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TERNARY WEAK AMENABILITY OF THE BIDUAL OF A JB*-TRIPLE

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Abstract. Beside the triple product induced by ultrapowers on the bidual of a JB*-triple, we assign a triple product to the bidual, E^{**}, of a JB-triple system E, and we show that, under some mild conditions, it makes E^{**} a JB-triple system. To study ternary n-weak amenability of E^{**}, we need to improve the module structures in the category of JB-triple systems and their iterated duals, which lead us to introduce a new type of ternary module. We then focus on the main question: when does ternary n-weak amenability of E^{**} imply the same property for E? In this respect, we show that if the bidual of a JB*-triple E is ternary n-weakly amenable, then E is ternary n-quasiweakly amenable. However, for a general JB-triple system, the results are slightly different for $n = 1$ and $n \geq 2$, and the case $n = 1$ requires some additional assumptions.

1. Introduction

Ternary weak amenability of JB*-triples has been recently introduced and extensively studied by Ho, Peralta, and Russo [13]. In contrast to the result for C*-algebras, where it is known that every C*-algebra is weakly amenable (see [11, Theorem 1.10]), it has been shown that a general C*-algebra is not necessarily ternary weakly amenable. However, the situation is different in the commutative case, where the same authors have proved that every commutative C^*-algebra is ternary weakly amenable (see [13, Proposition 3.11]).

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In the realm of derivations on Banach algebras, there is a natural question concerning the relationship between weak amenability of iterated duals of a Banach algebra, which leads to another question on lifting derivations from one algebra to its bidual (see [3] and some of the references therein, such as [6], [7], [10], [19], [23]). It is natural to raise the same question in the context of ternary derivations on a Jordan Banach-triple system ("JB-triple system" for short). To address this issue, we need to explore conditions on JB-triple systems and ternary derivations under which the second transpose (or some appropriate extensions) of a ternary derivation is again a ternary derivation. We also need to lift the triple product of JB-triple systems and module actions of Banach ternary modules to their bidual spaces.

Concerning the bidual of a JB-triple system, a result of Dineen [9, Theorem 1], [8, Corollary 11] shows that the bidual of a JB*-triple is again a JB*-triple. However, Dineen's approach relies more on the general properties of Banach spaces and ultrapowers but does not involve a detailed computation of the triple product. Besides his approach, we give an alternative construction by mimicking Arens's method [1] and by using the Aron–Berner extensions [2]. The new construction produces six generally different triple products on the bidual, E^{**}, of a JB-triple system E. However, in this paper we deal with only one of these triple products, say $\pi^{*\bar{*}\bar{*}*}$, on E^{**}, where π denotes the triple product of E. Contrary to what happens for Arens products, E^{**} is not generally a JB-triple system under this triple product. The main problem is that it is not symmetric in the outer variables. In this paper, we additionally assume that this triple product is symmetric in the outer variables; for example, this is the case for a JB*-triple. In other words, we will study those JB-triple systems (E, π) whose bidual $(E^{**}, \pi^{*\bar{*}\bar{*}*})$ is again a JB-triple system.

To present a reasonable definition of ternary n-weak amenability ($n \in \mathbb{N}$) for JB-triple systems, we improve the construction of ternary module structures and ternary derivations into iterated duals of a JB-triple system E. For this purpose, we will introduce two types of Banach ternary E-modules as well as two types of ternary derivations, adjusting the definition to even and odd duals of E, as ternary E-modules.

We say that a JB-triple system E is *ternary n-weakly amenable* ($n \in \mathbb{N}$) if every (continuous) ternary derivation from E to its nth dual, $E^{(n)}$, is inner (see [13, p. 1117] for the case $n = 1$). The main aim of the present article is to answer the following question:

Does ternary n-weak amenability of E^{**} imply ternary n-weak amenability of E?

In trying to answer this question, we are led to define ternary n-quasiweak amenability, which is somehow weaker than ternary n-weak amenability. We then explore the above question for two different situations: one for the special case of $n = 1$ and the other for $n \geq 2$. In the case $n \geq 2$, we will show that for a general JB-triple system, n-weak amenability of E^{**} implies ternary n-quasiweak amenability of E (see Theorem 4.8); however, the situation is slightly different for the case $n = 1$. In the latter case, we also present some partial positive answers under extra assumptions (see Corollaries 4.4 and 4.6).

It should be noted that, in the context of Banach algebras, the same question was treated by Barootkoob and by Ebrahimi Vishki, the third author of this note, in [3], where they answered it positively for every $n \geq 2$. However, to the best of our knowledge, no example is yet known for whether this fails if one considers the case $n = 1$ instead. More explicitly, the question for the case $n = 1$ is a long-standing open problem in the theory of weak amenability of Banach algebras, which was first raised in [10].

Our article is organized as follows. In the first part of Section 2, we recall some definitions and basic facts about JB-triple systems and JB*-triples, which will be applied in the rest of the paper. In the second part, we describe how a bounded tri(conjugate)linear map on Banach spaces can be extended to the bidual spaces. We devote the last part of this section to the bidual of a JB-triple system. (We equip the bidual E^{**} of a JB-triple system E with a triple product that we will use in the rest of the paper.)

In Section 3, we deal with ternary modules. For a JB-triple system E we introduce two types of ternary E-modules and study their interaction. We also present appropriate ternary module structures on the iterated duals of E, and we derive some essential facts about them which will be helpful for reaching the main results of this paper.

Motivated by two notions of ternary modules given in Section 3, we start Section 4 by introducing two types of ternary derivations. The main results of the paper are gathered in this section. Trying to prove the ternary n-weak amenability of a JB-triple system when its dual space enjoys the same property, we are led to two different situations: one for the case $n = 1$, the other for $n \geq 2$. The first part of Section 4 is devoted to deriving the results for $n = 1$. In the second part, we extend the results for $n \geq 2$. We will see that the restrictions we encounter for the case $n = 1$ will be relaxed for $n \geq 2$. Some questions are left undecided in Section 5.

2. Jordan triples and their biduals

2.1. **JB-triple systems and JB*-triples.** Let E be a complex (respectively, real) vector space. A complex (respectively, real) *triple product* on E is a mapping

$$\pi : E \times E \times E \to E, \qquad \pi(a,b,c) = [a,b,c], \quad (a,b,c \in E),$$

which is bilinear and symmetric in the outer variables and conjugate-linear (respectively, linear) in the middle variable, satisfying the so-called *Jordan identity*:

$$\begin{aligned}\big[a,b,[c,d,e]\big] &= \big[[a,b,c],d,e\big] - \big[c,[b,a,d],e\big] \\ &\quad + \big[c,d,[a,b,e]\big], \quad (a,b,c,d,e \in E). \qquad (1)\end{aligned}$$

Then the pair (E,π) is called a *Jordan triple*.

For the sake of convenience, in the rest of the paper we will be dealing with *complex* JB-triple systems.

A *subtriple* of a Jordan triple E is a subspace F of E satisfying $[F,F,F] \subseteq F$. A *triple ideal* of E is a subspace I of E which satisfies $[E,E,I]+[E,I,E] \subseteq I$.

For a Jordan triple (E,π), when E is a Banach space and π is continuous, we say that E is a *JB-triple system*.

A *JB*-triple* is a JB-triple system E satisfying the following axioms:

(1) For any a in E, the mapping $x \mapsto [a,a,x]$ is a Hermitian operator on E with nonnegative spectrum;
(2) $\|[a,a,a]\| = \|a\|^3$ for all a in A.

For example, every C*-algebra is a JB*-triple with respect to the triple product $[a,b,c]=\frac{1}{2}(ab^*c+cb^*a)$. A JB*-algebra with Jordan product $\circ$ is a JB*-triple under the triple product $[a,b,c]=(a\circ b^*)\circ c+(c\circ b^*)\circ a-(a\circ c)\circ b^*$. A Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ is a JB*-triple when endowed with the triple product $[a,b,c]=\frac{1}{2}(\langle a,b\rangle c+\langle c,b\rangle a)$. The Banach space $B(H,K)$ of all bounded linear operators between two Hilbert spaces, H,K, is also an example of a JB*-triple with triple product $[S,T,U]=\frac{1}{2}(ST^*U+UT^*S)$.

A JB*-triple which is a dual Banach space is called a *JBW*-triple*. It is well known that the triple product of a JBW*-triple is separately weak*-continuous (see [4, Theorem 2.1], [17, Theorem 2.11]). Basic examples of JBW*-triples are given by the so-called *Carton factors* (see [14]).

We refer the readers to [22] and [5] for the basic background on JB*-triples and JB-algebras.

2.2. **Extension of tri(conjugate)linear mappings.** Let X,Y,Z, and W be normed spaces, and let $f: X\times Y\times Z\to W$ be a continuous map which is linear or conjugate-linear in each of its variables. Define the *transpose* f^* of f by

$$f^*: W^*\times X\times Y\to Z^*, \qquad \big\langle f^*(w^*,x,y),z\big\rangle = \big\langle w^*, f(x,y,z)\big\rangle$$

whenever f is linear in the third variable, and define the *conjugate transpose* $f^{\bar{*}}$ of f by

$$f^{\bar{*}}: W^*\times X\times Y\to Z^*, \qquad \big\langle f^{\bar{*}}(w^*,x,y),z\big\rangle = \overline{\big\langle w^*, f(x,y,z)\big\rangle}$$

whenever f is conjugate-linear in the third variable. It is easy to see that both maps f^* and $f^{\bar{*}}$ are w^*-w^*-continuous in the first variable. It is not hard to see that

$$f'''' = \Big(\big((f')'\big)'\Big)': X^{**}\times Y^{**}\times Z^{**}\to W^{**},$$

where $\prime$ denotes $*$ or $\bar{*}$ depending on linearity or conjugate linearity of f in the corresponding variable, is an extension of f, and the following assignments are w^*-w^*-continuous maps:

$$\begin{aligned}
&\cdot \mapsto f''''(\cdot,y^{**},z^{**}), \quad (y^{**}\in Y^{**}, z^{**}\in Z^{**}),\\
&\cdot \mapsto f''''(x,\cdot,z^{**}), \quad (x\in X, z^{**}\in Z^{**}),\\
&\cdot \mapsto f''''(x,y,\cdot), \quad (x\in X, y\in Y).
\end{aligned}$$

To explore the separate weak* continuity of f'''' in more depth, we write

$$\begin{aligned} Z_{l,1}(f) &= \big\{x^{**} \in X^{**} : \cdot \mapsto f''''(x^{**}, \cdot, z^{**}) \text{ is } w^*\text{-}w^*\text{-continuous for every} \\ &\qquad z^{**} \in Z^{**}\big\}, \\ Z_{l,2}(f) &= \big\{x^{**} \in X^{**} : \cdot \mapsto f''''(x^{**}, y, \cdot) \text{ is } w^*\text{-}w^*\text{-continuous for every } y \in Y\big\}, \end{aligned}$$

and we call $Z_l(f) = Z_{l,1}(f) \cap Z_{l,2}(f)$ the *left topological center* of f.

In this article, we will be dealing with the above-mentioned extension in various situations, especially $\pi^{*\bar{*}\bar{*}*}$ for the extension of the triple product π of a JB-triple system. However, in the most general setting, every such map $f : X \times Y \times Z \to W$ has six generally different extensions from $X^{**} \times Y^{**} \times Z^{**}$ to W^{**} (see Aron and Berner [2]; more explicit properties of these extensions are investigated in the forthcoming work [16]).

2.3. **Bidual of a JB-triple system.** A result of Dineen [9, Theorem 1] shows that the bidual of a JB*-triple is itself a JB*-triple. Barton and Timoney [4, Theorem 1.4] show that the extended triple product on the bidual of a JB*-triple is separately w^*-continuous. Thus the bidual of a JB*-triple is a JBW*-triple. Their method relies on the so-called *local reflexivity principle* (see [12, Proposition 6.6]). This is based on a finite-dimensional analysis of subspaces of the second dual of a Banach space, which implies that the bidual E^{**} of a Banach space E can be isometrically embedded via a map $J_{\mathcal{U}} : E^{**} \longrightarrow (E)_{\mathcal{U}}$ into the ultrapower $(E)_{\mathcal{U}}$ of E for a suitable ultrafilter $\mathcal{U}$ such that $J_{\mathcal{U}}(E^{**})$ is the range of a contractive projection P on $(E)_{\mathcal{U}}$, and the restriction of $J_{\mathcal{U}}$ to E is the canonical embedding of E into $(E)_{\mathcal{U}}$ (see [8, Proposition 5] and [12, Proposition 6.7]). In the case where E is a JB*-triple, since the ultrapower $(E)_{\mathcal{U}}$ carries a natural triple product, say $[\cdot,\cdot,\cdot]_{\mathcal{U}}$, given componentwise by the triple product of E, the formula

$$[x^{**}, y^{**}, z^{**}] = P\big(\big[J_{\mathcal{U}}(x^{**}), J_{\mathcal{U}}(y^{**}), J_{\mathcal{U}}(z^{**})\big]_{\mathcal{U}}\big)$$

inherits a triple product on E^{**}, making it a JBW*-triple with a separately w^*-continuous triple product (see [5, Corollary 3.3.5]). This idea was explored by Iochum and Loupias in [15], where they assign a Banach algebra product to the bidual of a Banach algebra and compare it with the so-called *Arens products* (see [1]), constructed by duality so that they are two natural, generally different, extensions of the original product.

Here, by using the duality method of Arens [1] and of Aron and Berner [2] (as described in 2.2), we consider the extension

$$\pi^{*\bar{*}\bar{*}*} : E^{**} \times E^{**} \times E^{**} \to E^{**}$$

of the triple product $\pi : E \times E \times E \to E$ for a JB-triple system E. An easy verification reveals that $\pi^{*\bar{*}\bar{*}*}$ is bilinear in the outer variables and conjugate-linear in the middle one. Further, it can be readily checked that $(E^{**}, \pi^{*\bar{*}\bar{*}*})$ is a JB-triple system if and only if $\pi^{*\bar{*}\bar{*}*}$ is symmetric in the outer variables; that is,

$$\pi^{*\bar{*}\bar{*}*}(x^{**}, y^{**}, z^{**}) = \pi^{*\bar{*}\bar{*}*}(z^{**}, y^{**}, x^{**}) \quad (\text{for all } x^{**}, y^{**}, z^{**} \text{ in } E^{**}). \qquad (2)$$

When this property holds, the JB-triple system E is called *regular*. It is easy to check that E is regular in the case where $\pi^{*\bar{*}\bar{*}*}$ is separately weak*-continuous, and it is worth noting that every JB*-triple enjoys this property. However, to the best of our knowledge, we do not know a regular JB-triple system which is not a JB*-triple. Similarly, we can extend the triple product π to a triple product $\pi^{[n]}$ on the $(2n)$-dual $E^{(2n)}$ of a JB-triple system E, where $\pi^{[n]}$ is defined inductively by the following formulae:

$$\pi^{[1]} = \pi^{*\bar{*}\bar{*}*}, \qquad \pi^{[n+1]} = \pi^{[n]*\bar{*}\bar{*}*}, \quad (n \in \mathbb{N}). \tag{3}$$

At this moment we do not know whether the bidual of a regular JB-triple system is a regular JB-triple system. It is again worth mentioning that every JB*-triple is permanently regular; that is, $(E^{(2n)}, \pi^{[n]})$ is a JB*-triple for each $n \in \mathbb{N}$, and the triple product $\pi^{[n]}$ is separately weak*-continuous.

Hereafter we will be dealing with regular JB-triple systems, and, for the sake of convenience, all JB-triple systems in this note are usually assumed to be permanently regular.

Although in the rest of this article we will be dealing with merely the triple product $\pi^{*\bar{*}\bar{*}*}$, it should be noted that, by changing the usual ordering in the involved three variables, we can obtain six generally different triple products on E^{**}. More explicitly, corresponding to each permutation on a three-elements set, we can assign a triple product to the bidual E^{**} of E. The coincidence of these triple products would be of interest in its own right.

3. Ternary modules

In [13] Ho et al. introduced the notion of a ternary module which intended to explore the weak amenability problem for Jordan triples. To explore the ternary n-weak amenability problem in the context of Jordan triples, we need to introduce another type of ternary module. We call the triple modules defined in [13] *ternary modules of type* (I), while the new ones are called *ternary modules of type* (II).

Definition 3.1. Let E be a Jordan triple, and let X be a complex vector space. We consider the following mappings and axioms:

$$\begin{aligned} \pi_1 &: X \times E \times E \to X, & \pi_1(x,a,b) &= [x,a,b]_1; \\ \pi_2 &: E \times X \times E \to X, & \pi_2(a,x,b) &= [a,x,b]_2; \\ \pi_3 &: E \times E \times X \to X, & \pi_3(a,b,x) &= [a,b,x]_3. \end{aligned}$$

(1) π_1 is linear in the first and second variables and is conjugate-linear in the third variable; π_2 is conjugate-linear in each variable; π_3 is conjugate-linear in the first variable and is linear in the second and third variables.

(1)′ Each of the mappings π_1, π_2, and π_3 is linear in the first and third variables, and is conjugate-linear in the second variable.

(2) Also $[x,b,a]_1 = [a,b,x]_3$ and $[a,x,b]_2 = [b,x,a]_2$ for every $a, b \in E$ and $x \in X$.

(3) Let $[\cdot,\cdot,\cdot]$ denote any of the mappings $[\cdot,\cdot,\cdot]_1, [\cdot,\cdot,\cdot]_2, [\cdot,\cdot,\cdot]_3$ for the triple product of E. Then the (Jordan) identity

$$\big[a,b,[c,d,e]\big] = \big[[a,b,c],d,e\big] - \big[c,[b,a,d],e\big] + \big[c,d,[a,b,e]\big] \tag{4}$$

holds for every a, b, c, d, e, where one of them is in X and the others are in E.

When the mappings π_1, π_2, and π_3 satisfy the axioms (1), (2), and (3), X is called *a ternary E-module of type* (I), and when they satisfy the axioms (1)′, (2), and (3), X is called *a ternary E-module of type* (II). We usually write (X, π_1, π_2, π_3) for a ternary E-module X with the module actions π_1, π_2, and π_3.

It is now easy to see that every (complex) Jordan triple (E, π) is a (complex) ternary E-module of type (II) when π is considered its module actions; that is, $\pi_1 = \pi_2 = \pi_3 := \pi$. This resolves what Ho et al. [13, Section 2.2] claimed: it is problematical whenever every complex Jordan triple E is a complex ternary E-module for a suitable triple product.

Note that the identity (4) of Definition 3.1 consists of five identities regarding the position of the module element. We usually write the expression “ternary E-module” without declaring the type whenever a statement is true for both types or whenever the type is clear from the context.

When E is a JB-triple system, X is a Banach space, and the module actions π_1, π_2, and π_3 are continuous, we say that X is a *Banach ternary E-module.*

A subspace Y of a ternary E-module X is called a *ternary E-submodule* if $[Y, E, E]_1 \subseteq Y$ and $[E, Y, E]_2 \subseteq Y$.

Hereafter, to simplify notation, the module actions $[\cdot,\cdot,\cdot]_1, [\cdot,\cdot,\cdot]_2, [\cdot,\cdot,\cdot]_3$ and the triple product of E will be denoted by $[\cdot,\cdot,\cdot]$, and its meaning will be clear from the context.

Applying twice identity (4) and using axiom (2) of Definition 3.1, for every a, b, c, d, e where one of them is in X and the others are in E, we have

$$\begin{aligned}\big[c, b, [a, d, e]\big] &= \big[[c, b, a], d, e\big] - \big[a, [b, c, d], e\big] + \big[a, d, [c, b, e]\big]\\ &= \underbrace{\big[e, d, [c, b, a]\big]} - \big[a, [b, c, d], e\big] + \big[a, d, [c, b, e]\big]\\ &= \Big(\big[[e, d, c], b, a\big] - \big[c, [d, e, b], a\big] + \big[c, b, [e, d, a]\big]\Big)\\ &\quad - \big[a, [b, c, d], e\big] + \big[a, d, [c, b, e]\big],\end{aligned}$$

which proves the following identity:

$$\big[a, [b, c, d], e\big] = \big[a, b, [c, d, e]\big] - \big[a, [b, e, d], c\big] + \big[a, d, [e, b, c]\big]. \tag{5}$$

We will make use of this identity in the proof of Proposition 3.2, stating that the dual space of a ternary module is again a ternary module. Before proceeding we need to introduce more notation.

For a continuous map $f : X \times Y \times Z \to W$, which is linear or conjugate-linear in each of its variables, the following three maps provide a tool to construct ternary module actions. Define

$$f^1 : W^* \times X \times Y \to Z^*, \qquad \big\langle f^1(w^*, x, y), z\big\rangle = \big\langle w^*, f(x, y, z)\big\rangle$$

whenever f is linear in the third variable,

$$f^2 : X \times W^* \times Z \to Y^*, \qquad \big\langle f^2(x, w^*, z), y\big\rangle = \overline{\big\langle w^*, f(x, y, z)\big\rangle}$$

whenever f is conjugate-linear in the second variable, and

$$f^3 : Y \times Z \times W^* \to X^*, \qquad \big\langle f^3(y,z,w^*), x \big\rangle = \big\langle w^*, f(x,y,z) \big\rangle$$

whenever f is linear in the first variable. Note that $f^1 = f^*$.

The next result shows that if X is a Banach ternary E-module of type (I) (resp., (II)), then its dual space X^* is a Banach ternary E-module of type (II) (resp., (I)). In particular, if $(X, \pi_1, \pi_2, \pi_3) = (E, \pi, \pi, \pi)$ is a JB-triple system considered an E-module of type (II), then $(E^*, \pi^1, \pi^2, \pi^3)$ coincides with the definition of E^* as an E-module given in [13, page 1114], and more precisely in the following proposition.

Proposition 3.2. *Let E be a JB-triple system. If (X, π_1, π_2, π_3) is a Banach ternary E-module of type (I) (resp., (II)), then $(X^*, \pi_3^1, \pi_2^2, \pi_1^3)$ is a Banach ternary E-module of type (II) (resp., (I)).*

Proof. Let (X, π_1, π_2, π_3) be a Banach ternary E-module of type (I). We show that π_3^1, π_2^2, and π_1^3 serve as module actions for X^*, and we define a Banach ternary E-module structure of type (II). Rewriting the actions π_1, π_2, π_3 and π_3^1, π_2^2, π_1^3 in the usual bracket notation of Definition 3.1, for every $x^* \in X^*, a, b \in E$ and $x \in X$, we have $[x^*, a, b]_1(x) = x^*([a,b,x]_3), [a, x^*, b]_2(x) = \overline{x^*([a,x,b]_2)}$, and $[a, b, x^*]_3(x) = x^*([x,a,b]_1)$.

Axiom $(1)'$ of Definition 3.1 is easily checkable. Let $x^* \in X^*$, and let $a, b \in E$. For every $x \in X$, we have

$$\begin{aligned}
[x^*, a, b]_1(x) &= x^*\big([a,b,x]_3\big) = x^*\big([x,b,a]_1\big) = [b,a,x^*]_3(x),\\
[a, x^*, b]_2(x) &= \overline{x^*\big([a,x,b]_2\big)} = \overline{x^*\big([b,x,a]_2\big)} = [b, x^*, a]_2(x).
\end{aligned}$$

This proves axiom (2) of Definition 3.1. To prove axiom (3), we just prove one of the five identities of part (3). A similar calculation can be applied for the other identities. Let $x^* \in X^*, a, b, c, d \in E$, and let $x \in X$. Identity (5) implies that

$$\begin{aligned}
&\big[x^*, a, [b,c,d]\big]_1(x)\\
&\quad = x^*\big(\big[a, [b,c,d], x\big]_3\big)\\
&\quad = x^*\big(\big[a, b, [c,d,x]_3\big]_3\big) - x^*\big(\big[a, [b,x,d]_2, c\big]_2\big) + x^*\big(\big[a, d, [x,b,c]_1\big]_3\big)\\
&\quad = [x^*, a, b]_1\big([c,d,x]_3\big) - \overline{[a, x^*, c]_2\big([b,x,d]_2\big)} + [x^*, a, d]_1\big([x,b,c]_1\big)\\
&\quad = \big[[x^*, a, b]_1, c, d\big]_1(x) - \big[b, [a, x^*, c]_2, d\big]_2(x) + \big[b, c, [x^*, a, d]_1\big]_3(x).
\end{aligned}$$

Since the above identity holds for every $x \in X$, we obtain the following desired identity:

$$\big[x^*, a, [b,c,d]\big]_1 = \big[[x^*, a, b]_1, c, d\big]_1 - \big[b, [a, x^*, c]_2, d\big]_2 + \big[b, c, [x^*, a, d]_1\big]_3.$$

A similar argument can be applied to show that X^* is a Banach ternary E-module of type (I) whenever X is a Banach ternary E-module of type (II). $\square$

As every JB-triple system (E, π) is a Banach ternary E-module of type (II), under its own triple product π as module actions, Proposition 3.2 shows that $(E^*, \pi^1, \pi^2, \pi^3)$ is a Banach ternary E-module of type (I) and that $(E^{**}, \pi^{31},$

$\pi^{22}, \pi^{13})$ is a Banach ternary E-module of type (II), and so forth. This procedure shows that the iterated dual space $E^{(n)}$ is a Banach ternary E-module of type (I) with module actions

$$\pi^{13131\cdots 31}, \pi^{222\cdots 2} \text{ and } \pi^{31313\cdots 13} \tag{6}$$

whenever the integer n is *odd* and that it is a Banach ternary E-module of type (II) with module actions

$$\pi^{3131\cdots 31}, \pi^{222\cdots 2} \text{ and } \pi^{1313\cdots 13} \tag{7}$$

whenever the integer n is *even*.

Remark 3.3. For a permanently regular JB-triple system (E, π), it is easy to check that

$$\begin{aligned} &\pi^{31} = \pi^{*\bar{*}\bar{*}*}|_{E^{**}\times E\times E}, \qquad \pi^{22} = \pi^{*\bar{*}\bar{*}*}|_{E\times E^{**}\times E} = \pi^{22} \qquad \text{and} \\ &\pi^{13} = \pi^{*\bar{*}\bar{*}*}|_{E\times E\times E^{**}} \end{aligned} \tag{8}$$

or, more generally for E-module actions of $E^{(2n)}$, that

$$\begin{aligned} &\pi^{3131\cdots 31} = \pi^{[n]}|_{E^{(2n)}\times E\times E}, \qquad \pi^{222\cdots 2} = \pi^{[n]}|_{E\times E^{(2n)}\times E} \qquad \text{and} \\ &\pi^{1313\cdots 13} = \pi^{[n]}|_{E\times E\times E^{(2n)}}, \end{aligned} \tag{9}$$

where $\pi^{[n]}$ is defined in (3). In the sequel, for the sake of simplicity, by a slight abuse of the notation we will denote any of the three E-module actions of $E^{(2n)}$ by $\pi^{[n]}$ without declaring its restriction to the corresponding spaces as given in (9). We therefore write $(E^{(2n)}, \pi^{[n]}, \pi^{[n]}, \pi^{[n]})$ for the case where $E^{(2n)}$ is considered a ternary E-module, and we write $(E^{(2n)}, \pi^{[n]})$ for $E^{(2n)}$ as a JB-triple system. We then simply have $(E^{(2n+1)}, \pi^{[n]1}, \pi^{[n]2}, \pi^{[n]3})$ for $E^{(2n+1)}$ as a ternary E-module. We write $(E^{(2n+2)}, \pi^{[n]31}, \pi^{[n]22}, \pi^{[n]13})$ for $E^{(2n+2)}$ as a ternary E-module, which, by the above convenience, enjoys the same module actions as $(E^{(2n+2)}, \pi^{[n+1]}, \pi^{[n+1]}, \pi^{[n+1]})$.

Furthermore, since $(E^{**}, \pi^{[1]})$ is a JB-triple system, it is a Banach ternary E^{**}-module of type (II) under its own triple product $\pi^{[1]}$ as module actions. It follows that $(E^{(3)}, \pi^{[1]1}, \pi^{[1]2}, \pi^{[1]3})$ and $(E^{(4)}, \pi^{[1]31}, \pi^{[1]22}, \pi^{[1]13})$ and, more generally, for every $n \in \mathbb{N}$, $E^{(2n+1)}$ and $E^{(2n+2)}$ are Banach ternary E^{**}-modules. More generally, their module actions can be obtained inductively somewhat similarly to what we have discussed above.

The following result will be used frequently in the rest of this article.

Proposition 3.4. *Let E be a permanently regular JB-triple system. For every $n \in \mathbb{N}$, $E^{(n)}$ is a Banach ternary E^{**}-submodule of $E^{(n+2)}$. In particular, E^* is a Banach ternary E^{**}-submodule of E^{***}.*

Proof. Let π denote the triple product of E. Then $\pi^{[1]}$ is the triple product of E^{**}. We will show that the restriction of the E^{**}-module actions of $E^{(n+2)}$ to $E^{(n)}$ coincides with the E^{**}-module actions of $E^{(n)}$, which proves the desired result.

By Remark 3.3 this is clear for even integers. Let $n = 2k+1$ ($k \geq 1$) be an odd integer. The E^{**}-module actions of $E^{(n+2)} = (E^{**})^{(n)}$ are

$$\pi^{[k+1]1} = (\pi^{[1]})^{[k]1}, \qquad \pi^{[k+1]2} = (\pi^{[1]})^{[k]2}, \qquad \text{and} \qquad \pi^{[k+1]3} = (\pi^{[1]})^{[k]3}.$$

We examine the action $\pi^{[1][k]1}$. The others are the same. Let $a^{**}, b^{**} \in E^{**}$, $a^{(2k+1)} \in E^{(2k+1)}$, and let $a^{(2k+2)} \in E^{(2k+2)}$. Then

$$\begin{aligned}\big\langle \pi^{[1][k]1}(a^{(2k+1)}, a^{**}, b^{**}), a^{(2k+2)} \big\rangle &= \big\langle a^{(2k+1)}, \pi^{[1][k]}(a^{**}, b^{**}, a^{(2k+2)}) \big\rangle \\ &= \big\langle \pi^{[1][k-1]13}(a^{**}, b^{**}, a^{(2k+2)}), a^{(2k+1)} \big\rangle \\ &= \big\langle a^{(2k+2)}, \pi^{[1][k-1]1}(a^{(2k+1)}, a^{**}, b^{**}) \big\rangle,\end{aligned}$$

showing that

$$\pi^{[1][k-1]1} = \pi^{[1][k]1}|_{E^{(2k+1)} \times E^{**} \times E^{**}}.$$

For the special case where $n = 1$, let $a^{**}, b^{**} \in E^{**}$, let $a^* \in E^*$, and let (e_λ^{**}) be a net in E^{**} which w^*-converges to e^{**}. Then

$$\begin{aligned}\lim_\lambda \big\langle \pi^{[1]1}(a^*, a^{**}, b^{**}), e_\lambda^{**} \big\rangle &= \lim_\lambda \big\langle a^*, \pi^{[1]}(a^{**}, b^{**}, e_\lambda^{**}) \big\rangle \\ &= \big\langle a^*, \pi^{[1]}(a^{**}, b^{**}, e^{**}) \big\rangle \\ &= \big\langle \pi^{[1]1}(a^*, a^{**}, b^{**}), e^{**} \big\rangle,\end{aligned}$$

which shows that $\pi^{[1]1}(a^*, a^{**}, b^{**})$ is a w^*-continuous element of $E^{***}$and thus an element of E^*. The same arguments show that $\pi^{[1]2}(a^{**}, a^*, b^{**})$ and $\pi^{[1]3}(a^{**}, b^{**}, a^*)$ are in E^*. $\square$

The same argument given in the proof of the preceding proposition shows that every $n \in \mathbb{Z}^+$, $E^{(n)}$ is a Banach ternary E-submodule of $E^{(n+2)}$ and that the module actions of $E^{(n+2)}$ are extensions of the module actions of $E^{(n)}$.

We close this section with the following remark concerning different ternary module actions on the third dual of a JB-triple system.

Remark 3.5. Let (E, π) be a JB-triple system. We can consider two different groups of E^{**}-actions on E^{***}. The first group is

$$\pi^{*\bar{*}\bar{*}*1}, \pi^{*\bar{*}\bar{*}*2}, \pi^{*\bar{*}\bar{*}*3}, \tag{10}$$

which is obtained by dualizing the triple product $\pi^{*\bar{*}\bar{*}*}$ of E^{**}. The second group is

$$\pi^{1\bar{*}\bar{*}**}, \pi^{2\bar{*}**\bar{*}}, \pi^{3**\bar{*}\bar{*}}, \tag{11}$$

which is obtained by extending the actions of the ternary E-module $(E^*, \pi^1, \pi^2, \pi^3)$ to its bidual $(E^*)^{**}$.

As we have seen earlier (see Remark 3.3), $E^{***} = (E^{**})^*$ with actions (10) is an E^{**}-module. At this moment, we do not know whether or not $E^{***} = (E^*)^{**}$ with actions (11) is an E^{**}-module; however, as the following result demonstrates, their restrictions to E^* (as a ternary E^{**}-submodule of E^{***}) coincide.

Proposition 3.6. *Let (E,π) be a regular JB-triple system. Then*

$$\begin{aligned} \pi^{*\bar{*}\bar{*}*1}|_{E^*\times E^{**}\times E^{**}} &= \pi^{1\bar{*}\bar{*}**}|_{E^*\times E^{**}\times E^{**}}, \\ \pi^{*\bar{*}\bar{*}*2}|_{E^{**}\times E^{*}\times E^{**}} &= \pi^{2\bar{*}**\bar{*}}|_{E^{**}\times E^{*}\times E^{**}}, \qquad \text{and} \\ \pi^{*\bar{*}\bar{*}*3}|_{E^{**}\times E^{**}\times E^{*}} &= \pi^{3**\bar{*}\bar{*}}|_{E^{**}\times E^{**}\times E^{*}}, \end{aligned}$$

and both groups of actions (10) and (11) canonically present E^* as a Banach ternary E^{**}-module.

Proof. The first identity trivially holds since $\pi^1 = \pi^*$. To prove the second identity, let (a_α) and (b_β) be nets in E^{**}, w^*-converging to $a^{**}, b^{**} \in E^{**}$, respectively. For any $a^* \in E^*, c^{**} \in E^{**}$, by regularity of (E,π), we have

$$\begin{aligned} \left\langle \pi^{*\bar{*}\bar{*}*2}(a^{**},a^*,b^{**}), c^{**}\right\rangle &= \overline{\left\langle \pi^{*\bar{*}\bar{*}*}(a^{**},c^{**},b^{**}), a^*\right\rangle} \\ &= \lim_\alpha \lim_\beta \overline{\left\langle \pi^{*\bar{*}\bar{*}*}(a_\alpha,c^{**},b_\beta), a^*\right\rangle} \\ &= \lim_\alpha \lim_\beta \left\langle \pi^{*\bar{*}\bar{*}*2}(a_\alpha,a^*,b_\beta), c^{**}\right\rangle. \end{aligned}$$

Hence

$$\pi^{*\bar{*}\bar{*}*2}(a^{**},a^*,b^{**}) = w^* - \lim_\alpha w^* - \lim_\beta \pi^{*\bar{*}\bar{*}*2}(a_\alpha,a^*,b_\beta). \tag{12}$$

For any $a \in E$, we have

$$\begin{aligned} \left\langle \pi^{*\bar{*}\bar{*}*2}(a_\alpha,a^*,b_\beta), a\right\rangle &= \overline{\left\langle a^*, \pi^{*\bar{*}\bar{*}*}(a_\alpha,a,b_\beta)\right\rangle} \\ &= \overline{\left\langle a^*, \pi(a_\alpha,a,b_\beta)\right\rangle} = \left\langle \pi^2(a_\alpha,a^*,b_\beta), a\right\rangle. \end{aligned}$$

Since E^* is an E-submodule of E^{***}, we have $\pi^{*\bar{*}\bar{*}*2}(a_\alpha,a^*,b_\beta) \in E^*$. Hence the above calculation proves the equality $\pi^{*\bar{*}\bar{*}*2}(a_\alpha,a^*,b_\beta) = \pi^2(a_\alpha,a^*,b_\beta)$. Combining this with identity (12), we obtain

$$\pi^{*\bar{*}\bar{*}*2}(a^{**},a^*,b^{**}) = w^* - \lim_\alpha w^* - \lim_\beta \pi^2(a_\alpha,a^*,b_\beta). \tag{13}$$

Since $\pi^{2\bar{*}**\bar{*}}$ is an extension of π^2, we have

$$\pi^{2\bar{*}**\bar{*}}(a^{**},a^*,b^{**}) = w^* - \lim_\alpha w^* - \lim_\beta \pi^2(a_\alpha,a^*,b_\beta). \tag{14}$$

Comparing the identities (13) and (14), we establish the second identity. The third identity can be proved in a similar way. □

4. Ternary derivations and ternary weak amenability

The presence of two different types of ternary modules in Definition 3.1 makes it natural to consider two different types of derivations.

Definition 4.1. A *ternary derivation* from a JB-triple system E into a Banach ternary E-module of type (I) (resp., (II)) X is a continuous, conjugate-linear (resp., linear) mapping $D : E \to X$, satisfying

$$D\big([a,b,c]\big) = \big[D(a),b,c\big] + \big[a,D(b),c\big] + \big[a,b,D(c)\big]$$

for every a,b,c in E.

Note the conjugate linearity of a ternary derivation when its codomain is a ternary module of type (I) and the linearity of the one when its codomain is a ternary module of type (II) (see [13, Section 2.3]). The set of all ternary derivations from a JB-triple system E into a Banach ternary E-module X is denoted by $\mathcal{D}_t(E,X)$.

Let E be a JB-triple system, and let X be a Banach ternary E-module. Applying the axiom (3) in Definition 3.1, for every $b \in E$ and $x \in X$, we see that the mapping $\delta(b,x) : E \to X$, defined by

$$\delta(b,x)(a) = [b,x,a] - [x,b,a], \quad (a \in E),$$

is a ternary derivation. A finite sum of the above derivations is called a *ternary inner derivation*. The set of all ternary inner derivations from E to X is denoted by $\mathcal{I}nn_t(E,X)$. Obviously, we have $\mathcal{I}nn_t(E,X) \subseteq \mathcal{D}_t(E,X)$.

Now we can present some types of amenability notions in the context of JB-triple systems. A JB-triple system E is said to be *ternary amenable of type (I) (resp., (II))* if, for every Banach ternary E-module of type (I) (resp., (II)) X, we have

$$\mathcal{D}_t(E,X^*) = \mathcal{I}nn_t(E,X^*).$$

The JB-triple system E will be called *ternary weakly amenable* if $\mathcal{D}_t(E,E^*) = \mathcal{I}nn_t(E,E^*)$. We can extend this latter concept to any iterated dual of E. For any integer $n \in \mathbb{N}$, we say that E is *ternary n-weakly amenable* if $\mathcal{D}_t(E,E^{(n)}) = \mathcal{I}nn_t(E,E^{(n)})$. Note that ternary 1-weak amenability is nothing but ternary weak amenability (see [13]).

The most well-known amenability results for JB-triple systems are as follows. Meyberg [18] proved that every derivation on a finite-dimensional JB*-triple is inner. Recently Ho et al. showed that every commutative C*-algebra and every commutative JBW*-triple is ternary weakly amenable (see [13, Proposition 3.11 and Corollary 6.4]). In a more general case, they showed that every commutative JB*-triple is *almost* ternary weakly amenable (see [13, Theorem 6.7]). They also showed that a general C*-algebra is not necessarily ternary weakly amenable; however, it is weakly amenable in the sense of Banach algebras (see [11, Theorem 1.10]).

In the following we introduce a variant of ternary inner derivations which can be represented by some elements not from the corresponding spaces but from bidual spaces of them. Let E be a JB-triple system. Since $\delta(b^{**}, e^{(n+2)}) : E^{**} \to E^{(n+2)}$, $(n \in \mathbb{N})$, is a ternary derivation for every $b^{**} \in E^{**}$ and $e^{(n+2)} \in E^{(n+2)}$, it is not hard to see that the mapping $J^*_{n-1} \circ \delta(b^{**}, e^{(n+2)}) : E \to E^{(n)}$ is also a ternary derivation, where $J_{n-1} : E^{(n-1)} \to E^{(n+1)}$ denotes the canonical embedding. We call a finite sum of this type of derivation a *ternary quasi-inner derivation*. For any $n \in \mathbb{N}$, a Jordan-triple E is said to be *ternary n-quasiweakly amenable* if every continuous ternary derivation $D : E \to E^{(n)}$ is a ternary quasi-inner derivation.

4.1. **Ternary weak amenability of the bidual.** In this section we find some conditions under which a JB-triple system is ternary quasiweakly amenable whenever its bidual is ternary weakly amenable. To this end, we need to study those conditions under which a ternary derivation $D: E \to E^*$ can be lifted to a suitable ternary derivation to the biduals.

Before proceeding, it should be noted that in [13, Proposition 2.1] Ho et al. proved that, for a JB*-triple E, the second transpose of a ternary derivation $D: E \to E^*$ is again a ternary derivation (as it is used in (16) below). Their proof involves actions (11). As mentioned in Remark 3.5, it is not known whether they serve as E^{**}-module actions on E^{***}. However, in the circumstances of the quoted proposition, D is weakly compact, and so $D^{**}(E^{**}) \subseteq E^*$, and Proposition 3.6, together with the fact that E^* is E^{**}-submodule of E^{***} (see Proposition 3.4), can be applied to clarify the above question.

We commence with the following result specifying some conditions under which a ternary derivation can be lifted to bidual spaces. We denote by $J: E \to E^{**}$ the canonical embedding, and we also recall that $((E^{**})^*, \pi^{*\bar{*}\bar{*}*1}, \pi^{*\bar{*}\bar{*}*2}, \pi^{*\bar{*}\bar{*}*3})$ is our desired ternary E^{**}-module, induced naturally by dualizing the triple product $\pi^{*\bar{*}\bar{*}*}$ of E^{**} (see (10)).

Proposition 4.2. *Let (E,π) be a regular JB-triple system, and let $D: E \to E^*$ be a ternary derivation satisfying the following conditions:*

$$\pi^{*\bar{*}\bar{*}*2}\big(E, D^{**}(E^{**}), E\big) \subseteq E^* \qquad \textit{and} \qquad \pi^{*\bar{*}\bar{*}*3}\big(E^{**}, E, D^{**}(E^{**})\big) \subseteq E^*.$$

Then $J^* \circ D^{**}: E^{**} \to E^*(\subseteq (E^{**})^*)$ is a ternary derivation.

Proof. Let $D: E \to E^*$ be a ternary derivation, and let $a^{**}, b^{**}, c^{**} \in E^{**}$. We will show that

$$\begin{aligned} J^* \circ D^{**}\big(\pi^{*\bar{*}\bar{*}*}(a^{**}, b^{**}, c^{**})\big) = {} & \pi^{*\bar{*}\bar{*}*1}\big(J^* \circ D^{**}(a^{**}), b^{**}, c^{**}\big) \\ & + \pi^{*\bar{*}\bar{*}*2}\big(a^{**}, J^* \circ D^{**}(b^{**}), c^{**}\big) \\ & + \pi^{*\bar{*}\bar{*}*3}\big(a^{**}, b^{**}, J^* \circ D^{**}(c^{**})\big). \end{aligned} \tag{15}$$

Let $(a_\alpha), (b_\beta), (c_\gamma)$ be bounded nets in E which w^*-converge to a^{**}, b^{**}, c^{**}, respectively. Then (by the same argument used in the proof of [13, Proposition 2.1]; see also Remark 3.5) we have

$$\begin{aligned} & D^{**}\big(\pi^{*\bar{*}\bar{*}*}(a^{**}, b^{**}, c^{**})\big) \\ & = w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma D\big(\pi(a_\alpha, b_\beta, c_\gamma)\big) \\ & = w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma \pi^1\big(D(a_\alpha), b_\beta, c_\gamma\big) \\ & \quad + w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma \pi^2\big(a_\alpha, D(b_\beta), c_\gamma\big) \\ & \quad + w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma \pi^3\big(a_\alpha, b_\beta, D(c_\gamma)\big) \\ & = \pi^{1\bar{*}\bar{*}**}\big(D^{**}(a^{**}), b^{**}, c^{**})\big) + \pi^{2\bar{*}**\bar{*}}\big(a^{**}, D^{**}(b^{**}), c^{**}\big) \\ & \quad + \pi^{3**\bar{*}\bar{*}}\big(a^{**}, b^{**}, D^{**}(c^{**})\big). \end{aligned} \tag{16}$$

To obtain the desired result, by comparing identities (15) and (16), we need to establish the following identities:

(i) $J^* \circ \pi^{1\bar{*}\bar{*}**}(D^{**}(a^{**}), b^{**}, c^{**}) = \pi^{*\bar{*}\bar{*}*1}(J^* \circ D^{**}(a^{**}), b^{**}, c^{**})$,

(ii) $J^* \circ \pi^{2\bar{*}**\bar{*}}(a^{**}, D^{**}(b^{**}), c^{**}) = \pi^{*\bar{*}\bar{*}*2}(a^{**}, J^* \circ D^{**}(b^{**}), c^{**})$,
(iii) $J^* \circ \pi^{3**\bar{*}\bar{*}}(a^{**}, b^{**}, D^{**}(c^{**})) = \pi^{*\bar{*}\bar{*}*3}(a^{**}, b^{**}, J^* \circ D^{**}(c^{**}))$.

Let $a, b \in E$. For every $c \in E$, we have

$$\begin{aligned}\left\langle \pi^{1\bar{*}\bar{*}*2}\big(b, D^{**}(a^{**}), a\big), c\right\rangle &= \overline{\left\langle D^{**}(a^{**}), \pi^{1\bar{*}\bar{*}*}(b, c, a)\right\rangle}\\ &= \overline{\left\langle D^{**}(a^{**}), \pi(b, c, a)\right\rangle}\\ &= \overline{\left\langle J^* \circ D^{**}(a^{**}), \pi(b, c, a)\right\rangle}\\ &= \left\langle \pi^{1\bar{*}\bar{*}*2}\big(b, J^* \circ D^{**}(a^{**}), a\big), c\right\rangle.\end{aligned}$$

Since $\pi^{1\bar{*}\bar{*}*2}(E, D^{**}(E^{**}), E) \subseteq E^*$, we obtain the equality

$$\pi^{1\bar{*}\bar{*}*2}\big(b, D^{**}(a^{**}), a\big) = \pi^{1\bar{*}\bar{*}*2}\big(b, J^* \circ D^{**}(a^{**}), a\big). \tag{17}$$

Since in general we have $\pi^{1\bar{*}\bar{*}*3}(E^{**}, E^{**}, E^*) \subseteq E^*$ from the above identity (17) and from the hypothesis, $\pi^{1\bar{*}\bar{*}*3}(E^{**}, E, D^{**}(E^{**})) \subseteq E^*$, we have

$$\begin{aligned}&\left\langle J^* \circ \pi^{1\bar{*}\bar{*}**}\big(D^{**}(a^{**}), b^{**}, c^{**}\big), a\right\rangle\\ &\quad= \left\langle D^{**}(a^{**}), \pi^{1\bar{*}\bar{*}*}(b^{**}, c^{**}, a)\right\rangle\\ &\quad= \left\langle \pi^{1\bar{*}\bar{*}*3}\big(c^{**}, a, D^{**}(a^{**})\big), b^{**}\right\rangle\\ &\quad= \lim_\beta \left\langle \pi^{1\bar{*}\bar{*}*3}\big(c^{**}, a, D^{**}(a^{**})\big), b_\beta\right\rangle\\ &\quad= \lim_\beta \left\langle D^{**}(a^{**}), \pi^{1\bar{*}\bar{*}*}(b_\beta, c^{**}, a)\right\rangle\\ &\quad= \lim_\beta \overline{\left\langle \pi^{1\bar{*}\bar{*}*2}\big(b_\beta, D^{**}(a^{**}), a\big), c^{**}\right\rangle}\\ &\quad= \lim_\beta \overline{\left\langle \pi^{1\bar{*}\bar{*}*2}\big(b_\beta, J^* \circ D^{**}(a^{**}), a\big), c^{**}\right\rangle}\\ &\quad= \lim_\beta \left\langle J^* \circ D^{**}(a^{**}), \pi^{1\bar{*}\bar{*}*}(b_\beta, c^{**}, a)\right\rangle\\ &\quad= \lim_\beta \left\langle \pi^{1\bar{*}\bar{*}*3}\big(c^{**}, a, J^* \circ D^{**}(a^{**})\big), b_\beta\right\rangle\\ &\quad= \left\langle \pi^{1\bar{*}\bar{*}*3}\big(c^{**}, a, J^* \circ D^{**}(a^{**})\big), b^{**}\right\rangle\\ &\quad= \left\langle J^* \circ D^{**}(a^{**}), \pi^{1\bar{*}\bar{*}*}(b^{**}, c^{**}, a)\right\rangle\\ &\quad= \left\langle \pi^{1\bar{*}\bar{*}**}\big(J^* \circ D^{**}(a^{**}), b^{**}, c^{**}\big), a\right\rangle\\ &\quad= \left\langle \pi^{*\bar{*}\bar{*}*1}\big(J^* \circ D^{**}(a^{**}), b^{**}, c^{**}\big), a\right\rangle.\end{aligned}$$

Since by Proposition 3.4 E^* is a ternary E^{**}-submodule of E^{***}, we have $\pi^{*\bar{*}\bar{*}*1}(J^* \circ D^{**}(a^{**}), b^{**}, c^{**}) \in E^*$, and the above equality implies the identity (i).

Let $a, b \in E$. For every $c \in E$, we have

$$\begin{aligned}\left\langle \pi^{*\bar{*}\bar{*}*1}\big(D^{**}(b^{**}), b, a\big), c\right\rangle &= \left\langle D^{**}(b^{**}), \pi(b, a, c)\right\rangle\\ &= \left\langle J^* \circ D^{**}(b^{**}), \pi(b, a, c)\right\rangle\\ &= \left\langle \pi^*\big(J^* \circ D^{**}(b^{**}), b, a\big), c\right\rangle.\end{aligned}$$

Since $\pi^{*\bar{*}\bar{*}\bar{*}1}(D^{**}(E^{**}),E,E) = \pi^{*\bar{*}\bar{*}\bar{*}3}(E,E,D^{**}(E))$, applying the assumption $\pi^{*\bar{*}\bar{*}\bar{*}3}(E,E,D^{**}(E)) \subseteq E^*$, we obtain the following identity:

$$\pi^{*\bar{*}\bar{*}\bar{*}1}\big(D^{**}(b^{**}),b,a\big) = \pi^*\big(J^* \circ D^{**}(b^{**}),b,a\big).$$

Applying this identity, we have

$$\begin{aligned}
&\big\langle J^* \circ \pi^{2\bar{*}**\bar{*}}\big(a^{**},D^{**}(b^{**}),c^{**}\big),a\big\rangle\\
&\quad= \lim_\alpha \lim_\beta \lim_\gamma \big\langle \pi^2\big(a_\alpha,D(b_\beta),c_\gamma\big),a\big\rangle\\
&\quad= \lim_\alpha \lim_\beta \lim_\gamma \overline{\big\langle D(b_\beta),\pi(a_\alpha,a,c_\gamma)\big\rangle}\\
&\quad= \lim_\alpha \lim_\beta \overline{\big\langle \pi^{*\bar{*}\bar{*}\bar{*}}(a_\alpha,a,c^{**}),D(b_\beta)\big\rangle}\\
&\quad= \lim_\alpha \lim_\beta \overline{\big\langle \pi^{*\bar{*}\bar{*}\bar{*}1}\big(D(b_\beta),a_\alpha,a\big),c^{**}\big\rangle}\\
&\quad= \lim_\alpha \overline{\big\langle \pi^{*\bar{*}\bar{*}\bar{*}1}\big(D^{**}(b^{**}),a_\alpha,a\big),c^{**}\big\rangle}\\
&\quad= \lim_\alpha \overline{\big\langle \pi^*\big(J^* \circ D^{**}(b^{**}),a_\alpha,a\big),c^{**}\big\rangle}\\
&\quad= \lim_\alpha \overline{\big\langle \pi^{*\bar{*}\bar{*}\bar{*}*}\big(J^* \circ D^{**}(b^{**}),a_\alpha,a\big),c^{**}\big\rangle}\\
&\quad= \lim_\alpha \overline{\big\langle J^* \circ D^{**}(b^{**}),\pi^{*\bar{*}\bar{*}\bar{*}}(a_\alpha,a,c^{**})\big\rangle}\\
&\quad= \overline{\big\langle J^* \circ D^{**}(b^{**}),\pi^{*\bar{*}\bar{*}\bar{*}}(a^{**},a,c^{**})\big\rangle}\\
&\quad= \big\langle \pi^{*\bar{*}\bar{*}\bar{*}2}\big(a^{**},J^* \circ D^{**}(b^{**}),c^{**}\big),a\big\rangle.
\end{aligned}$$

Proposition 3.4 implies that $\pi^{*\bar{*}\bar{*}\bar{*}2}(a^{**},J^* \circ D^{**}(b^{**}),c^{**}) \in E^*$, and so we obtain identity (ii).

For every $a \in E$, we have

$$\begin{aligned}
&\big\langle J^* \circ \pi^{3**\bar{*}\bar{*}}\big(a^{**},b^{**},D^{**}(c^{**})\big),a\big\rangle\\
&\quad= \lim_\alpha \lim_\beta \lim_\gamma \big\langle \pi^3\big(a_\alpha,b_\beta,D(c_\gamma)\big),a\big\rangle\\
&\quad= \lim_\alpha \lim_\beta \lim_\gamma \big\langle D(c_\gamma),\pi(a,a_\alpha,b_\beta)\big\rangle\\
&\quad= \lim_\alpha \lim_\beta \big\langle D^{**}(c^{**}),\pi(a,a_\alpha,b_\beta)\big\rangle\\
&\quad= \lim_\alpha \lim_\beta \big\langle J^* \circ D^{**}(c^{**}),\pi(a,a_\alpha,b_\beta)\big\rangle\\
&\quad= \big\langle J^* \circ D^{**}(c^{**}),\pi^{*\bar{*}\bar{*}\bar{*}}(a,a^{**},b^{**})\big\rangle\\
&\quad= \big\langle \pi^{*\bar{*}\bar{*}\bar{*}3}\big(a^{**},b^{**},J^* \circ D^{**}(c^{**})\big),a\big\rangle.
\end{aligned}$$

Again Proposition 3.4 implies that $\pi^{*\bar{*}\bar{*}\bar{*}3}(a^{**},b^{**},J^* \circ D^{**}(c^{**})) \in E^*$, and so the above calculations prove identity (iii). $\square$

Now we can prove one of the main results of this paper, which is a ternary version of [3, Theorem 2].

Theorem 4.3. *Let (E, π) be a regular JB-triple system. If every ternary derivation $D : E \to E^*$ satisfies $D^{**}(E^{**}) \subseteq Z_l(\pi^*)$, then ternary weak amenability of E^{**} implies ternary quasiweak amenability of E.*

Proof. Let $D : E \to E^*$ be a ternary derivation. Let $a^{**}, b^{**} \in E^{**}, a, b \in E$, and let (e_λ^{**}) be a net in E^{**} which w^*-converges to $e^{**} \in E^{**}$.

Since by hypothesis $D^{**}(a^{**}) \in Z_{l,1}(\pi^*)$, we have

$$\begin{aligned}
&\lim_\lambda \big\langle \pi^{*\bar{*}\bar{*}*3}\big(b^{**}, a, D^{**}(a^{**})\big), e_\lambda^{**}\big\rangle \\
&\quad = \lim_\lambda \big\langle D^{**}(a^{**}), \pi^{*\bar{*}\bar{*}*}(e_\lambda^{**}, b^{**}, a)\big\rangle \\
&\quad = \lim_\lambda \big\langle \pi^{*\bar{*}\bar{*}***}\big(D^{**}(a^{**}), e_\lambda^{**}, b^{**}\big), a\big\rangle \\
&\quad = \big\langle \pi^{*\bar{*}\bar{*}***}\big(D^{**}(a^{**}), e^{**}, b^{**}\big), a\big\rangle \\
&\quad = \big\langle D^{**}(a^{**}), \pi^{*\bar{*}\bar{*}*}(e^{**}, b^{**}, a)\big\rangle \\
&\quad = \big\langle \pi^{*\bar{*}\bar{*}*3}\big(b^{**}, a, D^{**}(a^{**})\big), e^{**}\big\rangle.
\end{aligned}$$

This shows that $\pi^{*\bar{*}\bar{*}*3}(b^{**}, a, D^{**}(a^{**}))$ is a w^*-continuous element of E^{***}. Hence it belongs to E^*; that is, $\pi^{*\bar{*}\bar{*}*3}(E^{**}, E, D^{**}(E^{**})) \subseteq E^*$.

Since $D^{**}(a^{**}) \in Z_{l,2}(\pi^*)$, we have

$$\begin{aligned}
\lim_\lambda \big\langle \pi^{*\bar{*}\bar{*}*2}\big(a, D^{**}(a^{**}), b\big), e_\lambda^{**}\big\rangle &= \lim_\lambda \overline{\big\langle D^{**}(a^{**}), \pi^{*\bar{*}\bar{*}*}(a, e_\lambda^{**}, b)\big\rangle} \\
&= \lim_\lambda \overline{\big\langle \pi^{*\bar{*}\bar{*}***}\big(D^{**}(a^{**}), a, e_\lambda^{**}\big), b\big\rangle} \\
&= \overline{\big\langle \pi^{1\bar{*}\bar{*}***}\big(D^{**}(a^{**}), a, e^{**}\big), b\big\rangle} \\
&= \overline{\big\langle D^{**}(a^{**}), \pi^{*\bar{*}\bar{*}*}(a, e^{**}, b)\big\rangle} \\
&= \big\langle \pi^{*\bar{*}\bar{*}*2}\big(a, D^{**}(a^{**}), b\big), e^{**}\big\rangle.
\end{aligned}$$

Hence $\pi^{*\bar{*}\bar{*}*2}(a, D^{**}(a^{**}), b) \in E^*$; that is, $\pi^{*\bar{*}\bar{*}*2}(E, D^{**}(E^{**}), E) \subseteq E^*$.

Now Proposition 4.2 implies that $J^* \circ D^{**} : E^{**} \to (E^{**})^*$ is a ternary derivation. Since E^{**} is ternary weakly amenable, there exist $b_i^{**} \in E^{**}$ and $e_i^{***} \in E^{***} (1 \le i \le m)$ such that $J^* \circ D^{**} = \sum_{i=1}^m \delta(b_i^{**}, e_i^{***})$. Therefore, $D = \sum_{i=1}^m J^* \circ \delta(b_i^{**}, e_i^{***})$, which proves the ternary quasiweak amenability of E. $\square$

It is known that every ternary derivation $D : E \to E^*$, where E is a JB*-triple, is automatically bounded (see [21, Corollary 15]) and weakly compact (see [20, Lemma 5]). In particular, we have $D^{**}(E^{**}) \subseteq E^*$, and so D satisfies $D^{**}(E^{**}) \subseteq E^* \subseteq Z_l(\pi^*)$. We thus arrive at the following result as an immediate consequence of Theorem 4.3.

Corollary 4.4. *For every JB*-triple E, ternary weak amenability of E^{**} implies ternary quasiweak amenability of E.*

In the following result, we provide some alternative conditions under which a similar conclusion to that in Theorem 4.3 can be obtained.

Theorem 4.5. *Let (E, π) be a regular JB-triple system. If*

$$\pi^{*\bar{*}\bar{*}*}(E^{**}, E^{**}, E) \subseteq E \qquad \textit{and} \qquad \pi^{*\bar{*}\bar{*}*}(E, E, E^{**}) \subseteq E,$$

then ternary weak amenability of E^{**} implies ternary quasiweak amenability of E.

Proof. Let $D : E \to E^*$ be a ternary derivation. First we show that $J^* \circ D^{**} : E^{**} \to E^{***}$ is also a ternary derivation. To this end, we only need to establish the identities (i), (ii), and (iii) in the proof of Proposition 4.2. Then the last paragraph of the proof of Theorem 4.3 will also work here and completes the proof.

Let $a^{**}, b^{**}, c^{**} \in E^{**}$, and let $a \in E$. Since by definition we have $\pi^{1\bar{*}\bar{*}**} = \pi^{*\bar{*}\bar{*}**} = \pi^{*\bar{*}\bar{*}*1}$, and by hypothesis $\pi^{*\bar{*}\bar{*}*}(b^{**}, c^{**}, a) \in E$, then

$$\begin{aligned}
\left\langle J^* \circ \pi^{1\bar{*}\bar{*}**}\left(D^{**}(a^{**}), b^{**}, c^{**}\right), a\right\rangle &= \left\langle D^{**}(a^{**}), \pi^{*\bar{*}\bar{*}*}(b^{**}, c^{**}, a)\right\rangle \\
&= \left\langle J^* \circ D^{**}(a^{**}), \pi^{*\bar{*}\bar{*}*}(b^{**}, c^{**}, a)\right\rangle \\
&= \left\langle \pi^{*\bar{*}\bar{*}*1}\left(J^* \circ D^{**}(a^{**}), b^{**}, c^{**}\right), a\right\rangle.
\end{aligned}$$

This proves the identity (i).

Let (a_α), (b_β), (c_γ) be bounded nets in E which w^*-converge to a^{**}, b^{**}, c^{**}, respectively. Since for every a_α we have $\pi^{*\bar{*}\bar{*}*}(a_\alpha, a, c^{**}) \in E$, then

$$\begin{aligned}
&\left\langle J^* \circ \pi^{2\bar{*}**\bar{*}}\left(a^{**}, D^{**}(b^{**}), c^{**}\right), a\right\rangle \\
&\quad = \lim_\alpha \lim_\beta \lim_\gamma \left\langle \pi^2\left(a_\alpha, D(b_\beta), c_\gamma\right), a\right\rangle \\
&\quad = \lim_\alpha \lim_\beta \lim_\gamma \overline{\left\langle D(b_\beta), \pi(a_\alpha, a, c_\gamma)\right\rangle} \\
&\quad = \lim_\alpha \overline{\left\langle D^{**}(b^{**}), \pi^{*\bar{*}\bar{*}*}(a_\alpha, a, c^{**})\right\rangle} \\
&\quad = \lim_\alpha \overline{\left\langle J^* \circ D^{**}(b^{**}), \pi^{*\bar{*}\bar{*}*}(a_\alpha, a, c^{**})\right\rangle} \\
&\quad = \overline{\left\langle J^* \circ D^{**}(b^{**}), \pi^{*\bar{*}\bar{*}*}(a^{**}, a, c^{**})\right\rangle} \\
&\quad = \left\langle \pi^{*\bar{*}\bar{*}*2}\left(a^{**}, J^* \circ D^{**}(b^{**}), c^{**}\right), a\right\rangle,
\end{aligned}$$

which proves the identity (ii).

The proof of identity (iii) is the same as the one given in the proof of Proposition 4.2. □

As a consequence we present the following corollary.

Corollary 4.6. *Let E be a regular JB-triple system such that E is a triple ideal of E^{**}. Then ternary weak amenability of E^{**} implies ternary quasiweak amenability of E.*

Proof. Let E be a triple ideal of E^{**}. Then by definition

$$\pi^{*\bar{*}\bar{*}*}(E^{**}, E^{**}, E) + \pi^{*\bar{*}\bar{*}*}(E^{**}, E, E^{**}) \subseteq E,$$

and this supports both inclusions of Theorem 4.5. □

4.2. **Ternary n-weak amenability of the bidual.**

Here we extend the results of Section 4.1 for $n \geq 2$. Surprisingly, in this case, the restrictive conditions required in Theorems 4.3 and 4.5 can be relaxed. Throughout this section, for any $n \in \mathbb{N}$ we denote by $J_n : E^{(n)} \to E^{(n+2)}$ the canonical embedding of $E^{(n)}$ into $E^{(n+2)}$.

In the following result, which is the n-analogue of Proposition 4.2, we show that, for $n \geq 2$, every ternary derivation $D : E \to E^{(n)}$ can be lifted to an appropriate ternary derivation on the bidual spaces.

Proposition 4.7. *Let E be a permanently regular JB-triple system, $n \in \mathbb{N}$.*

(a) *If $D : E \to E^{(2n)}$ is a ternary derivation, then $D^{**} : E^{**} \to E^{(2n+2)}$ is a ternary derivation.*

(b) *If $D : E \to E^{(2n+1)}$ is a ternary derivation, then $J_{2n}^* \circ D^{**} : E^{**} \to E^{(2n+3)}$ is a ternary derivation.*

Proof. Let π be the triple product of a permanently regular JB-triple system E. Then we recall that $(E^{(2n)}, \pi^{[n]}, \pi^{[n]}, \pi^{[n]})$ and $(E^{(2n+2)}, \pi^{[n+1]}, \pi^{[n+1]}, \pi^{[n+1]})$ are Banach ternary E^{**}-modules. Let $D : E \to E^{(2n)}$ be a ternary derivation, and let (a_α), (b_β), and (c_γ) be nets in E^{**}, w^*-converging to $a^{**}, b^{**}, c^{**} \in E^{**}$, respectively. Then we have

$$
\begin{aligned}
&D^{**}\big(\pi^{*\bar{*}\bar{*}*}(a^{**}, b^{**}, c^{**})\big)\\
&\quad = w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma D\big(\pi(a_\alpha, b_\beta, c_\gamma)\big)\\
&\quad = w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma \pi^{[n]}\big(D(a_\alpha), b_\beta, c_\gamma\big)\\
&\qquad + w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma \pi^{[n]}\big(a_\alpha, D(b_\beta), c_\gamma\big)\\
&\qquad + w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma \pi^{[n]}\big(a_\alpha, b_\beta, D(c_\gamma)\big)\\
&\quad = \pi^{[n+1]}\big(D^{**}(a^{**}), b^{**}, c^{**}\big) + \pi^{[n+1]}\big(a^{**}, D^{**}(b^{**}), c^{**}\big)\\
&\qquad + \pi^{[n+1]}\big(a^{**}, b^{**}, D^{**}(c^{**})\big).
\end{aligned}
$$

Thus $D^{**} : E^{**} \to E^{(2n+2)}$ is a ternary derivation as required in (a).

To prove (b), first recall that for any $n \in \mathbb{N}$, by Remark 3.3, $\pi^{[n]1}, \pi^{[n]2}, \pi^{[n]3}$ are E-module actions of $E^{(2n+1)}$, and $\pi^{[n+1]1}, \pi^{[n+1]2}, \pi^{[n+1]3}$ are E^{**}-module actions of $E^{(2n+3)}$. Note that the latter mappings are also E-module actions of $E^{(2n+3)}$.

Let $a^{**}, b^{**}, c^{**} \in E^{**}$. We want to show that

$$
\begin{aligned}
&J_{2n}^* \circ D^{**}\big(\pi^{*\bar{*}\bar{*}*}(a^{**}, b^{**}, c^{**})\big)\\
&\quad = \pi^{[n+1]1}\big(J_{2n}^* \circ D^{**}(a^{**}), b^{**}, c^{**}\big)\\
&\qquad + \pi^{[n+1]2}\big(a^{**}, J_{2n}^* \circ D^{**}(b^{**}), c^{**}\big) + \pi^{[n+1]3}\big(a^{**}, b^{**}, J_{2n}^* \circ D^{**}(c^{**})\big).
\end{aligned}
$$

A double limiting process argument—similar to what has been used in the proof of Proposition 4.2—can be applied here to show that we only need to establish the following identities:

(i) $J_{2n}^* \circ \pi^{[n]1\bar{*}\bar{*}***}(D^{**}(a^{**}), b^{**}, c^{**}) = \pi^{[n+1]1}(J_{2n}^* \circ D^{**}(a^{**}), b^{**}, c^{**})$,

(ii) $J_{2n}^* \circ \pi^{[n]2\bar{*}***\bar{*}}(a^{**}, D^{**}(b^{**}), c^{**}) = \pi^{[n+1]2}(a^{**}, J_{2n}^* \circ D^{**}(b^{**}), c^{**})$,
(iii) $J_{2n}^* \circ \pi^{[n]3***\bar{*}\bar{*}}(a^{**}, b^{**}, D^{**}(c^{**})) = \pi^{[n+1]3}(a^{**}, b^{**}, J_{2n}^* \circ D^{**}(c^{**}))$.

For every $a^{(2n)} \in E^{(2n)}$, we have

$$\begin{aligned}
&\left\langle J_{2n}^* \circ \pi^{[n]1\bar{*}\bar{*}***}\left(D^{**}(a^{**}), b^{**}, c^{**}\right), a^{(2n)}\right\rangle \\
&\quad = \left\langle D^{**}(a^{**}), \pi^{[n]1\bar{*}\bar{*}*}(b^{**}, c^{**}, a^{(2n)})\right\rangle \\
&\quad = \left\langle D^{**}(a^{**}), \pi^{[n]}(b^{**}, c^{**}, a^{(2n)})\right\rangle \\
&\quad = \left\langle J_{2n}^* \circ D^{**}(a^{**}), \pi^{[n]}(b^{**}, c^{**}, a^{(2n)})\right\rangle \\
&\quad = \left\langle J_{2n}^* \circ D^{**}(a^{**}), \pi^{[n+1]}(b^{**}, c^{**}, a^{(2n)})\right\rangle \\
&\quad = \left\langle \pi^{[n+1]1}\left(J_{2n}^* \circ D^{**}(a^{**}), b^{**}, c^{**}\right), a^{(2n)}\right\rangle,
\end{aligned}$$

which proves (i).

(It is worth mentioning that the second equality in the above calculations is not valid in the case where $n = 1$).

Let (a_α), (b_β), and (c_γ) be bounded nets in E which are w^*-converging to a^{**}, b^{**}, and c^{**}, respectively. For every $a^{(2n)} \in E^{(2n)}$, we have

$$\begin{aligned}
&\left\langle J_{2n}^* \circ \pi^{[n]2\bar{*}***\bar{*}}\left(a^{**}, D^{**}(b^{**}), c^{**}\right), a^{(2n)}\right\rangle \\
&\quad = \lim_\alpha \lim_\beta \lim_\gamma \left\langle \pi^{[n]2}\left(a_\alpha, D(b_\beta), c_\gamma\right), a^{(2n)}\right\rangle \\
&\quad = \lim_\alpha \lim_\beta \lim_\gamma \overline{\left\langle D(b_\beta), \pi^{[n]}(a_\alpha, a^{(2n)}, c_\gamma)\right\rangle} \\
&\quad = \lim_\alpha \lim_\beta \overline{\left\langle \pi^{[n+1]}(a_\alpha, a^{(2n)}, c^{**}), D(b_\beta)\right\rangle} \\
&\quad = \lim_\alpha \lim_\beta \overline{\left\langle \pi^{[n+1]1}\left(D(b_\beta), a_\alpha, a^{(2n)}\right), c^{**}\right\rangle} \\
&\quad = \lim_\alpha \overline{\left\langle \pi^{[n+1]1}\left(D^{**}(b^{**}), a_\alpha, a^{(2n)}\right), c^{**}\right\rangle} \\
&\quad = \lim_\alpha \overline{\left\langle D^{**}(b^{**}), \pi^{[n+1]}(a_\alpha, a^{(2n)}, c^{**})\right\rangle} \\
&\quad = \lim_\alpha \overline{\left\langle D^{**}(b^{**}), \pi^{[n]}(a_\alpha, a^{(2n)}, c^{**})\right\rangle} \\
&\quad = \lim_\alpha \overline{\left\langle J_{2n}^* \circ D^{**}(b^{**}), \pi^{[n]}(a_\alpha, a^{(2n)}, c^{**})\right\rangle} \\
&\quad = \overline{\left\langle J_{2n}^* \circ D^{**}(b^{**}), \pi^{[n+1]}(a^{**}, a^{(2n)}, c^{**})\right\rangle} \\
&\quad = \left\langle \pi^{[n+1]2}\left(a^{**}, J_{2n}^* \circ D^{**}(b^{**}), c^{**}\right), a^{(2n)}\right\rangle.
\end{aligned}$$

Proposition 3.4 implies that $\pi^{[n+1]2}(a^{**}, J_{2n}^* \circ D^{**}(b^{**}), c^{**}) \in E^{(2n+1)}$, and so we obtain identity (ii).

Let (a_α), (b_β), and (c_γ) be bounded nets in E which are w^*-converging to a^{**}, b^{**}, and c^{**}, respectively. For every $a^{(2n)} \in E^{(2n)}$, we have

$$\begin{aligned}
&\left\langle J_{2n}^* \circ \pi^{[n]3***\bar{*}\bar{*}}\left(a^{**}, b^{**}, D^{**}(c^{**})\right), a^{(2n)}\right\rangle \\
&\quad = \lim_\alpha \lim_\beta \lim_\gamma \left\langle \pi^{[n]3}\left(a_\alpha, b_\beta, D(c_\gamma)\right), a^{(2n)}\right\rangle
\end{aligned}$$

$$= \lim_\alpha \lim_\beta \lim_\gamma \big\langle D(c_\gamma), \pi^{[n]}(a^{(2n)}, a_\alpha, b_\beta)\big\rangle$$
$$= \lim_\alpha \lim_\beta \big\langle D^{**}(c^{**}), \pi^{[n]}(a^{(2n)}, a_\alpha, b_\beta)\big\rangle$$
$$= \lim_\alpha \lim_\beta \big\langle J^*_{2n} \circ D^{**}(c^{**}), \pi^{[n]}(a^{(2n)}, a_\alpha, b_\beta)\big\rangle$$
$$= \big\langle J^*_{2n} \circ D^{**}(c^{**}), \pi^{[n+1]}(a^{(2n)}, a^{**}, b^{**})\big\rangle$$
$$= \big\langle \pi^{[n+1]3}\big(a^{**}, b^{**}, J^*_{2n} \circ D^{**}(c^{**})\big), a^{(2n)}\big\rangle.$$

Another application of Proposition 3.4 yields the identity (iii) from the above calculations. □

Now we can prove the main result of this section, which is a ternary version of [3, Theorem 1].

Theorem 4.8. *Let E be a permanently regular JB-triple system. Then for every integer $n \geq 2$, ternary n-weak amenability of E^{**} implies ternary n-quasiweak amenability of E.*

Proof. First we prove the theorem for even integers. Let $D : E \longrightarrow E^{(2n)}$ be a ternary derivation. Then $D^{**} : E^{**} \to E^{(2n+2)}$ is a ternary derivation by Proposition 4.7. Since E^{**} is ternary $2n$-weakly amenable, there exist $b_i^{**} \in E$ and $e_i^{(2n+2)} \in E^{(2n+2)} (1 \leq i \leq m)$ such that $D^{**} = \sum_{i=1}^m \delta(b_i^{**}, e_i^{(2n+2)})$. Therefore, $D = \sum_{i=1}^m J^*_{2n-1} \circ \delta(b_i^{**}, e_i^{(2n+2)})$, which shows the ternary $2n$-quasiweak amenability of E. This proves the theorem for even integers.

For odd integers, let E^{**} be ternary $(2n+1)$-weakly amenable, and let $D : E \to E^{(2n+1)}$ be a ternary derivation. Proposition 4.7 shows that $J^*_{2n} \circ D^{**} : E^{**} \to E^{(2n+3)}$ is a ternary derivation. Since E^{**} is ternary $(2n+1)$-weakly amenable, there exist $b_i^{**} \in E^{**}$ and $e_i^{(2n+3)} \in E^{(2n+3)}$ $(1 \leq i \leq m)$ such that $J^*_{2n} \circ D^{**} = \sum_{i=1}^m \delta(b_i^{**}, e_i^{(2n+3)})$. Therefore, $D = \sum_{i=1}^m J^*_{2n} \circ \delta(b_i^{**}, J^*_{2n}(e_i^{(2n+3)}))$, which shows the ternary $(2n+1)$-quasiweak amenability of E. □

5. Some questions

We conclude the paper with the following undecided questions.

(a) Is a commutative C*-algebra ternary n-weakly amenable for any $n \in \mathbb{N}$? (The case $n = 1$ has been positively answered by Ho et al. [13, Proposition 2.1]).

(b) For a general (regular) JB-algebra E, does ternary weak amenability of E^{**} imply ternary-quasiweak amenability of E? (See Theorem 4.8 for $n \geq 2$, and some partial positive answers in Theorems 4.3 and 4.5 and their subsequent corollaries.)

(c) Let E be a JB-triple system, and let X be a Banach ternary E-module. Is the bidual X^{**} a ternary E^{**}-module? As has been discussed in Remark 3.5, we do not know the answer even in the special case $X = E^*$ (see the actions (11)); however, the answer is positive for the case $X = E^{(2n)}, (n \in \mathbb{N})$, and it is quite easy to verify that $E^{(2n+2)}$ is a ternary

E^{**}-module under the actions induced by $\pi^{[n+1]}$, where π is the triple product of E.

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References

1. R. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** (1951), 839–848. Zbl 0044.32601. MR0045941. DOI 10.2307/2031695. 677, 680
2. R. M. Aron and P. D. Berner, *A Hahn-Banach extension theorem for analytic mappings*, Bull. Soc. Math. France, **106** (1978), 3–24. Zbl 378.46043. MR0508947. 677, 680
3. S. Barootkoob and H. R. Ebrahimi Vishki, *Lifting derivations and n-weak amenability of the second dual of a Banach algebra*, Bull. Aust. Math. Soc. **83** (2011), no. 1, 122–129. Zbl 1270.46042. MR2765419. DOI 10.1017/S0004972710001838. 677, 678, 690, 695
4. T. J. Barton and R. M. Timoney, *Weak***-continuity of Jordan triple products and its applications*, Math. Scand. **59** (1986), no. 2, 177–191. Zbl 0621.46044. MR0884654. DOI 10.7146/math.scand.a-12160. 679, 680
5. Ch.-H. Chu, *Jordan structures in geometry and analysis*, Cambridge Tracts in Math. **190**, Cambridge Univ. Press, Cambridge, 2012. Zbl 1238.17001. MR2885059. 679, 680
6. H. G. Dales, F. Ghahramani, and N. Grønbæk, *Derivations into iterated duals of Banach algebras*, Studia Math. **128** (1998), no. 1, 19–54. Zbl 0903.46045. MR1489459. 677
7. H. G. Dales, A. Rodríguez-Palacios, and M. V. Velasco, *The second transpose of a derivation*, J. Lond. Math. Soc. **64** (2001), no. 3, 707–721. Zbl 1023.46051. MR1865558. DOI 10.1112/S0024610701002496. 677
8. S. Dineen, *Complete holomorphic vector fields on the second dual of a Banach space*, Quart. J. Math. Scand. **59** (1986), 131–142. Zbl 0625.46055. MR0873493. 677, 680
9. S. Dineen, "The second dual of a JB*-triple system" in *Complex Analysis, Functional Analysis and Approximation Theory (Campinas, 1984)*, North-Holland Math. Stud. **125**, Notas Mat. **110**, North-Holland, Amsterdam, 1986, 67–69. Zbl 0653.46053. MR0893410. 677, 680
10. F. Ghahramani, R. J. Loy, and G. A. Willis, *Amenability and weak amenability of the second conjugate Banach algebras*, Proc. Amer. Math. Soc. **124** (1996), no. 5, 1489–1497. Zbl 0851.46035. MR1307520. DOI 10.1090/S0002-9939-96-03177-2. 677, 678
11. U. Haagerup and N. J. Laustsen, "Weak amenability of C*-algebras and a theorem of Goldstein" in *Banach Algebras '97 (Blaubeuren, 1997)*, de Gruyter, Berlin, 1998, 223–243. Zbl 1034.46506. MR1656608. 676, 687
12. S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. **313** (1979), 72–104. Zbl 0427.46009. MR0552464. 680
13. T. Ho, A. M. Peralta, and B. Russo, *Ternary weakly amenable C***-algebras and JB***-triples*, Quart. J. Math. **64** (2013), no. 4, 1109–1139. Zbl 1308.46074. MR3151607. DOI 10.1093/qmath/has032. 676, 677, 681, 682, 683, 687, 688, 695
14. G. Horn, *Klassifikation der JBW***-tripel von typ I*, Ph.D. dissertation, University of Tübingen, Tübingen, Germany, 1984. 679
15. B. Iochum and G. Loupias, *Arens regularity and local reflexivity principle for Banach algebras*, Math. Ann. **284** (1989), no. 1, 23–40. Zbl 0648.46044. MR0995379. DOI 10.1007/BF01443502. 680
16. A. A. Khosravi and H. R. Ebrahimi Vishki, *Aron-Berner extensions of trilinear maps with application to the biduals of a JB***-triple*, in preparation. 680

17. J. Martínez and A. M. Peralta, *Separate weak*-continuity of the triple product in dual real JB*-triples*, Math. Z. **234** (2000), no. 4, 635–646. Zbl 0977.17032. MR1778403. DOI 10.1007/s002090000094. 679
18. K. Meyberg, *Lecture notes on triple systems*, lecture notes, Univ. of Virginia, Charlottesville, VA, 1972. Zbl 0253.17017. MR0340353. 687
19. S. Mohammadzadeh and H. R. Ebrahimi Vishki, *Arens regularity of module actions and the second adjoint of a derivation*, Bull. Aust. Math. Soc. **77** (2008), no. 3, 465–476. Zbl 1159.46027. MR2454977. DOI 10.1017/S0004972708000403. 677
20. A. M. Peralta and A. R. Palacios, *Grothendieck's inequalities for real and complex JBW*-triples*, Proc. Lond. Math. Soc., **83** (2001), no. 3, 605–625. Zbl 1037.46058. MR1851084. DOI 10.1112/plms/83.3.605. 691
21. A. M. Peralta and B. Russo, *Automatic countinuity of triple derivations on C*-algebras and JB*-triples*, J. Algebra, **399** (2014), 960–977. Zbl 1332.46055. MR3144622. 691
22. B. Russo, "Structure of JB*-triples" in *Jordan Algebras (Oberwolfach, 1992)*, de Gruyter, Berlin, 1994, 209–280. Zbl 0805.46072. MR1293321. 679
23. Y. Zhang, *Weak amenability of module extension of Banach algebras*, Trans. Amer. Math. Soc. **354** (2002), 1431–1451. Zbl 1008.46019. MR1926868. 677

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