Banach J. Math. Anal. 11 (2017), no. 3, 536–553
http://dx.doi.org/10.1215/17358787-2017-0005
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

TOEPLITZ ALGEBRAS ARISING FROM ESCAPE POINTS OF INTERVAL MAPS

C. CORREIA RAMOS,[1] NUNO MARTINS,[2] and PAULO R. PINTO[2*]

Communicated by J. A. Ball

Abstract. We generate a representation of the Toeplitz C^*-algebra $\mathcal{T}_{A_f}$ on a Hilbert space H_x that encodes the orbit of an escape point $x \in I$ of a Markov interval map f, with transition matrix A_f. This leads to a family of representations of $\mathcal{T}_{A_f}$ labeled by points in all intervals I. The underlying dynamics of the interval map are used in the study of this family.

1. Introduction

The Toeplitz algebra $\mathcal{T}_A$ of a finite $(0,1)$-matrix A is an extension of the Cuntz–Krieger algebra $\mathcal{O}_A$ in which the Cuntz–Krieger relations are replaced by inequalities (see [8], [10], [11]). In [4] and [3], we produced and studied orbit representations π_x of the Cuntz–Krieger algebra $\mathcal{O}_{A_f}$ associated with the transition $(0,1)$-matrix A_f arising from a Markov interval map f. Using β-transformations $f(x) = \beta x \pmod 1$, we were able to recover Bratteli and Jorgensen permutation representations of the Cuntz algebra (where $A_f = (a_{ij})$ is the full matrix $a_{ij} = 1$ for all i's and j's) (see [2]). The representation π_x acts on a Hilbert space H_x that naturally arises from the generalized orbit of a point x, provided x remains in the domain of f under the iteration of f.

In the following we consider the case where the point x is in the *escape set* of f (i.e., $f^k(x) \in I$ does not belong to the domain of f for a certain $k \in \mathbb{N}$). We can likewise define a Hilbert space H_x from the backward orbit of x and partial

Received Apr. 19, 2016; Accepted Aug. 23, 2016.
First published online Apr. 29, 2017.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46L55; Secondary 37B10, 37E05, 46L05.
Keywords. Toeplitz algebra, interval maps, orbit representations.

isometries on H_x. If x is in the escape set, we prove that these operators do define a representation ν_x of the Toeplitz algebra $\mathcal{T}_{A_f}$ on H_x (which is no longer a representation of the Cuntz–Krieger algebra $\mathcal{O}_{A_f}$). The backward orbit of x has a natural structure of a rooted tree, where the root is the point in the escape set. We show that these rooted trees together with the underlying dynamics can be used to understand when two such representations of $\mathcal{T}_{A_f}$ are unitarily equivalent. We show that these representations ν_x can be recast in a Fock-like space F_x, as in the original spirit of [7, Section 5.2].

Therefore, we now have a representation of $\mathcal{T}_{A_f}$ for each point in I (either a point is in the escape set or it remains in the domain of f for any iteration of f). We show that the representations π_x of $\mathcal{T}_{A_f}$ arising from those of $\mathcal{O}_{A_f}$ produced in [4] and [3] are not unitarily equivalent to the representations ν_y of $\mathcal{T}_{A_f}$ when y is in the escape set.

This article is organized as follows. In Section 2, we review the definitions of the operator algebras (Cuntz–Krieger and Toeplitz algebra) that we consider in the main portion of the article, together with the main tools from the interval map dynamical systems side. In particular, we review the class of Markov interval maps $\mathcal{M}(I)$ as in Definition 2.1 and the transition matrix A_f that codifies the transitions among the subintervals $I_1, \ldots, I_n$.

In Section 3, we extend the symbolic dynamics to the case of interval maps with escape sets and, in particular, we introduce in Definition 3.2 a matrix $\widehat{A}_f$ which extends A_f and which is labeled by both the subintervals $I_1, \ldots, I_n$ and the escape subintervals $E_1, \ldots, E_{n-1}$, adding one more symbol for each escape subinterval. Then we provide some examples of such interval maps (with nonempty escape sets) and draw some conclusions in the rooted tree structures of the backward orbit of escape set points.

In Section 4.1, we fix an interval map $f \in \mathcal{M}(I)$ with transition $n \times n$ matrix A_f and a point $x \in E_f$ in the escape set of f. Then in (4.1) we define the partial isometries $T_1, \ldots, T_n$ acting on the Hilbert space H_x attached to the backward orbit of x. We then show in Lemma 4.1 that indeed these operators generate a representation ν_x of the Toeplitz algebra $\mathcal{T}_{A_f}$. Since the matrix A_f is aperiodic, we prove in Proposition 4.4 that $\nu_x(\mathcal{T}_{A_f})$ is an extension of the universal Cuntz–Krieger algebra $\mathcal{O}_{A_f}$ by the C^*-algebra $K(H_x)$ of the compacts operators on H_x.

In Section 4.2, if we are given two points x and y in E_f, then Theorem 4.5 shows that the associated Toeplitz representations ν_x and ν_y are unitarily equivalent whenever the rooted trees of x and y are isomorphic. We also remark that if y is not in the escape set, then the associated representation π_y of the Toeplitz algebra $\mathcal{T}_{A_f}$ that arises from the Cuntz–Krieger algebra $\mathcal{O}_{A_f}$ on H_y is not unitarily equivalent to any representation ν_x for $x \in E_f$.

In Section 4.3, we relate our representations ν_x of the Toeplitz algebra with another one that is constructed in a Fock space F_x associated with the matrix A_f, remarking that they are unitarily equivalent.

2. Background material

In this section, we provide some useful tools, starting with the operator algebras we obtain from dynamical systems that underline the interval maps we consider in this article.

2.1. **Toeplitz C^*-algebra from a finite matrix.** A representation π of a $*$-algebra $\mathcal{A}$ on a complex Hilbert space H is a $*$-homomorphism $\pi\colon \mathcal{A} \to B(H)$ into the $*$-algebra $B(H)$ of bounded linear operators on H. Usually, representations are studied up to unitary equivalence. Two representations $\pi : \mathcal{A} \to B(H)$ and $\widetilde{\pi} : \mathcal{A} \to B(\widetilde{H})$ are (unitarily) equivalent if there is a unitary operator $U : H \to \widetilde{H}$ (i.e., if U is a surjective isometry) such that $U\pi(a) = \widetilde{\pi}(a)U$ for every $a \in \mathcal{A}$.

A representation $\pi : \mathcal{A} \to B(H)$ of some $*$-algebra is said to be *irreducible* if there is no nontrivial subspace of H invariant with respect to all operators $\pi(a)$ with $a \in \mathcal{A}$. Then (see, e.g., [12, Proposition 3.13.2]) π is irreducible if and only if each nonzero vector $\xi \in H$ is cyclic for $\pi(\mathcal{A})$; that is, if $\overline{\pi(\mathcal{A})\xi} = H$. The representation is called *faithful* if it is injective.

Here we deal with a special family of $*$-algebras defined as follows. Let $A = (a_{ij})$ be an $n \times n$ $(0, 1)$-matrix such that each row and column has at least one nonzero entry. The Cuntz–Krieger algebra $\mathcal{O}_A$ associated with the matrix A satisfying a condition (I) was defined in [5] as *the* universal C^*-algebra generated by (nonzero) partial isometries $s_1, \dots, s_n$ satisfying

$$s_i^* s_i = \sum_j a_{ij} s_j s_j^*, \quad i = 1, \dots, n, \qquad \sum_i s_i s_i^* = 1, \tag{2.1}$$

where 1 denotes the identity. In [1], an Huef and Raeburn introduced the universal C^*-algebra associated to any finite $(0, 1)$-matrix with no zero row or column (following [6], which is a faithful realization of an Huef and Raeburn's universal Cuntz–Krieger algebra). Readers familiar with the work of Cuntz and Krieger will note that $s_i^* s_j = 0$ for $i \neq j$ and $s_i s_j = a_{ij} s_i s_j$. Moreover, the range projections $p_i = s_i s_i^*$ and support projections $q_i = s_i^* s_i$ obey the following:

$$p_i p_j = \delta_{ij} p_i, \qquad q_i q_j = q_j q_i, \qquad q_i s_j = a_{ij} s_j. \tag{2.2}$$

The C^*-algebra $\mathcal{O}_A$ is uniquely determined by the relations (2.1) if A satisfies Cuntz and Krieger's condition (I) (see [5, Theorem 2.13]). A special case is the Cuntz algebra $\mathcal{O}_n$ when A is full: $a_{ij} = 1$ for all i and j.

For the same $(0, 1)$-matrix A, a Toeplitz–Cuntz–Krieger A-family in a C^*-algebra B is a pair (t, q) where $t\colon i \mapsto t_i$ assigns to each $i = 1, \dots, n$ a partial isometry $t_i \in B$, and $q : i \mapsto q_i$ assigns to each $i = 1, \dots, n$ a projection $q_i \in B$ such that (see [13]):

(TCK1) $\quad t_i^* t_i \perp t_j^* t_j$ for all $i, j = 1, \dots, n$ with $i \neq j$,

(TCK2) $\quad t_i^* t_i = q_i$ for each $i = 1, \dots, n$,

(TCK3) $\quad \displaystyle\sum_{j=1}^{n} a_{ij} t_j t_j^* \leq t_i^* t_i$ for each i.

Then $\sum_j t_j t_j^* \leq 1$ and $(t_i^* t_i)(t_j t_j^*) = t_j t_j^*$ if $a_{ij} = 1$. Readers acquainted with the work of Cuntz and Krieger [5] will notice that (TCK1), (TCK2), and (TCK3) are satisfied by the partial isometries of the Cuntz–Krieger algebra $\mathcal{O}_A$ (see (2.2)). Moreover, there is a C^*-algebra $\mathcal{T}_A$ generated by a Toeplitz–Cuntz–Krieger A-family (t, q) which is universal (see [10]) in the sense that, given any Toeplitz–Cuntz–Krieger A-family (s, p) in a C^*-algebra B, there exists a homomorphism $\pi_{t,q} : \mathcal{T}_A \to B$ such that $\pi_{t,q}(t_i) = s_i$ and $\pi_{t,q}(q_i) = p_i$ for all i. This universal C^*-algebra $\mathcal{T}_A$ is called the *Toeplitz algebra of the matrix* A.

It is clear that the Cuntz–Krieger algebra $\mathcal{O}_A$ is a quotient of the Toeplitz algebra $\mathcal{T}_A$. In particular, a representation of $\mathcal{O}_A$ gives rise to a representation of $\mathcal{T}_A$.

The Toeplitz algebra $\mathcal{T}_A$ can be nicely described in the context of graph algebras, as was done in [11]: for that we note that, for any $n \times n$ matrix $A = (a_{ij})$ with entries in $\{0, 1\}$, we can construct a directed graph $G_A = (G_A^1, G_A^0, r, s)$ such that the vertex set $G_A^0 = \{1, \dots, n\}$ and $G_A^1 = \{e_{ij} : s(e_{ij}) = i \text{ and } r(e_{ij}) = j \text{ if } a_{ij} = 1\}$ (i.e., we draw an edge e_{ij} from i to j if and only if $a_{ij} = 1$), where s and r are the source and range maps, respectively. For completeness, we briefly provide the definition of the graph C^*-algebra $C^*(E)$ associated to this graph G_A or to a more general row-finite (directed) graph E: it is the universal C^*-algebra generated by partial isometries s_e with $e \in E^1$ and mutually orthogonal projections q_v with $v \in E^0$ such that

$$s_e^* s_e = q_{r(e)}, \qquad q_v = \sum_{e \in E^1 : s(e) = v} s_e s_e^* \quad \text{if and only if } s^{-1}(v) \neq \emptyset.$$

Then, thanks to [11, Theorem 3.7], the above Toeplitz algebra $\mathcal{T}_A$ is canonically isomorphic to the graph algebra $C^*(\widehat{G}_A)$, where $\widehat{G}_A$ extends G_A by adding a sink v' for every $v \in G_A^0$ as well as edges to this sink from each vertex (in G_A^0) that feeds into v.

Note that a representation π of the Cuntz–Krieger algebra $\mathcal{O}_A$ on a Hilbert space H is a representation of the Toeplitz algebra $\mathcal{T}_A$ (by the universality of $\mathcal{T}_A$) on the same Hilbert space H.

2.2. **Symbolic dynamics for interval maps.** Let $\Gamma = \{c_0, c_1^-, c_1^+, \dots, c_{n-1}^-, c_{n-1}^+, c_n\}$ be an ordered set of $2n$ real numbers such that

$$c_0 < c_1^- \leq c_1^+ < c_2^- \leq \cdots < c_{n-1}^- \leq c_{n-1}^+ < c_n. \tag{2.3}$$

Given Γ as above, we define the collection of closed intervals $C_\Gamma = \{I_1, \dots, I_n\}$ with

$$I_1 = [c_0, c_1^-], \dots, I_j = [c_{j-1}^+, c_j^-], \dots, I_n = [c_{n-1}^+, c_n]. \tag{2.4}$$

We also consider the collection of open intervals $\{E_1, \dots, E_{n-1}\}$, each one defined by

$$E_1 = \,]c_1^-, c_1^+[, \dots, E_{n-1} = \,]c_{n-1}^-, c_{n-1}^+[\tag{2.5}$$

in such a way that $I := [c_0, c_n] = (\bigcup_{j=1}^n I_j) \cup (\bigcup_{j=1}^{n-1} E_j)$.

We now consider the interval maps for which we can construct partitions of the interval I as in (2.3), (2.4), and (2.5).

Definition 2.1 (see [4, Definition 1]). Let $I \subset \mathbb{R}$ be an interval. A map f is in the class $\mathcal{M}(I)$ if it satisfies the properties (P1), (P2), (P3′), (P4) presented below, and it is in the class $\mathcal{PL}(I)$ if it satisfies the properties (P1), (P2), (P3), (P4).

(P1) [Existence of a finite partition in the domain of f] There is a partition $C = \{I_1, \ldots, I_n\}$ of closed intervals with $\#(I_i \cap I_j) \leq 1$ for $i \neq j$, $\mathrm{dom}(f) = \bigcup_{j=1}^n I_j \subset I$, and $\mathrm{im}(f) = I$.

(P2) [Markov property] For every $i = 1, \ldots, n$ the set $f(I_i) \cap (\bigcup_{j=1}^n I_j)$ is a nonempty union of intervals from C.

(P3) [Piecewise linear and expansive map] We have $f_{|I_j} \in \mathcal{C}^1(I_j)$, $|f'_{|I_j}(x)| = d_j > 1$, for every $x \in I_j$, $j = 1, \ldots, n$.

(P3′) [Expansive map] We have $f_{|I_j} \in \mathcal{C}^1(I_j)$, monotone and $|f'_{|I_j}(x)| > b > 1$ for every $x \in I_j, j = 1, \ldots, n$, and some b.

(P4) [Aperiodicity] For every interval I_j with $j = 1, \ldots, n$ there is a natural number q such that $\mathrm{dom}(f) \subset f^q(I_j)$.

Clearly, $\mathcal{PL}(I) \subset \mathcal{M}(I)$. The minimal partition C satisfying Definition 2.1 is denoted by C_f. We remark that the Markov property (P2) allows us to encode the transitions between the intervals in the so-called (Markov) transition $n \times n$ matrix $A_f = (a_{ij})$, defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } f(I_i) \supset I_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

A map $f \in \mathcal{M}(I)$ uniquely determines (together with the minimal partition $C_f = \{I_1, \ldots, I_n\}$):

(i) the f-invariant set $\Omega_f := \{x \in I : f^k(x) \in \mathrm{dom}(f) \text{ for all } k = 0, 1, \ldots\}$;

(ii) the collection of open intervals $\{E_1, \ldots, E_{n-1}\}$ such that $\overline{I \setminus \bigcup_{j=1}^n I_j} = \bigcup_{j=1}^{n-1} E_j$;

(iii) the transition matrix $A_f = (a_{ij})_{i,j=1,\ldots,n}$.

For the proof of (i) see [9]; (ii) is straightforward; (iii) is a consequence of the Markov property (P2) as in Definition 2.1.

Matrices A for which there exists a positive integer m such that all the entries of A^m are nonzero are called *aperiodic* (or *primitive*). We note that the matrix A_f is aperiodic (thus irreducible) because $f \in \mathcal{M}(I)$ (see Definition 2.1).

Definition 2.2. The *address map* $\mathrm{ad} : \bigcup_{j=1}^n \mathrm{int}(I_j) \to \{1, 2, \ldots, n\}$ is defined as follows: $\mathrm{ad}(x) = i$ if $x \in I_i$, where $\mathrm{int}(I_j)$ denotes the interior of I_j. The *itinerary map* $\mathrm{it}_f : \bigcup_{j=1}^n \mathrm{int}(I_j) \to \{1, 2, \ldots, n\}^{\mathbb{N}_0}$ is defined as $\mathrm{it}_f(x) = \mathrm{ad}(x)\,\mathrm{ad}(f(x)) \times \mathrm{ad}(f^2(x)) \cdots$.

Note that Ω_f is the set of points that remain in $\mathrm{dom}(f)$ under iteration of f, and is usually called a *cookie-cutter set* (see [9]). The open set

$$E_f := I \setminus \Omega_f = \bigcup_{k=0}^{\infty} f^{-k}\Big(\bigcup_{j=1}^{n-1} E_j\Big) \tag{2.7}$$

is usually called the *escape set*. Every point in E_f will eventually fall, under iteration of f, into the interior of some interval E_j (where f is not defined) and the iteration process will end. We may say that x is the escape set E_f of f if and only there is $k \in \mathbb{N}$ such that $f^k(x) \notin \operatorname{dom}(f)$. If $c_j^- = c_j^+$, for some j, then $E_j = \emptyset$ and c_j is a singular point, either a critical point or a discontinuity point of f.

Note that $E_f = \emptyset$ if and only if $\bigcup_{j=1}^{n-1} E_j = \emptyset$. Moreover, we assume that Γ is invariant under f, which means that each boundary point $c_i^{\pm}$ is sent to a boundary point $c_j^{\pm}$.

As in [4], we will consider the equivalence relation R_f defined by

$$R_f = \big\{(x,y) : f^n(x) = f^m(y) \text{ for some } n,m \in \mathbb{N}_0\big\}. \tag{2.8}$$

In that previous work, we considered R_f restricted to Ω_f. Here, however, we extend R_f to the whole interval I. The relation R_f is a countable equivalence relation in the sense that the equivalence class $R_f(x)$ of $x \in I$ is a countable set. We denote $x \sim y$ whenever $(x,y) \in R_f$.

3. Dynamics of escape orbit points

Let $f \in \mathcal{M}(I)$ such that $E_f \neq \emptyset$. This means that there is at least a nonempty open interval $E_j =]c_j^-, c_j^+[$, with $c_j^- \neq c_j^+$, with $j \in \{1,\dots,n-1\}$. The nonempty open subinterval E_j is called an *escape interval*. Now, to define orbit representations associated with escape orbits, that is, associated to points $y \in E_f$, we need to introduce some preliminary notions, from a dynamical point of view. For every $y \in E_f$ there is a least natural number $\tau(y)$ such that $f^{\tau(y)}(y) \notin \operatorname{dom}(f)$, which means that $f^{\tau(y)}(y) \in E_j$ for some j such that $E_j \neq \emptyset$.

Notation 3.1. The final escape point, for the orbit of y, is denoted by $e(y) := f^{\tau(y)}(y)$ and the final escape interval index is denoted by $\iota(y)$; that is, if $f^{\tau(y)}(y) \in E_j$, then $\iota(y) = j$.

In other words, for a point $y \in E_f$ the orbit terminates at $e(y) = f^{\tau(y)}(y) \in E_{\iota(y)}$, since f is not defined on $E_{\iota(y)}$. This means that the generalized orbit $R_f(y)$ is essentially the backward orbit of $e(y)$. Although $e(y)$ does not belong to the domain of f, we impose the condition that it belongs to $R_f(y)$ as a special point. The last point in the orbit belonging to the domain of f is $f^{\tau(y)-1}(y)$.

Therefore, for every $x \in I$ we have a generalized orbit $R_f(x)$ with a natural graph structure simple to describe: the vertices are the points of $R_f(x)$, and there is a directed edge, $y \to z$, between two points y, z, if and only if $z = f(y)$. The graph structure of $R_f(x)$ depends on the following. If $x \in \Omega_f$, then $R_f(x)$ has a graph structure without a preferred vertex. If $x \in E_f$, then $R_f(x)$ has a natural structure of a rooted tree. The root of $R_f(x)$ is precisely $e(x)$, a point with no outgoing edge. Thus, the generalized escape orbits can be parameterized by the points of the escape intervals. More precisely, every point x in $\bigcup_{j=1}^{n-1} E_j$ gives origin to a unique generalized orbit. Now, consider the preimage set of x, $f^{-1}(x)$. Since $f \in \mathcal{M}(I)$, the set $f^{-1}(x)$ is finite. We call the points in $f^{-1}(x)$ the (domain) endpoints of $R_f(x)$, and $e(x)$ is the final escape point of $R_f(x)$. Now, let us denote

$f_j^{-1} : f(I_j) \to \mathrm{dom}(f)$, the inverse branch of f whose domain is $f(I_j)$. Naturally, we have $f \circ f_j^{-1} = \mathrm{id}_{|I_j}$ the identity function on I_j. In this case, the preimages of x are enumerated by

$$f_1^{-1}(x), \dots, f_n^{-1}(x)$$

whenever $f_j^{-1}(x)$ exists, that is, $f_j^{-1}(x) \neq \emptyset$, $j = 1, \dots, n$.

In order to describe symbolically the escape orbits, we extend the symbol space adding a symbol for each escape interval E_j, which will represent an end for the symbolic sequence. For each escape interval E_j we associate a symbol $\widehat{j}$ to distinguish the symbol associated with the interval partition I_j. That is, we consider the symbols ordered by

$$1 < \widehat{1} < 2 < \widehat{2} < \cdots < n-1 < \widehat{n-1} < n. \tag{3.1}$$

If E_j is not an interval, that is $E_j = \emptyset$, then there is no symbol $\widehat{j}$. Moreover, we define

$$\Sigma_{E_f} = \big\{\widehat{j} : E_j \neq \emptyset, j \in \{1, \dots, n-1\}\big\}. \tag{3.2}$$

The address map ad (see Definition 2.2) is extended to the escape set E_f with $\mathrm{ad}(x) := \widehat{j} \in \Sigma_{E_f}$ if $x \in E_j$. Therefore, the address map is defined for all points of the interval I except the points of the boundary of the subintervals $I_1, \dots, I_n$ (see (2.3) and (2.4)); thus

$$\mathrm{ad} : I \setminus \{c_0, c_i^{\pm}, c_n : i = 1, \dots, n-1\} \longrightarrow \{1, \dots, n\} \cup \Sigma_{E_f}.$$

The itinerary map (see Definition 2.2) is also extended such that

$$\mathrm{it}_f(x) = \mathrm{ad}(x)\,\mathrm{ad}\big(f(x)\big) \cdots \mathrm{ad}\big(f^{\tau(x)-1}(x)\big)\,\mathrm{ad}\big(e(x)\big). \tag{3.3}$$

The itinerary of a point $x \in E_f$ is always a finite word terminating in a symbol $\widehat{j} \in \Sigma_{E_f}$.

An admissible escape word is a word occurring as the itinerary of an escape point $x \in E_f$. These words are formed by

$$\xi = \xi_1 \xi_2 \cdots \xi_k \widehat{j}$$

such that $a_{\xi_i \xi_{i+1}} = 1$ for $i = 1, 2, \dots, k-1$, and terminating in an escape symbol $\widehat{j}$.

Thus we have an index $\{1, \dots, n\} \cup \Sigma_{E_f}$ which is ordered in (3.1) and (3.2). To deal with the possible transitions from Markov transition intervals to escape intervals, we define a matrix $\widehat{A}_f$ as follows.

Definition 3.2. Given the transition matrix A_f as in (2.6), we define a matrix $\widehat{A}_f = (\widehat{a}_{ij})$ indexed by $\{1, \dots, n\} \cup \Sigma_{E_f}$ such that

$$\widehat{a}_{ij} = \begin{cases} a_{ij} & \text{if } i, j \in \{1, \dots, n\}, \\ 1 & \text{if } i \in \{1, \dots, n\}, j \in \Sigma_{E_f} \text{ and } I_i \cap f^{-1}(E_j) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

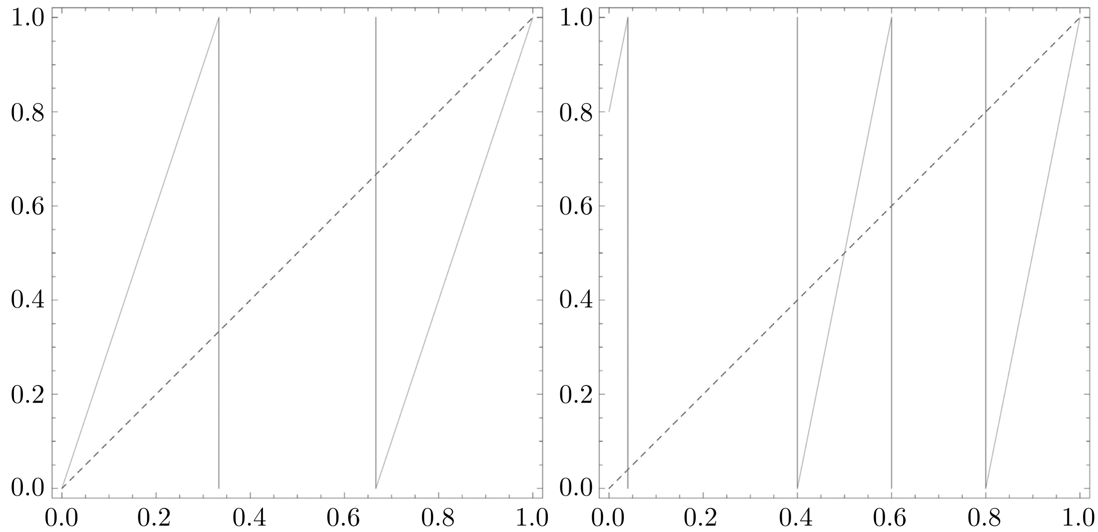

FIGURE 1. Graphs of the functions of Examples 3.5 and 3.6.

Two graphs are isomorphic if there is a one-to-one correspondence between the vertices, preserving the edges. In particular, for a pair of isomorphic rooted trees their roots are in correspondence.

Lemma 3.3. *Let $y, z \in E_f$. If $\mathrm{ad}(f^{-1}(e(y))) = \mathrm{ad}(f^{-1}(e(z)))$, then $R_f(y)$ is isomorphic to $R_f(z)$ as graphs.*

Proof. Consider $y, z \in E_f$ and $\mathrm{ad}(f^{-1}(e(y))) = \mathrm{ad}(f^{-1}(e(z)))$. Let us build explicitly the isomorphism between $R_f(y)$ and $R_f(z)$. Consider the map $h : R_f(y) \to R_f(z)$ defined such that $h(e(y)) = e(z)$. Next, choose an element $j \in \mathrm{ad}(f^{-1}(e(y)))$ and set $h(f_j^{-1}(e(y))) = f_j^{-1}(e(z))$ which exists since $j \in \mathrm{ad}(f^{-1}(e(z)))$. Since $\mathrm{ad}(f^{-1}(e(y))) = \mathrm{ad}(f^{-1}(e(z)))$, we have

$$\mathrm{ad}\big(f^{-1} \circ f^{-1}\big(e(y)\big)\big) = \mathrm{ad}\big(f^{-1} \circ f^{-1}\big(e(z)\big)\big).$$

Next, choose $i \in \mathrm{ad}(f^{-2}(e(y)))$ such that $a_{ij} = 1$. Then set $h(f_i^{-1} \circ f_j^{-1}(e(y))) = f_i^{-1} \circ f_j^{-1}(e(z))$. With this process we associate every point in $R_f(y)$ to the point in $R_f(z)$ which has the same itinerary. Therefore, h is an isomorphism of $R_f(y)$ to $R_f(z)$. $\square$

This allows us to identify the points in $R_f(y)$ with admissible words that finish in the symbol $\mathrm{ad}(e(y))$. The condition $\mathrm{ad}(f^{-1}(e(y))) = \mathrm{ad}(f^{-1}(e(z)))$ is not a necessary condition. There are special cases for which a certain symbolic symmetry implies an isomorphism between the trees, despite the difference in the symbols, as we can see in the following example.

Example 3.4. Let f be such that

$$A_f = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \widehat{A}_f = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The escape symbols set is $\Sigma_{E_f} = \{\widehat{1}, \widehat{2}\}$. The possible endings for escape orbits are $1\widehat{1}$, $2\widehat{1}$, $2\widehat{2}$, and $3\widehat{2}$, and therefore

$$\mathrm{ad}\big(f^{-1}\big(e(y)\big)\big) \in \{1, 2\} \qquad \text{and} \qquad \mathrm{ad}\big(f^{-1}\big(e(z)\big)\big) \in \{2, 3\}$$

for points $y \in E_1$ and $z \in E_2$. However, the trees $R_f(y)$ and $R_f(z)$ are isomorphic through $h\colon R_f(y) \to R_f(z)$, which changes $1 \to 3$ within admissible words.

Example 3.5. Let $f(x) = 3x (\mathrm{mod}\, 1)$ (see the left-hand side of Figure 1). Consider the domain of f given by $\mathrm{dom}(f) = I_1 \cup I_2$, with $I_1 = [0, 1/3]$ and $I_2 = [2/3, 1]$. In this case, we have $E_1 =]1/3, 2/3[$. The transition matrix associated with f is

$$A_f = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The escape symbols set is $\Sigma_{E_f} = \{\widehat{1}\}$ and the address map is $\mathrm{ad} : [0, 1] \setminus \{0, 1/3, 2/3, 1\} \to \{1, \widehat{1}, 2\}$. The matrix $\widehat{A}_f$ is given by

$$\widehat{A}_f = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

For points x, y in the escape set E_f, we have $\iota(x) = \iota(y) = 1$. Moreover, $\mathrm{ad}(f^{-1}(e(x))) = \mathrm{ad}(f^{-1}(e(y))) \in \{1, 2\}$. Therefore, $R_f(x)$ is isomorphic to $R_f(y)$ (see Notation 3.1). In particular, they are both isomorphic to the tree in the left-hand side of Figure 2.

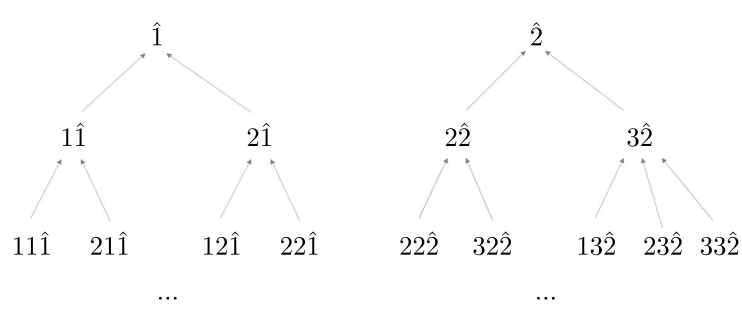

FIGURE 2. Rooted trees, part I.

Example 3.6. Let f be the map (see the right-hand side of Figure 1)

$$f(x) = \begin{cases} 5x + 4/5 & \text{if } x \leq 1/25, \\ 5x - 2 & \text{if } 2/5 \leq x \leq 3/5, \\ 5x - 4 & \text{if } 4/5 \leq x \leq 1. \end{cases}$$

The domain of f is given by $\mathrm{dom}(f) = I_1 \cup I_2 \cup I_3$, with $I_1 = [0, 1/25]$, $I_2 = [2/5, 3/5]$, $I_3 = [4/5, 1]$. The escape intervals are $E_1 =]1/25, 2/5[$ and $E_2 =]3/5, 4/5[$. The escape symbols set is $\Sigma_E = \{\widehat{1}, \widehat{2}\}$ and the transition matrix associated with f is

$$A_f = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The address map is $\mathrm{ad} : [0,1] \setminus \{0, 1/25, 2/5, 3/5, 4/5, 1\} \to \{1, \widehat{1}, 2, \widehat{2}, 3\}$. The matrix $\widehat{A}_f$ is given by

$$\widehat{A}_f = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

For points x, y in the escape set E_f, we have either $\iota(x) = \iota(y) = 1$, $\iota(x) = 1$ and $\iota(y) = 2$, $\iota(x) = 2$ and $\iota(y) = 1$, or $\iota(x) = \iota(y) = 2$. Nevertheless, $\mathrm{ad}(f^{-1}(e(x))) = \mathrm{ad}(f^{-1}(e(y))) \in \{2, 3\}$. Therefore, we have one rooted tree, up to isomorphism. We present two (isomorphic) trees, one with root $\widehat{1}$ and the other with root $\widehat{2}$ (see the left-hand side of Figure 4 and the right-hand side of Figure 2, respectively). Note that they are essentially the same tree.

Example 3.7. Let f be the map (see Figure 3)

$$f(x) = \begin{cases} 5x + 2/5 & \text{if } x \leq 3/25, \\ 5x - 2 & \text{if } 2/5 \leq x \leq 3/5, \\ 5x - 4 & \text{if } 4/5 \leq x \leq 1. \end{cases}$$

The domain of f is given by $\mathrm{dom}(f) = I_1 \cup I_2 \cup I_3$, with $I_1 = [0, 3/25]$, $I_2 = [2/5, 3/5]$, $I_3 = [4/5, 1]$. The escape intervals are $E_1 =]3/25, 2/5[$ and $E_2 =]3/5, 4/5[$. The escape symbols set is $\Sigma_E = \{\widehat{1}, \widehat{2}\}$ and the matrix associated with f is

$$A_f = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

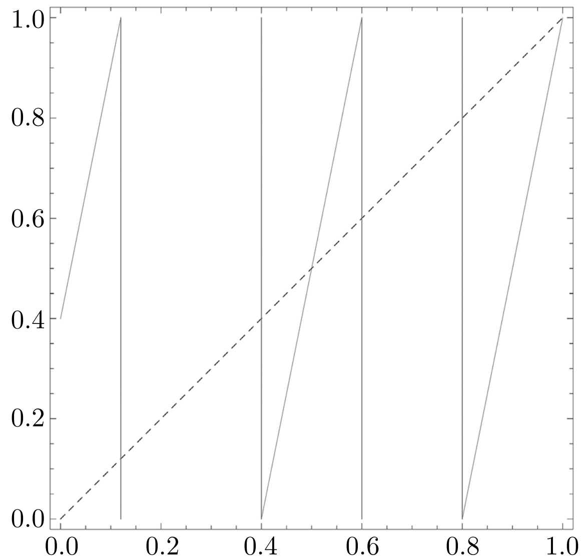

FIGURE 3. Graph of the function in Example 3.7.

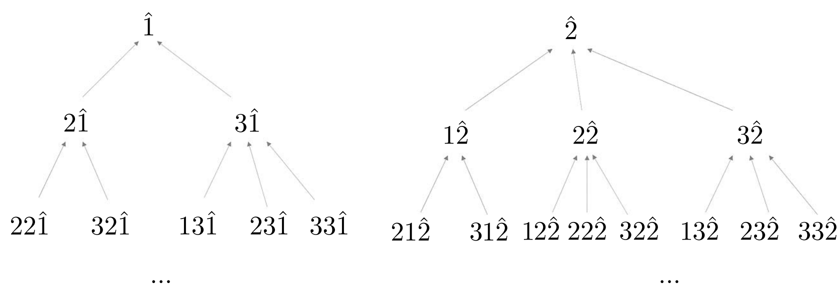

FIGURE 4. Rooted trees, part II.

The address map is ad : $[0,1] \setminus \{0, 3/25, 2/5, 3/5, 4/5, 1\} \to \{1, \widehat{1}, 2, \widehat{2}, 3\}$. The matrix $\widehat{A}_f$ is given by

$$\widehat{A}_f = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

For points x, y in the escape set E_f, we have either $\iota(x) = \iota(y) = 1$, $\iota(x) = 1$ and $\iota(y) = 2$, $\iota(x) = 2$ and $\iota(y) = 1$, or $\iota(x) = \iota(y) = 2$. If $y \in E_1$, then $\mathrm{ad}(f^{-1}(y)) \in \{2, 3\}$. If $z \in E_2$, then $\mathrm{ad}(f^{-1}(z)) \in \{1, 2, 3\}$. Therefore, we have two nonisomorphic rooted trees: $R_f(y)$ if $e(y) \in E_1$ (see the left-hand side of Figure 4) and $R_f(x)$ if $e(x) \in E_2$ (see the right-hand side of Figure 4).

4. Toeplitz representations on orbit spaces

In this section, we provide representations of the Toeplitz algebra $\mathcal{T}_{A_f}$ associated to points in the escape set of the underlying Markov interval map, and compare these representations with the previous ones considered by the authors.

4.1. **Toeplitz–Cuntz–Krieger A_f-families.** Let H_x, with $x \in E_f$, be the Hilbert space with canonical base

$$\{|z\rangle : f^k(z) = e(x), k \in \mathbb{N}_0\}$$

associated with $R_f(x)$. Note that there is a special vector basis which is $|e(x)\rangle$. The rank one projection on the 1-dimensional space $\mathbb{C}|z\rangle$ is denoted by P_z, or as usual in Dirac notation, $P_z = |z\rangle\langle z|$. Note also that $H_x = H_y = H_{e(x)}$ if for some $k \in \mathbb{N}$, $f^k(y) = f^{\tau(x)}(x) = e(x)$ (see Notation 3.1).

Let $x \in E_f$, and let T_i, $i = 1, \dots, n$ be defined by

$$T_i|y\rangle = \chi_{f(I_i)}(y)\big|f_i^{-1}(y)\big\rangle \quad \text{for } y \in R_f(x), \tag{4.1}$$

where χ_B denotes the characteristic function on a set B. Its adjoint T_i^* is given by

$$T_i^*|y\rangle = \chi_{I_i}(y)\big|f(y)\big\rangle.$$

In particular, $T_i^*|x\rangle = 0$ for all $i = 1, \dots, n$, and $T_i|x\rangle = 0$ if there are no transitions from the interval I_i to the escape interval E_j. If there is any transition, then $T_i|x\rangle = |z\rangle$ such that $z \in I_i$ and $f(z) = x$.

Lemma 4.1. *Let $x \in E_f$. Then the partial isometries T_j, $j = 1, \dots, n$, defined in (4.1) satisfy the following relations:*

(a) $\sum_{j=1}^n T_jT_j^* + P_{e(x)} = 1$, *and*

(b) $\sum_{j=1}^n a_{kj}T_jT_j^* + \widehat{a}_{k\iota(x)}P_{e(x)} = T_k^*T_k$ *for all* $k = 1, \dots, n$,

where $A_f = (a_{ij})$ is the transition matrix of f and ι is as in Notation 3.1.

Proof. Consider $T_jT_j^*$ acting on a vector $|y\rangle$ with $y \in R_f(x) \setminus \{e(x)\}$ in the canonical basis, and let $r \in \{1, \dots, n\}$ such that $y \in I_r$. Then

$$T_iT_i^*|y\rangle = \chi_{I_i}(y)T_i\big|f(y)\big\rangle = \chi_{I_i}(y)\chi_{f(I_i)}\big(f(y)\big)\big|f_i^{-1} \circ f(y)\big\rangle = \chi_{I_i}(y)|y\rangle.$$

Thus

$$\sum_{i=1}^n T_iT_i^*|y\rangle = \sum_{i=1}^n \chi_{I_i}(y)|y\rangle = \chi_{I_r}(y)|y\rangle = |y\rangle. \tag{4.2}$$

Now, let $T_k^*T_k$ act on the same vector $|y\rangle$:

$$T_k^*T_k|y\rangle = \chi_{f(I_k)}(y)T_k^*\big|f_k^{-1}(y)\big\rangle = \chi_{f(I_k)}(y)\chi_{I_k}\big(f_k^{-1}(y)\big)|y\rangle = \chi_{f(I_k)}(y)|y\rangle,$$

since $\chi_{I_k}(f_k^{-1}(y)) = 1$. The condition $\chi_{f(I_k)}(y) = 1$ is equivalent to the existence of a preimage of y in I_k; therefore, $y \in I_r$ means that $a_{kr} = 1$. On the other hand,

$$\sum_{j=1}^n a_{kj}T_jT_j^*|y\rangle = \sum_{j=1}^n a_{kj}\chi_{I_j}(y)|y\rangle = a_{kr}|y\rangle = |y\rangle. \tag{4.3}$$

Since $P_{e(x)}|y\rangle = 0$, (4.2) and (4.3) show statements (a) and (b) of the lemma for all vectors $|y\rangle \in H_x$ with $y \in R_f(x) \setminus \{e(x)\}$.

It remains to check (a) and (b) for the vector $|e(x)\rangle$.

For that we first note that $T_j^*|e(x)\rangle = 0$ for all j and

$$\Big(\sum_j T_j T_j^* + P_{e(x)}\Big)\big|e(x)\big\rangle = \sum_j T_j T_j^* \big|e(x)\big\rangle + \big|e(x)\big\rangle = \big|e(x)\big\rangle.$$

This implies that (a) holds.

Statement (b) is also readily checked for the vector $|e(x)\rangle$ since $T_k^*|e(x)\rangle = 0$ for all k, and $T_k^* T_k|e(x)\rangle = |e(x)\rangle$ if and only if $f^{-1}(e(x)) \in I_k$ (and this holds if and only if $\widehat{a}_{k\iota(x)} = 1$ as in Definition 3.2). □

4.2. **The unitary equivalence of Toeplitz algebra representations.** Let $f \in \mathcal{M}(I)$ with $E_f \neq \emptyset$. Thanks to Lemma 4.1 and the definition of $\mathcal{T}_{A_f}$, the following is immediate.

Proposition 4.2. *The partial isometries defined on H_x as in (4.1) yield a representation of the Toeplitz algebra $\mathcal{T}_{A_f}$.*

Definition 4.3. We denote by ν_x the representation of the Toeplitz algebra $\mathcal{T}_{A_f}$ yielded in Proposition 4.2.

Note that the matrix A_f is aperiodic (all the entries of A_f^m are nonzero for some m) because $f \in \mathcal{M}(I)$ (see Definition 2.1); thus A_f is an irreducible matrix.

Proposition 4.4. *The concrete C^*-algebra $\nu_x(\mathcal{T}_{A_f})$ is an extension of the (universal) Cuntz–Krieger algebra $\mathcal{O}_{A_f}$ by the C^*-algebra $K(H_x)$ of compact operators on H_x, that is, the short sequence*

$$0 \longrightarrow K(H_x) \longrightarrow \nu_x(\mathcal{T}_{A_f}) \longrightarrow \mathcal{O}_{A_f} \longrightarrow 0$$

is exact.

Proof. If $\mu = \mu_1 \cdots \mu_m \iota(x)$ is an admissible escape word, then we can find a vector $|y\rangle$ in the canonical basis of H_x by putting $y = f_{\mu_1}^{-1} \circ \cdots \circ f_{\mu_m}^{-1}(e(x))$. In this way $|y\rangle = T_\mu|e(x)\rangle$, where $T_\mu = T_{\mu_1} \cdots T_{\mu_m}$. Thus $T_\mu^* T_\mu|e(x)\rangle = |e(x)\rangle$. Since $\sum_{j=1}^n T_j T_j^* + P_{e(x)} = 1$, by Lemma 4.1, the finite rank projection $P_{e(x)}$ belongs to $\nu_x(\mathcal{T}_{A_f})$. Moreover, if $\eta = \eta_1 \cdots \eta_r \iota(x)$ is an admissible escape word and $z = f_{\eta_1}^{-1} \circ \cdots \circ f_{\eta_r}^{-1}(e(x))$, then

$$T_\mu P_{e(x)} T_\eta^* |z\rangle = T_\mu P_{e(x)} T_\eta^* T_\eta \big|e(x)\big\rangle = T_\mu P_{e(x)} \big|e(x)\big\rangle = T_\mu \big|e(x)\big\rangle = |y\rangle,$$

such that $T_\mu P_{e(x)} T_\eta^*$ is a rank one partial isometry from the vector $|z\rangle$ to $|y\rangle$. Thus the C^*-algebra generated by the operators $T_\mu P_{e(x)} T_\eta^*$ is nothing but the C^*-algebra of all compact operators $K(H_x)$ on H_x. In particular, $K(H_x) \subset \nu_x(\mathcal{T}_{A_f})$.

On the other hand, we may consider the (surjective) quotient map $q : \nu_x(\mathcal{T}_{A_f}) \to \nu_x(\mathcal{T}_{A_f})/K(H_x)$ and also denote $q(T_j)$ by S_j. In that case, Lemma 4.1 implies that $\sum_{j=1}^n S_j S_j^* = 1$ and $\sum_{j=1}^n a_{kj} S_j S_j^* = S_k^* S_k$ for all $k = 1, \ldots, n$, which are the relations of the Cuntz–Krieger algebra associated to the matrix A_f. Since A_f is an irreducible matrix (note that $f \in \mathcal{M}(I)$ as in Definition 2.1), the Cuntz–Krieger algebra $\mathcal{O}_{A_f}$ is simple (see [5]) so the image of every (nonzero) representation of $\mathcal{O}_{A_f}$ is isomorphic to the universal C^*-algebra $\mathcal{O}_{A_f}$. Thus $\mathcal{O}_{A_f} \simeq \nu_x(\mathcal{T}_{A_f})/K(H_x)$. This finishes the proof. $\square$

Given two points x and y in E_f, we now investigate when the Toeplitz algebra representations ν_x and ν_y (see Definition 4.3) are unitarily equivalent.

Theorem 4.5. *The representations ν_x, ν_y, with x, $y \in E_f$, are unitarily equivalent if and only if $R_f(x)$ and $R_f(y)$ are isomorphic as rooted trees.*

Proof. If $R_f(x)$ and $R_f(y)$ are isomorphic, then there is a homeomorphism h, preserving edges, $h : R_f(x) \to R_f(y)$, such that $h(e(y)) = e(x)$. This homeomorphism induces the unitary operator $U : H_x \to H_y$ such that $U|z\rangle = |w\rangle$ if and only if $h(z) = w$, with $z \in R_f(x)$ and $w \in R_f(y)$. Moreover, since the map h preserves edges the unitary U satisfies $U\nu_x(t_i) = \nu_y(t_i)U$. In fact, let $z \in R_f(x)$ with $z \in I_j$. The outgoing edge connects z to $f(z)$. The ingoing edges come from the points $f_i^{-1}(z) \in I_i$ with $i = 1, \dots, n$ such that $a_{ij} = 1$. This means that $\nu_x(t_j^*)|z\rangle = |f(z)\rangle$ and $\nu_x(t_r^*)|z\rangle = 0$ with $r \neq j$, and that $\nu_x(t_i)|z\rangle = |f_i^{-1}(z)\rangle$ for i such that $a_{ij} = 1$ and $\nu_x(t_i)|z\rangle = 0$ with $a_{ij} = 0$. Therefore, the preservation of edges means that $h(z) \in I_j$ and has an outgoing edge to $f(h(z))$ and that the ingoing edges of $h(z)$ are $f_i^{-1}(h(z)) \in I_i$ with $i = 1, \dots, n$ such that $a_{ij} = 1$. So $\nu_y(t_j^*)|h(z)\rangle = |f(h(z))\rangle$ and $\nu_y(t_r^*)|h(z)\rangle = 0$ with $r \neq j$, and $\nu_y(t_i)|h(z)\rangle = |f_i^{-1}(h(z))\rangle$ for i such that $a_{ij} = 1$ and $\nu_y(t_i)|h(z)\rangle = 0$ with $a_{ij} = 0$. Thus we obtain

$$\nu_y(t_j^*)\big|h(z)\big\rangle = \big|f\big(h(z)\big)\big\rangle \iff \nu_y(t_j^*)U|z\rangle = U\nu_x(t_j^*)|z\rangle$$

and

$$\nu_y(t_i)\big|h(z)\big\rangle = \big|f_i^{-1}\big(h(z)\big)\big\rangle \iff \nu_y(t_i)U|z\rangle = U\nu_x(t_i)|z\rangle.$$

Now, suppose that the representations ν_x, ν_y, with x, $y \in E_f$, are unitarily equivalent. This means that there is $U \in B(H_x, H_y)$ such that

$$U\nu_x(a) = \nu_y(a)U \quad \text{for every } a \in \mathcal{T}_{A_f}.$$

First note that U sends the vacuum vector $|e(x)\rangle$ to the vacuum vector $|e(y)\rangle$, that is, $U|e(x)\rangle = |e(y)\rangle$. In fact, since $U\nu_x(t_i^*)|e(x)\rangle = 0$ for all $i = 1, \dots, n$, $|e(y)\rangle$ is the only vector, up to scalar multiplication, in H_y satisfying $\nu_y(t_i^*)|e(y)\rangle = 0$ for all $i = 1, \dots, n$. With the same reasoning we conclude that for every $z \in R_f(x)$, with $\mathrm{it}_f(z) = \xi_1\xi_2 \cdots \xi_k\widehat{\iota}$, there is a unique $w \in R_f(y)$ such that $U|z\rangle = |w\rangle$. In fact, consider $\mathrm{it}_f(z) = \xi_1\xi_2 \cdots \xi_k\widehat{\iota}$, with

$$|z\rangle = \nu_x(t_{\xi_1\xi_2\cdots\xi_k\widehat{\iota}})\big|e(x)\big\rangle.$$

Then

$$\begin{aligned} U|z\rangle &= U\nu_x(t_{\xi_1\xi_2\cdots\xi_k\widehat{\iota}})\big|e(x)\big\rangle = \nu_x(t_{\xi_1\xi_2\cdots\xi_k\widehat{\iota}})U\big|e(x)\big\rangle \\ &= \nu_x(t_{\xi_1\xi_2\cdots\xi_k\widehat{\iota}})\big|e(y)\big\rangle = |w\rangle. \end{aligned}$$

Let h be the map sending $z \mapsto w$. The map h is one-to-one since U is unitary (and invertible). The map h preserves edges. In fact, if $z_1 = f_i^{-1}(z_2)$ in $R_f(x)$, then $\nu_x(t_i)|z_1\rangle = |z_2\rangle$. Therefore, $U\nu_x(t_i)|z_1\rangle = |z_2\rangle$ since $U\nu_x(a) = \nu_y(a)U$ for every $a \in \mathcal{T}_{A_f}$. Therefore, $R_f(x)$ and $R_f(y)$ are isomorphic as rooted trees. $\square$

If $x \in E_f$, then the partial isometries $T_1, \dots, T_n$ defined in (4.1) provide our new family of Toeplitz–Cuntz–Krieger operators. If $y \in \Omega_f$, then the family of partial isometries $S_1, \dots, S_n$ defined as

$$S_i|z\rangle = \chi_{f(I_i)}(z)\big|f_i^{-1}(z)\big\rangle, \quad \text{for all } z \in R_f(y), \tag{4.4}$$

gives rise to a representation π_y of the Cuntz–Krieger algebra $\mathcal{O}_{A_f}$ on H_y as in [4, Theorem 6]. Therefore $S_1, \dots, S_n$ gives a representation of the Toeplitz algebra $\mathcal{T}_{A_f}$ on the same Hilbert space H_y.

Proposition 4.6. *Let $x \in E_f$ and $y \in \Omega_f$. Then the representations ν_x and π_y of the Toeplitz algebra $\mathcal{T}_{A_f}$ are not unitarily equivalent.*

Proof. If $f^{-1}(x) = \emptyset$, then the result is clear as $H_x = \mathbb{C}|e(x)\rangle$.

So we assume that $f^{-1}(x) \neq \emptyset$ and the existence of a unitary operator $U : H_y \to H_x$ (i.e., a surjective isometry) such that $US_iU^* = T_i$ for all $i = 1, \dots, n$. On the other hand, $T_i|e(x)\rangle = 0$ for all $i = 1, \dots, n$, hence $US_iU^* = T_i$ implies that $S_iU^*|e(x)\rangle = 0$ for all i.

Now, write $U^*|e(x)\rangle$ as a linear combination in the canonical basis of H_y: $U^*|e(x)\rangle = \sum_z c_z|z\rangle$ for scalars c_z. Since $U^*|e(x)\rangle \neq 0$, there exists z_0 such that $c_{z_0} \neq 0$ and so we may find k such that $z_0 \in f(I_k)$ and $c_{z_0} \neq 0$. Then by the continuity of S_k, we obtain

$$0 = S_kU^*\big|e(x)\big\rangle = \sum_z c_zS_k|z\rangle \neq 0$$

because $S_k|z_0\rangle = |f_k^{-1}(z_0)\rangle \neq 0$. □

Theorem 4.7. *Let $x \in E_f$. Let ν_x be the corresponding Toeplitz algebra $\mathcal{T}_{A_f}$ representation associated to x. Then ν_x is irreducible.*

Proof. Note that every element in the basis of H_x is cyclic. Let $\xi \in H_x$ be a nonzero vector. Then we aim to prove that ξ is cyclic, and for that it is enough to find a sequence P_n in $\nu_x(\mathcal{T}_{A_f})$ such that $P_n\xi \to |x_0\rangle$ for some $x_0 \in R_f(x)$, since $|x_0\rangle$ is a cyclic vector. Choose $x_0 \in R_f(x)$ such that the inner product between ξ and $e(x)$ is nonzero, that is, $\langle\xi|x_0\rangle \neq 0$. Let $\mathrm{it}_f(x_0) = \alpha_1\alpha_2\cdots\alpha_k\iota(x)$ be the itinerary of the point x_0, and let $\alpha_{(k)} := (\alpha_1, \dots, \alpha_k)$ be its kth prefix with $k < \tau(x)$ (see Notation 3.1). For every $n \in \mathbb{N}$, let $P_n := T_{\alpha_{(n)}}T^*_{\alpha_{(n)}}$, and denote by $H_{x_0}(n)$ the range of the projection P_n. Then $(P_n)_{n\in\mathbb{N}}$ is a decreasing sequence of orthogonal projections and it strongly converges to P, the orthogonal projection onto $\bigcap_{n=1}^\infty H_{x_0}(n)$. Since f is expansive, x_0 is the unique point z such that $\mathrm{it}_f(z) = \alpha_1\alpha_2\cdots\alpha_k\iota(x)$, hence $\bigcap_{n=1}^\infty H_{x_0}(n) = \mathbb{C}|x_0\rangle$. Since $|x_0\rangle$ is a norm one vector, we conclude that $P\xi = \langle\xi|x_0\rangle|x_0\rangle$. □

4.3. **Fock space representations.** Consider the Fock space defined as follows. An admissible escape word is a word occurring as the itinerary of an escape point $x \in E_f$. Let $\iota = \iota(x)$, to simplify notation. These words are formed by

$$\xi = \xi_1\xi_2\cdots\xi_k\widehat{\iota}$$

such that $a_{\xi_i\xi_{i+1}} = 1$ for $i = 1, 2, \ldots, k-1$, and terminating on an escape symbol $\widehat{\iota} = \widehat{\iota(x)}$ such that $\widehat{a}_{\xi_k\widehat{\iota(x)}} = 1$. Let $\Lambda_k = \{\xi_1\xi_2\cdots\xi_k : a_{\xi_j\xi_{j+1}} = 1\}$, and let $\widehat{\Lambda}_k = \{\xi\widehat{\iota} : \xi \in \Lambda_k, \widehat{a}_{\xi_k\widehat{\iota}} = 1\}$.

Let F_0 be the Hilbert space generated by $\{|\emptyset\rangle_{\widehat{\iota}}\}$, the vacuum vector, let F_1 be the Hilbert space generated by $\{|i\rangle_{\widehat{\iota}} : \widehat{a}_{i\widehat{\iota}} = 1\}$, let F_2 be the Hilbert space generated by $\{|\xi_1\xi_2\rangle_{\widehat{\iota}} : a_{\xi_1\xi_2} = 1, \widehat{a}_{\xi_2\widehat{\iota}} = 1\}, \ldots$, let F_k be the Hilbert space generated by $\{|\xi\rangle_{\widehat{\iota}} : \xi \in \Lambda_k, \widehat{a}_{\xi_k\widehat{\iota}} = 1\}$. Finally, the Fock space, for $x \in E_f$, is

$$F_x := \bigoplus_{k=0}^{\infty} F_k.$$

Since the dependence of F_x on x appears only on the final symbol $\widehat{\iota}$ associated with the escape interval E_ι, naturally $F_x = F_y$ if and only if $\iota(x) = \iota(y)$.

Note that, from a purely symbolic point of view, A_f and Λ_k are not sufficient to determine the escape words since it is necessary to specify how the regular states transit to the escape states, which is accomplished by the matrix $\widehat{A_f}$, as we can see in the next example.

Example 4.8. Consider again the interval map f as in Example 3.6. We have that the possible endings for the admissible escape words are $2\widehat{1}$, $3\widehat{1}$, $2\widehat{2}$, and $3\widehat{2}$. We can have a map g with $A_g = A_f$ such that

$$\hat{A}_g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

In this case, the possible endings are only $2\widehat{1}$ and $3\widehat{1}$.

We now define the *creation operator* $\widetilde{T}_i$ as

$$\widetilde{T}_i|\xi\rangle_{\widehat{\iota}} := \begin{cases} |i\xi\rangle_{\widehat{\iota}} & \text{if } i\xi \in \Lambda_{k+1}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.5}$$

whose adjoint is the following annihilation operator

$$\widetilde{T}_i^*|\xi_1\cdots\xi_k\rangle_{\widehat{\iota}} = \delta_{i\xi_1}|\xi_2\cdots\xi_k\rangle_{\widehat{\iota}}.$$

There is a relation between the spaces F_x, H_x and between the operators T_i and $\widetilde{T}_i$. In fact, let $z \in R_f(x)$, and consider $V|z\rangle = |\xi_1\xi_2\cdots\xi_k\rangle_{\widehat{\iota}}$, with $\mathrm{it}_f(z) = \xi_1\xi_2\cdots\xi_k\widehat{\iota}$. In this case, V is a unitary operator and $V^*|\xi_1\xi_2\cdots\xi_k\rangle_{\widehat{\iota}} = T_{\xi_1\xi_2\cdots\xi_k}|e(x)\rangle$. In fact, $|z\rangle = T_{\xi_1\xi_2\cdots\xi_k}|e(x)\rangle$ is the only vector satisfying $\mathrm{it}_f(z) = \xi_1\xi_2\cdots\xi_k\widehat{\iota}$. Therefore,

$$VT_iV^* = \widetilde{T}_i. \tag{4.6}$$

The Fock space is explicitly based on the symbolic structure of the generalized orbit.

If the generalized orbit is seen as a graph, then the basis vectors of F_x can be seen as the vertices of the graph. That is, each admissible finite word $\xi_2\cdots\xi_k\widehat{\iota}$ labels one vertex and one basis vector $|\xi_2\cdots\xi_k\rangle_{\widehat{\iota}}$. There is an edge between two vertices if and only if their labels differ in the first symbol. If the generalized orbits $R_f(x)$, $R_f(y)$ for two different points x, y in the escape set (in different escape intervals necessarily) are isomorphic as graphs, we may expect that there is a unitary operator U relabeling the basis vectors, and therefore maintaining the structure of the Fock spaces F_x and F_y.

Acknowledgments. We thank the anonymous referees for useful remarks and suggestions.

Correia Ramos's work was partially supported by the Centro de Investigação em Matemática e Aplicações of the Universidade de Évora (CIMA–UE) and by FCT/Portugal project PEst-OE/MAT/UI0117/2014. Martins's and Pinto's work was partially supported by the Fundação para a Ciência e a Tecnologia (FCT/Portugal) project UID/MAT/04459/2013.

References

1. A. an Huef and I. Raeburn, *The ideal structure of Cuntz-Krieger algebras*, Ergodic Theory Dynam. Systems **17** (1997), no. 3, 611–624. Zbl 0886.46061. MR1452183. DOI 10.1017/S0143385797079200. 538
2. O. Bratteli and P. E. T. Jorgensen, *Iterated function systems and permutation representations of the Cuntz algebra*, Mem. Amer. Math. Soc. **139** (1999), no. 663. Zbl 0935.46057. MR1469149. 536
3. C. Correia Ramos, N. Martins, and P. R. Pinto, *Interval maps from Cuntz-Krieger algebras*, J. Math. Anal. Appl. **374** (2011), no. 2, 347–354. Zbl 1211.46063. MR2729225. DOI 10.1016/j.jmaa.2010.09.045. 536, 537
4. C. Correia Ramos, N. Martins, P. R. Pinto, and J. Sousa Ramos, *Cuntz-Krieger algebras representations from orbits of interval maps*, J. Math. Anal. Appl. **341** (2008), no. 2, 825–833. Zbl 1144.46050. MR2398251. DOI 10.1016/j.jmaa.2007.10.059. 536, 537, 540, 541, 550
5. J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), no. 3, 251–268. Zbl 0434.46045. MR0561974. DOI 10.1007/BF01390048. 538, 539, 549
6. D. E. Evans, *Gauge actions on $\mathcal{O}_A$*, J. Operator Theory **7** (1982), no. 1, 79–100. Zbl 0483.46047. MR0650194. 538
7. D. E. Evans, *The C^*-algebra of topological Markov chains*, lectures at Tokyo Metropolitan University, Tokyo, Japan, 1983. 537
8. R. Exel and M. Laca, *Cuntz-Krieger algebras for infinite matrices*, J. Reine Angew. Math. **512** (1999), 119–172. Zbl 0932.47053. MR1703078. DOI 10.1515/crll.1999.051. 536
9. K. Falconer, *Techniques in Fractal Geometry*, Wiley, Chichester, 1997. Zbl 0869.28003. MR1449135. 540
10. N. J. Fowler and I. Raeburn, *The Toeplitz algebra of a Hilbert bimodule*, Indiana Univ. Math. J. **48** (1999), no. 1, 155–181. Zbl 0938.47052. MR1722197. DOI 10.1512/iumj.1999.48.1639. 536, 539
11. P. S. Muhly and M. Tomforde, *Adding tails to C^*-correspondences*, Doc. Math. **9** (2004), 79–106. Zbl 1049.46045. MR2054981. 536, 539
12. G. K. Pedersen, *C^*-algebras and Their Automorphism Groups*, London Math. Soc. Monogr. **14**, Academic Press, London, 1979. Zbl 0416.46043. MR0548006. 538
13. A. Sims and S. B. G. Webster, *A direct approach to co-universal algebras associated to directed graphs*, Bull. Malays. Math. Sci. Soc. (2) **33** (2010), no. 2, 211–220. Zbl 1194.46077. MR2666424. 538

[1]Centro de Investigação em Matemática e Aplicações, Department of Mathematics, Universidade de Évora, R. Romão Ramalho, 59, 7000-671 Évora, Portugal.
E-mail address: ccr@uevora.pt

[2]Department of Mathematics, CAMGSD, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal.
E-mail address: nmartins@math.tecnico.ulisboa.pt; ppinto@math.tecnico.ulisboa.pt