $f\in K^\infty_\theta$, $g\in K^\infty_\alpha$, we have

$$\langle A^{\theta,\alpha}_{\psi+\bar\chi}f,g\rangle=\sum_{n=0}^{\infty}\big(\langle f,S^n_\theta k_0^\theta\rangle\langle S^n_\alpha\psi,g\rangle+\langle f,S^n_\theta\chi\rangle\langle S^n_\alpha k_0^\alpha,g\rangle\big). \tag{5.2}$$

Indeed, we obtain (5.2) as in the proof of [19, Lemma 4.2] by taking $\psi \in K_\alpha^2$ and $\chi \in K_\theta^2$ instead of $\psi, \chi \in K_u^2$ and S_α, S_θ^* instead of S_u, S_u^*, respectively. The convergence $\langle A_{\psi+\bar\chi}^{\theta,\alpha}(S_\theta^*)^{N+1} f, S_\alpha^{*N+1} g\rangle \to 0$ as $N \to \infty$ follows from

$$\begin{aligned}\left\langle S_\alpha^N A_{\psi+\bar\chi}^{\theta,\alpha}(S_\theta^*)^N f, g\right\rangle &= \left\langle S_\alpha^N A_\psi^{\theta,\alpha}(S_\theta^*)^N f, g\right\rangle + \left\langle S_\alpha^N A_{\bar\chi}^{\theta,\alpha}(S_\theta^*)^N f, g\right\rangle \\ &= \left\langle (S_\theta^*)^N f, A_{\bar\psi}^{\alpha,\theta}(S_\alpha^*)^N g\right\rangle + \left\langle S_\alpha^N P_\alpha A_{\bar\chi}^{\theta}(S_\theta^*)^N f, g\right\rangle \\ &= \left\langle (S_\theta^*)^N f, (S_\theta^*)^N A_{\bar\psi}^{\alpha,\theta} g\right\rangle + \left\langle (S_\theta^*)^N A_{\bar\chi}^{\theta} f, P_\alpha (S_\alpha^*)^N g\right\rangle \\ &= \left\langle (S_\theta^*)^N f, (S_\theta^*)^N A_{\bar\psi}^{\alpha,\theta} g\right\rangle + \left\langle (S_\theta^*)^N A_{\bar\chi}^{\theta} f, (S_\alpha^*)^N g\right\rangle,\end{aligned}$$

where the last expression tends to zero, when $N \to \infty$, by the strong convergence $(S^*)^N \to 0$.

Now assume that a bounded operator $A\colon K_\theta^2 \to K_\alpha^2$ satisfies (5.1). For any positive integer n we have

$$S_\alpha^n A S_\theta^{*n} - S_\alpha^{n+1} A S_\theta^{*(n+1)} = (S_\alpha^n \psi \otimes S_\theta^n k_0^\theta) + (S_\alpha^n k_0^\alpha \otimes S_\theta^n \chi).$$

Adding for $n = 0, 1, \ldots, N$, we obtain

$$A = \sum_{n=0}^{N} (S_\alpha^n \psi \otimes S_\theta^n k_0^\theta + S_\alpha^n k_0^\alpha \otimes S_\theta^n \chi) + S_\alpha^{N+1} A S_\theta^{*(N+1)}.$$

Taking into account that $S_\theta^{*n} \to 0$ in the strong operator topology, we get

$$A = \sum_{n=0}^{\infty} (S_\alpha^n \psi \otimes S_\theta^n k_0^\theta + S_\alpha^n k_0^\alpha \otimes S_\theta^n \chi); \tag{5.3}$$

hence, by comparing (5.2) and (5.3), we conclude that the right-hand side of (5.3) is equal to $A_{\psi+\bar\chi}^{\theta,\alpha}$. □

We can obtain a similar characterization for operators from $\mathcal{T}(\alpha,\theta)$ by taking adjoints in (5.1). Namely, we have the following.

Corollary 5.2. *Let α, θ be nonconstant inner functions such that $\alpha \le \theta$, and let $A\colon K_\alpha^2 \to K_\theta^2$ be a bounded operator. Then $A \in \mathcal{T}(\alpha,\theta)$ if and only if there are $\psi \in K_\alpha^2$, $\chi \in K_\theta^2$ such that*

$$A - S_\theta A S_\alpha^* = k_0^\theta \otimes \psi + \chi \otimes k_0^\alpha.$$

6. Second characterization in terms of rank 2 operators

Sarason also obtained a characterization for truncated Toeplitz operators belonging to $\mathcal{T}(\theta)$ involving the function $\tilde{k}_0^\theta = C_\theta k_0^\theta$ instead of k_0^θ by a simple application of the conjugation C_θ to the result of Theorem 5.1 in the case $\alpha = \theta$. Here we will show that an analogous result holds for operators belonging to $\mathcal{T}(\theta,\alpha)$, $\alpha \le \theta$. We cannot, however, use the same reasoning for $\alpha \ne \theta$ because, as we will see, the case of asymmetric truncated Toeplitz operators is more complex. The relation between a symbol of an asymmetric truncated Toeplitz operator and a rank 2 operator appearing in (6.1) is more involved.

Theorem 6.1. *Let α, θ be nonconstant inner functions such that $\alpha \leq \theta$, and let $A\colon K_\theta^2 \to K_\alpha^2$ be a bounded operator. Then $A \in \mathcal{T}(\theta, \alpha)$ if and only if there are $\mu \in K_\alpha^2$ and $\nu \in K_\theta^2$ such that*

$$A - S_\alpha^* A S_\theta = \mu \otimes \tilde{k}_0^\theta + \tilde{k}_0^\alpha \otimes \nu. \tag{6.1}$$

Moreover, if $A = A_{\psi+\bar{\chi}}^{\theta,\alpha}$ with $\psi \in K_\alpha^2$ and $\chi \in K_\theta^2$, then A satisfies (6.1) with

$$\mu = C_\alpha P_\alpha\left(\frac{\bar{\theta}}{\bar{\alpha}}\chi\right), \quad \nu = C_\alpha \psi + S^*(\alpha P_{\frac{\theta}{\alpha}}\chi). \tag{6.2}$$

Proof. Assume first that $A \in \mathcal{T}(\theta, \alpha)$. By Theorem 5.1, there are $\psi \in K_\alpha^2$ and $\chi \in K_\theta^2$ such that

$$A - S_\alpha A S_\theta^* = \psi \otimes k_0^\theta + k_0^\alpha \otimes \chi;$$

hence

$$S_\alpha^*(A - S_\alpha A S_\theta^*)S_\theta = S_\alpha^*\psi \otimes S_\theta^* k_0^\theta + S_\alpha^* k_0^\alpha \otimes S_\theta^*\chi.$$

Using $I_{K_\alpha^2} - S_\alpha^* S_\alpha = \tilde{k}_0^\alpha \otimes \tilde{k}_0^\alpha$ and $I_{K_\theta^2} - S_\theta^* S_\theta = \tilde{k}_0^\theta \otimes \tilde{k}_0^\theta$, after some calculations we get

$$\begin{aligned} A - S_\alpha^* A S_\theta = {}& \tilde{k}_0^\alpha \otimes A^*\tilde{k}_0^\alpha + A\tilde{k}_0^\theta \otimes \tilde{k}_0^\theta - \langle A\tilde{k}_0^\theta, \tilde{k}_0^\alpha\rangle \tilde{k}_0^\alpha \otimes \tilde{k}_0^\theta \\ & - S_\alpha^*\psi \otimes S_\theta^* k_0^\theta - S_\alpha^* k_0^\alpha \otimes S_\theta^*\chi. \end{aligned}$$

Thus, by using (2.3), we obtain

$$\begin{aligned} A - S_\alpha^* A S_\theta = {}& \big(A\tilde{k}_0^\theta - \langle A\tilde{k}_0^\theta, \tilde{k}_0^\alpha\rangle \tilde{k}_0^\alpha + \theta(0) S_\alpha^*\psi\big) \otimes \tilde{k}_0^\theta \\ & + \tilde{k}_0^\alpha \otimes \big(A^*\tilde{k}_0^\alpha + \alpha(0) S_\theta^*\chi\big). \end{aligned} \tag{6.3}$$

For the reverse implication, if we assume that a bounded operator A satisfies the equality (6.1) for $\mu \in K_\alpha^2$, $\nu \in K_\theta^2$, then

$$S_\alpha A S_\theta^* - S_\alpha S_\alpha^* A S_\theta S_\theta^* = S_\alpha\mu \otimes S_\theta \tilde{k}_0^\theta + S_\alpha \tilde{k}_0^\alpha \otimes S_\theta \nu.$$

By using $I_{K_\alpha^2} - S_\alpha S_\alpha^* = k_0^\alpha \otimes k_0^\alpha$, $I_{K_\theta^2} - S_\theta S_\theta^* = k_0^\theta \otimes k_0^\theta$ and (2.3), we obtain

$$\begin{aligned} A - S_\alpha A S_\theta^* = {}& A k_0^\theta \otimes k_0^\theta + k_0^\alpha \otimes A^* k_0^\alpha - \langle A k_0^\theta, k_0^\alpha\rangle k_0^\alpha \otimes k_0^\theta \\ & + \overline{\theta(0)} S_\alpha\mu \otimes k_0^\theta + \alpha(0) k_0^\alpha \otimes S_\theta\nu; \end{aligned}$$

hence

$$A - S_\alpha A S_\theta^* = \big(A k_0^\theta - \langle A k_0^\theta, k_0^\alpha\rangle k_0^\alpha + \overline{\theta(0)} S_\alpha \mu\big) \otimes k_0^\theta + k_0^\alpha \otimes \big(A^* k_0^\alpha + \overline{\alpha(0)} S_\theta \nu\big),$$

which, by Theorem 5.1, implies that $A \in \mathcal{T}(\theta, \alpha)$.

If $A = A_{\psi+\bar{\chi}}^{\theta,\alpha}$ with $\psi \in K_\alpha^2$, $\chi \in K_\theta^2$, then we can obtain formulas for μ and ν in terms of ψ and χ by using Proposition 3.6 and equality (6.3). Namely, we can take

$$\begin{aligned} \mu &= A_{\psi+\bar{\chi}}^{\theta,\alpha}\tilde{k}_0^\theta - \langle A_\psi^{\theta,\alpha}\tilde{k}_0^\theta, \tilde{k}_0^\alpha\rangle \tilde{k}_0^\alpha + \theta(0) S_\alpha^*\psi, \quad \text{and} \\ \nu &= A_{\bar{\psi}+\chi}^{\alpha,\theta}\tilde{k}_0^\alpha - \overline{\langle A_{\bar{\chi}}^{\theta,\alpha}\tilde{k}_0^\theta, \tilde{k}_0^\alpha\rangle} \tilde{k}_0^\theta + \alpha(0) S_\theta^*\chi. \end{aligned}$$

Since $\psi \in K_\alpha^2$, we have $\alpha\bar{\psi} \in H_0^2$; thus $(\alpha\bar{\psi})(0) = 0$, and, since $C_\alpha S_\alpha^* = S_\alpha C_\alpha$ (see [19, Lemma 2.1]), we then get

$$\begin{aligned}\langle S_\alpha^*\psi, \tilde{k}_0^\alpha\rangle &= \langle k_0^\alpha, C_\alpha S_\alpha^*\psi\rangle = \langle k_0^\alpha, S_\alpha C_\alpha\psi\rangle = \langle k_0^\alpha, S_\alpha \alpha\bar{z}\bar{\psi}\rangle \\ &= \langle k_0^\alpha, \alpha\bar{\psi}\rangle = \big\langle 1 - \overline{\alpha(0)}\alpha, \alpha\bar{\psi}\big\rangle = \langle 1, \alpha\bar{\psi}\rangle - \overline{\alpha(0)}\langle 1, \bar{\psi}\rangle = -\overline{\alpha(0)}\psi(0).\end{aligned}$$

Thus by (3) and (4) in Proposition 3.6 we have

$$\begin{aligned}\langle A_\psi^{\theta,\alpha}\tilde{k}_0^\theta, \tilde{k}_0^\alpha\rangle &= \left\langle \frac{\theta}{\alpha}(0)\psi(0)\tilde{k}_0^\alpha - \theta(0)S_\alpha^*\psi, \tilde{k}_0^\alpha\right\rangle = \frac{\theta}{\alpha}(0)\psi(0)\|\tilde{k}_0^\alpha\|^2 + \theta(0)\overline{\alpha(0)}\psi(0) \\ &= \frac{\theta}{\alpha}(0)\psi(0)\big(1 - \big|\alpha(0)\big|^2\big) + \theta(0)\overline{\alpha(0)}\psi(0) = \frac{\theta}{\alpha}(0)\psi(0).\end{aligned}$$

Taking into account that $\chi = \chi_{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha}\chi_\alpha$, where $\chi_{\frac{\theta}{\alpha}} \in K_{\frac{\theta}{\alpha}}^2$ and $\chi_\alpha \in K_\alpha^2$, we have

$$\langle A_{\bar{\chi}}^{\theta,\alpha}\tilde{k}_0^\theta, \tilde{k}_0^\alpha\rangle = \left\langle C_\alpha P_\alpha\left(\frac{\bar{\theta}}{\bar{\alpha}}\chi\right), \tilde{k}_0^\alpha\right\rangle = \langle k_0^\alpha, \chi_\alpha\rangle,$$

and it then follows that

$$\mu = C_\alpha\chi_\alpha \in K_\alpha^2.$$

Moreover, taking into account (7) and (8) in Proposition 3.6, it follows that

$$\nu = C_\alpha\psi + S_\theta^*(\alpha\chi_{\frac{\theta}{\alpha}}) \in K_\theta^2. \qquad \square$$

By taking adjoints in (6.1) we obtain the following similar characterization for operators from $\mathcal{T}(\alpha,\theta)$.

Corollary 6.2. *Let α, θ be nonconstant inner functions such that $\alpha \leq \theta$, and let $A\colon K_\alpha^2 \to K_\theta^2$ be a bounded operator. Then $A \in \mathcal{T}(\alpha,\theta)$ if and only if there are $\mu \in K_\alpha^2$, $\nu \in K_\theta^2$ such that*

$$A - S_\theta^* A S_\alpha = \tilde{k}_0^\theta \otimes \mu + \nu \otimes \tilde{k}_0^\alpha.$$

It is clear that, if an asymmetric truncated Toeplitz operator A satisfies equation (6.1) with some μ, ν, then that equation is also satisfied if μ, ν are replaced by

$$\mu' = \mu + \bar{b}\tilde{k}_0^\alpha, \qquad \nu' = \nu - b\tilde{k}_0^\theta, \tag{6.4}$$

respectively, for any $b \in \mathbb{C}$. On the other hand, it is also true that the symbol of $A = A_{\psi+\bar{\chi}}^{\theta,\alpha} \in \mathcal{T}(\theta,\alpha)$ is not unique, and by Corollary 4.5 we can replace $\psi \in K_\alpha^2$ and $\chi \in K_\theta^2$ with

$$\psi' = \psi + ck_0^\alpha \in K_\alpha^2, \qquad \chi' = \chi - \bar{c}k_0^\theta \in K_\theta^2, \tag{6.5}$$

respectively, for any $c \in \mathbb{C}$. Using (6.2), it is easy to see that the following relation between the freedom of choice holds: of μ, ν on the one hand and of ψ, χ on the other.

Corollary 6.3. *Let $\mu \in K_\alpha^2$ and $\nu \in K_\theta^2$ be defined by (6.2) for given $\psi \in K_\alpha^2$ and $\chi \in K_\theta^2$, and let $\mu' \in K_\alpha^2$ and $\nu' \in K_\theta^2$ be defined analogously for $\psi' \in K_\alpha^2$ and $\chi' \in K_\theta^2$. If (6.5) holds, then*

$$\mu' = \mu - c\frac{\theta}{\alpha}(0)\tilde{k}_0^\alpha, \qquad \nu' = \nu + \bar{c}\overline{\frac{\theta}{\alpha}}(0)\tilde{k}_0^\theta.$$

The examples below illustrate the result of Theorem 6.1 in the case of Toeplitz matrices.

Example 6.4. Let us consider $\alpha = z^2$, $\theta = z^5$, and a Toeplitz operator $A = A^{z^5}_{\psi+\bar{\chi}}$. If we assume that $\psi = a_0 + a_1 z$ and that $\chi = \bar{b}_0 + \bar{b}_{-1}z + \bar{b}_{-2}z^2 + \bar{b}_{-3}z^3 + \bar{b}_{-4}z^4 = (\bar{b}_0 + \bar{b}_{-1}z + \bar{b}_{-2}z^2) + z^3(\bar{b}_{-3} + \bar{b}_{-4}z)$, then $C_{z^2}\psi = \bar{a}_1 + \bar{a}_0 z$, $C_{z^2}P_{z^2}\bar{z}^3\chi = b_{-4} + b_{-3}z$, and $S^*(z^2(\bar{b}_0 + \bar{b}_{-1}z + \bar{b}_{-2}z^2)) = \bar{b}_0 z + \bar{b}_{-1}z^2 + \bar{b}_{-2}z^3$. Note that $A - S^*_{z^2}AS_{z^5}$ has a matrix representation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & b_{-4} \\ a_1 & a_0 + b_0 & b_{-1} & b_{-2} & b_{-3} \end{pmatrix}$$

which can be expressed as

$$(b_{-4} + b_{-3}z) \otimes z^4 + z \otimes \big(\bar{a}_1 + (\bar{a}_0 + \bar{b}_0)z + \bar{b}_{-1}z^2 + \bar{b}_{-2}z^3\big).$$

On the other hand, let $A - S^*_{z^2}AS_{z^5}$ have a matrix representation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & b_0 \\ a_0 & a_1 & a_2 & a_3 & a_4 + b_1 \end{pmatrix}$$

which can be expressed as

$$\mu \otimes z^4 + z \otimes \nu = (b_0 + b_1 z) \otimes z^4 + z \otimes (\bar{a}_0 + \bar{a}_1 z + \bar{a}_2 z^2 + \bar{a}_3 z^3 + \bar{a}_4 z^4).$$

Note that if $\nu = \nu_{z^2} + z^2\nu_{z^3} = (\bar{a}_0 + \bar{a}_1 z) + z^2(\bar{a}_2 + \bar{a}_3 z + \bar{a}_4 z^2)$, then $\psi = C_{z^2}P_{z^2}\nu = a_1 + a_0 z$, and $\chi = \bar{a}_2 z + \bar{a}_3 z^2 + (\bar{b}_1 + \bar{a}_4)z^3 + \bar{b}_0 z^4$. Hence by Theorem 5.1 we have

$$A - S_{z^2}AS^*_{z^5} = (a_1 + a_0 z) \otimes 1 + 1 \otimes \big(\bar{a}_2 z + \bar{a}_3 z^2 + (\bar{b}_1 + \bar{a}_4)z^3 + \bar{b}_0 z^4\big). \tag{6.6}$$

Requiring that ν_{z^3} be orthogonal to z^2 (see the proof of Theorem 8.5) determines that $a_4 = 0$.

On the other hand, we have some freedom in defining ψ and χ, namely, that $\psi_1 = s + a_0 z$ and $\chi_1 = \bar{t} + \bar{a}_2 z + \bar{a}_3 z^2 + (\bar{b}_1 + \bar{a}_4)z^3 + \bar{b}_0 z^4$ also satisfy (6.6) if we assume that $t + s = a_1$.

Example 6.5. Let us now take $\alpha = z^3$, $\theta = z^3((\lambda - z)/(1 - \bar{\lambda}z))^2$, $\lambda \in \mathbb{D}$, and let us consider the operator $A = A^{\theta,\alpha}_{\psi+\bar{\chi}}$, where $\psi = a_0 + a_1 z + a_2 z^2 \in K_\alpha^2$, and $\chi = (\bar{b}_0 + \bar{b}_1 z + \bar{b}_2 z^2 + \bar{b}_3 z^3 + \bar{b}_4 z^4)(1 - \bar{\lambda}z)^{-2} \in K_\theta^2$ (see [12, Corollary 5.7.3]). Then by Theorem 6.1

$$A - S^*_\alpha AS_\theta = \mu \otimes (\lambda^2 z^2 - 2\lambda z^3 + z^4)(1 - \bar{\lambda}z)^{-2} + z^2 \otimes \nu,$$

where $\mu = b_4 + (b_3 + 2\bar{\lambda}b_4)z + (b_2 + 3\bar{\lambda}^2 b_4 + \bar{\lambda}b_3)z^2$ and $\nu = (\bar{a}_2 + (\bar{a}_1 - 2\bar{\lambda}\bar{a}_2)z + (\bar{b}_0 + \bar{a}_0 - 2\bar{\lambda}\bar{a}_1 + \bar{\lambda}\bar{a}_2)z^2 + (\bar{b}_1 + \bar{\lambda}^2\bar{a}_1 - 2\bar{\lambda}\bar{a}_0)z^3 + \bar{\lambda}^2\bar{a}_0 z^4)(1 - \bar{\lambda}z)^{-2}$.

7. Characterizations in terms of rank 1 operators

Our aim now is to describe the classes of symbols of an operator $A \in \mathcal{T}(\theta,\alpha)$ for which the right-hand side of (6.1) is a rank 1 operator. The corresponding question regarding the equation (5.1) is trivial by Corollary 4.5, since the right-hand side of (5.1) is a rank 1 operator if and only if $\psi = c\cdot k_0^\alpha$ or $\chi = c\cdot k_0^\theta$ with $c\in\mathbb{C}$. In the case $\alpha=\theta$ the question regarding the equality (6.1) also has an easy answer, since the relation between the symbols in (5.1) and (6.1) is $\psi = C_\theta\nu$ and $\chi = C_\theta\mu$. For $\alpha\neq\theta$ we need the following lemma.

Lemma 7.1. *Let $\alpha\leq\theta$ be nonconstant inner functions, and let $A^{\theta,\alpha}_{\psi+\bar\chi}\in\mathcal{T}(\theta,\alpha)$. Assume that $\psi\in K^2_\alpha$ and that $\chi = \chi_{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha}\chi_\alpha \in K^2_\theta = K^2_{\frac{\theta}{\alpha}}\oplus\frac{\theta}{\alpha}K^2_\alpha$. Then*

1. $P_{\mathbb{C}\tilde k_0^\theta} S_\theta^*(\alpha\chi_{\frac{\theta}{\alpha}}) = -\overline{\frac{\theta}{\alpha}(0)}\chi_{\frac{\theta}{\alpha}}(0)\|\tilde k_0^\theta\|^{-2}\tilde k_0^\theta,$
2. $P_{\mathbb{C}\tilde k_0^\theta} C_\alpha\psi = \overline{\frac{\theta}{\alpha}(0)\psi(0)}\|\tilde k_0^\theta\|^{-2}\tilde k_0^\theta,$
3. $P_\alpha S_\theta^*(\alpha\chi_{\frac{\theta}{\alpha}}) = P_{\mathbb{C}\tilde k_0^\alpha} S_\theta^*(\alpha\chi_{\frac{\theta}{\alpha}}) = \chi_{\frac{\theta}{\alpha}}(0)\tilde k_0^\alpha,$
4. $(P_\theta - P_\alpha)S_\theta^*(\alpha\chi_{\frac{\theta}{\alpha}}) = \alpha S^*\chi_{\frac{\theta}{\alpha}},$
5. $P_{\mathbb{C}\tilde k_0^\alpha} C_\alpha P_\alpha(\chi_{\frac{\bar\theta}{\bar\alpha}}) = \overline{\chi_\alpha(0)}\|\tilde k_0^\alpha\|^{-2}\tilde k_0^\alpha,$ *and*
6. $P_{\mathbb{C}\tilde k_0^\alpha} C_\alpha\psi = \overline{\psi(0)}\|\tilde k_0^\alpha\|^{-2}\tilde k_0^\alpha.$

Proof. To prove (1), it is enough to calculate

$$\begin{aligned}\left\langle S^*(\alpha\chi_{\frac{\theta}{\alpha}}),\tilde k_0^\theta\right\rangle &= \left\langle \bar z\big(\alpha\chi_{\frac{\theta}{\alpha}} - \alpha(0)\chi_{\frac{\theta}{\alpha}}(0)\big), \bar z\big(\theta-\theta(0)\big)\right\rangle\\ &= \left\langle \chi_{\frac{\theta}{\alpha}}, \frac{\theta}{\alpha}\right\rangle - \alpha(0)\chi_{\frac{\theta}{\alpha}}(0)\langle 1,\theta\rangle - \overline{\theta(0)}\langle\alpha\chi_{\frac{\theta}{\alpha}},1\rangle + \alpha(0)\overline{\theta(0)}\chi_{\frac{\theta}{\alpha}}(0)\\ &= -\overline{\frac{\theta}{\alpha}(0)}\chi_{\frac{\theta}{\alpha}}(0).\end{aligned}$$

To show (2), note that by Proposition 2.2

$$\begin{aligned}\langle C_\alpha\psi,\tilde k_0^\theta\rangle &= \langle C_\alpha\psi, P_\alpha\tilde k_0^\theta\rangle = \left\langle C_\alpha\psi, \frac{\theta}{\alpha}(0)\tilde k_0^\alpha\right\rangle\\ &= \left\langle C_\alpha\frac{\theta}{\alpha}(0)\tilde k_0^\alpha,\psi\right\rangle = \overline{\frac{\theta}{\alpha}(0)}\langle k_0^\alpha,\psi\rangle = \overline{\frac{\theta}{\alpha}(0)}\,\overline{\psi(0)}.\end{aligned}$$

The equalities in (3) follow from

$$\begin{aligned}P_\alpha S_\theta^*(\alpha\chi_{\frac{\theta}{\alpha}}) &= P_\alpha(\bar z\alpha\chi_{\frac{\theta}{\alpha}}) = \alpha P^-\bar\alpha\bar z\alpha\chi_{\frac{\theta}{\alpha}} - P^-\bar z\alpha\chi_{\frac{\theta}{\alpha}} = \alpha P^-\bar z\chi_{\frac{\theta}{\alpha}} - \alpha(0)\chi_{\frac{\theta}{\alpha}}(0)\bar z\\ &= \alpha\chi_{\frac{\theta}{\alpha}}(0)\bar z - \alpha(0)\chi_{\frac{\theta}{\alpha}}(0)\bar z = \chi_{\frac{\theta}{\alpha}}(0)\tilde k_0^\alpha = P_{\mathbb{C}\tilde k_0^\alpha}S_\theta^*(\alpha\chi_{\frac{\theta}{\alpha}}).\end{aligned}$$

To see (4) it is enough to calculate

$$\begin{aligned}(P_\theta-P_\alpha)S_\theta^*(\alpha\chi_{\frac{\theta}{\alpha}}) &= \alpha P_{\frac{\theta}{\alpha}}\bar\alpha S_\theta^*(\alpha\chi_{\frac{\theta}{\alpha}}) = \alpha P_{\frac{\theta}{\alpha}}\big(\bar\alpha\bar z\big(\alpha\chi_{\frac{\theta}{\alpha}} - \alpha(0)\chi_{\frac{\theta}{\alpha}}(0)\big)\big)\\ &= \alpha P_{\frac{\theta}{\alpha}}(\bar z\chi_{\frac{\theta}{\alpha}}) = \alpha S^*_{\frac{\theta}{\alpha}}\chi_{\frac{\theta}{\alpha}} = \alpha S^*\chi_{\frac{\theta}{\alpha}}.\end{aligned}$$

The proof of (5) follows from $\langle C_\alpha P_\alpha(\chi_{\frac{\bar\theta}{\bar\alpha}}),\tilde k_0^\alpha\rangle = \langle k_0^\alpha,\chi_\alpha\rangle = \overline{\chi_\alpha(0)}$, and, to show that (6) holds, note that $\langle C_\alpha\psi,\tilde k_0^\alpha\rangle = \langle k_0^\alpha,\psi\rangle = \overline{\psi(0)}$. □

Theorem 7.2. *Let $\alpha \le \theta$ be nonconstant inner functions, and let $A^{\theta,\alpha}_{\psi+\bar\chi} \in \mathcal{T}(\theta,\alpha)$, where $\psi \in K^2_\alpha$, and $\chi \in K^2_\theta$. Then*

(1) $A^{\theta,\alpha}_{\psi+\bar\chi} - S^*_\alpha A^{\theta,\alpha}_{\psi+\bar\chi} S_\theta = \mu \otimes \tilde k^\theta_0$ *for $\mu \in K^2_\alpha$ if and only if there is $s \in \mathbb{C}$ such that $\psi = s k^\alpha_0$, $P_{\frac{\theta}{\alpha}}\chi = -\bar s k_0^{\frac{\theta}{\alpha}}$, and*

(2) $A^{\theta,\alpha}_{\psi+\bar\chi} - S^*_\alpha A^{\theta,\alpha}_{\psi+\bar\chi} S_\theta = \tilde k^\alpha_0 \otimes \nu$ *for $\nu \in K^2_\theta$ if and only if $P_\alpha(\chi\frac{\bar\theta}{\bar\alpha}) = \mathrm{const} \cdot k^\alpha_0$.*

Proof. Assume first that $A^{\theta,\alpha}_{\psi+\bar\chi} - S^*_\alpha A^{\theta,\alpha}_{\psi+\bar\chi} S_\theta = \mu \otimes \tilde k^\theta_0$ for $\mu \in K^2_\alpha$. Now, the right-hand side of the equation (6.1) reduces to $\mu \otimes \tilde k^\theta_0$ if and only if $\nu = c \cdot \tilde k^\theta_0$ with $c \in \mathbb{C}$, which is equivalent to $\nu - P_{\mathbb{C}\tilde k^\theta_0}\nu = 0$. Let $\chi = \chi_{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha}\chi_\alpha \in K^2_\theta = K^2_{\frac{\theta}{\alpha}} \oplus \frac{\theta}{\alpha} K^2_\alpha$. By formulas (6.2) and Lemma 7.1 we thus have that

$$
\begin{aligned}
0 &= C_\alpha\psi + S^*_\theta(\alpha\chi_{\frac{\theta}{\alpha}}) + \overline{\frac{\theta}{\alpha}(0)}\big(\chi_{\frac{\theta}{\alpha}}(0) - \overline{\psi(0)}\big)\|k^\theta_0\|^{-2}\tilde k^\theta_0 \\
&= \left(C_\alpha\psi + \left(\chi_{\frac{\theta}{\alpha}}(0) + \left|\frac{\theta}{\alpha}(0)\right|^2\big(\chi_{\frac{\theta}{\alpha}}(0) - \overline{\psi(0)}\big)\|k^\theta_0\|^{-2}\right)\tilde k^\alpha_0\right) \\
&\quad + \alpha\left(S^*_\theta\chi_{\frac{\theta}{\alpha}} + \overline{\frac{\theta}{\alpha}(0)}\big(\chi_{\frac{\theta}{\alpha}}(0) - \overline{\psi(0)}\big)\|k^\theta_0\|^{-2}\tilde k_0^{\frac{\theta}{\alpha}}\right).
\end{aligned}
\tag{7.1}
$$

Since the right-hand side of (7.1) is an orthogonal sum, each term must be zero. Thus, for $s = -\overline{\chi_{\frac{\theta}{\alpha}}(0)} - |\frac{\theta}{\alpha}(0)|^2(\overline{\chi_{\frac{\theta}{\alpha}}(0)} - \psi(0))\|k^\theta_0\|^{-2}$, we get $\psi = s k^\alpha_0$. Moreover, $S^*_\theta\chi_{\frac{\theta}{\alpha}} = -\overline{\frac{\theta}{\alpha}(0)}(\chi_{\frac{\theta}{\alpha}}(0) - \overline{\psi(0)})\|k^\theta_0\|^{-2}\tilde k_0^{\frac{\theta}{\alpha}}$; hence, proceeding as in (8.7), we get

$$
\chi_{\frac{\theta}{\alpha}} = \chi_{\frac{\theta}{\alpha}}(0) k_0^{\frac{\theta}{\alpha}} - \overline{\frac{\theta}{\alpha}(0)}\big(\chi_{\frac{\theta}{\alpha}}(0) - \overline{\psi(0)}\big)\|k^\theta_0\|^{-2} S_{\frac{\theta}{\alpha}}\tilde k_0^{\frac{\theta}{\alpha}} = -\bar s k_0^{\frac{\theta}{\alpha}}
$$

by Lemma 2.3.

The converse is immediate from (6.2). The proof of (2) is analogous. □

Remark 7.3. When the right-hand side of the characterization (5.1) reduces to a rank 1 operator const $\cdot\, k^\alpha_0 \otimes k^\theta_0$, this operator can immediately be expressed in terms of the symbol $\psi + \bar\chi$ as

$$
\begin{aligned}
\mathrm{const}\cdot k^\alpha_0 \otimes k^\theta_0 &= P_{\mathbb{C}k^\alpha_0}\psi \otimes k^\theta_0 + k^\alpha_0 \otimes P_{\mathbb{C}k^\theta_0}\chi \\
&= \big(\psi(0)\|\tilde k^\alpha_0\|^{-2} + \overline{\chi(0)}\|\tilde k^\theta_0\|^{-2}\big)k^\alpha_0 \otimes k^\theta_0.
\end{aligned}
$$

It might be of independent interest to consider the case when the right-hand side in the equation (6.1) reduces to a rank 1 operator const $\cdot\, \tilde k^\alpha_0 \otimes \tilde k^\theta_0$. By equations (6.2) and Lemma 7.1(1)–(2),(5), this operator can be expressed in terms of the symbol $\psi + \bar\chi$ as

$$
\begin{aligned}
\mathrm{const}\cdot \tilde k^\alpha_0 \otimes \tilde k^\theta_0 &= P_{\mathbb{C}\tilde k^\alpha_0}\mu \otimes \tilde k^\theta_0 + \tilde k^\alpha_0 \otimes P_{\mathbb{C}\tilde k^\theta_0}\nu \\
&= \left(\overline{\chi_\alpha(0)}\|\tilde k^\alpha_0\|^{-2} + \frac{\theta}{\alpha}(0)\big(\psi(0) - \overline{\chi_{\frac{\theta}{\alpha}}(0)}\|\tilde k^\theta_0\|^{-2}\big)\right)\tilde k^\alpha_0 \otimes \tilde k^\theta_0.
\end{aligned}
$$

A similar question can be asked regarding the case when the right-hand side of the equation (6.1) reduces to a rank 1 operator const $\cdot\, \tilde k^\alpha_0 \otimes \tilde k^\alpha_0$. By equations (6.2), Proposition 2.2(4), and Lemma 7.1(3),(5)–(6), we have

$$\begin{aligned}\text{const}\cdot \tilde{k}_0^\alpha \otimes \tilde{k}_0^\alpha &= P_{\mathbb{C}\tilde{k}_0^\alpha}\mu \otimes \tilde{k}_0^\alpha + \tilde{k}_0^\alpha \otimes P_{\mathbb{C}\tilde{k}_0^\alpha}\nu \\ &= \left(\overline{\chi(0)} - \overline{\chi_{\frac{\theta}{\alpha}}(0)}\big|\alpha(0)\big|^2 + \psi(0)\right)\|k_0^\alpha\|^{-2}\tilde{k}_0^\alpha \otimes \tilde{k}_0^\alpha.\end{aligned}$$

8. An inverse problem: From the operator to the symbol

In the case of a classical Toeplitz operator T_φ on H^2, the (unique) symbol φ can be obtained from the operator by the formula $\lim_{n\to\infty} \bar{z}^n T_\varphi z^n$. In the case of a truncated Toeplitz operator, that is, of the form $A_\varphi^{\alpha,\theta}$ with $\alpha = \theta$, one can obtain a symbol belonging to $H^2 + \overline{H^2}$ from the action of A_φ^θ on k_0^θ and $\tilde{k}_0^\theta$ (see [4]). A similar result can be obtained for an asymmetric truncated Toeplitz operator $A \in \mathcal{T}(\theta,\alpha)$ by considering the action of the operator A and its adjoint on reproducing kernel functions of the same kind.

Note first that, if A has a symbol $\psi + \bar{\chi}$ with $\psi \in K_\alpha^2, \chi \in K_\theta^2$, writing $\chi = \chi_{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha}\chi_\alpha$, where $\chi_{\frac{\theta}{\alpha}} \in K^2_{\frac{\theta}{\alpha}}$, and $\chi_\alpha \in K_\alpha^2$, then by Proposition 3.6 we have the following equations on L^2:

$$\begin{cases}\psi - \overline{\theta(0)}\alpha\bar{\chi}_\alpha = Ak_0^\theta - \overline{\chi(0)}k_0^\alpha + \overline{\theta(0)\chi_\alpha(0)}\alpha \\ -\theta(0)\bar{z}\psi + \bar{z}\alpha\bar{\chi}_\alpha = A\tilde{k}_0^\theta - \bar{z}\frac{\theta}{\alpha}(0)\psi(0)\alpha.\end{cases} \tag{8.1}$$

Taking the scalar product with k_0^α and $\tilde{k}_0^\alpha$, respectively, we obtain

$$\begin{cases}\psi(0) + \overline{\theta(0)}\alpha(0)\overline{\chi_\alpha(0)} + \|k_0^\alpha\|^2\overline{\chi(0)} = \langle Ak_0^\theta, k_0^\alpha\rangle \\ \frac{\theta}{\alpha}(0)\psi(0) + \overline{\chi_\alpha(0)} = \langle A\tilde{k}_0^\theta, \tilde{k}_0^\alpha\rangle,\end{cases} \tag{8.2}$$

and, by using the notation $a = \psi(0)$, $b = \overline{\chi_\alpha(0)}$, $c = \overline{\chi(0)}$ in (8.2), we obtain the following system of equations for the unknowns a, b, c:

$$\begin{cases}a + \overline{\theta(0)}\alpha(0)b + \|k_0^\alpha\|^2 c = \langle Ak_0^\theta, k_0^\alpha\rangle \\ -\frac{\theta}{\alpha}(0)a + b = \langle A\tilde{k}_0^\theta, \tilde{k}_0^\alpha\rangle.\end{cases} \tag{8.3}$$

Hence we have proven the following.

Lemma 8.1. *Let $A \in \mathcal{T}(\theta,\alpha), A = A_{\psi+\bar{\chi}}$, where $\psi \in K_\alpha^2, \chi \in K_\theta^2$, and $\chi = \chi_{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha}\chi_\alpha$, with $\chi_{\frac{\theta}{\alpha}} \in K^2_{\frac{\theta}{\alpha}}$ and $\chi_\alpha \in K_\alpha^2$. Then the values $\psi(0)$, $\overline{\chi_\alpha(0)}$, $\overline{\chi_{\frac{\theta}{\alpha}}(0)}$, and $\overline{\chi(0)} = \overline{\chi_{\frac{\theta}{\alpha}}(0)} + \overline{\frac{\theta}{\alpha}(0)\chi_\alpha(0)}$ can be determined from $\langle Ak_0^\theta, k_0^\alpha\rangle$, $\langle A\tilde{k}_0^\theta, \tilde{k}_0^\alpha\rangle$ as a solution to the system (8.3). The solution is unique if we fix a concrete value of $\psi(0)$ or $\chi(0)$.*

Note that by Corollary 4.5 we can always choose a concrete value of either $\psi(0)$ or $\chi(0)$.

From equations (8.1) we cannot obtain $\chi_{\frac{\theta}{\alpha}}$; hence we need another equality, which we formulate by using Proposition 3.6(5–6) and applying C_θ. We get

$$C_\theta(A_{\bar{\psi}}^{\alpha,\theta}k_0^\alpha) = \psi(0)\tilde{k}_0^\theta - \alpha(0)\left(\frac{\theta}{\alpha}\bar{z}\psi - \frac{\theta}{\alpha}(0)\psi(0)\bar{z}\right),$$

and by Proposition 2.3

$$C_\theta(A_\chi^{\alpha,\theta}k_0^\alpha) = \alpha\bar{z}\bar{\chi}_\alpha + \theta\bar{z}\bar{\chi}_{\frac{\theta}{\alpha}} - \alpha(0)\frac{\theta}{\alpha}\bar{z}\bar{\chi}_{\frac{\theta}{\alpha}}.$$

The operator A must satisfy the condition

$$-\alpha(0)\frac{\theta}{\alpha}\bar{z}\psi + \alpha\bar{z}\bar{\chi}_\alpha + \big(\alpha - \alpha(0)\big)\bar{z}\frac{\theta}{\alpha}\bar{\chi}_{\frac{\theta}{\alpha}} = C_\theta A^* k_0^\alpha - \psi(0)\bar{z}\theta. \tag{8.4}$$

To obtain a symbol for the operator A, we take the analytic functions $\psi = X$, $\alpha\bar{\chi}_\alpha = Y$, and $\frac{\theta}{\alpha}\bar{\chi}_{\frac{\theta}{\alpha}} = Z$ as unknowns, and we substitute $a = \psi(0)$, $b = \overline{\chi_\alpha(0)}$, $c = \overline{\chi(0)}$. Then from (8.1) and (8.4) we get the system of equations

$$\begin{cases} X - \overline{\theta(0)}Y = Ak_0^\theta - ck_0^\alpha + b\overline{\theta(0)}\alpha, \\ -\theta(0)X + Y = zA\tilde{k}_0^\theta - \frac{\theta}{\alpha}(0)a\alpha, \\ -\alpha(0)\frac{\theta}{\alpha}X + Y + \big(\alpha - \alpha(0)\big)Z = zC_\theta A^* k_0^\alpha - a\theta. \end{cases} \tag{8.5}$$

Note that the determinant of the matrix of the coefficients of the system (8.5) is $\|k_0^\theta\|^2(\alpha - \alpha(0)) \neq 0$; hence the solution of the system (8.5) is unique.

We have thus the following.

Theorem 8.2. *Let $\alpha \leq \theta$ be nonconstant inner functions, and let $A \in \mathcal{T}(\theta, \alpha)$. Suppose that a certain value is given for either $\psi(0)$ or $\chi(0)$, and let a, b, c be the corresponding solutions to (8.3). If X, Y, Z satisfy the linear system (8.5), then $A = A^{\theta,\alpha}_{\psi+\bar{\chi}}$, where $\psi = X$, and $\chi = \frac{\theta}{\alpha}\bar{Z} + \theta\bar{Y}$.*

Proof. If A has the symbol $\psi + \bar{\chi}$ with fixed value $\psi(0)$ or $\chi(0)$, then ψ and χ are uniquely determined. Since, as shown above, $X = \psi$, $Y = \alpha\bar{\chi}_\alpha$, $Z = \frac{\theta}{\alpha}\bar{\chi}_{\frac{\theta}{\alpha}}$ satisfy the system (8.5), the result follows from the uniqueness of the solution to that system. □

The characterizations of asymmetric truncated Toeplitz operators in terms of operators of rank 2 at most, as obtained in previous sections, allow us also to obtain a symbol for the operator.

In fact, regarding the first characterization, it follows from the proof of Theorem 5.1 that, if A is a bounded operator and satisfies the equality (5.1), then $A = A^{\theta,\alpha}_{\psi+\bar{\chi}}$. Note that by Corollary 4.5 we know that ψ and χ are not unique, and we can adjust the value of either ψ or χ at the origin.

For $\alpha = \theta$ the characterization (6.1) of truncated Toeplitz operators in Theorem 6.1 reduces to that of Sarason (see [19, Remark, p. 501]). In that case the relation between ψ, χ in the symbol of $A^\theta_{\psi+\bar{\chi}}$ and μ, ν is given by the conjugation C_θ, namely, $\mu = C_\theta\chi$ and $\nu = C_\theta\psi$. Thus one can also immediately associate a symbol of the form $\psi+\bar{\chi}$ to a truncated Toeplitz operator satisfying that equality. In the asymmetric case, however, Theorem 6.1 unveils a more complex connection between the rank 2 operator on the right-hand side of (6.1) and the symbols of $A^{\theta,\alpha}_{\psi+\bar{\chi}}$. Finding a symbol in terms of μ and ν for an operator A satisfying equality (6.1) is more difficult.

To solve that problem in the case of asymmetric truncated Toeplitz operators, we start with two auxiliary results.

Lemma 8.3. *Let $\psi \in K_\alpha^2$, $\chi \in K_\theta^2$. Assume that $\chi = \chi_{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha}\chi_\alpha$ according to the decomposition $K_\theta^2 = K_{\frac{\theta}{\alpha}}^2 \oplus \frac{\theta}{\alpha}K_\alpha^2$. If*

$$\mu = C_\alpha P_\alpha\left(\frac{\bar\theta}{\bar\alpha}\chi\right) + \bar b\tilde k_0^\alpha, \qquad \nu = C_\alpha\psi + S^*(\alpha P_{\frac{\theta}{\alpha}}\chi) - b\tilde k_0^\theta$$

for fixed $b \in \mathbb{C}$, then

$$\psi = C_\alpha\nu_\alpha - \left(\overline{\chi_{\frac{\theta}{\alpha}}(0)} - \bar b\overline{\frac{\theta}{\alpha}(0)}\right)k_0^\alpha,$$

$$\chi_\alpha = C_\alpha\mu - bk_0^\alpha, \qquad \chi_{\frac{\theta}{\alpha}} = S_{\frac{\theta}{\alpha}}\nu_{\frac{\theta}{\alpha}} + \left(\chi_{\frac{\theta}{\alpha}}(0) - b\frac{\theta}{\alpha}(0)\right)k_0^{\frac{\theta}{\alpha}},$$

where $\nu = \nu_\alpha + \alpha\nu_{\frac{\theta}{\alpha}}$ according to the decomposition $K_\theta^2 = K_\alpha^2 \oplus \alpha K_{\frac{\theta}{\alpha}}^2$.

Proof. Let us calculate using Proposition 2.1:

$$\begin{aligned}
P_\alpha\big(S^*(\alpha\chi_{\frac{\theta}{\alpha}})\big) &= P_\alpha\big(\bar z\big(\alpha\chi_{\frac{\theta}{\alpha}} - \alpha(0)\chi_{\frac{\theta}{\alpha}}(0)\big)\big) = P_\alpha(\bar z\alpha\chi_{\frac{\theta}{\alpha}})\\
&= \alpha P^-\bar\alpha(\bar z\alpha\chi_{\frac{\theta}{\alpha}}) - \alpha(0)\chi_{\frac{\theta}{\alpha}}(0)\bar z\\
&= \alpha P^-(\bar z\chi_{\frac{\theta}{\alpha}}) - \alpha(0)\chi_{\frac{\theta}{\alpha}}(0)\bar z = \big(\alpha\chi_{\frac{\theta}{\alpha}}(0) - \alpha(0)\chi_{\frac{\theta}{\alpha}}(0)\big)\bar z\\
&= \chi_{\frac{\theta}{\alpha}}(0)\tilde k_0^\alpha.
\end{aligned}$$

Hence, since $C_\alpha\psi \in K_\alpha^2$ and $\tilde k_0^\theta = \frac{\theta}{\alpha}(0)\tilde k_0^\alpha + \alpha\tilde k_0^{\frac{\theta}{\alpha}}$ (Proposition 2.2), we have

$$P_\alpha\nu = C_\alpha\psi + \left(\chi_{\frac{\theta}{\alpha}}(0) - b\frac{\theta}{\alpha}(0)\right)\tilde k_0^\alpha;$$

thus

$$\psi = C_\alpha P_\alpha\nu - C_\alpha\left(\left(\chi_{\frac{\theta}{\alpha}}(0) - b\frac{\theta}{\alpha}(0)\right)\tilde k_0^\alpha\right) = C_\alpha\nu_\alpha - \left(\overline{\chi_{\frac{\theta}{\alpha}}(0)} - \bar b\overline{\frac{\theta}{\alpha}(0)}\right)k_0^\alpha.$$

Now let us consider χ. Since $\mu = P_\alpha C_\alpha(\chi\frac{\bar\theta}{\bar\alpha}) + \bar b\tilde k_0^\alpha = C_\alpha\chi_\alpha + \bar b\tilde k_0^\alpha$, we have

$$\chi_\alpha = C_\alpha(\mu - \bar b\tilde k_0^\alpha) = C_\alpha\mu - bk_0^\alpha.$$

On the other hand, by Proposition 2.1,

$$\begin{aligned}
\nu_{\frac{\theta}{\alpha}} &= P_{\frac{\theta}{\alpha}}(\bar\alpha\nu) = P_{\frac{\theta}{\alpha}}\big(\bar\alpha S^*(\alpha\chi_{\frac{\theta}{\alpha}}) - b\bar\alpha\tilde k_0^\theta\big) = P_{\frac{\theta}{\alpha}}\big(\bar\alpha\bar z\big(\alpha\chi_{\frac{\theta}{\alpha}} - \alpha(0)\chi_{\frac{\theta}{\alpha}}(0)\big)\big) - b\tilde k_0^{\frac{\theta}{\alpha}}\\
&= P_{\frac{\theta}{\alpha}}(\bar z\chi_{\frac{\theta}{\alpha}}) - \alpha(0)\chi_{\frac{\theta}{\alpha}}(0)P_{\frac{\theta}{\alpha}}(\bar\alpha\bar z) - b\tilde k_0^{\frac{\theta}{\alpha}} = P_{\frac{\theta}{\alpha}}(\bar z\chi_{\frac{\theta}{\alpha}}) - b\tilde k_0^{\frac{\theta}{\alpha}}\\
&= \frac{\theta}{\alpha}P^-\left(\frac{\bar\theta}{\bar\alpha}\bar z\chi_{\frac{\theta}{\alpha}}\right) - P^-(\bar z\chi_{\frac{\theta}{\alpha}}) - b\tilde k_0^{\frac{\theta}{\alpha}} = \bar z\chi_{\frac{\theta}{\alpha}} - \bar z\chi_{\frac{\theta}{\alpha}}(0) - b\tilde k_0^{\frac{\theta}{\alpha}}\\
&= S_{\frac{\theta}{\alpha}}^*\chi_{\frac{\theta}{\alpha}} - b\tilde k_0^{\frac{\theta}{\alpha}} \qquad (8.6)
\end{aligned}$$

since $\frac{\bar{\theta}}{\bar{\alpha}}\chi_{\frac{\theta}{\alpha}} \perp H^2$. Hence, using a proper defect operator and (2.3), we obtain

$$\begin{aligned} \chi_{\frac{\theta}{\alpha}} &= (S_{\frac{\theta}{\alpha}} S^*_{\frac{\theta}{\alpha}} + k_0^{\frac{\theta}{\alpha}} \otimes k_0^{\frac{\theta}{\alpha}})\chi_{\frac{\theta}{\alpha}} \\ &= S_{\frac{\theta}{\alpha}} S^*_{\frac{\theta}{\alpha}}\chi_{\frac{\theta}{\alpha}} + (k_0^{\frac{\theta}{\alpha}} \otimes k_0^{\frac{\theta}{\alpha}})\chi_{\frac{\theta}{\alpha}} \end{aligned} \tag{8.7}$$

$$S_{\frac{\theta}{\alpha}}\nu_{\frac{\theta}{\alpha}} + bS_{\frac{\theta}{\alpha}}\tilde{k}_0^{\frac{\theta}{\alpha}} + \chi_{\frac{\theta}{\alpha}}(0)k_0^{\frac{\theta}{\alpha}} = \left(\chi_{\frac{\theta}{\alpha}}(0) - b\frac{\theta}{\alpha}(0)\right)k_0^{\frac{\theta}{\alpha}} + S_{\frac{\theta}{\alpha}}\nu_{\frac{\theta}{\alpha}}. \qquad \square$$

Lemma 8.4. *Let $A \in \mathcal{T}(\theta, \alpha)$ satisfy the equation*

$$A - S^*_\alpha A S_\theta = \mu \otimes \tilde{k}_0^\theta + \tilde{k}_0^\alpha \otimes \nu$$

for $\mu \in K^2_\alpha$, $\nu \in K^2_\theta$. Then μ and ν can be chosen such that $P_{\frac{\theta}{\alpha}}(\bar{\alpha}\nu)$ is orthogonal to $\tilde{k}_0^{\frac{\theta}{\alpha}}$. In this case, μ and ν are uniquely determined.

Proof. According to the decomposition $K^2_\theta = K^2_\alpha \oplus \alpha K^2_{\frac{\theta}{\alpha}}$ we can write $\nu = P_\alpha \nu + \alpha P_{\frac{\theta}{\alpha}}\bar{\alpha}\nu = \nu_\alpha + \alpha\nu_{\frac{\theta}{\alpha}}$. By Proposition 2.2 recall that $\tilde{k}_0^\theta = \frac{\theta}{\alpha}(0)\tilde{k}_0^\alpha + \alpha\tilde{k}_0^{\frac{\theta}{\alpha}}$. Let us define

$$\mu' = \mu + \|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^{-2}\overline{\langle \nu_{\frac{\theta}{\alpha}}, \tilde{k}_0^{\frac{\theta}{\alpha}}\rangle}\tilde{k}_0^\alpha, \qquad \nu' = \nu - \|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^{-2}\langle \nu_{\frac{\theta}{\alpha}}, \tilde{k}_0^{\frac{\theta}{\alpha}}\rangle\tilde{k}_0^\theta$$

(see (6.4)). We have

$$A - S^*_\alpha A S_\theta = \mu' \otimes \tilde{k}_0^\theta + \tilde{k}_0^\alpha \otimes \nu', \qquad \mu' \in K^2_\alpha, \quad \nu' \in K^2_\theta.$$

Moreover, $\nu'_{\frac{\theta}{\alpha}}$ is orthogonal to $\tilde{k}_0^{\frac{\theta}{\alpha}}$ since

$$\langle \nu'_{\frac{\theta}{\alpha}}, \tilde{k}_0^{\frac{\theta}{\alpha}}\rangle = \left\langle \nu_{\frac{\theta}{\alpha}} - \|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^{-2}\langle \nu_{\frac{\theta}{\alpha}}, \tilde{k}_0^{\frac{\theta}{\alpha}}\rangle\tilde{k}_0^{\frac{\theta}{\alpha}}, \tilde{k}_0^{\frac{\theta}{\alpha}}\right\rangle = 0.$$

To show uniqueness it is enough to note that $\mu \otimes \tilde{k}_0^\theta + \tilde{k}_0^\alpha \otimes \nu = 0$ if and only if $\mu = 0$ and $\nu = 0$. Assume then that $\mu \otimes \tilde{k}_0^\theta + \tilde{k}_0^\alpha \otimes \nu = 0$, which is equivalent to $\mu = c_1\tilde{k}_0^\alpha$, $\nu = c_2\tilde{k}_0^\theta$, and $c_1 + \bar{c}_2 = 0$ for $c_1, c_2 \in \mathbb{C}$. Then by Proposition 2.2(4),

$$P_{\frac{\theta}{\alpha}}\bar{\alpha}\nu = c_2 P_{\frac{\theta}{\alpha}}\bar{\alpha}\tilde{k}_0^\theta = c_2\tilde{k}_0^{\frac{\theta}{\alpha}}.$$

Since $P_{\frac{\theta}{\alpha}}\bar{\alpha}\nu \perp \tilde{k}_0^{\frac{\theta}{\alpha}}$, we then obtain that $c_2 = 0$, which implies that $c_1 = 0$; hence $\mu = 0$ and $\nu = 0$. $\square$

When investigating symbols of the asymmetric truncated Toeplitz operator, it is worth keeping in mind Corollary 4.5, which states that it is enough to find one of them.

Theorem 8.5. *Let α, θ be nonconstant inner functions such that $\alpha \leq \theta$, and let A be a bounded operator satisfying*

$$A - S^*_\alpha A S_\theta = \mu \otimes \tilde{k}_0^\theta + \tilde{k}_0^\alpha \otimes \nu \tag{8.8}$$

for $\mu \in K_\alpha^2$, $\nu \in K_\theta^2$. *Then* $A = A^{\theta,\alpha}_{\psi+\bar{\chi}}$, *where*

$$\psi = C_\alpha P_\alpha(\nu - c\tilde{k}_0^\theta) = C_\alpha P_\alpha \nu - \bar{c}\overline{\frac{\theta}{\alpha}(0)}k_0^\alpha \in K_\alpha^2 \quad \textit{and}$$

$$\begin{aligned}\chi &= S_{\frac{\theta}{\alpha}} P_{\frac{\theta}{\alpha}} \bar{\alpha}(\nu - c\tilde{k}_0^\theta) + \frac{\theta}{\alpha} C_\alpha(\mu + \bar{c}\tilde{k}_0^\alpha) \\ &= \left(S_{\frac{\theta}{\alpha}} P_{\frac{\theta}{\alpha}} \bar{\alpha}\nu + c\frac{\theta}{\alpha}(0)k_0^{\frac{\theta}{\alpha}}\right) + \frac{\theta}{\alpha}(C_\alpha \mu + ck_0^\alpha) \in K_\theta^2 = K^2_{\frac{\theta}{\alpha}} \oplus \frac{\theta}{\alpha}K_\alpha^2 \quad \textit{with} \\ c &= \langle P_{\frac{\theta}{\alpha}}\bar{\alpha}\nu, \tilde{k}_0^{\frac{\theta}{\alpha}}\rangle \|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^{-2}.\end{aligned}$$

Proof. Let $\mu_0 = \mu + \bar{c}\tilde{k}_0^\alpha$, $\nu_0 = \nu - c\tilde{k}_0^\theta$ with $c = \langle P_{\frac{\theta}{\alpha}}\bar{\alpha}\nu, \tilde{k}_0^{\frac{\theta}{\alpha}}\rangle \|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^{-2}$. We have $P_{\frac{\theta}{\alpha}}(\bar{\alpha}\nu_0) \perp \tilde{k}_0^{\frac{\theta}{\alpha}}$, and (8.8) is also satisfied; that is,

$$A - S_\alpha^* A S_\theta = \mu_0 \otimes \tilde{k}_0^\theta + \tilde{k}_0^\alpha \otimes \nu_0.$$

On the other hand, $A \in \mathcal{T}(\theta, \alpha)$, so $A = A^{\theta,\alpha}_{\psi+\bar{\chi}}$ with $\psi \in K_\alpha^2$ and $\chi \in K_\theta^2$ and, by (6.2) and (6.4), there are also $\mu' = C_\alpha P_\alpha(\chi \frac{\bar{\theta}}{\bar{\alpha}}) + \bar{b}\tilde{k}_0^\alpha \in K_\alpha^2$ and $\nu' = C_\alpha \psi + S_\theta^*(\alpha P_{\frac{\theta}{\alpha}}\chi) - b\tilde{k}_0^\theta \in K_\theta^2$ $(b \in \mathbb{C})$ such that

$$A - S_\alpha^* A S_\theta = \mu' \otimes \tilde{k}_0^\theta + \tilde{k}_0^\alpha \otimes \nu'.$$

Take

$$b = -\chi_{\frac{\theta}{\alpha}}(0)\overline{\frac{\theta}{\alpha}(0)}\|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^{-26},$$

where $\chi_{\frac{\theta}{\alpha}} = P_{\frac{\theta}{\alpha}}\chi$. Using calculations as in (8.6), we have $P_{\frac{\theta}{\alpha}}(\bar{\alpha}(\nu' - b\tilde{k}_0^\theta)) \perp \tilde{k}_0^{\frac{\theta}{\alpha}}$ because

$$\begin{aligned}\left\langle P_{\frac{\theta}{\alpha}}\bar{\alpha}(\nu' - b\tilde{k}_0^\theta), \tilde{k}_0^{\frac{\theta}{\alpha}}\right\rangle &= \langle S^*\chi_{\frac{\theta}{\alpha}}, \tilde{k}_0^{\frac{\theta}{\alpha}}\rangle - b\langle P_{\frac{\theta}{\alpha}}\bar{\alpha}\tilde{k}_0^\theta, \tilde{k}_0^{\frac{\theta}{\alpha}}\rangle \\ &= \left\langle \bar{z}\left(\chi_{\frac{\theta}{\alpha}} - \chi_{\frac{\theta}{\alpha}}(0)\right), \bar{z}\left(\frac{\theta}{\alpha} - \frac{\theta}{\alpha}(0)\right)\right\rangle - b\langle \tilde{k}_0^{\frac{\theta}{\alpha}}, \tilde{k}_0^{\frac{\theta}{\alpha}}\rangle \\ &= -\chi_{\frac{\theta}{\alpha}}(0)\overline{\frac{\theta}{\alpha}(0)} - b\|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^2 = 0;\end{aligned}$$

hence, by Lemma 8.4, $\mu_0 = \mu'$ and $\nu_0 = \nu'$. On the other hand, by Lemma 8.3,

$$\psi = C_\alpha P_\alpha \nu_0 - \left(\overline{\chi_{\frac{\theta}{\alpha}}(0)} + \overline{\chi_{\frac{\theta}{\alpha}}(0)}\left|\frac{\theta}{\alpha}(0)\right|^2 \|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^{-2}\right)k_0^\alpha = C_\alpha P_\alpha \nu_0 - \overline{\chi_{\frac{\theta}{\alpha}}(0)}\|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^{-2}k_0^\alpha$$

and

$$\begin{aligned}\chi &= S_{\frac{\theta}{\alpha}} P_{\frac{\theta}{\alpha}}(\bar{\alpha}\nu_0) + \chi_{\frac{\theta}{\alpha}}(0)\left(1 + \frac{|\frac{\theta}{\alpha}(0)|^2}{\|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^2}\right)k_0^{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha}\left(C_\alpha \mu_0 + \chi_{\frac{\theta}{\alpha}}(0)\frac{\overline{\frac{\theta}{\alpha}(0)}}{\|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^2}k_0^\alpha\right) \\ &= S_{\frac{\theta}{\alpha}} P_{\frac{\theta}{\alpha}}(\bar{\alpha}\nu_0) + \chi_{\frac{\theta}{\alpha}}(0)\|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^{-2}k_0^{\frac{\theta}{\alpha}} + \frac{\theta}{\alpha}\left(C_\alpha \mu_0 + \chi_{\frac{\theta}{\alpha}}(0)\overline{\frac{\theta}{\alpha}(0)}\|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^{-2}k_0^\alpha\right) \\ &= S_{\frac{\theta}{\alpha}} P_{\frac{\theta}{\alpha}}(\bar{\alpha}\nu_0) + \frac{\theta}{\alpha}C_\alpha \mu_0 + \chi_{\frac{\theta}{\alpha}}(0)\|\tilde{k}_0^{\frac{\theta}{\alpha}}\|^{-2}k_0^\theta.\end{aligned}$$

Thus, taking Corollary 4.5 into account, we conclude that $A = A^{\theta,\alpha}_{\psi'+\overline{\chi'}}$ with $\psi' = C_\alpha P_\alpha \nu_0 = C_\alpha P_\alpha(\nu - c\tilde{k}_0^\theta)$ and that $\chi' = S_{\frac{\theta}{\alpha}} P_{\frac{\theta}{\alpha}}(\bar{\alpha}\nu_0) + \frac{\theta}{\alpha} C_\alpha \mu_0 = S_{\frac{\theta}{\alpha}} P_{\frac{\theta}{\alpha}}(\bar{\alpha}(\nu - c\tilde{k}_0^\theta)) + \frac{\theta}{\alpha} C_\alpha(\mu + \bar{c}\tilde{k}_0^\alpha)$. $\square$

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