Banach J. Math. Anal. 11 (2017), no. 2, 335–347
http://dx.doi.org/10.1215/17358787-0000006X
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

NORM ESTIMATES FOR RANDOM POLYNOMIALS ON THE SCALE OF ORLICZ SPACES

ANDREAS DEFANT[1*] and MIECZYSŁAW MASTYŁO[2]

Communicated by D. Werner

Abstract. We prove an upper bound for the supremum norm of homogeneous Bernoulli polynomials on the unit ball of finite-dimensional complex Banach spaces. This result is inspired by the famous Kahane–Salem–Zygmund inequality and its recent extensions; in contrast to the known results, our estimates are on the scale of Orlicz spaces instead of ℓ_p-spaces. Applications are given to multidimensional Bohr radii for holomorphic functions in several complex variables, and to the study of unconditionality of spaces of homogenous polynomials in Banach spaces.

1. Introduction

The famous Kahane–Salem–Zygmund inequality on the maximum modulus of random polynomials in several complex variables (see [6, Theorem 4, Chapter 6]) states that, given a polynomial $P(z) = \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha z^\alpha$ on $\mathbb{C}^n$, there is a choice of signs $\varepsilon_\alpha = \pm 1, \alpha \in \mathbb{N}_0^n$, for which

$$\sup_{z\in\mathbb{D}^n}\Big|\sum_{\alpha\in\mathbb{N}_0^n} \varepsilon_\alpha c_\alpha z^\alpha\Big| \leq C\Big(n\log(\deg P)\sum_{\alpha\in\mathbb{N}_0^n}|c_\alpha|^2\Big)^{1/2},$$

where $C > 0$ is a universal constant, $\deg P := \max\{|\alpha|; c_\alpha \neq 0\}$ is the degree of P, and $\mathbb{D}$ is the open unit disk in $\mathbb{C}$. Here, as usual for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ denotes the αth monomial and $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

Received Feb. 2, 2016; Accepted May 30, 2016.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46B70; Secondary 47A53.
Keywords. homogeneous polynomials, interpolation spaces, Orlicz spaces.

The inequality above is a fundamental tool to assure the existence of polynomials with small supremum norms on the n-dimensional polydisk $\mathbb{D}^n$, and it has found deep applications in many areas of modern analysis. Of particular importance is the case of m-homogeneous polynomials $P(z) = \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha z^\alpha$ in $\mathbb{C}^n$, that is, polynomials such that all monomial coefficients $c_\alpha = 0$ when $|\alpha| < m = \deg P$.

In recent years, many different types of extensions of the Kahane–Salem–Zygmund inequality were obtained where the supremum is taken over various Reinhard domains $R \subset \mathbb{C}^n$ (as, e.g., the unit ball $B_{\ell_p^n}$ of the Banach space ℓ_p^n, $1 \le p < \infty$) instead of the unit polydisk $B_{\ell_\infty^n} = \mathbb{D}^n$ (see [1]).

We highlight an estimate of Bayart from [1] which motivates large parts of what we intend to do: Given $1 < p \le 2$, it states that, for every m-homogeneous polynomial $P(z) = \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha z^\alpha$ on $\mathbb{C}^n$, every Banach space $X = (\mathbb{C}^n, \|\cdot\|)$, and every choice $(c_\alpha)_{|\alpha|=m}$ of scalars, there are signs $(\varepsilon_\alpha)_{|\alpha|=m}$ such that

$$\sup_{z\in B_X}\Big|\sum_{|\alpha|=m} \varepsilon_\alpha c_\alpha z^\alpha\Big| \le C(n\log m)^{1/q} \sup_{|\alpha|=m} \frac{|c_\alpha|}{\left(\frac{m!}{\alpha!}\right)^{1/p}} \sup_{z\in B_X}\Big(\sum_{k=1}^n |z_k|^p\Big)^{m/p}, \tag{1.1}$$

where $C > 0$ is a constant depending only on p. Here and in what follows later, we have $1/q := 1 - 1/p$ and $\alpha! := \alpha_1!\cdots\alpha_n!$ for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. Estimates of this type have been used, for example, in studies of multidimensional Bohr radii, Bohnenblust–Hille inequalities, unconditionality in spaces of m-homogenous polynomials on Banach spaces, and the modern theory of Dirichlet series.

We point out that the approaches given in [1]–[3], and [4] are based on the scale of ℓ_p-spaces. The main aim of this paper is to find homogeneous polynomials with small sup-norms on the unit ball of arbitrary finite-dimensional Banach spaces where the upper estimates for the norms involve, instead of ℓ_p-norms, the more general Orlicz norms. Our tools are mainly based on the Calderón–Lozanovskii interpolation theory and the methods developed in the cited works, in particular the one used by Bayart. We give two applications.

2. Main result and preliminaries

We will consider complex Banach spaces X, and denote their duals by X^*. In particular, we are interested in Banach sequence spaces $(X(I), \|\cdot\|_X)$ of $\mathbb{C}$-valued sequences $(x_i)_{i\in I}$ which are defined over arbitrarily given (index) sets I. For each m, $n \in \mathbb{N}$ the following two finite index sets will be of special interest:

$$\mathcal{M}(m,n) = \big\{\mathbf{i} = (i_1, \ldots, i_m); 1 \le i_k \le n\big\} = \{1, \ldots, n\}^m,$$
$$\mathcal{J}(m,n) = \big\{\mathbf{j} = (j_1, \ldots, j_m) \in \mathcal{M}(m,n); 1 \le j_1 \le j_2 \le \cdots \le j_m \le n\big\}.$$

For indices $\mathbf{i}, \mathbf{j} \in \mathcal{M}(m,n)$, we write $\mathbf{i} \sim \mathbf{j}$ whenever there exists a permutation σ of $\{1, \ldots, m\}$ such that $(i_1, \ldots, i_m) = (j_{\sigma(1)}, \ldots, j_{\sigma(m)})$. Obviously, $\sim$ defines an equivalence relation on $\mathcal{M}(m,n)$. We denote by $[\mathbf{i}]$ the equivalence class of $\mathbf{i}$, and we put $|[\mathbf{i}]| := \operatorname{card}[\mathbf{i}]$. For every finite subset $\{x_1^*, \dots, x_n^*\}$ in the dual X^* of a Banach space X and $\mathbf{j} \in \mathcal{M}(m,n)$, we define the function $x_{\mathbf{j}}^* : X \to \mathbb{C}$ by

$$x_{\mathbf{j}}^*(x) = x_{j_1}^*(x) \cdots x_{j_m}^*(x), \quad x \in X.$$

We note that $\mathbf{i} \sim \mathbf{j}$ implies that $x_{\mathbf{i}}^* = x_{\mathbf{j}}^*$. Given an n-dimensional Banach space $X = (\mathbb{C}^n, \|\cdot\|)$, we write $\{e_1, \dots, e_n\}$ for its canonical basis, and we write $\{e_1^*, \dots, e_n^*\}$ for the corresponding dual basis in X^*.

Our study of Kahane–Salem–Zygmund type estimates motivates the following definition: Let $\psi\colon \mathbb{N}_{>1} \times \mathbb{N} \to [1,\infty)$ and $\phi\colon \mathbb{N} \to [1,\infty)$ be given increasing functions. Then a Banach sequence space E (modeled on $\mathbb{N}$) is said to be (ψ,ϕ)-admissible provided there is a constant $C>0$ such that, for every choice of scalars $(c_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ with $m>1$ and $n\geq 1$, and functionals $x_1^*, \dots, x_n^* \in X^*$ on an n-dimensional Banach space X, there exists a choice of signs $\varepsilon_{\mathbf{j}} = \pm 1$, $\mathbf{j} \in \mathcal{J}(m,n)$ such that

$$\begin{aligned}\sup_{z\in B_X}\Big|\sum_{\mathbf{j}\in\mathcal{J}(m,n)} \varepsilon_{\mathbf{j}} c_{\mathbf{j}} x_{\mathbf{j}}^*(z)\Big| \\ \leq C\psi(m,n) \sup_{\mathbf{j}\in\mathcal{J}(m,n)} \frac{|c_{\mathbf{j}}|}{\phi(|[\mathbf{j}]|)} \sup_{z\in B_X}\Big\|\sum_{k=1}^n x_k^*(z) e_k\Big\|_E^m.\end{aligned}$$

The following observation motivates our definition.

Remark 2.1. Let E be a (ψ,ϕ)-admissible Banach sequence space. Then, for every Banach space $X = (\mathbb{C}^n, \|\cdot\|)$ and every choice $(c_\alpha)_{|\alpha|=m}$ of scalars, there exist signs $(\varepsilon_\alpha)_{|\alpha|=m}$ such that

$$\sup_{z\in B_X}\Big|\sum_{|\alpha|=m} \varepsilon_\alpha c_\alpha z^\alpha\Big| \leq C\psi(m,n) \sup_{|\alpha|=m} \frac{|c_\alpha|}{\phi(\frac{m!}{\alpha!})} \sup_{z\in B_X}\Big\|\sum_{k=1}^n z_k e_k\Big\|_E^m,$$

where $C = C(\psi,\phi,E)$.

Since $|[\mathbf{j}]| = \frac{m!}{\alpha!}$ for each $\mathbf{j} \in \mathcal{J}(m,n)$, the preceding estimate is an immediate consequence of the definition of (ψ,ϕ)-admissability of E applied to the dual basis $e_1^*, \dots, e_n^* \in X^*$ of the canonical basis $\{e_1, \dots, e_n\}$ of X.

Let us give our first example.

Example 2.2. Let $\psi(m,n) = 1$ and $\phi(n) = n$ for $m>1, n\geq 1$. Then $E = \ell_1$ is (ψ,ϕ)-admissible.

Proof. Take a family $(c_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ of scalars, and extend it to the index set $\mathcal{M}(m,n)$ by defining $c_{\mathbf{i}} = c_{\mathbf{j}}$ whenever $\mathbf{i} \sim \mathbf{j}$. Then we clearly have

$$\begin{aligned}\sup_{z\in B_X}\Big|\sum_{\mathbf{j}\in\mathcal{J}(m,n)} \varepsilon_{\mathbf{j}} c_{\mathbf{j}} x_{\mathbf{j}}^*(z)\Big| &\leq \sup_{z\in B_X} \sum_{\mathbf{j}\in\mathcal{J}(m,n)} \big|c_{\mathbf{j}} x_{\mathbf{j}}^*(z)\big| \\ &= \sup_{z\in B_X} \sum_{\mathbf{j}\in\mathcal{M}(m,n)} \frac{|c_{\mathbf{j}}|}{|[\mathbf{j}]|} \big|x_{\mathbf{j}}^*(z)\big|\end{aligned}$$

$$\leq \sup_{\mathbf{j}\in\mathcal{M}(m,n)} \frac{|c_{\mathbf{j}}|}{|[\mathbf{j}]|} \sup_{z\in B_X} \sum_{\mathbf{j}\in\mathcal{M}(m,n)} \big|x_{\mathbf{j}}^*(z)\big|$$

$$= \sup_{\mathbf{j}\in\mathcal{M}(m,n)} \frac{|c_{\mathbf{j}}|}{|[\mathbf{j}]|} \sup_{z\in B_X} \Big(\sum_{k=1}^{n} \big|x_k^*(z)\big|\Big)^m,$$

which is what we aimed for. □

Inspired by the work of Boas from [2], Bayart in [1, Theorem 5.1] proved a strong extension of the preceding example. The following theorem reformulates his result in terms of our notion of (ψ,ϕ)-admissability of sequence spaces E.

Theorem 2.3. *Define for $1 < p \leq 2$ the functions $\psi(m,n) = (\log m)^{1/q} n^{1/q}$ and $\phi(n) = n^{1/p}$ where $m > 1, n \geq 1$, and $1/p + 1/q = 1$. Then $E = \ell_p$ is (ψ,ϕ)-admissible.*

Note that Theorem 2.3 includes (1.1) as a simple corollary.

Our main aim is to continue this work within the framework of Orlicz spaces. We will describe some class of Orlicz sequence spaces $E = \ell_\varphi$ and two associated functions ψ and ϕ such that E is (ψ,ϕ)-admissible. Before we formulate our result we recall some of the basics of the theory of Orlicz function spaces and more generally Calderón–Lozanovskii spaces. By $\mathcal{U}$ we denote the set of all positive, concave, and positively homogeneous continuous functions $\psi\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ for which $\psi(0,0) = 0$. Let $(\Omega,\mu) = (\Omega,\Sigma,\mu)$ be a σ-finite and complete measure space, and let (X_0,X_1) be a couple of Banach lattices on this measure space. For a given function $\psi \in \mathcal{U}$, the Calderón–Lozanovskii space $\psi(X_0,X_1)$ consists of all $f \in L^0(\mu)$ such that $|f| \leq \lambda\psi(|f_0|,|f_1|)$ μ-a.e. for some $f_j \in X_j$ with $\|f_j\|_{X_j} \leq 1$, $j = 0,1$. Equipped with the norm

$$\|f\|_{\psi(X_0,X_1)} = \inf\big\{\lambda > 0; |f| \leq \lambda\psi\big(|f_0|,|f_1|\big), \|f_0\|_{X_0} \leq 1, \|f_1\|_{X_1} \leq 1\big\},$$

the space $\psi(X_0,X_1)$ forms a Banach lattice.

Calderón–Lozanovskii spaces are closely related to Orlicz spaces. A function $\varphi\colon [0,\infty) \to [0,\infty)$ is an Orlicz function whenever it is increasing, convex, and left-continuous with $\varphi(0) = 0$. Let φ^{-1} be the right-continuous inverse of φ. Define ψ by $\psi(s,t) = t\varphi^{-1}(s/t)$ for $s \geq 0$, $t > 0$, and $\psi(0,0) = 0$. Then $\psi \in \mathcal{U}$, and, for any measure space (Ω,μ), the space $\psi(L_1,L_\infty)$ coincides isometrically with the Orlicz space

$$L_\varphi := \big\{f \in L^0(\mu); \varphi\big(|f|/\lambda\big) \in L_1(\mu)\big\},$$

where the norm on L_φ is given by

$$\|f\|_{L_\varphi} = \inf\Big\{\lambda > 0; \int_\Omega \varphi\big(|f|/\lambda\big)\, d\mu \leq 1\Big\}.$$

If on a set Ω we consider $\Sigma = 2^\Omega$ and the counting measure μ, then we write $\ell_\varphi(\Omega)$ instead of L_φ, and we write ℓ_φ for short whenever $\Omega = \mathbb{N}$.

For every Orlicz function $\varphi\colon [0,\infty) \to [0,\infty)$, we define the associated conjugate function φ_* by the formula

$$\varphi_*(t) = \sup\big\{st - \varphi(s); s \geq 0\big\}, \quad t \geq 0.$$

The function φ_* may take values 0 and ∞ on some intervals. Clearly, φ_* fulfils the Young inequality:

$$st \leq \varphi(s) + \varphi_*(t), \quad s, t \geq 0.$$

An Orlicz function φ is said to be an N-function if

$$\lim_{t\to 0+} \frac{\varphi(t)}{t} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

Note that φ_* takes value zero only at zero if and only if the first of the preceding conditions holds, and it is finite valued if and only if the second condition holds. Moreover, if φ is an N-function, then this is also true for φ_*, and the following estimates hold (see [7], [10]):

$$s \leq \varphi^{-1}(s)\varphi_*^{-1}(s) \leq 2s, \quad s \geq 0. \tag{2.1}$$

For more details on the theory of Orlicz spaces, we refer to [7], [8], and [10].

Before we state our main result, it should be pointed out that Example 2.2 explains why throughout the rest of this paper we consider only Orlicz functions φ which satisfy $\lim_{t\to 0+} \varphi(t)/t = 0$. The reason is that $\lim_{t\to 0+} \varphi(t)/t > 0$ is equivalent to $\ell_\varphi = \ell_1$ (up to equivalence of norms), and hence this case in fact is covered by Example 2.2.

Theorem 2.4. *Let φ be an N-function such that $t \mapsto \varphi(\sqrt{t})$ is equivalent to a concave function, $\varphi(st) \leq \varphi(s)\varphi(t)$ whenever $0 < s, t \leq 1$, and, for some $K \geq 1$, $\varphi(st) \leq K\varphi(s)\varphi(t)$ whenever $0 < s \leq 1 \leq t < \infty$. Then there exists a constant $C > 0$ such that, for any n-dimensional Banach space X, any finite set of functionals $x_1^*, \ldots, x_n^* \in X^*$, and any family $(c_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ of scalars, there exists a choice of signs $(\varepsilon_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ for which*

$$\begin{aligned}\sup_{z\in B_X}\Big|&\sum_{\mathbf{j}\in\mathcal{J}(m,n)} \varepsilon_{\mathbf{j}} c_{\mathbf{j}} x_{\mathbf{j}}^*(z)\Big| \\ &\leq C\varphi_*^{-1}(n\log m) \sup_{\mathbf{j}\in\mathcal{J}(m,n)} \frac{|c_{\mathbf{j}}|}{\varphi^{-1}(|[\mathbf{j}]|)} \sup_{z\in B_X} \Big\|\sum_{k=1}^n x_k^*(z)e_k\Big\|_{\ell_\varphi}^m.\end{aligned}$$

Equivalently, ℓ_φ is (ψ,ϕ)-admissible, where $\psi(m,n) = \varphi_*^{-1}(n\log m)$ and $\phi(n) = \varphi^{-1}(n)$. Moreover, for all $m \geq 2$ and $n \geq 1$, we have

$$\frac{n\log m}{\varphi^{-1}(n\log m)} \leq \psi(m,n) \leq \frac{2n\log m}{\varphi^{-1}(n\log m)}.$$

We conclude this section with the remark that in the case $1 < p \leq 2$ and $\varphi(t) = t^p$, for all $t \geq 0$, we recover Bayart's theorem (our Theorem 2.3).

3. The proof

In the proof of the main Theorem 2.4, we use the following two lemmas. The first one is obvious, and so we omit its proof.

Lemma 3.1. *Let φ be an Orlicz function, and let $(\Omega_j, \mathcal{A}_j, \mu_j)$ be measure spaces for each $1 \le j \le m$. If $\varphi(st) \le \varphi(s)\varphi(t)$ for every $s, t > 0$, then the multiplication operator $\otimes$ defined on $L_\varphi(\mu_1) \times \cdots \times L_\varphi(\mu_m)$ by*

$$\otimes(f_1, \ldots, f_m)(\omega_1, \ldots, \omega_m) = f_1(\omega_1) \cdots f_m(\omega_m)$$

for every $f_j \in L_\varphi(\mu_j)$, $\omega_j \in \Omega_j$, and each $1 \le j \le m$ is a bounded contraction from $L_\varphi(\mu_1) \times \cdots \times L_\varphi(\mu_m)$ into $L_\varphi(\mu_1 \times \cdots \times \mu_m)$. If in addition all measure spaces are purely atomic with counting measures, then the above statement is also true provided that $\varphi(st) \le \varphi(s)\varphi(t)$ for all $s, t \in (0, 1]$.

The second lemma, which seems to be of independent interest, is our main technical tool. We note that, for $1 < p \le 2$ and $\varphi(t) = t^p$, $t \ge 0$, this result is well known (see, e.g., [11]) and is the key ingredient for the proof of Bayart's theorem (our Theorem 2.3).

Lemma 3.2. *Let φ be an N-function such that $t \mapsto \varphi(\sqrt{t})$ is equivalent to a concave function and $\varphi(t)\varphi(1/t) \ge c$ for all $t \ge 1$ and some $c > 0$. Let L_Φ be the Orlicz space of functions on the nonatomic probability space $(\Omega, \mathcal{A}, \mathbb{P})$ generated by the Orlicz function $\Phi(t) = \exp(\varphi_*(t)) - 1$, $t \ge 0$. Then there is a constant $C > 0$ such that, for every sequence (ε_i) of independent Bernoulli random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ and every $(x_i) \in \ell_\varphi$, we have*

$$\Big\| \sum_{i=1}^{\infty} \varepsilon_i x_i \Big\|_{L_\Phi} \le C \big\| (x_i) \big\|_{\ell_\varphi}.$$

Proof. Without loss of generality, we may assume that $t \mapsto \varphi(\sqrt{t})$ is concave on $\mathbb{R}_+$, and thus the function $t \mapsto \varphi(t)/t^2$ is nonincreasing. This implies that the function $t \mapsto \varphi^{-1}(t^2)/t$ is quasiconcave, and hence equivalent to a concave function, say, ρ:

$$\varphi^{-1}(t^2)/t \asymp \rho(t).$$

Put $\psi(s, t) := t\rho(s/t)$ for $s \ge 0$, $t > 0$, and $\psi(s, 0) = 0$ for $s \ge 0$ and $t = 0$. Clearly, $\psi \in \mathcal{U}$ and

$$\varphi^{-1}(t) \asymp \psi(t, \sqrt{t}).$$

Now take some sequence $\boldsymbol{\varepsilon} = (\varepsilon_i)$ of independent Bernoulli random variables on $(\Omega, \mathcal{A}, \mathbb{P})$. For every $(x_i) \in \ell_2$, the series

$$T_{\boldsymbol{\varepsilon}}(x_i) = \sum_{i=1}^{\infty} \varepsilon_i x_i$$

converges almost surely, and so $T_{\boldsymbol{\varepsilon}}$ defines a linear operator from ℓ_2 into the linear space $L^0(\mathbb{P})$ of all $\mathcal{A}$-measurable functions on Ω. Clearly, $T_\varepsilon \colon \ell_1 \to L_\infty(\mathbb{P})$ is bounded with norm 1. On the other hand, we know from [12, p. 342] that $T_\varepsilon \colon \ell_2 \to L_{\varphi_2}(\mathbb{P})$ is bounded with $\|T_\varepsilon\| \le \sqrt{8e}$, where $\varphi_2(t) = \exp(t^2) - 1$ for all $t \ge 0$. Altogether, $T_{\boldsymbol{\varepsilon}}$ can be viewed as a bounded linear operator from the Banach couple (ℓ_1, ℓ_2) into the Banach couple $(L_\infty, L_{\varphi_2})$. Applying the interpolation theorem for Calderón–Lozanovskii spaces (see [9]) we conclude that, for any $\psi \in \mathcal{U}$,

$$T_{\boldsymbol{\varepsilon}} \colon \psi(\ell_1, \ell_2) \to \psi(L_\infty, L_{\varphi_2})$$

with $\|T_{\boldsymbol{\varepsilon}}\| \le 2K_G\sqrt{8e}$, where $K_G > 1$ is the Grothendieck constant.

It is known that $\psi(L_p, L_q) = L_\Upsilon$ for any $1 \leq p, q < \infty$, where $\Upsilon^{-1}(t) = \psi(t^{1/p}, t^{1/q})$ (see again [9]). Since $\varphi^{-1}(t) \asymp \psi(t, \sqrt{t})$, we have up to equivalence of norms that

$$\psi(\ell_1, \ell_2) = \ell_\varphi.$$

On the other hand, it is easy to verify that $\psi(L_\infty, L_\phi) = L_\Psi$ for any Orlicz function ϕ, where Ψ is given by $\Psi^{-1}(t) = \psi(1, \phi^{-1}(t))$ for all $t \geq 0$. Hence

$$\psi(L_\infty, L_{\varphi_2}) = L_\Psi,$$

where $\Psi^{-1}(t) = \psi(1, \ln^{1/2}(1+t))$ for all $t \geq 0$.

Combining the above, we conclude that the operator $T_{\boldsymbol{\varepsilon}} \colon \ell_\varphi \to L_\Psi$ is bounded, where $C > 0$ is a universal constant. In other terms, for each sequence $(x_i) \in \ell_\varphi$, we have

$$\Big\| \sum_{i=1}^{\infty} \varepsilon_i x_i \Big\|_{L_\Psi} \leq C \big\| (x_i) \big\|_{\ell_\varphi},$$

and hence, to finish the proof, it is enough to show that $L_\Psi(\mathbb{P}) \hookrightarrow L_\Phi(\mathbb{P})$ is a continuous inclusion. To see this, we recall that, for the N-function φ, we have (by (2.1))

$$\frac{s}{\varphi^{-1}(s)} \leq \varphi_*^{-1}(s) \leq \frac{2s}{\varphi^{-1}(s)}, \quad s > 0.$$

We assume without loss of generality that $\varphi(1) = 1$. The left-hand side of the above inequality combined with our hypothesis $c \leq \varphi(s)\varphi(1/s)$ for all $s \geq 1$ yields (by $0 < c \leq 1$)

$$cs\varphi^{-1}(1/s) \leq \varphi_*^{-1}(s), \quad s \geq 1.$$

Since $\varphi^{-1}(t) \asymp \psi(t, \sqrt{t})$ and $\Psi^{-1}(t) = \psi(1, \ln^{1/2}(1+t))$ for all $t \geq 0$,

$$\Psi^{-1}(t) \asymp \ln(1+t)\varphi^{-1}\Big(\frac{1}{\ln(1+t)}\Big), \quad t \geq 0.$$

This equivalence combined with $\Phi^{-1}(t) = \varphi_*^{-1}(\ln(1+t))$ for all $t \geq 0$ gives $c\Psi^{-1}(t) \leq \Phi^{-1}(t)$ for large t ($t \geq e - 1$). Thus $\Phi(ct) \leq \Psi(t)$ for large t, and so the required continuous inclusion follows. $\square$

We are now ready to give the proof of Theorem 2.4.

Proof of Theorem 2.4. For a given sequence $(\varepsilon_{\mathbf{j}})_{\mathbf{j} \in \mathcal{J}(m,n)}$ of independent, identically distributed Bernoulli variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a set $\{x_1^*, \ldots, x_n^*\}$ of functionals on X^*, we define the random polynomials on X by

$$P(\omega, z) = \sum_{\mathbf{j} \in \mathcal{J}(m,n)} \varepsilon_{\mathbf{j}}(\omega) c_{\mathbf{j}} x_{\mathbf{j}}^*(z), \quad \omega \in \Omega, z \in X.$$

Let $\Phi(t) := \exp(\varphi_*(t)) - 1$ for all $t \geq 0$. Then, applying Lemma 3.2, we find that there exists a constant $C > 1$ such that, for all $z \in B_X$,

$$\big\| P(\cdot, z) \big\|_{L_\Phi} = \Big\| \sum_{\mathbf{j} \in \mathcal{J}(m,n)} \varepsilon_{\mathbf{j}}(\cdot) c_{\mathbf{j}} x_{\mathbf{j}}^*(z) \Big\|_{L_\Phi} \leq C \big\| \big(c_{\mathbf{j}} x_{\mathbf{j}}^*(z)\big) \big\|_{\ell_\varphi(\mathcal{J}(m,n))}.$$

Recall that by the assumption on φ there exists $K \geq 1$ such that $\varphi(st) \leq K\varphi(s)\varphi(t)$ for all $0 < s \leq 1 \leq t < \infty$. We claim that this implies the following estimate for every $z \in B_X$:

$$\big\|\big(c_{\mathbf{j}} x^*_{\mathbf{j}}(z)\big)\big\|_{\ell_\varphi(\mathcal{J}(m,n))} \leq K\Big(\sup_{\mathbf{j}\in\mathcal{J}(m,n)} \frac{|c_{\mathbf{j}}|}{\varphi^{-1}(|[\mathbf{j}]|)}\Big)\big\|\big(x^*_{\mathbf{j}}(z)\big)\big\|_{\ell_\varphi(\mathcal{M}(m,n))}.$$

Indeed, to see this inequality, we may assume without loss of generality that $\|(x^*_{\mathbf{j}}(z))\|_{\ell_\varphi(\mathcal{M}(m,n))} \leq 1$. Hence

$$\sum_{\mathbf{j}\in\mathcal{M}(m,n)} \varphi\big(\big|x^*_{\mathbf{j}}(z)\big|\big) \leq 1,$$

and so $|x^*_{\mathbf{j}}(z)| \leq \varphi^{-1}(1)$ for each $\mathbf{j} \in \mathcal{J}(m,n)$. Now, if we use our hypotheses on φ, then we see that $|x^*_{\mathbf{j}}(z)| \leq 1$ and

$$\varphi\big(\varphi^{-1}\big(\big|[\mathbf{j}]\big|\big)\big|x^*_{\mathbf{j}}(z)\big|\big) \leq K\big|[\mathbf{j}]\big|\varphi\big(\big|x^*_{\mathbf{j}}(z)\big|\big), \quad \mathbf{j} \in \mathcal{J}(m,n).$$

Put $\lambda := \sup_{\mathbf{j}\in\mathcal{J}(m,n)} \frac{|c_{\mathbf{j}}|}{\varphi^{-1}(|[\mathbf{j}]|)}$. Then $|c_{\mathbf{j}}| \leq \lambda\varphi^{-1}(|[\mathbf{j}]|)$ for each $\mathbf{j} \in \mathcal{J}(m,n)$. Since φ is convex and $K \geq 1$, $\varphi(s/K) \leq \varphi(s)/K$ for all $s > 0$. Thus the inequalities above imply

$$\begin{aligned}
\sum_{\mathbf{j}\in\mathcal{J}(m,n)} \varphi\Big(\frac{|c_{\mathbf{j}}||x^*_{\mathbf{j}}(z)|}{K\lambda}\Big) &= \sum_{\mathbf{j}\in\mathcal{J}(m,n)} \sum_{\mathbf{i}\in[\mathbf{j}]} \varphi\Big(\frac{|c_{\mathbf{j}}||x^*_{\mathbf{j}}(z)|}{K\lambda}\Big)\frac{1}{|[\mathbf{j}]|} \\
&\leq \frac{1}{K}\sum_{\mathbf{j}\in\mathcal{M}(m,n)} \varphi\big(\varphi^{-1}\big(\big|[\mathbf{j}]\big|\big)\big|x^*_{\mathbf{j}}(z)\big|\big)\frac{1}{|[\mathbf{j}]|} \\
&\leq \sum_{\mathbf{j}\in\mathcal{M}(m,n)} \varphi\big(\big|x^*_{\mathbf{j}}(z)\big|\big) \leq 1,
\end{aligned}$$

and so this finally proves the above claim. Together with Lemma 3.1 (applied for purely atomic measure spaces), this gives that, for each $z \in B_X$,

$$\big\|P(\cdot,z)\big\|_{L_\Phi} \leq C \sup_{\mathbf{j}\in\mathcal{J}(m,n)} \frac{|c_{\mathbf{j}}|}{\varphi^{-1}(|[\mathbf{j}]|)} \sup_{x\in B_X}\Big\|\sum_{k=1}^n x^*_k(x)e_k\Big\|^m_{\ell_\varphi} =: A. \tag{3.1}$$

We fix $\omega \in \Omega$, and define the m-homogeneous polynomial $P_\omega := P(\omega,\cdot) : X \to \mathbb{C}$. Then there is a unique symmetric m-linear form $\check{P}_\omega$ on X for which $P_\omega(z) = \check{P}_\omega(z,\ldots,z)$ for all $z \in X$. By a polarization estimate of Harris [5] we know that

$$\sup_{v,\zeta\in B_X}\big|\check{P}_\omega(v,\zeta,\ldots,\zeta)\big| \leq \Big(\frac{m}{m-1}\Big)^{m-1}\sup_{x\in B_X}\big|P_\omega(x)\big|,$$

and hence, for all $z,u \in B_X$,

$$\begin{aligned}
\big|P(\omega,z) - P(\omega,u)\big| &\leq m\|z-u\|_X \sup_{v,\zeta\in B_X}\big|\check{P}_\omega(v,\zeta,\ldots,\zeta)\big| \\
&\leq m\Big(\frac{m}{m-1}\Big)^{m-1}\|z-u\|_X \sup_{x\in B_X}\big|P(\omega,x)\big|
\end{aligned}$$

$$\begin{aligned}
&\le m \sup_{t\ge 2}\Big(\frac{t}{t-1}\Big)^{t-1}\|z-u\|_X \sup_{x\in B_X}\big|P(\omega,x)\big| \\
&\le 3m\|z-u\|_X \sup_{x\in B_X}\big|P(\omega,x)\big|.
\end{aligned}$$

Since X is of dimension n, there exists a finite subset F of B_X (independent of ω) with $\operatorname{card}(F) \le (1+12m)^{2n}$ and such that, for every $z \in B_X$, there exists $u \in F$ with $\|z-u\|_X \le 1/(6m)$. Combining this with

$$\begin{aligned}
\big|P(\omega,z)\big| &\le \big|P(\omega,z)-P(\omega,u)\big| + \big|P(\omega,u)\big| \\
&\le \frac{1}{2}\sup_{x\in B_X}\big|P(\omega,x)\big| + \sup_{x\in F}\big|P(\omega,x)\big|,
\end{aligned}$$

we obtain

$$\sup_{x\in B_X}\big|P(\omega,x)\big| \le 2\sup_{x\in F}\big|P(\omega,x)\big|, \quad \omega\in\Omega. \tag{3.2}$$

From the estimate (3.1) we have that $\|P(\cdot,z)/A\|_{L_\Phi} \le 1$ for every $z \in B_X$. Thus it follows by Chebyshev's inequality that, for every $R > 0$,

$$\mathbb{P}\big(\big\{\omega\in\Omega; \big|P(\omega,z)\big| > R\big\}\big) \le \frac{1}{\Phi(\frac{R}{A})}, \quad z \in B_X,$$

and so we get

$$\begin{aligned}
\mathbb{P}\big(\big\{\omega\in\Omega; \sup_{x\in F}\big|P(\omega,z)\big| > R\big\}\big) &= \mathbb{P}\Big(\bigcup_{x\in F}\big\{\omega\in\Omega; \big|P(\omega,x)\big| > R\big\}\Big) \\
&\le \sum_{x\in F}\mathbb{P}\big(\big\{\omega\in\Omega; \big|P(\omega,x)\big| > R\big\}\big) \\
&\le \frac{\operatorname{card}(F)}{\Phi(\frac{R}{A})} \le \frac{(1+12m)^{2n}}{\Phi(\frac{R}{A})}.
\end{aligned}$$

Applying inequality (3.2) yields

$$\mathbb{P}\big(\big\{\omega\in\Omega; \sup_{x\in B_X}\big|P(\omega,x)\big| > 2R\big\}\big) \le \frac{(1+12m)^{2n}}{\Phi(\frac{R}{A})}.$$

We now observe that, for $R = \delta\varphi_*^{-1}(n\log m)A$ with $\delta \ge 1$,

$$\frac{(1+12m)^{2n}}{\Phi(\frac{R}{A})} \le \frac{(1+12m)^{2n}}{\exp(\delta n\log m)-1}.$$

This inequality allows us to find $\delta > 1$ independent of $n, m \in \mathbb{N}$ such that

$$\frac{(1+12m)^{2n}}{\Phi(\frac{R}{A})} < \frac{1}{2}.$$

Summarizing, we obtain $\mathbb{P}(\{\omega\in\Omega; \sup_{x\in B_X}|P(\omega,z)| > 2R\}) < 1/2$, and thus

$$\mathbb{P}\big(\big\{\omega\in\Omega; \sup_{x\in B_X}\big|P(\omega,z)\big| \le 2R\big\}\big) \ge \frac{1}{2}.$$

In consequence, we conclude that there exists $\omega \in \Omega$ such that

$$\sup_{z\in B_X}\big|P(\omega,z)\big| \le 2\delta C\varphi_*^{-1}(n\log m)\sup_{\mathbf{j}\in\mathcal{J}(m,n)}\frac{|c_{\mathbf{j}}|}{\varphi^{-1}(|[\mathbf{j}]|)}\sup_{z\in B_X}\Big\|\sum_{k=1}^{n}x_k^*(z)e_k\Big\|_{\ell_\varphi}^m,$$

and this completes the proof of the required inequality. To get the equivalent statement, it is enough to apply the inequalities from (2.1). □

4. Applications

Inspired by [1], we present some applications of the main result. As usual, we denote by $\mathcal{P}(^mX)$ the Banach space of all m-homogenous scalar-valued polynomials P on a Banach space X equipped with the norm

$$\|P\|_{\mathcal{P}(^mX)} = \sup\big\{\big|P(x)\big|; \|x\|\le 1\big\}.$$

Given an n-dimensional Banach space $X=(\mathbb{C}^n,\|\cdot\|)$, we exhibit some lower estimates of the unconditional basis constant of the monomials z^α, $\alpha\in\mathbb{N}_0^n$ denoted by

$$\chi_{\mathrm{mon}}\big(\mathcal{P}(^mX)\big).$$

We recall that a basis (x_n) of a Banach space X is said to be unconditional if there exists $C\ge 1$ such that

$$\Big\|\sum_{k=1}^{n}\theta_k\lambda_k x_k\Big\| \le C\Big\|\sum_{k=1}^{n}\lambda_k x_k\Big\|$$

for each $n\in\mathbb{N}$ and all $\lambda_1,\dots,\lambda_n,\theta_1,\dots,\theta_n\in\mathbb{C}$ with $|\theta_k|=1$; the best of these constants C is the unconditional basis constant of (x_n). For lower estimates of $\chi_{\mathrm{mon}}(\mathcal{P}(^mX))$, we again refer to [1], where former results of [3] and [4] were improved.

A simple consequence of our definitions gives the following abstract lower bound.

Lemma 4.1. *Let E be a (ψ,ϕ)-admissible Banach sequence space such that $(k/\phi(k))_k$ is nondecreasing. Then there is some constant $C>0$ such that, for each Banach space $X=(\mathbb{C}^n,\|\cdot\|)$ and each $m\ge 2$, we have*

$$\frac{C\phi(m!)}{\psi(m,n)m!}\Big(\frac{\sup_{z\in B_X}\sum_{k=1}^n|z_k|}{\sup_{z\in B_X}\|\sum_{k=1}^n z_k e_k\|_E}\Big)^m \le \chi_{\mathrm{mon}}\big(\mathcal{P}(^mX)\big).$$

Proof. It follows from the definition of $\chi_{\mathrm{mon}}(\mathcal{P}(^mX))$ that, for any choice of signs $\varepsilon_\alpha=\pm1, |\alpha|=m$, we have

$$\begin{aligned}\Big(\sup_{z\in B_X}\sum_{k=1}^{n}|z_k|\Big)^m &= \sup_{z\in B_X}\Big|\sum_{|\alpha|=m}\frac{m!}{\alpha!}z^\alpha\Big| = \sup_{z\in B_X}\Big|\sum_{|\alpha|=m}\frac{m!}{\alpha!}\varepsilon_\alpha\varepsilon_\alpha z^\alpha\Big|\\ &\le \chi_{\mathrm{mon}}\big(\mathcal{P}(^mX)\big)\sup_{z\in B_X}\Big|\sum_{|\alpha|=m}\frac{m!}{\alpha!}\varepsilon_\alpha z^\alpha\Big|.\end{aligned}$$

By Remark 2.1 we can find signs $(\varepsilon_\alpha)_{|\alpha|=m}$ such that

$$\sup_{z\in B_X}\Big|\sum_{|\alpha|=m}\frac{m!}{\alpha!}\varepsilon_\alpha z^\alpha\Big| \le C\psi(m,n)\sup_{|\alpha|=m}\frac{m!/\alpha!}{\phi(m!/\alpha!)}\sup_{z\in B_X}\Big\|\sum_{k=1}^n z_k e_k\Big\|_E^m.$$

Combining the above two inequalities with our hypothesis on ϕ yields the desired estimate. □

Now we are ready to give a lower bound of $\chi_{\mathrm{mon}}(\mathcal{P}(^mX))$ on the scale of Orlicz spaces ℓ_φ extending Bayart's theorem [1, Theorem 5.1], which is on the scale of ℓ_p-spaces for $p\in(1,2]$.

Corollary 4.2. *Assume that an N-function φ satisfies the conditions of Theorem 2.4. Then there exists a constant C such that, for every Banach space $X=(\mathbb{C}^n,\|\cdot\|)$ and each $m\ge 2$, we have*

$$\frac{C}{\varphi_*^{-1}(n\log m)\varphi_*^{-1}(m!)}\Big(\frac{\sup_{z\in B_X}\sum_{k=1}^n|z_k|}{\sup_{z\in B_X}\|\sum_{k=1}^n z_k e_k\|_{\ell_\varphi}}\Big)^m \le \chi_{\mathrm{mon}}\big(\mathcal{P}(^mX)\big).$$

Proof. From Theorem 2.4 it follows that ℓ_φ is (ψ,ϕ)-admissible with $\psi(m,n)=\varphi_*^{-1}(m,n)$ and $\phi(n)=\varphi^{-1}(n)$ for each $m\ge 2$, $n\ge 1$. Since

$$\frac{1}{2}\varphi_*^{-1}(t)\le t/\varphi^{-1}(t)\le\varphi_*^{-1}(t),\quad t>0,$$

the estimate follows immediately from Corollary 4.1. □

As an application we apply the preceding result to the n-dimensional Orlicz space ℓ_ϕ^n. Note that by the Köthe duality between Orlicz spaces we have

$$\sup_{z\in B_{\ell_\varphi^n}}\sum_{k=1}^n|z_k| = \Big\|\sum_{k=1}^n e_k\Big\|_{(\ell_\varphi)'} = n\varphi^{-1}(1/n),$$

and hence Corollary 4.2 gives the following lower bound for $\chi_{\mathrm{mon}}(\mathcal{P}(^m\ell_\varphi^n))$ in terms of the original Orlicz function φ instead of its conjugate function φ_*.

Corollary 4.3. *Assume that an N-function φ satisfies the conditions of Theorem 2.4. Then there exists a constant C such that, for each $n\ge 1$, $m\ge 2$, we have*

$$\chi_{\mathrm{mon}}\big(\mathcal{P}(^m\ell_\varphi^n)\big)\ge C\frac{\varphi^{-1}(n\log m)}{n\log m}\frac{\varphi^{-1}(m!)}{m!}\big(n\varphi^{-1}(1/n)\big)^m.$$

Finally, we indicate how these estimates can be applied to get upper bounds for multidimensional Bohr radii of certain Reinhardt domains in several complex variables. We recall that if the canonical basis $(e_k)_{k=1}^n$ forms a normalized 1-unconditional basis of the Banach space $X=(\mathbb{C}^n,\|\cdot\|)$, then the Bohr radius $K(B_X)$ is defined to be the supremum over all $r\in[0,1]$ such that, for each holomorphic function f on B_X, we have

$$\sum_\alpha\Big|\frac{\partial^\alpha f(0)}{\alpha!}z^\alpha\Big|\le\sup_{z\in B_X}|f(z)|.$$

We note that in [1] the following upper bound for $K(B_X)$ was given:

$$K(B_X) \leq C(\log n)^{1-1/p} \frac{\sup_{\|z\|_X \leq 1} (\sum_{k=1}^n |z_k|^p)^{1/p}}{\sup_{\|z\|_X \leq 1} \sum_{k=1}^n |z_k|}, \quad 1 < p \leq 2. \tag{4.1}$$

The case $p = 2$ had been shown earlier in [3] (see also [4]). The following link from [3, Theorem 2.2] shows how to use upper (lower) estimates for unconditional basis constants $\chi_{\mathrm{mon}}(\mathcal{P}(^m X))$ in order to get lower (upper) estimates for the Bohr radius $K(B_X)$:

$$\frac{1}{3} \frac{1}{\sup_m \chi_{\mathrm{mon}} \mathcal{P}(^m X)^{1/m}} \leq K(B_X) \leq \min\Big(\frac{1}{3}, \frac{1}{\sup_m \chi_{\mathrm{mon}} \mathcal{P}(^m X)^{1/m}}\Big).$$

Combining this inequality with the Corollaries 4.2 and 4.3, we obtain an extension of (4.1) to the scale of Orlicz spaces. We leave the details to the reader.

Acknowledgments. The authors are thankful to Gord Sinnamon for his comments which improved the article.

Mastyło's work was partially supported by the Foundation for Polish Science (FNP).

References

1. F. Bayart, *Maximum modulus of random polynomials*, Q. J. Math. **63** (2012), 21–39. Zbl 1248.46033. MR2889180. DOI 10.1093/qmath/haq026. 336, 338, 344, 345, 346
2. H. Boas, *Majorant series*, J. Korean Math. Soc. **37** (2000), 321–337. Zbl 0965.32001. MR1775963. 336, 338
3. A. Defant, D. García, and M. Maestre, *Bohr's power series theorem and local Banach space theory*, J. Reine Angew. Math. **557** (2003), 173–197. Zbl 1031.46014. MR1978407. DOI 10.1515/crll.2003.030. 336, 344, 346
4. A. Defant, D. García, and M. Maestre, *Maximum moduli of unimodular polynomials in several variables*, J. Korean Math. Soc. **41** (2004), 209–230. Zbl 1050.32003. MR2048710. DOI 10.4134/JKMS.2004.41.1.209. 336, 344, 346
5. L. A. Harris, "Bounds on the derivatives of holomorphic functions of vectors" in *Analyse Fonctionnelle et Applications (Rio de Janeiro, 1972)*, Actualités Aci. Indust. **1367**, Hermann, Paris, 1975, 145–163. Zbl 0315.46040. MR0477773. 342
6. J. P. Kahane, *Some Random Series of Functions*, 2nd ed., Cambridge Stud. Adv. Math. **5**, Cambridge Univ. Press, Cambridge, 1985. Zbl 0805.60007. MR0833073. 335
7. M. Krasnoselskii and Y. Rutickii, *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen, 1961. Zbl 0095.09103. MR0126722. 339
8. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, II*, Ergeb. Math. Grenzgeb. **97**, Springer, Berlin, 1979. Zbl 0403.46022. MR0540367. 339
9. V. I. Ovchinnikov, *Interpolation theorems resulting from Grothendieck's inequality* (in Russian), Funktsional Anal. i Prilozhen. **10** (1976), 45–54; English translation in Funkctional Anal. Appl. **10** (1976), 287–294. Zbl 0353.46020. MR0430813. 340, 341
10. M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Pure Appl. Math. **146**, Marcel Dekker, New York, 1991. Zbl 0724.46032. MR1113700. DOI 10.1080/03601239109372748. 339
11. V. A. Rodin and E. M. Semyonov, *Rademacher series in symmetric spaces*, Anal. Math. **1** (1975), no. 3, 207–222. Zbl 0315.46031. MR0388068. 340
12. A. Zygmund, *Trigonometric Series, I, II*, 2nd ed., Cambridge Univ. Press, New York, 1959. Zbl 0085.05601. MR0107776. 340

[1]Institut of Mathematics, Carl von Ossietzky University, Postfach 2503, D-26111 Oldenburg, Germany.
E-mail address: andreas.defant@uni-oldenburg.de

[2]Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland.
E-mail address: mastylo@math.amu.edu.pl