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A HYBRID SHRINKING PROJECTION METHOD FOR COMMON FIXED POINTS OF A FINITE FAMILY OF DEMICONTRACTIVE MAPPINGS WITH VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this article, we prove some properties of a demicontractive mapping defined on a nonempty closed convex subset of a Hilbert space. By using these properties, we obtain strong convergence theorems of a hybrid shrinking projection method for finding a common element of the set of common fixed points of a finite family of demicontractive mappings and the set of common solutions of a finite family of variational inequality problems in a Hilbert space. A numerical example is presented to illustrate the proposed method and convergence result. Our results improve and extend the corresponding results existing in the literature.

1. Introduction

Throughout this article, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, that C is a nonempty closed convex subset of H, and that P_C is the metric projection of H onto C. Let $T : C \to C$ be a mapping. The fixed point set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a mapping T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and that it is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - z\| \leq \|x - z\|$ for all $x \in C$ and $z \in F(T)$. Clearly, every

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nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive. It is well known that the fixed point set of a nonexpansive (quasi-nonexpansive) mapping in a Hilbert space is always closed and convex.

In 2011, Osilike and Isiogugu [10] introduced a class of nonlinear mappings which contains the class of nonexpansive mappings in a Hilbert space, namely, *strictly pseudo-nonspreading*, and obtained weak and strong convergence theorems of Halpern's type for this mapping in Hilbert spaces. Recall that a mapping T is called *strictly pseudo-nonspreading* if there exists $k \in [0,1)$ such that, for all $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\big\|(I-T)x - (I-T)y\big\|^2 + 2\langle x - Tx, y - Ty\rangle,$$

where I denotes the identity mapping. Note that if $k = 0$, a mapping T is called *nonspreading* (see [9]). Iterative methods for strictly pseudo-nonspreading mapping have been extensively investigated (see, e.g., [2], [16]).

As a generalization of the class of strictly pseudo-nonspreading mappings, the class of demicontractive mappings was introduced by Hicks and Kubicek [4] in 1977 and has been studied by several authors (see, e.g., [3]). Recall that a mapping $T : C \to C$ is said to be *demicontractive* if $F(T) \neq \emptyset$ and there exists $k \in [0,1)$ such that, for all $x \in C$ and for all $z \in F(T)$,

$$\|Tx - z\|^2 \leq \|x - z\|^2 + k\|x - Tx\|^2.$$

We call k the *contraction coefficient*.

Remark 1.1. From the above definitions, we have the following facts.

(i) Every nonspreading mapping with a nonempty fixed point set is quasi-nonexpansive.
(ii) Every strictly pseudo-nonspreading mapping with a nonempty fixed point set is demicontractive.
(iii) The class of demicontractive mappings includes the class of quasi-nonexpansive mappings.

We now give three examples for the class of demicontractive mappings.

Example 1.2. Let H be the real line, and let $C = [-1,1]$. Define a mapping $T : C \to C$ by

$$Tx = \begin{cases} \frac{4}{7}x\sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Obviously, $F(T) = \{0\}$. Also, for all $x \in C$, we have $|Tx - T0|^2 = |Tx|^2 = |\frac{4}{7}x\sin(\frac{1}{x})|^2 \leq |\frac{4x}{7}|^2 \leq |x|^2 \leq |x-0|^2 + k|x - Tx|^2$ for all $k \in [0,1)$. Therefore, T is demicontractive.

Example 1.3. Let H be the real line, and let $C = [-1,1]$. Define a mapping $T : C \to C$ by

$$Tx = \begin{cases} \frac{9-x}{10}, & x \in [-1,0), \\ \frac{x+9}{10}, & x \in [0,1]. \end{cases}$$

Obviously, $F(T)=\{1\}$. We will show that there exists $k \in [0,1)$ such that

$$|Tx-1|^2 \le |x-1|^2 + k|x-Tx|^2 \tag{1.1}$$

for all $x \in [-1,1]$. Then, it suffices to consider the following two cases.

Case 1: $x \in [-1,0)$. Then we have $Tx = \frac{9-x}{10}$. From the definition of T, we have $|Tx-1|^2 = |\frac{9-x}{10}-1|^2 = \frac{1}{10}|x+1|^2$ and $|x-Tx|^2 = |x-\frac{9-x}{10}|^2 = \frac{1}{100}|11x-9|^2 \ge 0$. So, there exists $k \in [0,1)$ such that

$$\begin{aligned} |x-1|^2 + k|x-Tx|^2 \ge |x-1|^2 &= x^2-2x+1 = x^2+2x+1-4x \\ &= |x+1|^2 - 4x. \end{aligned}$$

Since $x \in [-1,0)$, we have $-4x \ge 0$. Then we get $|x-1|^2 + k|x-Tx|^2 \ge |x+1|^2 \ge \frac{1}{10}|x+1|^2 = |Tx-1|^2$.

Case 2: $x \in [0,1]$. Then we have $Tx = \frac{x+9}{10}$. From the definition of T, we have $|Tx-1|^2 = |\frac{x+9}{10}-1|^2 = \frac{1}{10}|x-1|^2$ and $|x-Tx|^2 = |x-\frac{x+9}{10}|^2 = \frac{81}{100}|x-1|^2 \ge 0$. Then, there exists $k \in [0,1)$ such that

$$|x-1|^2 + k|x-Tx|^2 \ge |x-1|^2 \ge \frac{1}{10}|x-1|^2 = |Tx-1|^2.$$

Therefore, inequality (1.1) holds. Hence, T is demicontractive.

Example 1.4 ([3, p. 862])**.** Let H be the real line, and let $C=[-2,1]$. Define a mapping $T: C \to C$ by $Tx = -x^2 - x$ for all $x \in C$. This mapping is demicontractive but not quasi-nonexpansive.

Let $B: C \to H$ be a mapping. The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Bu, v-u\rangle \ge 0, \quad \forall v \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $\mathrm{VI}(C,B)$.

A mapping $B: C \to H$ is called *ϕ-inverse strongly monotone* (see [5]) if there exists a positive real number ϕ such that

$$\langle x-y, Bx-By\rangle \ge \phi\|Bx-By\|^2, \quad \forall x,y \in C.$$

The variational inequality theory, which was first introduced by Stampacchia [17] in 1964, has emerged as an interesting and fascinating branch of applied mathematics with a wide range of applications in economics, network analysis, optimizations, pure and applied sciences, and so on. In recent years, much attention has been given to developing efficient iterative methods for solving variational inequality problems (see, e.g., [1], [11]).

In 2003, Takahashi and Toyoda [20] introduced the following iterative scheme for finding a solution of the variational inequality problem in a Hilbert space. For an initial point $x_1 \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)TP_C(x_n - \eta_n Bx_n), \quad n \ge 1, \tag{1.3}$$

where $T: C \to C$ is a nonexpansive mapping, $B: C \to H$ is a ϕ-inverse strongly monotone mapping, and $\{\alpha_n\}$ is a sequence in (0,1). Under some restrictions on parameters, they proved that the sequence $\{x_n\}$ generated by (1.3) converges weakly to a point $u \in F(T) \cap \mathrm{VI}(C,B)$, where $u = \lim_{n\to\infty} P_{F(T)\cap \mathrm{VI}(C,B)} x_n$.

In 2008, Takahashi, Takeuchi, and Kubota [19] introduced the following iterative scheme, known as the *shrinking projection method*, for finding a fixed point of a nonexpansive mapping $T : C \to C$ in a Hilbert space. For an initial point $x_1 \in C$, $C_1 = C$, define sequences $\{x_n\}$ and $\{y_n\}$ recursively by

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{p \in C_n : \|y_n - p\| \le \|x_n - p\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. They proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a point $u = P_{F(T)}x_1$ under some suitable conditions.

In 2012, Kangtunyakarn [6] introduced the following iterative scheme based on the shrinking projection method for finding a common element of the set of solutions of variational inequality problems and the set of common fixed points of a finite family of nonspreading mappings $\{T_i\}_{i=1}^N$ in a Hilbert space. For an initial point $x_1 \in C$, $C_1 = C$, define sequences $\{x_n\}$ and $\{y_n\}$ recursively by

$$\begin{cases} y_n = \alpha_n x_n + \beta_n Sx_n + \gamma_n \sum_{i=1}^N \delta_n^i P_C(I - \eta B_i)x_n, \\ C_{n+1} = \{p \in C_n : \|y_n - p\| \le \|x_n - p\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 1, \end{cases} \tag{1.5}$$

where $S : C \to C$ is an S-mapping generated by $T_1, T_2, \dots, T_N$ and $\rho_1, \rho_2, \dots, \rho_N$, and where $\{B_i\}_{i=1}^N$ is a finite family of ϕ-inverse strongly monotone mappings of C into H. He proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a point $u = P_{\mathcal{F}}x_1$ under some suitable conditions, where $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N \mathrm{VI}(C, B_i)$.

The shrinking projection method is a popular method that plays an important role in studying the strong convergence for finding fixed points of nonlinear mappings. Many researchers developed the shrinking projection method for solving variational inequality problems and fixed point problems in Hilbert and Banach spaces (see, e.g., [8], [7]).

Motivated and inspired by the above results and related literature, we introduce a new hybrid shrinking projection method for finding a common element of the set of common fixed points of a finite family of demicontractive mappings and of the set of common solutions of a finite family of variational inequalities for ϕ-inverse strongly monotone mappings in a Hilbert space. Then we prove some strong convergence theorems which extend and improve the corresponding results of Takahashi and Toyoda [20], Takahashi, Takeuchi, and Kubota [19], Kangtunyakarn [6], and many others. Finally, we provide numerical examples in support of our main results.

2. Preliminaries

In this section, we give some useful lemmas to prove our main results. Let C be a nonempty closed convex subset of a real Hilbert space H. It is also known that a Hilbert space H satisfies *Opial's condition* (see [15, Lemma 1]); that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Let P_C be the metric projection of H onto C; that is, for $x \in H$, $P_C x$ satisfies the property $\|x - P_C x\| = \min_{y\in C} \|x - y\|$. It is well known that P_C is a nonexpansive mapping of H onto C.

Lemma 2.1 ([18, Lemma 3.1.3]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let B be a mapping of C into H. Let $u \in C$. Then, for $\eta > 0$, $u = P_C(I - \eta B)u$ if and only if $u \in \mathrm{VI}(C, B)$.*

Lemma 2.2 ([12, Lemma 1.3]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $P_C : H \to C$ be the metric projection. Given $x \in H$ and $z \in C$, we have $z = P_C x$ if and only if the following holds:*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.3 ([14, p. 375]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $P_C : H \to C$ be the metric projection. Then the following inequality holds:*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, y \in C.$$

Lemma 2.4 ([21, Lemma 1.1]). *Let H be a Hilbert space. Let $x_1, x_2, \ldots, x_N \in H$ and*
$\alpha_1, \alpha_2, \ldots, \alpha_N$ be real numbers in $[0, 1]$ such that $\sum_{i=1}^{N} \alpha_i = 1$. Then,

$$\Big\|\sum_{i=1}^{N} \alpha_i x_i\Big\|^2 = \sum_{i=1}^{N} \alpha_i \|x_i\|^2 - \sum_{1\leq i,j\leq N} \alpha_i\alpha_j \|x_i - x_j\|^2.$$

Lemma 2.5 ([13, Lemma 1.3]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x, y, z \in H$ and α a real number, the set $\{u \in C : \|y - u\|^2 \leq \|x - u\|^2 + \langle z, u\rangle + \alpha\}$ is closed and convex.*

3. Main results

In this section, we prove the strong convergence theorems for finding a common element of the set of common fixed points of a finite family of demicontractive mappings and the set of common solutions of a finite family of variational inequality problems in a Hilbert space, and we give a numerical example in support our main results.

We now present the following useful properties of a demicontractive mapping in a Hilbert space.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \to C$ be a demicontractive mapping with contraction coefficient k. Then the following hold:*

(i) $F(T) = \mathrm{VI}(C, I - T)$,
(ii) $F(T) = F(P_C(I - \lambda(I - T)))$ *for all* $\lambda > 0$,
(iii) $F(T)$ *is closed and convex,*

(iv) $P_C(I-\lambda(I-T))$ *is quasi-nonexpansive, for all* $\lambda \in (0, 1-k]$.

Proof. (i) It is easy to see that $F(T) \subseteq \mathrm{VI}(C, I-T)$. Assume that $z \in \mathrm{VI}(C, I-T)$ and $q \in F(T)$. Since $z \in \mathrm{VI}(C, I-T)$, we have $\langle y-z, (I-T)z\rangle \geq 0, \forall y \in C$. Consider

$$\begin{aligned} \|Tz-Tq\|^2 &= \big\|\big(I-(I-T)\big)z - \big(I-(I-T)\big)q\big\|^2 \\ &= \big\|(z-q) - \big((I-T)z-(I-T)q\big)\big\|^2 \\ &= \|z-q\|^2 + \big\|(I-T)z-(I-T)q\big\|^2 \\ &\quad - 2\big\langle z-q, (I-T)z-(I-T)q\big\rangle \\ &= \|z-q\|^2 + \big\|(I-T)z\big\|^2 - 2\big\langle z-q, (I-T)z\big\rangle. \end{aligned} \tag{3.1}$$

Since T is demicontractive, we have

$$\|Tz-Tq\|^2 \leq \|z-q\|^2 + k\big\|(I-T)z\big\|^2. \tag{3.2}$$

By (3.1) and (3.2), we have

$$(1-k)\big\|(I-T)z\big\|^2 \leq 2\big\langle z-q, (I-T)z\big\rangle = -2\big\langle q-z, (I-T)z\big\rangle \leq 0.$$

Then we have $z \in F(T)$, and so $\mathrm{VI}(C, I-T) \subseteq F(T)$. Hence, $F(T) = \mathrm{VI}(C, I-T)$.

(ii) From Lemmas 2.1 and 3.1, we have $F(T) = F(P_C(I-\lambda(I-T))), \forall \lambda > 0$.

(iii) To show that $F(T)$ is closed, we let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x$. Since T is demicontractive, we have $\|Tx-Tx_n\|^2 \leq \|x-x_n\|^2 + k\|x-Tx\|^2$. It follows that

$$\begin{aligned} \|x-Tx\| &\leq \|x-Tx_n\| + \|Tx_n-Tx\| \\ &\leq \|x-x_n\| + \sqrt{\|x-x_n\|^2 + k\|x-Tx\|^2}. \end{aligned}$$

By letting $n \to \infty$ in the above inequality, we get $(1-\sqrt{k})\|x-Tx\| \leq 0$. This implies by $k \in [0,1)$ that $\|x-Tx\| = 0$. Hence, $x = Tx$ so that $F(T)$ is closed.

Finally, we show that $F(T)$ is convex. Let $z = \alpha x + (1-\alpha)y$, where $x, y \in F(T)$ and $\alpha \in (0,1)$. We need to show that $z \in F(T)$. By the definition of T, we obtain

$$\begin{aligned} \|z-Tz\|^2 &= \big\|\alpha(x-Tz) + (1-\alpha)(y-Tz)\big\|^2 \\ &= \alpha\|Tz-x\|^2 + (1-\alpha)\|Tz-y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\ &\leq \alpha\big(\|z-x\|^2 + k\|z-Tz\|^2\big) + (1-\alpha)\big(\|z-y\|^2 + k\|z-Tz\|^2\big) \\ &\quad - \alpha(1-\alpha)\|x-y\|^2 \\ &= k\|z-Tz\|^2. \end{aligned}$$

Thus, $(1-k)\|z-Tz\|^2 = 0$. This implies that $z = Tz$ and hence $F(T)$ is convex.

(iv) Let $x \in C$ and $z \in F(T)$. Then, by (ii), we have $z \in F(P_C(I-\lambda(I-T)))$. This implies that

$$\begin{aligned} \big\|P_C\big(I-\lambda(I-T)\big)x - z\big\|^2 &\leq \big\|\big(I-\lambda(I-T)\big)x - z\big\|^2 \\ &= \big\|(x-z) - \lambda(I-T)x\big\|^2 \\ &= \|x-z\|^2 - 2\lambda\big\langle x-z, (I-T)x\big\rangle \end{aligned}$$

$$+ \lambda^2\big\|(I-T)x\big\|^2. \tag{3.3}$$

By the fact that T is demicontractive, we have

$$\|Tx - Tz\|^2 \le \|x-z\|^2 + k\big\|(I-T)x\big\|^2. \tag{3.4}$$

Since

$$\begin{aligned}\|Tx - Tz\|^2 &= \Big\|\big(I-(I-T)\big)x - \big(I-(I-T)\big)z\Big\|^2\\ &= \Big\|(x-z) - \big((I-T)x-(I-T)z\big)\Big\|^2\\ &= \|x-z\|^2 - 2\big\langle x-z,(I-T)x\big\rangle + \big\|(I-T)x\big\|^2,\end{aligned}$$

it follows by (3.4) that $(1-k)\|(I-T)x\|^2 \le 2\langle x-z,(I-T)x\rangle$. Therefore, by (3.3), we have

$$\begin{aligned}\big\|P_C\big(I-\lambda(I-T)\big)x - z\big\|^2 &\le \|x-z\|^2 - (1-k)\lambda\big\|(I-T)x\big\|^2 + \lambda^2\big\|(I-T)x\big\|^2\\ &= \|x-z\|^2 - \lambda(1-k-\lambda)\big\|(I-T)x\big\|^2.\end{aligned}$$

This implies by $\lambda \in (0,1-k]$ that $\|P_C(I-\lambda(I-T))x - z\| \le \|x-z\|$. Thus, $P_C(I-\lambda(I-T))$ is a quasi-nonexpansive mapping. □

We now prove the following strong convergence theorems.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of continuous and demicontractive mappings of C into itself with contraction coefficient k, and let $\{B_i\}_{i=1}^N$ be a finite family of ϕ-inverse strongly monotone mappings from C into H. Assume that $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N \mathrm{VI}(C,B_i) \neq \emptyset$. For an initial point $x_1 \in C$ with $C_1 = C$, define sequences $\{x_n\}$ and $\{y_n\}$ recursively by*

$$\begin{cases} w_n = \sum_{i=1}^N \sigma_i P_C(I-\eta B_i)x_n,\\ z_n = \sum_{i=1}^N \xi_i P_C(I-\lambda(I-T_i))x_n,\\ y_n = \alpha_n x_n + \beta_n z_n + \gamma_n w_n,\\ C_{n+1} = \{p \in C_n : \|y_n - p\| \le \|x_n - p\|\},\\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 1, \end{cases} \tag{3.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$, and suppose that the following conditions hold:

(i) $\sigma_i, \xi_i \in (0,1]$ *and* $\sum_{i=1}^N \sigma_i = \sum_{i=1}^N \xi_i = 1$;
(ii) $\eta \in [d,e]$ *for some* $d,e \in (0,2\phi)$ *and* $\lambda \in (0,1-k]$;
(iii) $0 < a \le \alpha_n, \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $u = P_{\mathcal{F}}x_1$.

Proof. Let $x, y \in C$. Since B_i is ϕ-inverse strongly monotone, for all $i = 1,2,\dots,N$, we have

$$\begin{aligned}&\big\|P_C(I-\eta B_i)x - P_C(I-\eta B_i)y\big\|^2\\ &\quad\le \big\|(I-\eta B_i)x - (I-\eta B_i)y\big\|^2\\ &\quad= \|x-y\|^2 - 2\eta\langle x-y, B_i x - B_i y\rangle + \eta^2\|B_i x - B_i y\|^2\end{aligned}$$

$$\begin{aligned}&\le \|x-y\|^2 - 2\eta\phi\|B_ix - B_iy\|^2 + \eta^2\|B_ix - B_iy\|^2\\&= \|x-y\|^2 - \eta(2\phi - \eta)\|B_ix - B_iy\|^2\\&\le \|x-y\|^2 - d(2\phi - e)\|B_ix - B_iy\|^2\\&\le \|x-y\|^2.\end{aligned}$$

This shows that $P_C(I-\eta B_i)$ is nonexpansive for all $i = 1, 2, \ldots, N$. It follows by Lemma 2.1 and the closedness and convexity of $F(P_C(I-\eta B_i))$ that $\mathrm{VI}(C, B_i) = F(P_C(I-\eta B_i))$ is closed and convex for all $i = 1, 2, \ldots, N$. Thus, $\bigcap_{i=1}^N \mathrm{VI}(C, B_i)$ is closed and convex. Then, by Lemma 3.1(iii), we have that $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N \mathrm{VI}(C, B_i)$ is also closed and convex; consequently, $P_{\mathcal{F}}x_1$ is well defined for every $x_1 \in C$.

Next, we divide the proof into five steps.

Step 1. We show that $\{x_n\}$ is well defined for every $n \in \mathbb{N}$.

Let $p \in \mathcal{F}$. Then, by Lemmas 2.1 and 3.1(ii), we have $P_C(I-\eta B_i)p = p$ and $P_C(I-\lambda(I-T_i))p = p$ for all $i = 1, 2, \ldots, N$. By Lemma 3.1(iv), we have

$$\begin{aligned}\|z_n - p\| &\le \sum_{i=1}^N \xi_i \big\|P_C\big(I-\lambda(I-T_i)\big)x_n - p\big\| \le \sum_{i=1}^N \xi_i\|x_n - p\|\\&= \|x_n - p\|.\end{aligned} \tag{3.6}$$

By the nonexpansiveness of $P_C(I-\eta B_i)$, we have

$$\|w_n - p\| \le \sum_{i=1}^N \sigma_i \big\|P_C(I-\eta B_i)x_n - p\big\| \le \sum_{i=1}^N \sigma_i\|x_n - p\| = \|x_n - p\|. \tag{3.7}$$

From the above, we get that $\|y_n - p\| \le \alpha_n\|x_n - p\| + \beta_n\|z_n - p\| + \gamma_n\|w_n - p\| \le \|x_n - p\|$. This shows that $p \in C_{n+1}$ and hence that $\mathcal{F} \subset C_{n+1} \subset C_n$. By Lemma 2.5, we observe that C_n is closed and convex. Hence, $P_{C_{n+1}}x_1$ is well defined for every $x_1 \in C$. Therefore, $\{x_n\}$ is well defined.

Step 2. We show that $\lim_{n\to\infty} x_n = q$ for some $q \in C$.

Since $\mathcal{F}$ is a nonempty closed convex subset of H, there exists a unique $\omega \in \mathcal{F}$ such that $\omega = P_{\mathcal{F}}x_1$. From $x_n = P_{C_n}x_1$ and $x_{n+1} \in C_{n+1} \subset C_n$, for all $n \ge 1$, we have $\|x_n - x_1\| \le \|x_{n+1} - x_1\|$ for all $n \ge 1$. On the other hand, by $\mathcal{F} \subset C_n$, we obtain that $\|x_n - x_1\| \le \|\omega - x_1\|$ for all $n \ge 1$. Hence, $\{\|x_n - x_1\|\}$ is bounded; so are $\{w_n\}$, $\{z_n\}$, and $\{y_n\}$. Therefore, $\lim_{n\to\infty}\|x_n - x_1\|$ exists. By the construction of the set C_n, we know that $x_m = P_{C_m}x_1 \in C_m \subset C_n$ for $m > n \ge 1$. This implies by Lemma 2.3 that

$$\|x_m - x_n\|^2 \le \|x_m - x_1\|^2 - \|x_n - x_1\|^2 \to 0, \quad \text{as } m, n \to \infty. \tag{3.8}$$

Since $\lim_{n\to\infty}\|x_n - x_1\|$ exists, it implies that $\{x_n\}$ is a Cauchy sequence. By the completeness of H and the closedness of C, we get that there exists an element $q \in C$ such that $\lim_{n\to\infty} x_n = q$.

Step 3. We show that $q \in \bigcap_{i=1}^N F(T_i)$.

From (3.8), we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.9}$$

Since $x_{n+1} \in C_{n+1}$, we get that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\|.$$

This implies by (3.9) that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.10}$$

Thus, $\lim_{n\to\infty} y_n = q$.

For $p \in \mathcal{F}$, by Lemma 2.4, (3.6), and (3.7), we have

$$\begin{aligned}\|y_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 + \gamma_n\|w_n - p\|^2 - \alpha_n\beta_n\|z_n - x_n\|^2 \\ &\quad - \alpha_n\gamma_n\|w_n - x_n\|^2 - \beta_n\gamma_n\|z_n - w_n\|^2 \\ &\leq \|x_n - p\|^2 - \alpha_n\beta_n\|z_n - x_n\|^2 - \alpha_n\gamma_n\|w_n - x_n\|^2 - \beta_n\gamma_n\|z_n - w_n\|^2.\end{aligned}$$

This implies by condition (iii) that

$$\begin{aligned}&a^2\|z_n - x_n\|^2 + a^2\|w_n - x_n\|^2 + a^2\|z_n - w_n\|^2 \\ &\quad\leq \alpha_n\beta_n\|z_n - x_n\|^2 + \alpha_n\gamma_n\|w_n - x_n\|^2 + \beta_n\gamma_n\|z_n - w_n\|^2 \\ &\quad\leq \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\quad\leq \|x_n - y_n\|\big(\|x_n - p\| + \|y_n - p\|\big).\end{aligned}$$

Then, by (3.10) and $a > 0$, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0, \qquad \lim_{n\to\infty} \|w_n - x_n\| = 0, \qquad \lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.11}$$

Since T_i is continuous for all $i = 1, 2, \ldots, N$, we have that $P_C(I - \lambda(I - T_i))$ is continuous for all $i = 1, 2, \ldots, N$. So, $\sum_{i=1}^{N} \xi_i P_C(I - \lambda(I - T_i))$ is continuous. By $\lim_{n\to\infty} x_n = q$, this implies that

$$\lim_{n\to\infty} \sum_{i=1}^{N} \xi_i P_C\big(I - \lambda(I - T_i)\big)x_n = \sum_{i=1}^{N} \xi_i P_C\big(I - \lambda(I - T_i)\big)q. \tag{3.12}$$

Since

$$\begin{aligned}&\Big\|\sum_{i=1}^{N} \xi_i P_C\big(I - \lambda(I - T_i)\big)q - q\Big\| \\ &\quad\leq \Big\|\sum_{i=1}^{N} \xi_i P_C\big(I - \lambda(I - T_i)\big)q - \sum_{i=1}^{N} \xi_i P_C\big(I - \lambda(I - T_i)\big)x_n\Big\| \\ &\qquad + \Big\|\sum_{i=1}^{N} \xi_i P_C\big(I - \lambda(I - T_i)\big)x_n - x_n\Big\| + \|x_n - q\| \\ &\quad= \Big\|\sum_{i=1}^{N} \xi_i P_C\big(I - \lambda(I - T_i)\big)q - \sum_{i=1}^{N} \xi_i P_C\big(I - \lambda(I - T_i)\big)x_n\Big\| + \|z_n - x_n\| \\ &\qquad + \|x_n - q\|,\end{aligned}$$

this implies by $\lim_{n\to\infty} x_n = q$, (3.11), and (3.12) that

$$\sum_{i=1}^{N} \xi_i P_C\big(I - \lambda(I - T_i)\big)q = q. \tag{3.13}$$

Next, we will show that $q \in \bigcap_{i=1}^{N} F(P_C(I - \lambda(I - T_i)))$. To show this, we let $z \in \bigcap_{i=1}^{N} F(P_C(I - \lambda(I - T_i)))$. Then, by (3.13) and Lemma 3.1(iv), we have

$$\begin{aligned}\|q - z\| &= \Big\|\sum_{i=1}^{N} \xi_i P_C\big(I - \lambda(I - T_i)\big)q - z\Big\| \leq \sum_{i=1}^{N} \xi_i \big\|P_C\big(I - \lambda(I - T_i)\big)q - z\big\| \\ &\leq \|q - z\|.\end{aligned}$$

This shows that $\|q - z\| = \sum_{i=1}^{N} \xi_i \|P_C(I - \lambda(I - T_i))q - z\|$. Thus,

$$\|q - z\| = \big\|P_C\big(I - \lambda(I - T_i)\big)q - z\big\|, \quad \forall i = 1, 2, \dots, N. \tag{3.14}$$

By Lemma 2.4, we obtain

$$\begin{aligned}&\Big\|\sum_{i=1}^{N} \xi_i P_C\big(I - \lambda(I - T_i)\big)q - z\Big\|^2 \\ &\quad = \sum_{i=1}^{N} \xi_i \big\|P_C\big(I - \lambda(I - T_i)\big)q - z\big\|^2 \\ &\qquad - \sum_{1\leq i,j\leq N} \xi_i\xi_j \big\|P_C\big(I - \lambda(I - T_i)\big)q - P_C\big(I - \lambda(I - T_j)\big)q\big\|^2.\end{aligned}$$

By (3.14), we have

$$\|q - z\|^2 = \|q - z\|^2 - \sum_{1\leq i,j\leq N} \xi_i\xi_j \big\|P_C\big(I - \lambda(I - T_i)\big)q - P_C\big(I - \lambda(I - T_j)\big)q\big\|^2,$$

and so $\sum_{1\leq i,j\leq N} \xi_i\xi_j \|P_C(I-\lambda(I-T_i))q - P_C(I-\lambda(I-T_j))q\|^2 = 0$. Since $\xi_i \in (0, 1]$, we get $\|P_C(I - \lambda(I - T_i))q - P_C(I - \lambda(I - T_j))q\|^2 = 0$ for all $i, j \in \{1, 2, \dots, N\}$. That is,

$$P_C\big(I - \lambda(I - T_i)\big)q = P_C\big(I - \lambda(I - T_j)\big)q, \quad \forall i, j \in \{1, 2, \dots, N\}.$$

Then, for all $j = 1, 2, \dots, N$, we have

$$q = \sum_{i=1}^{N} \xi_i P_C\big(I - \lambda(I - T_i)\big)q = \sum_{i=1}^{N} \xi_i P_C\big(I - \lambda(I - T_j)\big)q = P_C\big(I - \lambda(I - T_j)\big)q.$$

Thus, $q \in \bigcap_{i=1}^{N} F(P_C(I - \lambda(I - T_i)))$. By Lemma 3.1(ii), we obtain that $F(P_C(I - \lambda(I - T_i))) = F(T_i)$ for all $i = 1, 2, \dots, N$. Hence, $q \in \bigcap_{i=1}^{N} F(T_i)$.

Step 4. We show that $q \in \bigcap_{i=1}^{N} \mathrm{VI}(C, B_i)$.

In fact, since $P_C(I-\eta B_i)$ is nonexpansive, we have

$$\begin{aligned}\Big\|\sum_{i=1}^N \sigma_i P_C(I-\eta B_i)q - w_n\Big\| &= \Big\|\sum_{i=1}^N \sigma_i P_C(I-\eta B_i)q - \sum_{i=1}^N \sigma_i P_C(I-\eta B_i)x_n\Big\| \\ &= \Big\|\sum_{i=1}^N \sigma_i \big(P_C(I-\eta B_i)q - P_C(I-\eta B_i)x_n\big)\Big\| \\ &\le \sum_{i=1}^N \sigma_i \big\|P_C(I-\eta B_i)q - P_C(I-\eta B_i)x_n\big\| \\ &\le \|q-x_n\|. \end{aligned} \tag{3.15}$$

From $x_n \to q \in C$ as $n\to\infty$, it follows by (3.11) and (3.15) that

$$\begin{aligned}\Big\|\sum_{i=1}^N \sigma_i P_C(I-\eta B_i)q - q\Big\| &\le \Big\|\sum_{i=1}^N \sigma_i P_C(I-\eta B_i)q - w_n\Big\| + \|w_n - x_n\| \\ &\quad + \|x_n - q\| \\ &\le 2\|x_n-q\| + \|w_n-x_n\| \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

This shows that

$$\sum_{i=1}^N \sigma_i P_C(I-\eta B_i)q = q. \tag{3.16}$$

Next, we will show that $q \in \bigcap_{i=1}^N F(P_C(I-\eta B_i))$. To show this, we let $z \in \bigcap_{i=1}^N F(P_C(I-\eta B_i))$. Then, by (3.16), we have

$$\|q-z\| = \Big\|\sum_{i=1}^N \sigma_i P_C(I-\eta B_i)q - z\Big\| \le \sum_{i=1}^N \sigma_i \big\|P_C(I-\eta B_i)q - z\big\| \le \|q-z\|.$$

This shows that $\|q-z\| = \sum_{i=1}^N \sigma_i \|P_C(I-\eta B_i)q - z\|$. Thus,

$$\|q-z\| = \big\|P_C(I-\eta B_i)q - z\big\|, \quad \forall i = 1,2,\ldots,N. \tag{3.17}$$

By Lemma 2.4, we obtain

$$\begin{aligned}\Big\|\sum_{i=1}^N \sigma_i P_C(I-\eta B_i)q - z\Big\|^2 &= \sum_{i=1}^N \sigma_i \big\|P_C(I-\eta B_i)q - z\big\|^2 \\ &\quad - \sum_{1\le i,j\le N} \sigma_i\sigma_j \big\|P_C(I-\eta B_i)q - P_C(I-\eta B_j)q\big\|^2.\end{aligned}$$

By (3.17), we have $\|q-z\|^2 = \|q-z\|^2 - \sum_{1\le i,j\le N}\sigma_i\sigma_j\|P_C(I-\eta B_i)q - P_C(I-\eta B_j)q\|^2$, and so $\sum_{1\le i,j\le N}\sigma_i\sigma_j\|P_C(I-\eta B_i)q - P_C(I-\eta B_j)q\|^2 = 0$. Since $\sigma_i \in (0,1]$, we get $\|P_C(I-\eta B_i)q - P_C(I-\eta B_j)q\|^2 = 0$ for all $i,j \in \{1,2,\ldots,N\}$. That is,

$$P_C(I-\eta B_i)q = P_C(I-\eta B_j)q, \quad \forall i,j \in \{1,2,\ldots,N\}.$$

Then, for all $j = 1, 2, \ldots, N$, we have

$$q = \sum_{i=1}^{N} \sigma_i P_C(I - \eta B_i)q = \sum_{i=1}^{N} \sigma_i P_C(I - \eta B_j)q = P_C(I - \eta B_j)q.$$

Thus, $q \in \bigcap_{i=1}^{N} F(P_C(I - \eta B_i))$. By Lemma 2.1, we have $F(P_C(I - \eta B_i)) = \mathrm{VI}(C, B_i)$ for all $i = 1, 2, \ldots, N$. Then we have $q \in \bigcap_{i=1}^{N} \mathrm{VI}(C, B_i)$.

Step 5. Finally, we show that $q = u = P_{\mathcal{F}} x_1$. Since $x_n = P_{C_n} x_1$ and $\mathcal{F} \subset C_n$, we obtain $\langle x_1 - x_n, x_n - p\rangle \geq 0$ for all $p \in \mathcal{F}$. Thus, we get $\langle x_1 - q, q - p\rangle \geq 0$ for all $p \in \mathcal{F}$. This shows that $q = P_{\mathcal{F}} x_1 = u$.

By Steps 1–5, we can conclude that $\{x_n\}$ and $\{y_n\}$ converge strongly to $u = P_{\mathcal{F}} x_1$. This completes the proof. □

As a direct consequence of Theorem 3.2, we have the following corollary.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous and demicontractive mapping with contraction coefficient k, and let $B : C \to H$ be a ϕ-inverse strongly monotone mapping. Assume that $\mathcal{F} := F(T) \cap \mathrm{VI}(C, B) \neq \emptyset$. For an initial point $x_1 \in C$ with $C_1 = C$, define sequences $\{x_n\}$ and $\{y_n\}$ recursively by*

$$\begin{cases} y_n = \alpha_n x_n + \beta_n P_C(I - \lambda(I - T))x_n + \gamma_n P_C(I - \eta B)x_n, \\ C_{n+1} = \{p \in C_n : \|y_n - p\| \leq \|x_n - p\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1, \end{cases} \tag{3.18}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0, 1]$, and suppose that the following conditions hold:

(i) *$\eta \in [d, e]$ for some $d, e \in (0, 2\phi)$ and $\lambda \in (0, 1 - k]$;*
(ii) *$0 < a \leq \alpha_n, \beta_n, \gamma_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$.*

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $u = P_{\mathcal{F}} x_1$.

Proof. In Theorem 3.2, we put $N = 1$, $T_1 = T$, $B_1 = B$, $\sigma_1 = 1$, and $\xi_1 = 1$. Hence, we obtain the desired result from Theorem 3.2. □

Remark 3.4. It is known that the class of demicontractive mappings contains the classes of nonexpansive mappings, nonspreading mappings, quasi-nonexpansive mappings, and strictly pseudo-nonspreading mappings. Thus, Theorem 3.2 and Corollary 3.3 can be applied to these classes of mappings.

Finally, we give a numerical example supporting Theorem 3.2. All codes were written in Scilab.

Example 3.5. Let H be a real line with the Euclidean norm, and let $C = [0, 15]$. For all $x \in C$, we define mappings T_1, T_2, B_1, B_2 on C as follows:

$$T_1 x = \begin{cases} \frac{4}{7} x \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad T_2 x = -\frac{5x}{7}, \qquad B_1 x = \frac{x}{15}, \qquad B_2 x = \frac{x}{5}.$$

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by (3.5), where $\xi_1 = \xi_2 = \frac{1}{2}$, $\lambda = \frac{1}{6}$, $\sigma_1 = \frac{1}{4}$, $\sigma_2 = \frac{3}{4}$, $\eta = 4$, $\alpha_n = \frac{19}{27} + \frac{4}{135n}$, $\beta_n = \frac{10n-1}{135n}$, $\gamma_n = \frac{10n-1}{45n}$, for all $n \geq 1$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to zero.

Solution. Obviously, T_1 and T_2 are demicontractive mappings with contraction coefficient $\frac{2}{3}$, and B_1 and B_2 are 5-inverse strongly monotone mappings. It can be observed that all the assumptions of Theorem 3.2 are satisfied. It is also easy to see that $\bigcap_{i=1}^{2} F(T_i) \cap \bigcap_{i=1}^{2} \mathrm{VI}(C, B_i) = \{0\}$. For any arbitrary $x_1 \in C = C_1 = [0, 15]$, it follows by (3.5) that $0 \leq y_1 \leq x_1 \leq 15$. Then we have $C_2 = \{p \in C_1 : |y_1 - p| \leq |x_1 - p|\} = [0, \frac{x_1+y_1}{2}]$. Since $\frac{x_1+y_1}{2} \leq x_1$, we get $x_2 = P_{C_2}x_1 = \frac{x_1+y_1}{2}$. By continuing this process, we obtain that $C_{n+1} = \{p \in C_n : |y_n - p| \leq |x_n - p|\} = [0, \frac{x_n+y_n}{2}]$, and hence that $x_{n+1} = P_{C_{n+1}}x_1 = \frac{x_n+y_n}{2}$.

Now, we rewrite the algorithm (3.5) as follows:

$$
\begin{aligned}
&x_1 \in [0, 15],\\
&y_n = \begin{cases} (\frac{9470n+187}{11340n})x_n + (\frac{10n-1}{2835n})x_n \sin(\frac{1}{x_n}), & x_n \neq 0,\\ 0, & x_n = 0, \end{cases}\\
&x_{n+1} = \frac{x_n + y_n}{2}, \quad n \geq 1.
\end{aligned}
$$

The values of the sequences $\{x_n\}$ and $\{y_n\}$ with different n's are reported in Table 1.

Remark 3.6. Table 1 and Figure 1 show that the sequences $\{x_n\}$ and $\{y_n\}$ converge to zero, where $\{0\} = \bigcap_{i=1}^{2} F(T_i) \cap \bigcap_{i=1}^{2} \mathrm{VI}(C, B_i)$.

TABLE 1. The values of the sequences $\{x_n\}$ and $\{y_n\}$ in Example 3.5.

	$x_1 = 4$		$x_1 = 11.5$	
n	x_n	y_n	x_n	y_n
1	4.0000000	3.4094908	11.5000000	9.7964246
2	3.7047454	3.1276784	10.6482123	8.9834323
3	3.4162119	2.8750077	9.8158223	8.2545229
4	3.1456098	2.6432389	9.0351726	7.5859259
5	2.8944243	2.4300595	8.3105492	6.9709719
⋮	⋮	⋮	⋮	⋮
100	0.0008595	0.0007206	0.0024875	0.0020767
⋮	⋮	⋮	⋮	⋮
176	0.0000012	0.0000010	0.0000035	0.0000030
177	0.0000011	0.0000010	0.0000032	0.0000027
178	0.0000010	0.0000009	0.0000030	0.0000025
179	0.0000010	0.0000008	0.0000027	0.0000023
180	0.0000009	0.0000007	0.0000025	0.0000021

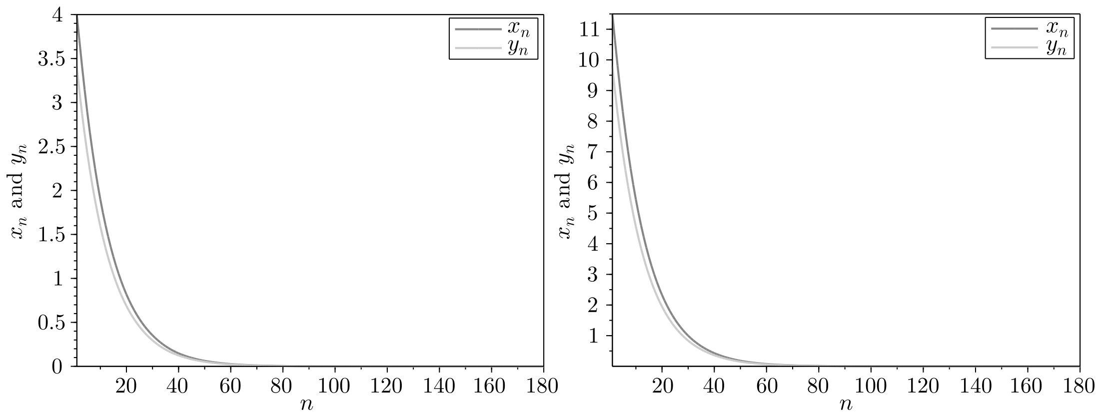

FIGURE 1. The convergence of the sequences $\{x_n\}$ and $\{y_n\}$ with initial values 4 and 11.5 in Example 3.5.

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