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A GENERALIZED HILBERT OPERATOR ACTING ON CONFORMALLY INVARIANT SPACES

DANIEL GIRELA* and NOEL MERCHÁN

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Abstract. If μ is a positive Borel measure on the interval $[0,1)$, we let $\mathcal{H}_\mu$ be the Hankel matrix $\mathcal{H}_\mu = (\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where, for $n = 0, 1, 2, \dots$, μ_n denotes the moment of order n of μ. This matrix formally induces the operator

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\infty}\mu_{n,k}a_k\Big)z^n$$

on the space of all analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, in the unit disk $\mathbb{D}$. This is a natural generalization of the classical Hilbert operator. The action of the operators H_μ on Hardy spaces has been recently studied. This article is devoted to a study of the operators H_μ acting on certain conformally invariant spaces of analytic functions on the disk such as the Bloch space, the space BMOA, the analytic Besov spaces, and the Q_s-spaces.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence in compact subsets. We also let H^p $(0 < p \leq \infty)$ be the classical Hardy spaces. (See [18] for notation and results regarding Hardy spaces.)

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*Corresponding author.
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If μ is a finite positive Borel measure on $[0,1)$ and $n = 0,1,2,\dots,$ we let μ_n denote the moment of order n of μ (i.e., $\mu_n = \int_{[0,1)} t^n \, d\mu(t)$), and we let $\mathcal{H}_\mu$ be the Hankel matrix $(\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$. The matrix $\mathcal{H}_\mu$ formally induces an operator, which will also be called $\mathcal{H}_\mu$, on spaces of analytic functions by its action on the Taylor coefficients: $a_n \mapsto \sum_{k=0}^{\infty} \mu_{n,k} a_k$, $n = 0,1,2,\dots$. To be precise, if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}ol(\mathbb{D})$, we define

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{\infty} \mu_{n,k} a_k \Big) z^n$$

whenever the right-hand side makes sense and defines an analytic function in $\mathbb{D}$.

If μ is the Lebesgue measure on $[0,1)$, then the matrix $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix $\mathcal{H} = ((n+k+1)^{-1})_{n,k\geq 0}$, which induces the classical Hilbert operator $\mathcal{H}$ which has been extensively studied recently (see [1], [13], [14], [16], [24]).

Galanopoulos and Peláez [20] described the measures μ, so that the generalized Hilbert operator $\mathcal{H}_\mu$ becomes well defined and bounded on H^1. Chatzifountas, Girela, and Peláez [12] extended this work describing those measures μ for which $\mathcal{H}_\mu$ is a bounded operator from H^p into H^q, $0 < p, q < \infty$. Obtaining an integral representation of H_μ plays a basic role in these works. If μ is as above, we will write

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz} \, d\mu(t) \tag{1.1}$$

throughout this article whenever the right-hand side makes sense and defines an analytic function in $\mathbb{D}$. It turns out that the operators H_μ and I_μ are closely related. In fact, the authors in [20] and [12] have characterized the measures μ for which the operator I_μ is well defined in H^p $(0 < p < \infty)$, and it is proved that for such measures we have $\mathcal{H}_\mu(f) = I_\mu(f)$ for all $f \in H^p$. These measures are Carleson-type measures.

If $I \subset \partial\mathbb{D}$ is an arc, then $|I|$ will denote the length of I. The *Carleson square* $S(I)$ is defined as $S(I) = \{re^{it} : e^{it} \in I, 1 - \frac{|I|}{2\pi} \leq r < 1\}$. If $s > 0$ and μ is a positive Borel measure on $\mathbb{D}$, we will say that μ is an *s-Carleson measure* if there exists a positive constant C such that

$$\mu\big(S(I)\big) \leq C|I|^s, \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

If μ satisfies $\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^s} = 0$, then we say that μ is a *vanishing s-Carleson measure*. A 1-Carleson measure (resp., vanishing 1-Carleson measure) will simply be called a *Carleson measure* (resp., *vanishing Carleson measure*). We recall that Carleson [11] proved that $H^p \subset L^p(d\mu)$ $(0 < p < \infty)$ if and only if μ is a Carleson measure. This result was extended by Duren [17] (see also [18, Theorem 9.4]) who proved that for $0 < p \leq q < \infty$, $H^p \subset L^q(d\mu)$ if and only if μ is a q/p-Carleson measure.

Following [32], if μ is a positive Borel measure on $\mathbb{D}$, $0 \leq \alpha < \infty$ and $0 < s < \infty$, then we say that μ is an *α-logarithmic s-Carleson measure* if there exists a positive constant C such that

$$\frac{\mu(S(I))(\log\frac{2\pi}{|I|})^\alpha}{|I|^s} \le C \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

If $\mu(S(I))(\log\frac{2\pi}{|I|})^\alpha = \mathrm{o}(|I|^s)$, as $|I| \to 0$, we say that μ is a *vanishing α-logarithmic s-Carleson measure*.

A positive Borel measure μ on $[0,1)$ can be seen as a Borel measure on $\mathbb{D}$ by identifying it with the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(A) = \mu\big(A \cap [0,1)\big) \quad \text{for any Borel subset } A \text{ of } \mathbb{D}.$$

In this way, a positive Borel measure μ on $[0,1)$ is an s-Carleson measure if and only if there exists a positive constant C such that

$$\mu\big([t,1)\big) \le C(1-t)^s, \quad 0 \le t < 1,$$

and we have similar statements for vanishing s-Carleson measures and for α-logarithmic s-Carleson and vanishing α-logarithmic s-Carleson measures.

Our main aim in this article is to study the operators $\mathcal{H}_\mu$ acting on conformally invariant spaces. It is a standard fact that the set of all disk automorphisms (i.e., all one-to-one analytic maps f of $\mathbb{D}$ onto itself), denoted $\mathrm{Aut}(\mathbb{D})$, coincides with the set of all Möbius transformations of $\mathbb{D}$ onto itself:

$$\mathrm{Aut}(\mathbb{D}) = \big\{\lambda\varphi_a : |a| < 1, |\lambda| = 1\big\},$$

where $\varphi_a(z) = (a-z)/(1-\overline{a}z)$. A space X of analytic functions in $\mathbb{D}$, defined via a seminorm ρ, is said to be *conformally invariant* or *Möbius invariant* if, whenever $f \in X$, then also $f \circ \varphi \in X$ for any $\varphi \in \mathrm{Aut}(\mathbb{D})$ and moreover, $\rho(f \circ \varphi) \le C\rho(f)$ for some positive constant C and all $f \in X$. (A great deal of information on conformally invariant spaces can be found in [5], [15], and [30].)

We begin our consideration with the Bloch space and BMOA (the space of analytic functions of bounded mean oscillation). The *Bloch space* $\mathcal{B}$ consists of all analytic functions f in $\mathbb{D}$ with bounded invariant derivative:

$$f \in \mathcal{B} \qquad \Leftrightarrow \qquad \|f\|_{\mathcal{B}} \overset{\text{def}}{=} \big|f(0)\big| + \sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\big|f'(z)\big| < \infty.$$

The *little Bloch space* $\mathcal{B}_0$ is the closure of the polynomials in the above norm of $\mathcal{B}$ and consists of all functions f analytic in $\mathbb{D}$ for which

$$\lim_{|z|\to 1}\big(1-|z|^2\big)\big|f'(z)\big| = 0.$$

A classical source for the Bloch space is [3]; see also [34]. Rubel and Timoney [30] proved that $\mathcal{B}$ is the biggest "natural" conformally invariant space. The space BMOA consists of those functions f in H^1 whose boundary values have bounded mean oscillation on the unit circle $\partial\mathbb{D}$ as defined by John and Nirenberg. There are many characterizations of BMOA functions. Let us mention the following.

If f is an analytic function in $\mathbb{D}$, then $f \in$ BMOA if and only if

$$\|f\|_{\mathrm{BMOA}} \overset{\text{def}}{=} \big|f(0)\big| + \|f\|_\star < \infty,$$

where

$$\|f\|_\star \stackrel{\text{def}}{=} \sup_{a\in\mathbb{D}} \big\|f\circ\varphi_a - f(a)\big\|_{H^2}.$$

It is clear that the seminorm $\|\cdot\|_\star$ is conformally invariant. If

$$\lim_{|a|\to 1} \big\|f\circ\varphi_a - f(a)\big\|_{H^2} = 0,$$

then we say that f belongs to the space VMOA (analytic functions of vanishing mean oscillation). (We mention [9] and [21] as general references for the spaces BMOA and VMOA.) Let us recall that

$$H^\infty \subsetneq \mathrm{BMOA} \subsetneq \bigcap_{0<p<\infty} H^p \qquad \text{and} \qquad \mathrm{BMOA} \subsetneq \mathcal{B}.$$

Other important Möbius invariant spaces are the analytic Besov spaces B^p ($1 < p < \infty$) and the Q_s-spaces ($s > 0$). These spaces will be considered in Section 3.

We close this section by noting that, as usual, we will be using the convention that $C = C(p,\alpha,q,\beta,\dots)$ denotes a positive constant which depends only upon the displayed parameters $p,\alpha,q,\beta\dots$ (which sometimes will be omitted) but the value of C may not necessarily be the same at different occurrences. Moreover, for two real-valued functions E_1, E_2 we will write $E_1 \lesssim E_2$ or $E_1 \gtrsim E_2$, if there exists a positive constant C independent of the arguments such that $E_1 \le CE_2$, respectively, $E_1 \ge CE_2$. If we have $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$ simultaneously, then we say that E_1 and E_2 are *equivalent* and we write $E_1 \asymp E_2$.

2. The operator $\mathcal{H}_\mu$ acting on BMOA and the Bloch space

We start by characterizing those μ's for which the operator I_μ is well defined in BMOA and in the Bloch space. It turns out that they coincide.

Theorem 2.1. *Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent.*

(i) *The measure μ satisfies $\int_{[0,1)} \log\frac{2}{1-t}\,d\mu(t) < \infty$.*
(ii) *For any given $f \in \mathcal{B}$, the integral in (1.1) converges for all $z \in \mathbb{D}$ and the resulting function $I_\mu(f)$ is analytic in $\mathbb{D}$.*
(iii) *For any given $f \in BMOA$, the integral in (1.1) converges for all $z \in \mathbb{D}$ and the resulting function $I_\mu(f)$ is analytic in $\mathbb{D}$.*

Proof. (i) $\Rightarrow$ (ii). It is well known (see [3, p. 13]) that there exists a positive constant C such that

$$\big|f(z)\big| \le C\|f\|_{\mathcal{B}} \log\frac{2}{1-|z|}, \quad (z\in\mathbb{D}), \text{for every } f\in\mathcal{B}. \tag{2.1}$$

Assume (i), and set $A = \int_{[0,1)} \log\frac{2}{1-t}\,d\mu(t)$. Using (2.1) we see that

$$\int_{[0,1)} \big|f(t)\big|\,d\mu(t) \le C\|f\|_{\mathcal{B}} \int_{[0,1)} \log\frac{2}{1-t}\,d\mu(t) = AC\|f\|_{\mathcal{B}}, \quad f\in\mathcal{B}. \tag{2.2}$$

This implies that

$$\int_{[0,1)} \frac{|f(t)|}{|1-tz|}\,d\mu(t) \le \frac{AC\|f\|_{\mathcal{B}}}{1-|z|}, \quad (z \in \mathbb{D}), f \in \mathcal{B}. \tag{2.3}$$

Using (2.2), (2.3), and Fubini's theorem, we see that if $f \in \mathcal{B}$, then

- for every $n \in \mathbb{N}$, the integral $\int_{[0,1)} t^n f(t)\,d\mu(t)$ converges absolutely and

$$\sup_{n\ge 0}\left|\int_{[0,1)} t^n f(t)\,d\mu(t)\right| < \infty;$$

- the integral $\int_{[0,1)} \frac{f(t)}{1-tz}\,d\mu(t)$ converges absolutely, and

$$\int_{[0,1)} \frac{f(t)}{1-tz}\,d\mu(t) = \sum_{n=0}^{\infty}\Big(\int_{[0,1)} t^n f(t)\,d\mu(t)\Big) z^n, \quad z \in \mathbb{D}.$$

Thus, if $f \in \mathcal{B}$, then $I_\mu(f)$ is a well-defined analytic function in $\mathbb{D}$ and

$$I_\mu(f)(z) = \sum_{n=0}^{\infty}\Big(\int_{[0,1)} t^n f(t)\,d\mu(t)\Big) z^n, \quad z \in \mathbb{D}.$$

(ii) $\Rightarrow$ (iii) is clear because BMOA $\subset \mathcal{B}$.

(iii) $\Rightarrow$ (i). Suppose (iii). Since the function $F(z) = \log\frac{2}{1-z}$ belongs to BMOA, $I_\mu(F)(z)$ is well defined for every $z \in \mathbb{D}$. In particular,

$$I_\mu(F)(0) = \int_{[0,1)} \log\frac{2}{1-t}\,d\mu(t)$$

is a complex number. Since μ is a positive measure and $\log\frac{2}{1-t} > 0$ for all $t \in [0,1)$, (i) follows. $\square$

Our next aim is characterizing the measures μ so that I_μ is bounded in BMOA or $\mathcal{B}$ and seeing whether or not I_μ and H_μ coincide for such measures. We have the following results.

Theorem 2.2. *Let μ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)}\log\frac{2}{1-t}\,d\mu(t) < \infty$. Then the following three conditions are equivalent.*

(i) *The measure ν defined by $d\nu(t) = \log\frac{2}{1-t}\,d\mu(t)$ is a Carleson measure.*

(ii) *The operator I_μ is bounded from $\mathcal{B}$ into BMOA.*

(iii) *The operator I_μ is bounded from BMOA into itself.*

Theorem 2.3. *Let μ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)}\log\frac{2}{1-t}\,d\mu(t) < \infty$. If the measure ν defined by $d\nu(t) = \log\frac{2}{1-t}\,d\mu(t)$ is a Carleson measure, then $\mathcal{H}_\mu$ is well defined on the Bloch space and*

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \textit{for all } f \in \mathcal{B}.$$

Theorem 2.2 and Theorem 2.3 together yield the following.

Theorem 2.4. *Let μ be a positive Borel measure on $[0,1)$ such that the measure ν defined by $d\nu(t) = \log\frac{2}{1-t}\,d\mu(t)$ is a Carleson measure. Then the operator $\mathcal{H}_\mu$ is bounded from $\mathcal{B}$ into BMOA.*

Proof of Theorem 2.2. Since $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$, (2.1) implies that

$$\int_{[0,1)} \left|f(t)\right| d\mu(t) < \infty, \quad \text{for all } f \in \mathcal{B},$$

and this implies that

$$\int_0^{2\pi} \int_{[0,1)} \left|\frac{f(t)g(e^{i\theta})}{1-re^{i\theta}t}\right| d\mu(t)\, d\theta < \infty, \quad 0 \le r < 1, f \in \mathcal{B}, g \in H^1.$$

Using this, Fubini's theorem, and Cauchy's integral representation of H^1-functions (see [18, Theorem 3.6]), we deduce that whenever $f \in \mathcal{B}$ and $g \in H^1$, we have

$$\begin{aligned}\int_0^{2\pi} I_\mu(f)(re^{i\theta})\overline{g(e^{i\theta})}\, d\theta &= \int_0^{2\pi} \Big(\int_{[0,1)} \frac{f(t)\, d\mu(t)}{1-re^{i\theta}t}\Big)\overline{g(e^{i\theta})}\, d\theta \\ &= \int_{[0,1)} f(t)\Big(\int_0^{2\pi} \frac{\overline{g(e^{i\theta})}\, d\theta}{1-re^{i\theta}t}\Big)\, d\mu(t) \\ &= \int_{[0,1)} f(t)\overline{g(rt)}\, d\mu(t), \quad 0 \le r < 1. \end{aligned} \tag{2.4}$$

(i) $\Rightarrow$ (ii). Assume that ν is a Carleson measure, and take $f \in \mathcal{B}$ and $g \in H^1$. Using (2.4) and (2.1), we obtain

$$\begin{aligned}\left|\int_0^{2\pi} I_\mu(f)(re^{i\theta})\overline{g(e^{i\theta})}\, d\theta\right| &= \left|\int_{[0,1)} f(t)\overline{g(rt)}\, d\mu(t)\right| \\ &\lesssim \|f\|_{\mathcal{B}} \int_{[0,1)} \left|g(rt)\right| \log\frac{2}{1-t}\, d\mu(t) \\ &= \|f\|_{\mathcal{B}} \int_{[0,1)} \left|g(rt)\right| d\nu(t). \end{aligned}$$

Since ν is a Carleson measure, we have

$$\int_{[0,1)} \left|g(rt)\right| d\nu(t) \lesssim \|g_r\|_{H^1} \le \|g\|_{H^1}.$$

Here, g_r is the function defined (as usual) by $g_r(z) = g(rz)$ ($z \in \mathbb{D}$). Thus, we have proved that

$$\left|\int_0^{2\pi} I_\mu(f)(re^{i\theta})\overline{g(e^{i\theta})}\, d\theta\right| \lesssim \|f\|_{\mathcal{B}}\|g\|_{H^1}, \quad f \in \mathcal{B}, g \in H^1.$$

Using Fefferman's duality theorem (see [21, Theorem 7.1]), we deduce that if $f \in \mathcal{B}$, then $I_\mu(f) \in \mathrm{BMOA}$ and

$$\left\|I_\mu(f)\right\|_{\mathrm{BMOA}} \lesssim \|f\|_{\mathcal{B}}.$$

(ii) $\Rightarrow$ (iii) is trivial because $\mathrm{BMOA} \subset \mathcal{B}$.

(iii) $\Rightarrow$ (i). Assume (iii). Then there exists a positive constant A such that $\|I_\mu(f)\|_{\mathrm{BMOA}} \le A\|f\|_{\mathrm{BMOA}}$ for all $f \in \mathrm{BMOA}$. Set

$$F(z) = \log \frac{2}{1-z}, \quad z \in \mathbb{D}.$$

It is well known that $F \in \mathrm{BMOA}$. Then $I_\mu(F) \in \mathrm{BMOA}$ and

$$\big\|I_\mu(F)\big\|_{\mathrm{BMOA}} \le A\|F\|_{\mathrm{BMOA}}.$$

Then using again Fefferman's duality theorem, we obtain that

$$\Big|\int_0^{2\pi} I_\mu(F)(re^{i\theta})\overline{g(e^{i\theta})}\, d\theta\Big| \lesssim \|g\|_{H^1}, \quad g \in H^1.$$

Using (2.4) and the definition of F, this implies that

$$\Big|\int_{[0,1)]} \overline{g(rt)} \log \frac{2}{1-t}\, d\mu(t)\Big| \lesssim \|g\|_{H^1}, \quad g \in H^1. \tag{2.5}$$

Take $g \in H^1$. Using Proposition 2 of [12], we know that there exists a function $G \in H^1$ with $\|G\|_{H^1} = \|g\|_{H^1}$ and such that

$$\big|g(s)\big| \le G(s), \quad \text{for all } s \in [0,1).$$

Using these properties and (2.5) for G, we obtain

$$\begin{aligned}\int_{[0,1)} \big|g(rt)\big| \log \frac{2}{1-t}\, d\mu(t) &\le \int_{[0,1)} G(rt) \log \frac{2}{1-t}\, d\mu(t)\\ &\le C\|G_r\|_{H^1} \le C\|G\|_{H^1} = C\|g\|_{H^1}\end{aligned}$$

for a certain constant $C > 0$, independent of g. Letting r tend to 1, it follows that

$$\int_{[0,1)} \big|g(t)\big| \log \frac{2}{1-t}\, d\mu(t) \lesssim \|g\|_{H^1}, \quad g \in H^1.$$

This is equivalent to saying that ν is a Carleson measure. □

It is worth noting that for μ and ν as in Theorem 2.1, ν being a Carleson measure is equivalent to μ being a 1-logarithmic 1-Carleson measure. Actually, we have the following more general result.

Proposition 2.5. *Let μ be a positive Borel measure on $[0,1)$, and let $s > 0$ and $\alpha \ge 0$. Let ν be the Borel measure on $[0,1)$ defined by*

$$d\nu(t) = \Big(\log \frac{2}{1-t}\Big)^\alpha d\mu(t).$$

Then the following two conditions are equivalent:

(a) *ν is an s-Carleson measure, and*
(b) *μ is an α-logarithmic s-Carleson measure.*

Proof. (a) ⇒ (b). Assume (a). Then there exists a positive constant C such that

$$\int_{[t,1)} \left(\log \frac{2}{1-u}\right)^{\alpha} d\mu(u) \le C(1-t)^s, \quad t \in [0,1).$$

Using this and the fact that the function $u \mapsto \log \frac{2}{1-u}$ is increasing in $[0,1)$, we obtain

$$\left(\log \frac{2}{1-t}\right)^{\alpha} \int_{[t,1)} d\mu(u) \le \int_{[t,1)} \left(\log \frac{2}{1-u}\right)^{\alpha} d\mu(u) \le C(1-t)^s, \quad t \in [0,1).$$

This shows that μ is an α-logarithmic s-Carleson measure.

(b) ⇒ (a). Assume (b). Then there exists a positive constant C such that

$$\left(\log \frac{2}{1-t}\right)^{\alpha} \mu\big([t,1)\big) \le C(1-t)^s, \quad 0 \le t < 1. \tag{2.6}$$

For $0 \le u < 1$, set $F(u) = \mu([0,u)) - \mu([0,1)) = -\mu([u,1))$. Integrating by parts and using (2.6), we obtain

$$\begin{aligned}
\nu\big([t,1)\big) &= \int_{[t,1)} \left(\log \frac{2}{1-u}\right)^{\alpha} d\mu(u) \\
&= \left(\log \frac{2}{1-t}\right)^{\alpha} \mu\big([t,1)\big) - \lim_{u \to 1^-} \left(\log \frac{2}{1-u}\right)^{\alpha} \mu\big([u,1)\big) \\
&\quad + \alpha \int_{[t,1)} \mu\big([u,1)\big) \left(\log \frac{2}{1-u}\right)^{\alpha-1} \frac{du}{1-u} \\
&= \left(\log \frac{2}{1-t}\right)^{\alpha} \mu\big([t,1)\big) + \alpha \int_{[t,1)} \mu\big([u,1)\big) \left(\log \frac{2}{1-u}\right)^{\alpha-1} \frac{du}{1-u} \\
&\le C(1-t)^s + C\alpha \int_t^1 \frac{(1-u)^{s-1}}{\log \frac{2}{1-u}}\, du \\
&\lesssim (1-t)^s, \quad 0 \le t < 1.
\end{aligned}$$

Thus, ν is an s-Carleson measure. □

The following lemma will be needed in the proof of Theorem 2.3.

Lemma 2.6. *Let μ be a positive Borel measure in $[0,1)$ such that the measure ν defined by $d\nu(t) = \log \frac{1}{1-t}\, d\mu(t)$ is a Carleson measure. Then the sequence of moments $\{\mu_n\}$ satisfies*

$$\mu_n = \mathrm{O}\Big(\frac{1}{n \log n}\Big), \quad \textit{as } n \to \infty.$$

Actually, we will prove the following more general result.

Lemma 2.7. *Suppose that $0 \le \alpha \le \beta$, $s \ge 1$, and let μ be a positive Borel measure on $[0,1)$ which is a β-logarithmic s-Carleson measure. Then*

$$\int_{[0,1)} t^k \left(\log \frac{2}{1-t}\right)^{\alpha} d\mu(t) = \mathrm{O}\Big(\frac{(\log k)^{\alpha-\beta}}{k^s}\Big), \quad \textit{as } k \to \infty.$$

Using Proposition 2.5, Lemma 2.6 follows taking $\alpha = 0$, $\beta = 1$, and $s = 1$ in Lemma 2.7.

Proof of Lemma 2.7. Arguing as in the proof of the implication (b) $\Rightarrow$ (a) of Proposition 2.5, integrating by parts, and using the fact that μ is a β-logarithmic 1-Carleson measure, we obtain

$$\begin{aligned}
&\int_{[0,1)} t^k \Big(\log \frac{2}{1-t}\Big)^{\alpha} d\mu(t) \\
&\quad = k \int_0^1 \mu\big([t,1)\big) t^{k-1} \Big(\log \frac{2}{1-t}\Big)^{\alpha} dt \\
&\qquad + \alpha \int_0^1 \mu\big([t,1)\big) t^{k} \Big(\log \frac{2}{1-t}\Big)^{\alpha-1} \frac{dt}{1-t} \\
&\quad \lesssim k \int_0^1 (1-t)^s t^{k-1} \Big(\log \frac{2}{1-t}\Big)^{\alpha-\beta} dt \\
&\qquad + \alpha \int_0^1 (1-t)^{s-1} t^{k} \Big(\log \frac{2}{1-t}\Big)^{\alpha-\beta-1} dt.
\end{aligned} \tag{2.7}$$

Now, we note that the weight functions

$$\omega_1(t) = (1-t)^s \Big(\log \frac{2}{1-t}\Big)^{\alpha-\beta} \qquad \text{and} \qquad \omega_2(t) = (1-t)^{s-1} \Big(\log \frac{2}{1-t}\Big)^{\alpha-\beta-1}$$

are regular in the sense of [29, p. 6] (see also [2, Example 2]). Then, using Lemma 1.3 of [29] and the fact that the ω_j's are also decreasing, we obtain

$$\begin{aligned}
\int_0^1 (1-t)^s t^{k-1} \Big(\log \frac{2}{1-t}\Big)^{\alpha-\beta} dt &\lesssim \int_{1-\frac{1}{k}}^1 (1-t)^s t^{k-1} \Big(\log \frac{2}{1-t}\Big)^{\alpha-\beta} dt \\
&\lesssim \frac{(\log k)^{\alpha-\beta}}{k^{s+1}}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 (1-t)^{s-1} t^{k} \Big(\log \frac{2}{1-t}\Big)^{\alpha-\beta-1} dt &\lesssim \int_{1-\frac{1}{n}}^1 (1-t)^{s-1} t^{k} \Big(\log \frac{2}{1-t}\Big)^{\alpha-\beta-1} dt \\
&\lesssim \frac{(\log k)^{\alpha-\beta-1}}{k^{s}}.
\end{aligned}$$

Using these two estimates in (2.7) yields

$$\int_{[0,1)} t^k \Big(\log \frac{2}{1-t}\Big)^{\alpha} d\mu(t) \lesssim \frac{(\log k)^{\alpha-\beta}}{k^s},$$

finishing the proof. □

We will also use the characterization of the coefficient multipliers from $\mathcal{B}$ into ℓ^1 obtained by Anderson and Shields in [4].

Theorem A. *A sequence* $\{\lambda_n\}_{n=0}^{\infty}$ *of complex numbers is a coefficient multiplier from* $\mathcal{B}$ *into* ℓ^1 *if and only if*

$$\sum_{n=1}^{\infty}\Big(\sum_{k=2^n+1}^{2^{n+1}}|\lambda_k|^2\Big)^{1/2}<\infty.$$

Bearing in mind Definition 1 of [4], Theorem A reduces to the case $p=1$ in Corollary 1 on p. 259 of [4].

We recall that if X is a space of analytic functions in $\mathbb{D}$ and Y is a space of complex sequences, a sequence $\{\lambda_n\}_{n=0}^{\infty}\subset\mathbb{C}$ is said to be a *multiplier* of X into Y if whenever $f(z)=\sum_{n=0}^{\infty}a_nz^n\in X$, one has that the sequence $\{\lambda_na_n\}_{n=0}^{\infty}$ belongs to Y. Thus, by saying that $\{\lambda_n\}_{n=0}^{\infty}$ is a *coefficient multiplier* from $\mathcal{B}$ into ℓ^1, we mean that

$$\text{if } f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{B},\quad\text{then }\sum_{n=0}^{\infty}|\lambda_na_n|<\infty.$$

Actually, using the closed graph theorem, we can assert the following. A complex sequence $\{\lambda_n\}_{n=0}^{\infty}$ is a multiplier from $\mathcal{B}$ to ℓ^1 if and only if there exists a positive constant C such that whenever $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{B}$, we have that $\sum_{n=0}^{\infty}|\lambda_na_n|\le C\|f\|_{\mathcal{B}}$.

Proof of Theorem 2.3. Suppose that ν is a Carleson measure. Then, using Lemma 2.6, we see that there exists $C>0$ such that

$$|\mu_n|\le\frac{C}{n\log n},\quad n\ge 2. \tag{2.8}$$

It is clear that

$$k^2\log^2k\ge 2^{2n}n^2(\log 2)^2,\quad\text{if } 2^n+1\le k\le 2^{n+1}\text{ for all } n.$$

Then it follows that

$$\sum_{n=1}^{\infty}\Big(\sum_{k=2^n+1}^{2^{n+1}}\frac{1}{k^2\log^2k}\Big)^{1/2}\lesssim\sum_{n=1}^{\infty}\Big(\frac{2^n}{n^22^{2n}}\Big)^{1/2}=\sum_{n=1}^{\infty}\frac{1}{n2^{n/2}}<\infty.$$

Using this, (2.8), and Theorem A, we obtain:

$$\text{The sequence of moments } \{\mu_n\}_{n=0}^{\infty} \text{ is a multiplier from } \mathcal{B} \text{ to } \ell^1. \tag{2.9}$$

Now take $f\in\mathcal{B}$, $f(z)=\sum_{n=0}^{\infty}a_nz^n$ $(z\in\mathbb{D})$. Using the simple fact that the sequence $\{\mu_n\}_{n=0}^{\infty}$ is a decreasing sequence of positive numbers and (2.9), we see that there exists $C>0$ such that

$$\sum_{k=0}^{\infty}|\mu_{n+k}a_k|\le\sum_{k=0}^{\infty}|\mu_ka_k|\le C\|f\|_{\mathcal{B}},\quad n=0,1,2,\dots. \tag{2.10}$$

This implies that $\mathcal{H}_\mu(f)(z)$ is well defined for all $z\in\mathbb{D}$ and that, in fact, $\mathcal{H}_\mu(f)$ is an analytic function in $\mathbb{D}$. Furthermore, since (2.10) also implies that we can interchange the order of summation in the expression defining $\mathcal{H}_\mu(f)(z)$, we have

$$\begin{aligned}\mathcal{H}_\mu(f)(z) &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\infty}\mu_{n+k}a_k\Big)z^n = \sum_{k=0}^{\infty}a_k\Big(\sum_{n=0}^{\infty}\mu_{n+k}z^n\Big)\\ &= \sum_{k=0}^{\infty}a_k\Big(\sum_{n=0}^{\infty}\int_{[0,1)}t^{n+k}z^n\,d\mu(t)\Big) = \sum_{k=0}^{\infty}\int_{[0,1)}\frac{a_kt^k}{1-tz}\,d\mu(t)\\ &= \int_{[0,1)}\frac{f(t)}{1-tz}\,d\mu(t) = I_\mu(f)(z), \quad z\in\mathbb{D}.\end{aligned}$$

□

We have the following result regarding compactness.

Theorem 2.8. *Let μ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)}\log\frac{2}{1-t}\,d\mu(t)<\infty$. If the measure ν defined by $d\nu(t)=\log\frac{2}{1-t}\,d\mu(t)$ is a vanishing Carleson measure, then*

(i) *the operator I_μ is a compact operator from $\mathcal{B}$ into BMOA,*
(ii) *the operator I_μ is a compact operator from BMOA into itself.*

Before embarking on the proof of Theorem 2.8, it is convenient to recall some facts about Carleson measures and to fix some notation. If μ is a Carleson measure on $\mathbb{D}$, we define the Carleson norm of μ, denoted $\mathcal{N}(\mu)$, as

$$\mathcal{N}(\mu)=\sup_{I \text{ subarc of } \partial\mathbb{D}}\frac{\mu(S(I))}{|I|}.$$

We let also $\mathcal{E}(\mu)$ denote the norm of the inclusion operator $i: H^1\to L^1(d\mu)$. It turns out that these quantities are equivalent: There exist two positive constants A_1, A_2 such that

$$A_1\mathcal{N}(\mu)\le\mathcal{E}(\mu)\le A_2\mathcal{N}(\mu), \quad \text{for every Carleson measure } \mu \text{ on } \mathbb{D}.$$

For a Carleson measure μ on $\mathbb{D}$ and $0<r<1$, we let μ_r be the measure on $\mathbb{D}$ defined by

$$d\mu_r(z)=\chi_{\{r<|z|<1\}}\,d\mu(z).$$

We have that μ is a vanishing Carleson measure if and only if

$$\mathcal{N}(\mu_r)\to 0, \quad \text{as } r\to 1.$$

Proof of Theorem 2.8. Since BMOA is continuously contained in the Bloch spaces, it suffices to prove (i). Suppose that ν is a vanishing Carleson measure. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Bloch functions with $\sup_{n\ge1}\|f_n\|_{\mathcal{B}}<\infty$ and such that $\{f_n\}\to 0$, uniformly on compact subsets of $\mathbb{D}$. We have to prove that $I_\mu(f_n)\to 0$ in BMOA. The condition $\sup_{n\ge1}\|f_n\|_{\mathcal{B}}<\infty$ implies that there exists a positive constant M such that

$$\big|f_n(z)\big|\le M\log\frac{2}{1-|z|}, \quad z\in\mathbb{D}, n\ge 1. \tag{2.11}$$

Recall that for $0<r<1$, ν_r is the measure defined by

$$d\nu_r(t)=\chi_{\{r<t<1\}}\,d\nu(t).$$

Since ν is a vanishing Carleson measure, we have that $\mathcal{N}(\nu_r) \to 0$, as $r \to 1$, or, equivalently,

$$\mathcal{E}(\nu_r) \to 0 \quad \text{as } t \to 1. \tag{2.12}$$

Take $g \in H^1$ and $r \in [0,1)$. Using (2.11) we have

$$\begin{aligned}\int_{[0,1)} |f_n(t)||g(t)|\,d\mu(t) &= \int_{[0,r)} |f_n(t)||g(t)|\,d\mu(t) + \int_{[r,1)} |f_n(t)||g(t)|\,d\mu(t)\\ &\le \int_{[0,r)} |f_n(t)||g(t)|\,d\mu(t) + M\int_{[r,1)} \log\frac{2}{1-t}|g(t)|\,d\mu(t)\\ &= \int_{[0,r)} |f_n(t)||g(t)|\,d\mu(t) + M\int_{[0,1)} |g(t)|\,d\nu_r(t)\\ &\le \int_{[0,r)} |f_n(t)||g(t)|\,d\mu(t) + M\mathcal{E}(\nu_r)\|g\|_{H^1}.\end{aligned}$$

Using (2.12) and the fact that $\{f_n\} \to 0$, uniformly on compact subsets of $\mathbb{D}$, it follows that

$$\lim_{n\to\infty} \int_{[0,1)} |f_n(t)||g(t)|\,d\mu(t) = 0, \quad \text{for all } g \in H^1.$$

Bearing in mind (2.4), this yields

$$\lim_{n\to\infty}\left(\lim_{r\to 1}\left|\int_0^{2\pi} I_\mu(f_n)(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta\right|\right) = 0, \quad \text{for all } g \in H^1.$$

By the duality relation $(H^1)^\star = \mathrm{BMOA}$, this is equivalent to saying that $I_\mu(f_n) \to 0$ in BMOA. $\square$

3. The operator $\mathcal{H}_\mu$ acting on Q_s-spaces and Besov spaces

If $0 \le s < \infty$, then we say that $f \in Q_s$ if f is analytic in $\mathbb{D}$ and

$$\|f\|_{Q_s} \stackrel{\text{def}}{=} \left(|f(0)|^2 + \rho_{Q_s}(f)^2\right)^{1/2} < \infty,$$

where

$$\rho_{Q_s}(f) \stackrel{\text{def}}{=} \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g(z,a)^s\,dA(z)\right)^{1/2}.$$

Here, $g(z,a)$ is the Green's function in $\mathbb{D}$, given by $g(z,a) = \log\left|\frac{1-\bar{a}z}{z-a}\right|$, while $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized area measure on $\mathbb{D}$. All Q_s-spaces ($0 \le s < \infty$) are conformally invariant with respect to the seminorm ρ_{Q_s} (see, e.g., [31, p. 1] or [15, p. 47]).

These spaces were introduced by Aulaskari and Lappan in [6] while looking for new characterizations of Bloch functions. They proved that for $s > 1$, Q_s is the Bloch space. Using one of the many characterizations of the space BMOA (see, e.g., [9, Theorem 5] or [21, Theorem 6.2]), we see that $Q_1 = \mathrm{BMOA}$. In the limit case $s = 0$, Q_s is the classical Dirichlet space $\mathcal{D}$ of those analytic functions f in $\mathbb{D}$ satisfying $\int_{\mathbb{D}} |f'(z)|^2\,dA(z) < \infty$.

It is well known that $\mathcal{D} \subset \mathrm{VMOA}$. Aulaskari, Xiao, and Zhao [8] proved that

$$\mathcal{D} \subsetneq Q_{s_1} \subsetneq Q_{s_2} \subsetneq \mathrm{BMOA}, \quad 0 < s_1 < s_2 < 1.$$

We mention the book [31] as an excellent reference for the theory of Q_s-spaces.

It is well known that the functions $F(z) = \log \frac{2}{1-z}$ belong to Q_s, for all $s > 0$ (in fact, it is proved in [7] that the univalent functions in all Q_s-spaces ($0 < s < \infty$) are the same). Using this, we can easily see that Theorem 2.1 and Theorem 2.4 can be improved as follows.

Theorem 3.1. *Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent.*

(i) *We have $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$.*

(ii) *For any given $s \in (0,\infty)$ and any $f \in Q_s$, the integral in (1.1) converges for all $z \in \mathbb{D}$ and the resulting function $I_\mu(f)$ is analytic in $\mathbb{D}$.*

We remark that condition (ii) with $s \geq 1$ includes the points (ii) and (iii) of Theorem 2.1.

Theorem 3.2. *Let μ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$. Then the following two conditions are equivalent.*

(i) *The measure ν defined by $d\nu(t) = \log \frac{2}{1-t}\, d\mu(t)$ is a Carleson measure.*

(ii) *For any given $s \in (0,\infty)$, the operator I_μ is bounded from Q_s into BMOA.*

We remark that (ii) with $s > 1$ reduces to condition (ii) of Theorem 2.2, while (ii) with $s = 1$ reduces to condition (iii) of Theorem 2.2.

These results cannot be extended to the limit case $s = 0$. Indeed, the function $F(z) = \log \frac{2}{1-z}$ does not belong to the Dirichlet space $\mathcal{D}$.

The Dirichlet space is one among the analytic Besov spaces. For $1 < p < \infty$, the *analytic Besov space* B^p is defined as the set of all functions f analytic in $\mathbb{D}$ such that

$$\|f\|_{B^p} \stackrel{\text{def}}{=} \left(\left|f(0)\right|^p + \rho_p(f)^p\right)^{1/p} < \infty,$$

where

$$\rho_p(f) = \left(\int_{\mathbb{D}} \left(1-|z|^2\right)^{p-2} \left|f'(z)\right|^p \, dA(z)\right)^{1/p}.$$

All B^p-spaces ($1 < p < \infty$) are conformally invariant with respect to the seminorm ρ_p (see [5, p. 112] or [15, p. 46]). We have that $\mathcal{D} = B^2$. (A lot of information on Besov spaces can be found in [5], [15], [23], [33], [34].) Let us recall that

$$B^p \subsetneq B^q \subsetneq \mathrm{VMOA}, \quad 1 < p < q < \infty.$$

From now on, if $1 < p < \infty$, we let p' denote the exponent conjugate to p, that is, p' is defined by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \in B^p$ ($1 < p < \infty$), then (see [23] or [33])

$$\left|f(z)\right| = \mathrm{o}\left(\left(\log \frac{1}{1-|z|}\right)^{1/p'}\right), \quad \text{as } |z| \to 1, \tag{3.1}$$

and there exists a positive constant $C > 0$ such that

$$\big|f(z)\big| \le C\|f\|_{B^p}\left(\log\frac{2}{1-|z|}\right)^{1/p'}, \quad z \in \mathbb{D}, f \in B^p. \tag{3.2}$$

Clearly, (3.1) or (3.2) imply that the function $F(z) = \log\frac{2}{1-z}$ does not belong to B^p ($1 < p < \infty$), a fact that we have already mentioned for $p = 2$. Our substitutes of Theorem 2.1 and Theorem 2.2 for Besov spaces are the following.

Theorem 3.3. *Let $1 < p < \infty$, and let μ be a positive Borel measure on $[0, 1)$. We have the following.*

(i) *If $\int_{[0,1)}(\log\frac{2}{1-t})^{1/p'}\,d\mu(t) < \infty$, then for any given $f \in B^p$, the integral in (1.1) converges for all $z \in \mathbb{D}$ and the resulting function $I_\mu(f)$ is analytic in $\mathbb{D}$.*

(ii) *If for any given $f \in B^p$, the integral in (1.1) converges for all $z \in \mathbb{D}$ and the resulting function $I_\mu(f)$ is analytic in $\mathbb{D}$, then $\int_{[0,1)}(\log\frac{2}{1-t})^{\gamma}\,d\mu(t) < \infty$ for all $\gamma < \frac{1}{p'}$.*

Theorem 3.4. *Suppose that $1 < p < \infty$, and let μ be a positive Borel measure on $[0, 1)$. Let ν be the measure defined by*

$$d\nu(t) = \left(\log\frac{2}{1-t}\right)^{1/p'} d\mu(t).$$

(i) *If ν is a Carleson measure, then the operator I_μ is bounded from B^p into BMOA.*

(ii) *If ν is a vanishing Carleson measure, then the operator I_μ is compact from B^p into BMOA.*

These results follow using the growth condition (3.2), the fact that if $\gamma < \frac{1}{p'}$, then the function $f(z) = (\log\frac{2}{1-z})^{\gamma}$ belongs to B^p (see [23, Theorem 1]), and with arguments similar to those used in the proofs of Theorem 2.1, Theorem 2.2, and Theorem 2.8. We omit the details.

Let us work next with the operator $\mathcal{H}_\mu$ directly. In order to study its action on the Besov spaces, we need some results on the Taylor coefficients of functions in B^p. The following result was proved by Holland and Walsh in [23, Theorem 2].

Theorem B.

(i) *Suppose that $1 < p \le 2$. Then there exists a positive constant C_p such that if $f \in B^p$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ ($z \in \mathbb{D}$), then*

$$\sum_{k=1}^{\infty} k^{p-1}|a_k|^p \le C_p\rho_p(f)^p.$$

(ii) *If* $2 \le p < \infty$, *then there exists* $C_p > 0$ *such that if* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$ *with* $\sum_{k=1}^{\infty} k^{p-1}|a_k|^p < \infty$, *then* $f \in B^p$ *and*

$$\rho_p(f)^p \le C_p \sum_{k=1}^{\infty} k^{p-1}|a_k|^p.$$

If $p \neq 2$, *the converses to (i) and (ii) are false.*

Theorem B is the analogue for Besov spaces of results of Hardy and Littlewood for Hardy spaces (Theorems 6.2 and 6.3 of [18]). In spite of the fact that the converse to (ii) is not true, the membership of f in B^p $(p > 2)$ implies some summability conditions on the Taylor coefficients $\{a_k\}$ of f. Indeed, Pavlović has proved the following result in [28, Theorem 2.3].

Theorem C. *Suppose that* $2 < p < \infty$. *Then there exists a positive constant* C_p *such that if* $f \in B^p$ *and* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$, *then*

$$\sum_{k=1}^{\infty} k|a_k|^p \le C_p \rho_p(f)^p.$$

These results allow us to obtain conditions on μ which are sufficient to ensure that $\mathcal{H}_\mu$ is well defined on the Besov spaces.

Theorem 3.5. *Let* μ *be a finite positive Borel measure on* $[0, 1)$.

(i) *If* $1 < p \le 2$ *and* $\sum_{k=1}^{\infty} \frac{\mu_k^{p'}}{k} < \infty$, *then the operator* $\mathcal{H}_\mu$ *is well defined in* B^p.

(ii) *If* $2 < p < \infty$ *and* $\sum_{k=1}^{\infty} \frac{\mu_k^{p'}}{k^{p'/p}} < \infty$, *then the operator* $\mathcal{H}_\mu$ *is well defined in* B^p.

Proof. Suppose that $1 < p < \infty$ and $f \in B^p$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$. Since the sequence of moments $\{\mu_n\}_{n=0}^{\infty}$ is clearly decreasing, we have

$$\sum_{k=1}^{\infty} |\mu_{n+k}||a_k| \le \sum_{k=1}^{\infty} |\mu_k||a_k|, \quad \text{for all } n \ge 0.$$

Consequently, we have the following.

(i) If $1 < p \le 2$ and $f \in B^p$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$, then

$$\sum_{k=1}^{\infty} |\mu_{n+k} a_k| \le \sum_{k=1}^{\infty} |\mu_k||a_k| = \sum_{k=1}^{\infty} k^{1-\frac{1}{p}}|a_k| \frac{\mu_k}{k^{1/p'}}, \quad n \ge 0.$$

Then using Hölder's inequality and Theorem B(i), we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} |\mu_{n+k} a_k| &\le \Big(\sum_{k=1}^{\infty} k^{p-1}|a_k|^p\Big)^{1/p} \Big(\sum_{k=1}^{\infty} \frac{|\mu_k|^{p'}}{k}\Big)^{1/p'} \\ &\le C\rho_p(f) \Big(\sum_{k=1}^{\infty} \frac{|\mu_k|^{p'}}{k}\Big)^{1/p'}, \quad n \ge 0.\end{aligned}$$

Then it is clear that the condition $\sum_{k=1}^{\infty} \frac{|\mu_k|^{p'}}{k} < \infty$ implies that the power series appearing in the definition of $\mathcal{H}_\mu(f)$ defines an analytic function in $\mathbb{D}$.

(ii) If $2 < p < \infty$ and $f \in B^p$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$, then

$$\sum_{k=1}^{\infty} |\mu_{n+k} a_k| \leq \sum_{k=1}^{\infty} |\mu_k||a_k| = \sum_{k=1}^{\infty} k^{\frac{1}{p}} |a_k| \frac{\mu_k}{k^{1/p}}, \quad n \geq 0.$$

Then using Hölder's inequality and Theorem B(ii), we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} |\mu_{n+k} a_k| &\leq \Big(\sum_{k=1}^{\infty} k|a_k|^p\Big)^{1/p} \Big(\sum_{k=1}^{\infty} \frac{|\mu_k|^{p'}}{k^{p'/p}}\Big)^{1/p'} \\ &\leq C\rho_p(f) \Big(\sum_{k=1}^{\infty} \frac{|\mu_k|^{p'}}{k^{p'/p}}\Big)^{1/p'}, \quad n \geq 0. \end{aligned}$$

Then we see that the condition $\sum_{k=1}^{\infty} \frac{|\mu_k|^{p'}}{k^{p'/p}} < \infty$ implies that the power series appearing in the definition of $\mathcal{H}_\mu(f)$ defines an analytic function in $\mathbb{D}$. □

Let us turn our attention to study when the operator $\mathcal{H}_\mu$ is bounded from B^p into itself. We mention that Bao and Wulan [10] considered an operator which is closely related to the operator $\mathcal{H}_\mu$ acting on the Dirichlet spaces $\mathcal{D}_\alpha$ $(\alpha \in \mathbb{R})$ which are defined as follows. For $\alpha \in \mathbb{R}$, the space $\mathcal{D}_\alpha$ consists of those functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in $\mathbb{D}$ for which

$$\|f\|_{\mathcal{D}_\alpha} \overset{\text{def}}{=} \Big(\sum_{n=0}^{\infty} (n+1)^{1-\alpha} |a_n|^2\Big)^{1/2} < \infty.$$

Let us remark that $\mathcal{D}_0$ is the Dirichlet space $\mathcal{D} = B^2$, while $\mathcal{D}_1 = H^2$. Bao and Wulan proved that if μ is a positive Borel measure on $[0,1)$ and $0 < \alpha < 2$, then the operator $\mathcal{H}_\mu$ is bounded from $\mathcal{D}_\alpha$ into itself if and only if μ is a Carleson measure. Let us remark that this does not include the case $\alpha = 0$. In fact, the following results are proved in [10].

Theorem D.

(i) *There exists a positive Borel measure μ on $[0,1)$ which is a Carleson measure but such that $\mathcal{H}_\mu(B^2) \not\subset B^2$.*

(ii) *Let μ be a positive Borel measure on $[0,1)$ such that the operator $\mathcal{H}_\mu$ is a bounded operator from B^2 into itself. Then μ is a Carleson measure.*

We can improve these results and, even more, we will obtain extensions of these improvements to all B^p-spaces $(1 < p < \infty)$. More precisely, we are going to prove the following results.

Theorem 3.6. *Suppose that $1 < p < \infty$ and $0 < \beta \leq \frac{1}{p}$. Then there exists a positive Borel measure μ on $[0,1)$ which is a β-logarithmic 1-Carleson measure but such that the operator $\mathcal{H}_\mu$ does not apply B^p into itself.*

Next we prove that μ being a β-logarithmic 1-Carleson measure for a certain β is a necessary condition for $\mathcal{H}_\mu$ being a bounded operator from B^p into itself.

Theorem 3.7. *Suppose that $1 < p < \infty$, and let μ be a positive Borel measure on $[0,1)$ such that the operator $\mathcal{H}_\mu$ is bounded from B^p into itself. Then μ is a γ-logarithmic 1-Carleson measure for any $\gamma < 1 - \frac{1}{p}$.*

Finally, we obtain a sufficient condition for the boundedness of $\mathcal{H}_\mu$ from B^p into itself.

Theorem 3.8. *Suppose that $1 < p < \infty$, $\gamma > 1$, and let μ be a positive Borel measure on $[0,1)$ which is a γ-logarithmic 1-Carleson measure. Then the operator $\mathcal{H}_\mu$ is a bounded operator from B^p into itself.*

We will need a number of results on Besov spaces, as well as some lemmas, to prove these last three theorems. First of all, we note that the Besov spaces can be characterized in terms of "dyadic blocks." In order to state this in a precise way, we need to introduce some notation. For a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in $\mathbb{D}$, define the polynomials $\Delta_j f$ as follows:

$$\begin{aligned} \Delta_j f(z) &= \sum_{k=2^j}^{2^{j+1}-1} a_k z^k, \quad \text{for } j \geq 1, \\ \Delta_0 f(z) &= a_0 + a_1 z. \end{aligned}$$

Mateljević and Pavlović [25, Theorem 2.1] (see also [27, Theorem C]) proved the following result.

Theorem E. *Let $1 < p < \infty$ and $\alpha > -1$. For a function f analytic in $\mathbb{D}$, we define*

$$Q_1(f) \stackrel{\text{def}}{=} \int_{\mathbb{D}} \left|f(z)\right|^p \left(1 - |z|\right)^{\alpha} dA(z), \qquad Q_2(f) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \|\Delta_n f\|_{H^p}^p.$$

Then $Q_1(f) \asymp Q_2(f)$.

Theorem E readily implies the following result.

Corollary 3.9. *Suppose that $1 < p < \infty$ and that f is an analytic function in $\mathbb{D}$. Then*

$$f \in B^p \qquad \Leftrightarrow \qquad \sum_{n=0}^{\infty} 2^{-n(p-1)} \|\Delta_n f'\|_{H^p}^p < \infty.$$

Furthermore,

$$\rho_p(f)^p \asymp \sum_{n=0}^{\infty} 2^{-n(p-1)} \|\Delta_n f'\|_{H^p}^p.$$

Using Corollary 3.9, we can prove that the converses of (i) and (ii) in Theorem B hold if the sequence of Taylor coefficients $\{a_n\}$ decreases to zero. This is the analogue for Besov spaces of the result proved in [22, Theorem 5] by Hardy and Littlewood for Hardy spaces (see also [27], [26, 7.5.9, p. 121], and [35, Chapter XII, Lemma 6.6]).

Theorem 3.10. *Suppose that $1 < p < \infty$, and let $\{a_n\}_{n=0}^{\infty}$ be a decreasing sequence of nonnegative numbers with $\{a_n\} \to 0$, as $n \to \infty$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. Then*

$$f \in B^p \qquad \Leftrightarrow \qquad \sum_{n=1}^{\infty} n^{p-1} a_n^p < \infty.$$

Furthermore, $\rho_p(f)^p \asymp \sum_{n=1}^{\infty} n^{p-1} a_n^p$.

Proof. For every n, we have

$$z(\Delta_n f')(z) = \sum_{k=2^n+1}^{2^{n+1}} k a_k z^k.$$

Since the sequence $\lambda = \{k\}_{k=0}^{\infty}$ is an increasing sequence of nonnegative numbers, using Lemma A of [27] we see that

$$\left\| z(\Delta_n f') \right\|_{H^p}^p \asymp 2^{np} \|\Delta_n f\|_{H^p}^p. \tag{3.3}$$

Now, set $h(z) = \sum_{n=0}^{\infty} z^n$ $(z \in \mathbb{D})$. Since the sequence $\tilde{\lambda} = \{a_n\}_{n=0}^{\infty}$ is a decreasing sequence of nonnegative numbers, using the second part of Lemma A of [27], we see that

$$a_{2^n}^p \|\Delta_n h\|_{H^p}^p \lesssim \|\Delta_n f\|_{H^p}^p \lesssim a_{2^{n-1}}^p \|\Delta_n h\|_{H^p}^p. \tag{3.4}$$

Note that $h(z) = \frac{1}{1-z}$ $(z \in \mathbb{D})$. Then it is well known that $M_p(r,h) \asymp (1-r)^{\frac{1}{p}-1}$ (recall that $1 < p < \infty$). Following the notation of [25], this can be written as $h \in H(p, \infty, 1 - \frac{1}{p})$. Then using Theorem 2.1 of [25] (see also [26, p. 120]), we deduce that $\|\Delta_n\|_{H^p}^p \asymp 2^{n(p-1)}$. Using this and (3.4), it follows that

$$2^{n(p-1)} a_{2^n}^p \lesssim \|\Delta_n f\|_{H^p}^p \lesssim 2^{n(p-1)} a_{2^{n-1}}^p. \tag{3.5}$$

Using Corollary 3.9, (3.3), and (3.5), we see that

$$\rho_p(f)^p \asymp \sum_{n=0}^{\infty} 2^{-n(p-1)} \|z\Delta_n f'\|_{H^p}^p \asymp \sum_{n=0}^{\infty} 2^n \|\Delta_n f\|_{H^p} \asymp \sum_{n=0}^{\infty} 2^{np} a_{2^n}^p.$$

Now, the fact that $\{a_n\}$ is decreasing implies that $\sum_{n=0}^{\infty} 2^{np} a_{2^n}^p \asymp \sum_{n=1}^{\infty} n^{p-1} a_n^p$ and, then it follows that $\rho_p(f)^p \asymp \sum_{n=1}^{\infty} n^{p-1} a_n^p$. ☐

Remark 3.11. If f is an analytic function in $\mathbb{D}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$, and $1 < p < \infty$, then any of the two conditions $f \in B^p$ and $\sum_{n=1}^{\infty} n^{p-1} |a_n|^p < \infty$ implies that $\{a_n\} \to 0$. Consequently, the condition $\{a_n\} \to 0$ can be omitted in the hypotheses of Theorem 3.10.

Suppose that $\beta \geq 0$, $s \geq 1$, $1 < p < \infty$, and that μ is a positive Borel measure on $[0, 1)$ which is a β-logarithmic s-Carleson measure. Using Lemma 2.7 and Theorem 3.5, it follows that $\mathcal{H}_\mu$ is well defined on B^p. Also, it is easy to see that $\int_{[0,1)} (\log \frac{2}{1-t})^{1/p'} \, d\mu(t) < \infty$, a fact that, using Theorem 3.3(i), shows that I_μ is also well defined in B^p. Using standard arguments, it then follows that I_μ and $\mathcal{H}_\mu$ coincide in B^p. Let us state this as a lemma.

Lemma 3.12. *Suppose that $\beta \geq 0$, $s \geq 1$, $1 < p < \infty$, and that μ is a positive Borel measure on $[0,1)$ which is a β-logarithmic s-Carleson measure. Then the operators $\mathcal{H}_\mu$ and I_μ are well defined in B^p and $\mathcal{H}_\mu(f) = I_\mu(f)$, for all $f \in B^p$.*

Proof of Theorem 3.6. Let μ be the Borel measure on $[0,1)$ defined by

$$d\mu(t) = \left(\log \frac{2}{1-t}\right)^{-\beta} dt.$$

Since the function $x \mapsto (\log \frac{2}{1-x})^{-\beta}$ is decreasing in $[0,1)$, we have

$$\mu\big([t,1)\big) = \int_t^1 \left(\log \frac{2}{1-x}\right)^{-\beta} dx \leq (1-t)\left(\log \frac{2}{1-t}\right)^{-\beta}, \quad 0 \leq t < 1.$$

Hence, μ is a β-logarithmic 1-Carleson measure. Then, taking $\alpha = 0$ in Lemma 2.7, we see that

$$\mu_k = \mathrm{O}\left(\frac{1}{k(\log k)^\beta}\right).$$

On the other hand,

$$\mu_k \geq \int_0^{1-\frac{1}{k}} t^k \left(\log \frac{2}{1-t}\right)^{-\beta} dt \gtrsim \frac{1}{(\log k)^\beta} \int_0^{1-\frac{1}{k}} t^k \, dt \gtrsim \frac{1}{k(\log k)^\beta}.$$

Thus, we have seen that μ is a β-logarithmic 1-Carleson measure which satisfies

$$\mu_n \asymp \frac{1}{n(\log n)^\beta}. \tag{3.6}$$

Take $p \in (1,\infty)$ and $\alpha > \frac{1}{p}$, and set

$$a_n = \frac{1}{(n+1)(\log(n+2))^\alpha}, \quad n = 0,1,2,\dots,$$

and

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$

Note that $\{a_n\} \downarrow 0$ and that $\sum_{n=0}^{\infty} n^{p-1}|a_n|^p < \infty$. Hence, $g \in B^p$.

We are now going to prove that $\mathcal{H}_\mu(g) \notin B^p$. This implies that $\mathcal{H}_\mu(B^p) \not\subset B^p$, proving the theorem. We have $\mathcal{H}_\mu(g)(z) = \sum_{n=0}^{\infty}(\sum_{k=0}^{\infty} \mu_{n+k} a_k) z^n$. Note that $a_k \geq 0$ for all k and that the sequence of moments $\{\mu_n\}$ is a decreasing sequence of nonnegative numbers. Then it follows that the sequence $\{\sum_{k=0}^{\infty} \mu_{n+k} a_k\}_{n=0}^{\infty}$ of the Taylor coefficients of $\mathcal{H}_\mu(g)$ is decreasing. Consequently, we have that

$$\mathcal{H}_\mu(g) \in B^p \qquad \Leftrightarrow \qquad \sum_{n=1}^{\infty} n^{p-1} \left|\sum_{k=0}^{\infty} \mu_{n+k} a_k\right|^p < \infty. \tag{3.7}$$

Using the definition of the sequence $\{a_k\}$, (3.6), and the simple inequalities $\frac{k}{n+k} \geq \frac{1}{n+1}$ and $\log(n+k) \leq (\log n)(\log k)$ which hold whenever $k, n \geq 10$, say, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} n^{p-1}\Big|\sum_{k=0}^{\infty} \mu_{n+k}a_k\Big|^p &\geq \sum_{n=10}^{\infty} n^{p-1}\Big|\sum_{k=10}^{\infty} \mu_{n+k}a_k\Big|^p \\ &\gtrsim \sum_{n=10}^{\infty} n^{p-1}\Big(\sum_{k=10}^{\infty}\Big[\frac{1}{(n+k)(\log(n+k))^{\beta}}\frac{1}{k(\log k)^{\alpha}}\Big]\Big)^p \\ &\gtrsim \sum_{n=10}^{\infty} \frac{1}{n(\log n)^{p\beta}}\Big(\sum_{k=10}^{\infty}\frac{1}{k^2(\log k)^{\alpha+\beta}}\Big)^p = \infty.\end{aligned}$$

Bearing in mind (3.7), this implies that $H_\mu(g) \notin B^p$, as desired. □

Proof of Theorem 3.7. Suppose that $1 < p < \infty$ and $\gamma < 1 - \frac{1}{p}$. Let μ be a positive Borel measure on $[0, 1)$ such that the operator $\mathcal{H}_\mu$ is a bounded operator from B^p into itself. Set $\alpha = 1 - \gamma$,

$$a_k = \frac{1}{k(\log k)^{\alpha}}, \quad k \geq 2,$$

and

$$f(z) = \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{D}.$$

Since $\alpha > \frac{1}{p}$, using Theorem 3.10 we see that $f \in B^p$. By our assumption, $H_\mu(f) \in B^P$, that is, $\|\mathcal{H}_\mu(f)\|_{B^p} < \infty$. We have

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\Big(\sum_{k=2}^{\infty} \mu_{n+k}a_k\Big)z^n.$$

Since $a_k \geq 0$ for all k and $\{\mu_n\}$ is a decreasing sequence of nonnegative numbers, it follows that the sequence $\{\sum_{k=2}^{\infty} \mu_{n+k}a_k\}_{n=0}^{\infty}$ is a decreasing sequence of nonnegative numbers. Then, using Theorem 3.10 we obtain

$$\begin{aligned}\|\mathcal{H}_\mu(f)\|_{B^p}^p &\gtrsim \sum_{n=1}^{\infty} n^{p-1}\Big(\sum_{k=2}^{\infty} \mu_{n+k}a_k\Big)^p \\ &\gtrsim \sum_{n=1}^{\infty} n^{p-1}\Big(\sum_{k=2}^{\infty} \frac{1}{k(\log k)^{\alpha}}\int_{[0,1)} x^{n+k}\,d\mu(x)\Big)^p \\ &\geq \sum_{n=1}^{\infty} n^{p-1}\Big(\sum_{k=2}^{\infty} \frac{1}{k(\log k)^{\alpha}}\int_{[t,1)} x^{n+k}\,d\mu(x)\Big)^p \\ &\geq \sum_{n=1}^{\infty} n^{p-1}\Big(\sum_{k=2}^{\infty} \frac{t^{n+k}}{k(\log k)^{\alpha}}\Big)^p \mu\big([t,1)\big)^p \\ &= \sum_{n=1}^{\infty} n^{p-1}t^{np}\Big(\sum_{k=2}^{\infty} \frac{t^{k}}{k(\log k)^{\alpha}}\Big)^p \mu\big([t,1)\big)^p, \quad \text{for all } t \in (0,1).\end{aligned}$$

Now, it is well known that $\sum_{k=2}^{\infty} \frac{t^k}{k(\log k)^\alpha} \asymp (\log \frac{2}{1-t})^{1-\alpha} = (\log \frac{2}{1-t})^{\gamma}$ (see [35, Volume I, p. 192]). Then it follows that

$$\begin{aligned}\big\|\mathcal{H}_\mu(f)\big\|_{B^p}^p &\gtrsim \Big(\log \frac{2}{1-t}\Big)^{\gamma p} \Big(\sum_{n=1}^{\infty} n^{p-1} t^{np}\Big) \mu\big([t,1)\big)^p \\ &\asymp \Big(\log \frac{2}{1-t}\Big)^{\gamma p} \frac{1}{(1-t)^p} \mu\big([t,1)\big)^p.\end{aligned}$$

Since $\|\mathcal{H}_\mu(f)\|_{B^p} < \infty$, this shows that μ is a γ-logarithmic 1-Carleson measure. $\square$

The following lemma will be used to prove Theorem 3.8. It is an adaptation of [19, Lemma 7] to our setting. The proof is very similar to that of the latter, but we include it for the sake of completeness.

Lemma 3.13. *Let p, γ, and μ be as in Theorem 3.8. Then, there exists a constant $C = C(p,\gamma,\mu) > 0$ such that if $f \in B^p$, $g(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathcal{H}ol(\mathbb{D})$, and we set*

$$h(z) = \sum_{k=0}^{\infty} c_k \Big(\int_0^1 t^{k+1} f(t)\, d\mu(t)\Big) z^k,$$

then

$$\|\Delta_n h\|_{H^p} \le C\Big(\int_0^1 t^{2^{n-2}+1} \big|f(t)\big|\, d\mu(t)\Big) \|\Delta_n g\|_{H^p}, \quad n \ge 3.$$

Proof. For each $n = 1, 2, \dots,$ define

$$\Upsilon_n(s) = \int_0^1 t^{2^n s+1} f(t)\, d\mu(t), \quad s \ge 0.$$

Clearly, Υ_n is a $C^\infty(0,\infty)$-function and

$$\big|\Upsilon_n(s)\big| \le \int_0^1 t^{2^{n-2}+1} \big|f(t)\big|\, d\mu(t), \quad s \ge \frac{1}{2}. \tag{3.8}$$

Furthermore, since $\sup_{0<x<1} (\log \frac{1}{x})^2 x^{1/2} = C(2) < \infty$, we have

$$\begin{aligned}\big|\Upsilon_n''(s)\big| &\le \int_0^1 \Big[\Big(\log \frac{1}{t^{2^n}}\Big)^2 t^{2^{n-1}}\Big] t^{2^n s+1-2^{n-1}} \big|f(t)\big|\, d\mu(t) \\ &\le C(2) \int_0^1 t^{2^n s+1-2^{n-1}} \big|f(t)\big|\, d\mu(t) \\ &\le C(2) \int_0^1 t^{2^{n-2}+1} \big|f(t)\big|\, d\mu(t), \quad s \ge \frac{3}{4}.\end{aligned} \tag{3.9}$$

Then, using (3.8) and (3.9), for each $n = 1, 2, \dots,$ we can take a function $\Phi_n \in C^\infty(\mathbb{R})$ with $\operatorname{supp}(\Phi_n) \in (\frac{3}{4}, 4)$, and such that

$$\Phi_n(s) = \Upsilon_n(s), \quad s \in [1,2],$$

and

$$A_{\Phi_n} = \max_{s\in\mathbb{R}} \big|\Phi_n(s)\big| + \max_{s\in\mathbb{R}} \big|\Phi_n''(s)\big| \le C \int_0^1 t^{2^{n-2}+1} \big|f(t)\big|\, d\mu(t).$$

Following the notation used in [19, p. 236], we can then write

$$\begin{aligned}\Delta_n h(z) &= \sum_{k=2^n}^{2^{n+1}-1} c_k\Big(\int_0^1 t^{k+1} f(t)\,d\mu(t)\Big) z^k \\ &= \sum_{k=2^n}^{2^{n+1}-1} c_k \Phi_n\Big(\frac{k}{2^n}\Big) z^k = W_{2^n}^{\Phi_n} * \Delta_n g(z).\end{aligned}$$

So by using Theorem B(iii) of [19], we have

$$\begin{aligned}\|\Delta_n h\|_{H^p} &= \|W_{2^n}^{\Phi_n} * \Delta_n g\|_{H^p} \le C_p A_{\Phi_n} \|\Delta_n g\|_{H^p} \\ &\le C\Big(\int_0^1 t^{2^{n-2}+1} \big|f(t)\big|\,d\mu(t)\Big) \|\Delta_n g\|_{H^p}.\end{aligned}$$ □

Proof of Theorem 3.8. By the closed graph theorem it suffices to show that $\mathcal{H}_\mu(B^p) \subset B^p$. Take $f \in B^p$. Since μ is a γ-logarithmic 1-Carleson measure, using Lemma 3.12, we see that

$$\mathcal{H}_\mu(f)(z) = I_\mu(f)(z) = \sum_{n=0}^{\infty}\Big(\int_{[0,1)} t^n f(t)\,d\mu(t)\Big) z^n, \quad z \in \mathbb{D}.$$

Also, using Corollary 3.9, we see that

$$\mathcal{H}_\mu(f) \in B^p \qquad \Leftrightarrow \qquad \sum_{n=1}^{\infty} 2^{-n(p-1)} \big\|\Delta_n\big(\mathcal{H}_\mu(f)'\big)\big\|_{H^p}^p < \infty. \tag{3.10}$$

Now, we have

$$\Delta_n\big(\mathcal{H}_\mu(f)'\big)(z) = \sum_{k=2^n}^{2^{n+1}-1} (k+1)\Big(\int_{[0,1)} t^{k+1} f(t)\,d\mu(t)\Big) z^k.$$

Using Lemma 3.13, we obtain that

$$\big\|\Delta_n\big(\mathcal{H}_\mu(f)'\big)\big\|_{H^p} \lesssim \Big(\int_{[0,1)} t^{2^{n-2}+1} \big|f(t)\big|\,d\mu(t)\Big) \|\Delta_n F\|_{H^p}$$

with $F(z) = \sum_{k=0}^{\infty}(k+1) z^k$ ($z \in \mathbb{D}$). Now, we have that $M_p(r,F) = \mathrm{O}\big(\frac{1}{(1-r)^{2-\frac{1}{p}}}\big)$, and then it follows that $\|\Delta_n F\|_{H^p} = \mathrm{O}(2^{n(2-\frac{1}{p})})$ (see, e.g., [25]). Using this and the estimate $|f(t)| \lesssim (\log\frac{2}{1-t})^{1/p'}$, we obtain

$$\big\|\Delta_n\big(\mathcal{H}_\mu(f)'\big)\big\|_{H^p} \lesssim 2^{n(2-\frac{1}{p})}\Big(\int_{[0,1)} t^{2^{n-2}+1}\Big(\log\frac{2}{1-t}\Big)^{1/p'} d\mu(t)\Big),$$

which, using the fact that μ is a γ-logarithmic 1-Carleson measure and Lemma 2.7, implies that

$$\big\|\Delta_n\big(\mathcal{H}_\mu(f)'\big)\big\|_{H^p} \lesssim 2^{n(2-\frac{1}{p})} 2^{-n} n^{\frac{1}{p'}-\gamma} = 2^{n/p'} n^{\frac{1}{p'}-\gamma}.$$

This, together with the fact that $\gamma > 1$, implies that

$$\sum_{n=1}^{\infty} 2^{-n(p-1)} \big\|\Delta_n\big(\mathcal{H}_\mu(f)'\big)\big\|_{H^p}^p \lesssim \sum_{n=1}^{\infty} 2^{-n(p-1)} 2^{np/p'} n^{p(1-\gamma)-1} = \sum_{n=1}^{\infty} n^{p(1-\gamma)-1} < \infty.$$

Bearing in mind (3.10), this shows that $\mathcal{H}_\mu(f) \in B^p$ and finishes the proof. □

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Análisis Matemático, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain.
E-mail address: girela@uma.es; noel@uma.es